Solution for Applied Econometric Time Series (Enders, 2014)

Lucca Simeoni Pavan *

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*Doutorando em Desenvolvimento Econômico, PPDGE/UFPR - warrenjax@gmail.com

Capítulo 1

Equações em diferença

1. a) Encontrando a solução de Jill:

$$\begin{aligned} y_t &= a_0 + a_1 y_{t-1} \\ &= a_0 + a_1(a_0 + a_1 y_{t-2}) \\ &= a_0 + a_0 a_1 + a_0 a_1^2 + \cdots + a_0 a_1^{t-1} + a_1^t y_0 \\ &= a_0 \sum_{i=0}^{t-1} a_1^i + a_1^t y_0. \end{aligned}$$

O primeiro termo da soma é uma progressão geométrica finita, então

$$a_0 \sum_{i=0}^{t-1} a_1^i = a_0 \frac{(1-a_1^t)}{(1-a_1)} = \frac{a_0}{(1-a_1)} - \frac{a_0 a_1^t}{(1-a_1)}.$$

Substituindo a P.G. finita na solução de Jill, temos

$$\begin{aligned} y_t &= \frac{a_0}{(1-a_1)} - \frac{a_0 a_1^t}{(1-a_1)} + a_1^t y_0 \\ &= \frac{a_0}{(1-a_1)} - a_1^t \left[y_0 - \frac{a_0}{(1-a_1)} \right]. \end{aligned}$$

Solução para Bill:
A função complementar é dada pela solução homogênea de $y_t = a_0 + a_1 y_{t-1}$.

$$\begin{aligned} y_t - a_1 y_{t-1} &= 0 \\ \text{se } \; y_t = Ab^t, \;\; y_{t-1} &= a_1 A b^{t-1} \\ \text{então } \; Ab^t - a_1 A b^{t-1} &= 0 \\ Ab^t(1 - a_1 b^{-1}) &= 0 \Rightarrow b = a_1 \end{aligned}$$

Com isso a função complementar se torna $y_t^c = A a_1^t$,
A solução particular pode ser $y_t^p = k$, então

$$\begin{aligned} k &= a_0 + a_1 k \\ k &= \frac{a_0}{1-a_1} \Rightarrow y_t^p = \frac{a_0}{1-a_1} \end{aligned}$$

então a solução geral é

$$y_t = y_t^p + y_t^c = \frac{a_0}{1-a_1} + Aa_1^t$$

Para encontrarmos a solução definida usamos a condição inicial y_0 em $t = 0$ na solução geral.

$$\begin{aligned} y_0 &= \frac{a_0}{1-a_1} + Aa_1^0 \\ &= \frac{a_0}{1-a_1} + A \Rightarrow A = y_0 - \frac{a_0}{1-a_1} \end{aligned}$$

Substituindo a constante arbitrária A na solução geral,

$$y_t = \frac{a_0}{1-a_1} + a_1^t \left[y_0 - \frac{a_0}{1-a_1} \right],$$

assim temos a igualdade entre as soluções de Jill e Bill.

b) Para Jill, se $a_1 = 1$, temos

$$\begin{aligned} y_t &= a_0 + y_{t-1} \\ &= a_0 + a_0 + y_{t-2} \\ &= a_0 + a_0 + \cdots + a_0 + y_0 \\ &= a_0 t + y_0 \end{aligned}$$

Método de Bill:

$$\begin{aligned} y_t^c = Ab^t \Rightarrow Ab^t &= Ab^{t-1} \\ b &= 1 \\ \Rightarrow y_t^c &= A \end{aligned}$$

$$\begin{aligned} y_t^p = kt \Rightarrow kt &= a_0 + k(t-1) \\ \Rightarrow a_0 &= k \\ \Rightarrow y_t^p = k + k(t-1) &= kt \end{aligned}$$

$$y_t = y_t^c + y_t^p = A + kt \text{ (solução geral)}$$

$$\begin{aligned} \text{em } \ t = 0, \ y_t &= y_0 \Rightarrow y_0 = A, \text{ então} \\ y_t &= y_0 + kt \end{aligned}$$

$$\begin{aligned} \text{para } t = 1, \ y_1 &= a_0 + y_0 \Rightarrow y_0 = y_1 - a_0 \\ \text{substituindo em } y_t &= y_0 + kt \text{ avaliada em } t = 1 \\ y_1 &= y_1 - a_0 + k \Rightarrow k = a_0 \\ y_t &= a_0 t + y_0 \text{ (solução definida)} \end{aligned}$$

2. a) solução homogênea

$$\begin{aligned}p_t^* - (1-\alpha)p_{t-1}^* &= 0\\ p_t^* &= Ab^t \Rightarrow Ab^t = (1-\alpha)Ab^{t-1}\\ &\Rightarrow b = (1-\alpha) \text{ e } p_t^* = A(1-\alpha)^t\end{aligned}$$

b) solução particular

$$\begin{aligned}p_t^* &= \alpha L p_t + (1-\alpha)L p_t^*\\ p_t^* &= \frac{\alpha L p_t}{1-(1-\alpha)L} \equiv \alpha p_t \sum_{i=0}^{\infty}(1-\alpha)^i L^{i+1}\end{aligned}$$

Somando a solução homogênea e particular

$$p_t^* = A(1-\alpha)^t + \alpha p_t \sum_{i=0}^{\infty}(1-\alpha)^i L^{i+1}$$

No período $t=0$, $p_t^* = p_0^*$, então

$$\begin{aligned}p_0^* &= A + \alpha p_0 \sum_{i=0}^{\infty}(1-\alpha)^i L^{i+1}\\ A &= p_0^* - \alpha p_0 \sum_{i=0}^{\infty}(1-\alpha)^i L^{i+1}\end{aligned}$$

No período inicial $p_0^* = p_0$, então a solução definida fica

$$\begin{aligned}p_t^* &= \left[p_0 - \alpha p_0 \sum_{i=0}^{\infty}(1-\alpha)^i L^{i+1}\right](1-\alpha)^t + \alpha p_t \sum_{i=0}^{\infty}(1-\alpha)^i L^{i+1}\\ p_t^* &= p_0(1-\alpha)^t + \alpha p_t \sum_{i=0}^{t-1}(1-\alpha)^i L^{i+1}\end{aligned}$$

Substituindo na equação em diferença original:
Antes temos,

$$p_{t-1}^* = p_0(1-\alpha)^{t-1} + \alpha p_t \sum_{i=0}^{t-1}(1-\alpha)^i L^{i+2}$$

Então

$$p_0(1-\alpha)^t + \alpha p_t \sum_{i=0}^{t-1}(1-\alpha)^i L^{i+1} = \alpha p_{t-1} + (1-\alpha)\Big[p_0(1-\alpha)^{t-1} + \alpha p_t \sum_{i=0}^{t-1}(1-\alpha)^i L^{i+2}\Big]$$
$$= \alpha p_{t-1} + (1-\alpha)p_0(1-\alpha)^{t-1} + \alpha p_t \sum_{i=0}^{t-1}(1-\alpha)^{i+1} L^{i+2}$$

tratando apenas o primeiro e o terceiro termo da expressão acima

$$\iff \alpha p_{t-1} + \alpha p_t \sum_{i=0}^{t-1}(1-\alpha)^{i+1}L^{i+2} = \alpha p_t L(1-\alpha)^0 + \alpha p_t \sum_{i=0}^{t-1}(1-\alpha)^{i+1}L^{i+2}$$
$$\equiv \alpha p_t \sum_{i=0}^{t-1}(1-\alpha)^i L^{i+1}$$
$$\Rightarrow p_0(1-\alpha)^t + \alpha p_t \sum_{i=0}^{t-1}(1-\alpha)^i L^{i+1} = p_0(1-\alpha)^t + \alpha p_t \sum_{i=0}^{t-1}(1-\alpha)^i L^{i+1}$$

Com isso os dois lados se tornam iguais, o que prova a veracidade da solução.

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3. a)

$$\begin{aligned} m_t &= m + \rho m_{t-1} + \varepsilon_t \\ \Rightarrow m_{t+1} &= m + \rho m_t + \varepsilon_{t+1} \\ m_{t+2} &= m + \rho m_{t+1} + \varepsilon_{t+2} \\ &\vdots \\ m_{t+n} &= m + \rho m_{t+n-1} + \varepsilon_{t+n} \end{aligned}$$

substituindo recursivamente o termo defasado

$$\begin{aligned} m_{t+1} &= m + \rho m_t + \varepsilon_{t+1} \\ m_{t+2} &= m + \rho(m + \rho m_t + \varepsilon_{t+1}) + \varepsilon_{t+2} = m + \rho m + \rho^2 m_t + \rho\varepsilon_{t+1} + \varepsilon_{t+2} \\ m_{t+3} &= m + \rho(m + \rho m + \rho^2 m_t + \rho\varepsilon_{t+1} + \varepsilon_{t+2}) + \varepsilon_{t+3} \\ &\equiv m + \rho m + \rho^2 m + \rho^3 m_t + \rho^2\varepsilon_{t+1} + \rho\varepsilon_{t+2} + \varepsilon_{t+3} \\ &\vdots \\ m_{t+n} &= \sum_{j=0}^{n-1}\rho^j m + \rho^n m_t + \sum_{i=0}^{n}\rho^{n-i}\varepsilon_{t+i} \end{aligned}$$

b)

$$\begin{aligned} E\Big[m_{t+n}\Big] &= E\Big[\sum_{j=0}^{n-1}\rho^j m + \rho^n m_t + \sum_{i=0}^{n}\rho^{n-i}\varepsilon_{t+i}\Big] \\ &= \sum_{j=0}^{n-1}\rho^j m + \rho^n m_t + E\Big[\sum_{i=0}^{n}\rho^{n-1}\varepsilon_{t+i}\Big] \\ &= \sum_{j=0}^{n-1}\rho^j m + \rho^n m_t + \sum_{i=0}^{n}\rho^{n-i}E\Big[\varepsilon_{t+i}\Big] \\ E\Big[m_{t+n}\Big] &= \sum_{j=0}^{n-1}\rho^j m + \rho^n m_t = \frac{1-\rho^{n-1}}{(1-\rho)}m + \rho^n m_t \end{aligned}$$

Como m_{t+n} depende somente de uma variável conhecida m_t e uma sequência de termos de erro $\{\varepsilon_1, \varepsilon_2, ...\varepsilon_{t+n}\}$ de média zero, um modelo univariado pode ser útil para prever a oferta monetária n períodos no futuro. Isto é possível estimando ρ por meio de técnicas univariadas de séries temporais.

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4. a) i.

$$y_t - 1.5y_{t-1} + 0.5y_{t-2} = 0$$
$$y_t = Ab^t \Rightarrow y_{t-1} = Ab^{t-1} \text{ e } y_{t-2} = Ab^{t-2}$$

$$Ab^t - 1.5Ab^{t-1} + 0.5Ab^{t-2} = 0$$
$$b^2 - 1.5b + 0.5 = 0$$

$$b_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}, \quad a_1 = -1.5, \quad a_2 = 0.5$$
$$\Rightarrow b_1 = \frac{1.5 + \sqrt{(-1.5)^2 - 4 \times 0.5}}{2} = 1$$
$$b_2 = \frac{1.5 - \sqrt{(-1.5)^2 - 4 \times 0.5}}{2} = 0.5$$

A solução homogênea fica $A_1 + A_2(0.5)^t$.

ii.

$$y_t - y_{t-2} = 0$$
$$y_t = Ab^t \Rightarrow y_{t-2} = Ab^{t-2}$$

$$Ab^t - Ab^{t-2} = 0, \Rightarrow b^2 - 1 = 0,$$
$$b_1 = 1 \text{ ou } b_2 = -1$$

iii.

$$y_t - 2y_{t-1} + y_{t-2} = 0$$
$$\Rightarrow Ab^t - 2Ab^{t-1} + Ab^{t-2} = 0$$
$$b^2 - 2b + 1 = 0$$
$$b_1 = b_2 = 1$$

como são raízes repetidas, a solução homogênea fica $A_1 + A_2 t$.

iv.

$$y_t - y_{t-1} - 0.25y_{t-2} + 0.25y_{t-3} = 0$$
$$\Rightarrow Ab^t - Ab^{t-1} - 0.25Ab^{t-2} + 0.25Ab^{t-3} = 0$$
$$[b^3 - b^2 - 0.25b + 0.25 = 0] \times 4$$
$$(2b)^2 b - (2b)^2 - 1b + 1 = 0$$
$$\equiv (b-1)(2b+1)(2b-1) = 0$$
$$\Rightarrow b_1 = 1, \quad b_2 = 0.5 \quad b_3 = -0.5$$

A solução homogênea fica $A_1 + A_2(0.5)^t + A_3(-0.5)^t$.

b) i.

$$\begin{aligned} y_t &= 1.5y_{t-1} - 0.5y_{t-2} + \varepsilon_t \\ \Rightarrow y_t &= 1.5Ly_t - 0.5L^2y_t + \varepsilon_t \\ y_t - 1.5Ly_t + 0.5L^2y_t &= \varepsilon_t \\ (1-L)(1-0.5L)y_t &= \varepsilon_t \\ y_t &= \frac{\varepsilon_t}{(1-L)(1-0.5L)} \end{aligned}$$

Embora a expressão $\varepsilon_t/(1-0.5L)$ seja convergente, a expressão $\varepsilon_t/(1-L)$ não é, portanto a solução retrospectiva é não convergente.

ii.

$$\begin{aligned} y_t &= y_{t-2} + \varepsilon_t \\ \Rightarrow y_t - L^2y_t &= \varepsilon_t \\ (1-L^2)y_t &= \varepsilon_t \\ (1-L)(1+L)y_t &= \varepsilon_t \\ y_t &= \frac{\varepsilon_t}{(1-L)(1+L)y_t} \end{aligned}$$

A expressão $\varepsilon_t/(1-L)$ não converge, portanto a solução retrospectiva é não convergente.

iii.

$$\begin{aligned} y_t &= 2y_{t-1} - y_{t-2} + \varepsilon_t \\ \Rightarrow y_t - 2Ly_t + L^2y_t &= \varepsilon_t \\ (1-2L+L^2)y_t &= \varepsilon_t \\ (1-L)(1-L)y_t &= \varepsilon_t \\ y_t &= \frac{\varepsilon_t}{(1-L)(1-L)} \end{aligned}$$

Portanto a solução não converge.

iv.

$$\begin{aligned} y_t &= y_{t-1} + 0.25y_{t-2} - 0.25y_{t-3} + \varepsilon_t \\ y_t - Ly_t - 0.25L^2y_t + 0.25L^3y_t &= \varepsilon_t \\ (1-L-0.25L^2+0.25L^3)y_t &= \varepsilon_t \\ (1-L)(1+0.5L)(1-0.5L)y_t &= \varepsilon_t \\ y_t &= \frac{\varepsilon_t}{(1-L)(1+0.5L)(1-0.5L)} \end{aligned}$$

que não converge devido à expressão $\varepsilon_t/(1-L)$.

c)

$$
\begin{aligned}
y_t &= 1.5y_{t-1} - 0.5y_{t-2} + \varepsilon_t \\
y_t - y_{t-1} &= 1.5y_{t-1} - y_{t-1} - 0.5y_{t-2} + \varepsilon_t \\
y_t - y_{t-1} &= 0.5y_{t-1} - 0.5y_{t-2} + \varepsilon_t \\
\Delta y_t &= 0.5\Delta y_{t-1} + \varepsilon_t
\end{aligned}
$$

$$
\begin{aligned}
\Delta y_t &= 0.5L\Delta y_t + \varepsilon_t \\
\Delta y_t - 0.5L\Delta y_t &= \varepsilon_t \\
(1 - 0.5L)\Delta y_t &= \varepsilon_t \\
\Delta y_t &= \frac{\varepsilon_t}{(1-0.5L)} \equiv \sum_{i=0}^{\infty}(0.5)^i \varepsilon_{t-i}
\end{aligned}
$$

d) ii.

$$
\begin{aligned}
y_t &= y_{t-2} + \varepsilon_t \\
y_t - y_{t-1} &= -y_{t-1} + y_{t-2} + \varepsilon_t \\
\Delta y_t &= -\Delta y_{t-1} + \varepsilon_t
\end{aligned}
$$

$$
\begin{aligned}
\Delta y_t &= -\Delta L y_t + \varepsilon_t \\
(1 + L)\Delta y_t &= \varepsilon_t \\
\Delta y_t &= \frac{\varepsilon_t}{(1+L)} \equiv \sum_{i=0}^{\infty}(-1)^i \varepsilon_{t-i}
\end{aligned}
$$

não converge, portanto não há solução.

iii.

$$
\begin{aligned}
y_t &= 2y_{t-1} - y_{t-2} + \varepsilon_t \\
y_t - y_{t-1} &= 2y_{t-1} - y_{t-1} - y_{t-2} + \varepsilon_t \\
\Delta y_t &= \Delta y_{t-1} + \varepsilon_t
\end{aligned}
$$

$$
\begin{aligned}
\Delta y_t &= \Delta L y_t + \varepsilon_t \\
(1 - L)\Delta y_t &= \varepsilon_t \\
\Delta y_t &= \frac{\varepsilon_t}{(1-L)} \equiv \sum_{i=0}^{\infty}(1)^i \varepsilon_{t-i}
\end{aligned}
$$

não converge, portanto não há solução.

iv.

$$\begin{aligned}
y_t &= y_{t-1} + 0.25y_{t-2} - 0.25y_{t-3} + \varepsilon_t \\
y_t - y_{t-1} &= 0.25y_{t-2} - 0.25y_{t-3} + \varepsilon_t \\
\Delta y_t &= 0.25\Delta y_{t-2} + \varepsilon_t
\end{aligned}$$

$$\begin{aligned}
\Delta y_t &= 0.25\Delta L^2 y_t + \varepsilon_t \\
(1 - 0.25L^2)\Delta y_t &= \varepsilon_t \\
\Delta y_t &= \frac{\varepsilon_t}{(1+0.5L)(1-0.5L)} = \frac{\varepsilon_t}{(1+0.5L)}\frac{1}{(1-0.5L)} \\
&\equiv \left[\sum_{i=0}^{\infty}(-0.5)^i \varepsilon_{t-i}\right]\left[\sum_{i=0}^{\infty}(0.5)^i(1)\right] = 2\sum_{i=0}^{\infty}(-0.5)^i\varepsilon_{t-i}
\end{aligned}$$

que converge, portanto esta solução existe.

e) Já foi realizado anteriormente.

f)

$$y_t = a_0 - y_{t-1} + \varepsilon_t$$

$$\begin{aligned}
y_0 &= y_0 \\
y_1 &= a_0 - y_0 + \varepsilon_1 \\
y_2 &= a_0 - y_1 + \varepsilon_2 = a_0 - (a_0 - y_0 + \varepsilon_1) + \varepsilon_2 = y_0 + \varepsilon_2 - \varepsilon_1 \\
y_3 &= a_0 - y_2 + \varepsilon_3 = a_0 - (y_0 + \varepsilon_2 - \varepsilon_1) + \varepsilon_3 = a_0 - y_0 + \varepsilon_3 - \varepsilon_2 + \varepsilon_1 \\
\Rightarrow y_t &= \frac{a_0 + \varepsilon_t}{1+L} \equiv \sum_{i=0}^{t-1}(-1)^i a_0 + \sum_{i=0}^{t-1}(-1)^i\varepsilon_{t-i}
\end{aligned}$$

não converge.

---

5. a) i.

$$\begin{aligned}
y_t &= 0.75y_{t-1} - 0.125y_{t-2} \\
\Rightarrow b^2 - 0.75b + 0.125 &= 0 \\
b_{1,2} &= \frac{-a_1 \pm \sqrt{d}}{2}, \;\; d = (a_1)^2 - 4a_2, \;\; a_1 = -0.75, \;\; a_2 = 0.125 \\
d &= (-0.75)^2 - 4 \times 0.125 = 0.0625 \\
b_1 &= \frac{0.75 + \sqrt{0.0625}}{2} = 0.5 \\
b_2 &= \frac{0.75 - \sqrt{0.0625}}{2} = 0.25
\end{aligned}$$

$$y_t = A_1 0.5^t + A_2 0.25^t$$

ii.

$$\begin{aligned}
y_t &= 1.5y_{t-1} - 0.75y_{t-2} \\
\Rightarrow b^2 - 1.5b + 0.75 &= 0 \\
b_{1,2} &= \frac{1.5 \pm \sqrt{(-1.5)^2 - 4 \times 0.75}}{2} \\
b_1 &= \frac{1.5 + \sqrt{-0.75}}{2} = 0.75 + i\sqrt{\frac{0.75}{4}} = 0.75 + i\sqrt{0.1875} \\
b_2 &= \frac{1.5 - \sqrt{-0.75}}{2} = 0.75 - i\sqrt{0.1875} \\
\\
y_t &= A_1(0.75 + i\sqrt{0.1875})^t + A_2(0.75 - i\sqrt{0.1875})^t
\end{aligned}$$

iii.

$$\begin{aligned}
y_t &= 1.8y_{t-1} - 0.81y_{t-2} \\
\Rightarrow b^2 - 1.8b + 0.81 &= 0 \\
b_{1,2} &= \frac{1.8 \pm \sqrt{(-1.8)^2 - 4 \times 0.81}}{2} \\
d &= 0 \\
b_1 &= b_2 = 0.9 \\
\\
y_t &= A_1(0.9)^t + A_2t(0.9)^t
\end{aligned}$$

iv.

$$\begin{aligned}
y_t &= 1.5y_{t-1} - 0.5625y_{t-2} \\
\Rightarrow b^2 - 1.5b + 0.5625 &= 0 \\
b_{1,2} &= \frac{1.5 \pm \sqrt{(-1.5)^2 - 4 \times 0.5625}}{2} \\
d &= 0 \\
b_1 &= b_2 = 0.75 \\
\\
y_t &= A_1(0.75)^t + A_2t(0.75)^t
\end{aligned}$$

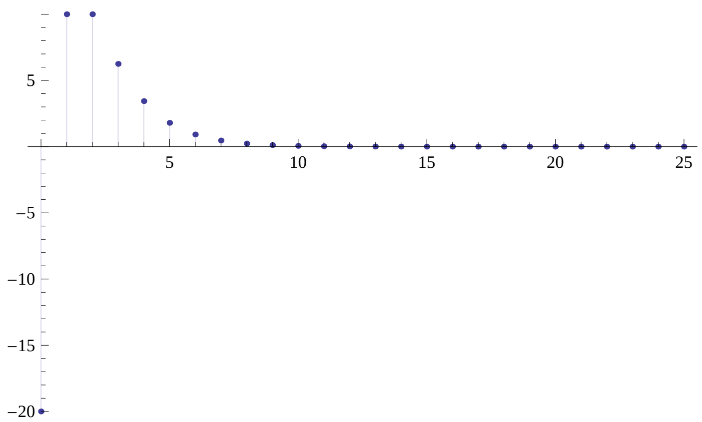
b) i.

$$
\begin{aligned}
y_t &= A_1 0.5^t + A_2 0.25^t, \quad y_1 = y_2 = 10 \\
\Rightarrow y_1 = 10 &= A_1 0.5^1 + A_2 0.25^1 \\
A_1 &= 20 - A_2 0.5
\end{aligned}
$$

$$
\begin{aligned}
y_2 = 10 &= (20 - A_2 0.5)(0.5)^2 + A_2(0.25)^2 \\
10 &= (20 - A_2 0.5)(0.25) + A_2(0.0.625) \\
10 &= (5 - A_2 0.125) + A_2(0.0.625) \\
5 &= -A_2 0.0625 \\
A_2 &= -80
\end{aligned}
$$

$$
\begin{aligned}
A_1 &= 20 - (-80)0.5 \\
A_1 &= 60
\end{aligned}
$$

$$y_t = 60(0.5)^t - 80(0.25)^t$$



ii.

$$y_t = A_1(0.75 + i\sqrt{0.1875})^t + A_2(0.75 - i\sqrt{0.1875})^t$$

$$(h \pm iv)^t = R^t(\cos(\theta t) \pm i\,\text{sen}(\theta t))$$

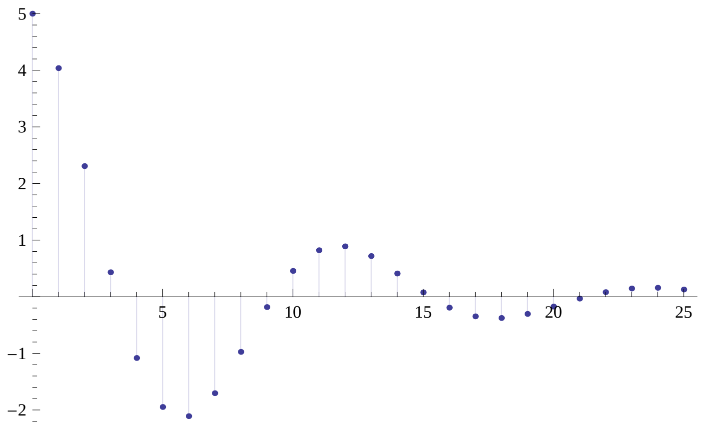
$$R = \sqrt{0.75}, \quad \cos(\theta) = \frac{1.5}{2\sqrt{0.75}} = \sqrt{0.75}, \quad \text{sen}(\theta) = \sqrt{1 - \frac{(-1.5)^2}{4(0.75)}} = 0.5 \to \theta = \frac{\pi}{6}$$

$$
\begin{aligned}
\Rightarrow y_t &= \sqrt{0.75}^t[A_1\{\cos(\theta t) + i\,\text{sen}(\theta t)\} + A_2\{\cos(\theta t) - i\,\text{sen}(\theta t)\}] \\
&= \sqrt{0.75}^t\{[A_1 + A_2]\cos(\theta t) + [A_1 - A_2]i\,\text{sen}(\theta t)\} \\
y_t &= \sqrt{0.75}^t\{A_5\cos(\frac{\pi}{6}t) + A_6\text{sen}(\frac{\pi}{6}t)\} \text{ (forma polar)}
\end{aligned}
$$

$$
\begin{aligned}
y_1 = y_2 = 10 \Rightarrow& \\
10 &= \sqrt{0.75}\{A_5 \cos(\frac{\pi}{6}) + A_6 \operatorname{sen}(\frac{\pi}{6})\} \\
\frac{10}{\sqrt{0.75}} &= A_5\sqrt{0.75} + A_6 0.5 \\
A_6 &= \frac{20}{\sqrt{0.75}} - A_5 2\sqrt{0.75}
\end{aligned}
$$

$$
\begin{aligned}
10 &= \sqrt{0.75}^2\{A_5 \cos(2\frac{\pi}{6}) + A_6 \operatorname{sen}(2\frac{\pi}{6})\} \\
10 &= 0.75\{A_5 \cos(\frac{\pi}{3}) + (\frac{20}{\sqrt{0.75}} - A_5 2\sqrt{0.75}) \operatorname{sen}(\frac{\pi}{3})\} \\
10 &= 0.75\{A_5\, 0.5 + (\frac{20}{\sqrt{0.75}} - A_5 2\sqrt{0.75})\, \sqrt{0.75}\} \\
15 &= A_5 0.5 + 20 - A_5 1.5 \Rightarrow A_5 = 5 \text{ e } A_6 = \frac{2\sqrt{3}}{3}
\end{aligned}
$$

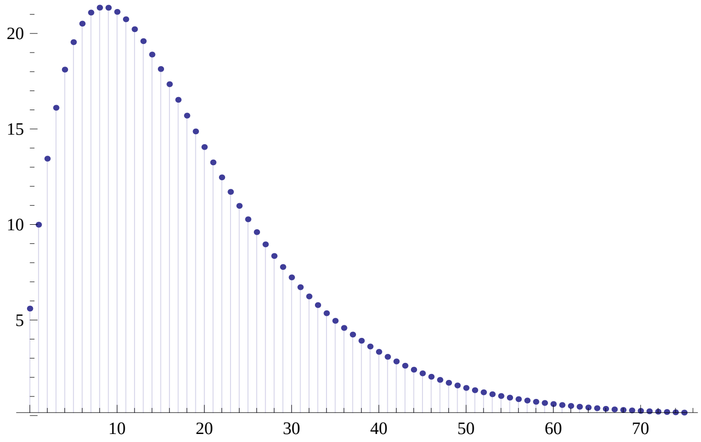
$$
y_t = \sqrt{0.75}^t\{5\cos(\frac{\pi}{6}t) + \frac{2\sqrt{3}}{3}\operatorname{sen}(\frac{\pi}{6}t)\}
$$



iii.

$$
\begin{aligned}
y_t &= A_1(0.9)^t + A_2 t(0.9)^t \\
y_1 \Rightarrow 10 &= A_1(0.9) + A_2(0.9) \\
A_1 &= \frac{10}{0.9} - A_2 \\
y_2 \Rightarrow 10 &= A_1(0.9)^2 + A_2 2(0.9)^2 \\
10 &= (\frac{10}{0.9} - A_2)^2 + A_2 1.62 = \frac{100}{0.81} - 2A_2\frac{10}{0.9} + A_2 1.62 \\
10 - \frac{100}{0.81} &= -A_2\frac{20}{0.9} + A_2 1.62 \\
-113.45679 &= -20,60222 A_2 \\
&\Rightarrow A_2 \approx 5.5 \text{ e } A_1 \approx 5.6
\end{aligned}
$$

$$
y_t = 5.6(0.9)^t + 5.5t(0.9)^t
$$

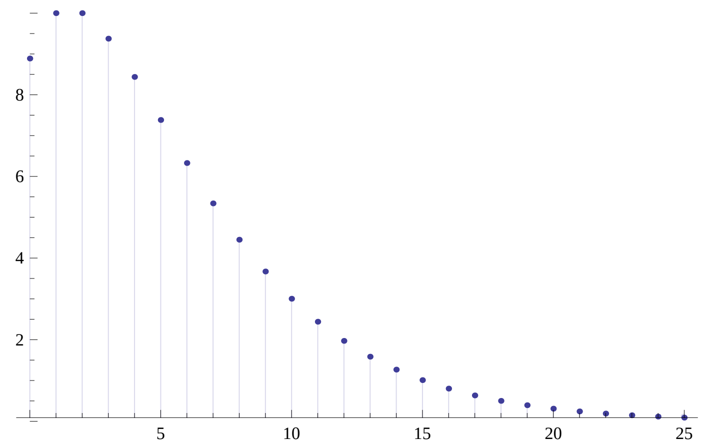


iv.

$$\begin{aligned}
y_t &= A_1(0.75)^t + A_2t(0.75)^t \\
y_1 \Rightarrow 10 &= A_1(0.75) + A_2(0.75) \\
A_1 &= \frac{40}{3} - A_2
\end{aligned}$$

$$\begin{aligned}
y_t &= (\frac{40}{3} - A_2)(0.75)^t + A_2t(0.75)^t \\
y_2 \Rightarrow 10 &= (\frac{40}{3} - A_2)(0.75)^2 + A_2 2(0.75)^2 \\
10 &= \frac{15}{2} - A_2\frac{9}{16} + A_2\frac{18}{16} \\
A_2 &= \frac{5}{2}\frac{16}{9} = \frac{40}{9} \text{ e } A_1 = \frac{80}{9}
\end{aligned}$$

$$y_t = \frac{80}{9}(0.75)^t + \frac{40}{9}t(0.75)^t$$



6. a)

$$
\begin{aligned}
y_t &= 1 + 0.7y_{t-1} - 0.1y_{t-2} + \varepsilon_t \\
y_t - 0.7y_{t_1} + 0.1y_{t-2} = 0 \Rightarrow b_{1,2} &= \frac{0.7 \pm \sqrt{0.7^2 - 4(0.1)}}{2} \\
b_1 &= 0.5 \\
b_2 &= 0.2 \\
y_t^c &= A_1^t 0.5 + A_2 0.2^t
\end{aligned}
$$

A solução particular teste neste caso é

$$
\begin{aligned}
y_t^p &= b_0 + b_1 t + b_2 t^2 + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i} \\
\Rightarrow b_0 + b_1 t + b_2 t^2 + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i} &= 1 + 0.7[b_0 + b_1(t-1) + b_2(t-1)^2 + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-1-i}] \\
&\quad - 0.1[b_0 + b_1(t-2) + b_2(t-2)^2 + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-2-i}] + \varepsilon_t
\end{aligned}
$$

Tratando primeiramente da parte estocástica da solução particular

$$
\begin{aligned}
\Rightarrow \alpha_0 &= 1 \\
\alpha_1 &= 0.7\alpha_0 \\
\alpha_2 &= 0.7\alpha_1 - 0.1\alpha_0 \\
\alpha_3 &= 0.7\alpha_2 - 0.1\alpha_1 \\
\alpha_4 &= 0.7\alpha_3 - 0.1\alpha_2 \\
&\vdots \\
\alpha_i &= 0.7\alpha_{i-1} - 0.1\alpha_{i-2}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \alpha_i - 0.7\alpha_{i-1} + 0.1\alpha_{i-2} &= 0 \\
\alpha_i &= A_1(0.2)^i + A_2(0.5)^i \\
\alpha_0 = 1 \Rightarrow A_1 &= 1 - A_2
\end{aligned}
$$

$$
\begin{aligned}
\alpha_i = (1 - A_2)(0.2)^i + A_2(0.5)^i & \\
\alpha_1 = 0.7 \Rightarrow 0.7 &= (1 - A_2)(0.2) + A_2(0.5) \\
0.7 &= 0.2 - A_2(0.2) + A_2(0.5) \\
0.5 &= A_2(0.3) \Rightarrow A_2 = \frac{5}{3} \text{ e } A_1 = -\frac{2}{3}
\end{aligned}
$$

$$
\alpha_i = -\frac{2}{3}(0.2)^i + \frac{5}{3}(0.5)^i
$$

Como neste problema não existe raiz unitária, não existe tendência na solução particular, então $b_1 = b_2 = 0$. Se $y_t^p = b_0$ e tratando $\varepsilon_{t-i} = 0$, podemos encontrar a parte determinística da solução particular.

$$\begin{aligned}
b_0 &= 1 + 0.7b_0 - 0.1b_0 \\
b_0 &= \frac{1}{(1 - 0.7 + 0.1)} = \frac{5}{2}
\end{aligned}$$

$$\Rightarrow y_t^p = \frac{5}{2} + \sum_{i=0}^{\infty} \left[\frac{5}{3}(0.5)^i - \frac{2}{3}(0.2)^i \right] \varepsilon_{t-i}$$

Então a solução geral fica:

$$y_t = y_t^c + y_t^p = A_1 0.5^t + A_2 0.2^t + \frac{5}{2} + \sum_{i=0}^{\infty} \left[\frac{5}{3}(0.5)^i - \frac{2}{3}(0.2)^i \right] \varepsilon_{t-i}$$

b) Função complementar

$$\begin{aligned}
y_t &= 1 - 0.3y_{t-1} + 0.1y_{t-2} + \varepsilon_t \\
b_1 &= \frac{-0.3 + \sqrt{(0.3)^2 + 4(0.1)}}{2} = 0.2 \\
b_2 &= \frac{-0.3 - \sqrt{(0.3)^2 + 4(0.1)}}{2} = -0.5
\end{aligned}$$

$$y_t^c = A_1(0.2)^t + A_2(-0.5)^t$$

solução particular determinística

$$\begin{aligned}
y_t^{p,d} = b_0 \Rightarrow b_0 &= 1 - 0.3b_0 + 0.1b_0 \\
y_t^{p,d} = b_0 &= \frac{1}{1 + 0.3 - 0.1} = 0.8333
\end{aligned}$$

solução particular estocástica

$$y_t^{p,e} = \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i}$$

substituindo na equação em diferença

$$\sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i} = 1 - 0.3 \left[\sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-1-i} \right] + 0.1 \left[\sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-2-i} \right] + \varepsilon_t$$

$$\begin{aligned}
\Rightarrow \alpha_0 &= 1 \\
\alpha_1 &= -0.3\alpha_0 = -0.3 \\
\alpha_2 &= -0.3\alpha_1 + 0.1\alpha_0 \\
&\vdots \\
\alpha_i &= -0.3\alpha_{i-1} + 0.1\alpha_{i-2}
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow \alpha_i &= A_1(0.2)^i + A_2(-0.5)^i \\
\alpha_0 = 1 &\Rightarrow A_1 = 1 - A_2 \\
\alpha_i &= (1-A_2)(0.2)^i + A_2(-0.5)^i \\
\alpha_1 = -0.3 \Rightarrow &- 0.3 = (1-A_2)(0.2) + A_2(-0.5) \\
&- 0.3 = 0.2 - A_2(0.2) + A_2(-0.5) \Rightarrow A_2 = \frac{5}{7} \text{ e } A_1 = \frac{2}{7} \\
\alpha_i &= \left(\frac{2}{7}\right)(0.2)^i + \left(\frac{5}{7}\right)(-0.5)^i \\
y_t^{p,e} &= \sum_{i=0}^{\infty}\left[\left(\frac{2}{7}\right)(0.2)^i + \left(\frac{5}{7}\right)(-0.5)^i\right]\varepsilon_{t-i}
\end{aligned}
$$

A solução geral fica

$$
y_t = y_t^c + y_t^{p,d} + y_t^{p,e} = A_1(0.2)^t + A_2(-0.5)^t + \frac{12}{10} + \sum_{i=0}^{\infty}\left[\left(\frac{2}{7}\right)(0.2)^i + \left(\frac{5}{7}\right)(-0.5)^i\right]\varepsilon_{t-i}
$$

---

7. $y_t = a_0 + a_2 y_{t-2} + \varepsilon_t$

   a)

$$
\begin{aligned}
y_t - a_2 y_{t-2} &= 0 \\
\Rightarrow b^2 - a_2 &= 0, \quad b_{1,2} = \pm\sqrt{a_2} \\
y_t^c &= A_1\sqrt{a_2}^t + A_2(-\sqrt{a_2})^t
\end{aligned}
$$

   b)

$$
y_t^p = b_0 + b_1 t + b_2 t^2 + \sum_{i=0}^{\infty}\alpha_i\varepsilon_{t-i}
$$

$$
\Rightarrow b_0 + b_1 t + b_2 t^2 + \sum_{i=0}^{\infty}\alpha_i\varepsilon_{t-i} = a_0 + a_2\left[b_0 + b_1 t + b_2 t^2 + \sum_{i=0}^{\infty}\alpha_i\varepsilon_{t-2-i}\right] + \varepsilon_t
$$

$$
\begin{aligned}
&\alpha_0 = 1 \text{ e } \alpha_1 = 0 \\
&\alpha_2 = a_2\alpha_0 = a_2 \\
&\alpha_3 = a_2\alpha_1 = 0 \\
&\alpha_4 = a_2\alpha_2 = a_2^2 \\
&\alpha_5 = a_2\alpha_3 = 0 \\
&\quad\vdots \\
&\alpha_i = a_2\alpha_{i-2}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \alpha_i &= A_1\sqrt{a_2}^i + A_2(-\sqrt{a_2})^i \\
\alpha_0 = 1 &\Rightarrow A_1 = 1 - A_2 \\
\alpha_i &= [1-A_2]\sqrt{a_2}^i + A_2(-\sqrt{a_2})^i \\
\alpha_1 = 0 &\Rightarrow 0 = [1-A_2]\sqrt{a_2} + A_2(-\sqrt{a_2}) \\
A_2 &= \frac{1}{2\sqrt{a_2}} \text{ e } A_1 = 1 - \frac{1}{2\sqrt{a_2}} \\
\alpha_i &= \left[1 - \frac{1}{2\sqrt{a_2}}\right]\sqrt{a_2}^i + \left[\frac{1}{2\sqrt{a_2}}\right](-\sqrt{a_2})^i
\end{aligned}
$$

$$
\begin{aligned}
b_0 + b_1 t + b_2 t^2 = a_0 + a_2 \Big[b_0 + b_1 t + b_2 t^2 \Big] \\
\Rightarrow b_0(1-a_2) + b_1(1-a_2)t + b_2(1-a_2)t^2 - a_0 = 0
\end{aligned}
$$

$$
\begin{aligned}
&\text{se } a_2 \neq 1 \Rightarrow b_1 = b_2 = 0 \text{ e } b_0 = \frac{a_0}{(1-a_2)} \\
&y_t^p = \frac{a_0}{(1-a_2)} + \sum_{i=0}^{\infty} \left\{ \left[1 - \frac{1}{2\sqrt{a_2}} \right] \sqrt{a_2}^i + \left[\frac{1}{2\sqrt{a_2}} \right] (-\sqrt{a_2})^i \right\} \varepsilon_{t-i}
\end{aligned}
$$

Se $a_2 = 1$ implica em $a_0 = 0$ e temos duas raízes unitárias 1 e -1. Com isso, o efeito do termo de erro não diminui à medida que o tempo passa e a solução geral não converge.

c)

$$
\begin{aligned}
y_t^p &= a_0 + a_2 L^2 y_t + \varepsilon_t \\
y_t^p &= \frac{a_0}{(1-a_2L^2)} + \frac{\varepsilon_t}{(1-a_2L^2)} \\
&\equiv \frac{a_0}{1-a_2} + \frac{1}{(1-\sqrt{a_2}L)} \frac{\varepsilon_t}{(1+\sqrt{a_2}L)} \\
y_t^p &= \frac{a_0}{1-a_2} + \frac{1}{(1-\sqrt{a_2})} \sum_{i=0}^{\infty} (-\sqrt{a_2})^i \varepsilon_{t-i}
\end{aligned}
$$

---

8. a)

$$
\begin{aligned}
y_t - y_{t-1} &= 0 \\
\Rightarrow Ab^t - Ab^{t-1} &= 0 \\
b = 1 &\Rightarrow y_t = A \equiv y_t = c
\end{aligned}
$$

b)

$$
\begin{aligned}
y_t - y_{t-1} &= a_0 \\
\Rightarrow y_t^c &= A \\
y_t^p &= kt \Rightarrow kt - k(t-1) = a_0 \\
k &= a_0 \Rightarrow y_t^p = a_0 t \\
y_t &= c + a_0 t
\end{aligned}
$$

c)

$$
\begin{aligned}
y_t - y_{t-2} &= 0 \\
\Rightarrow b^2 - 1 &= 0 \\
b_{1,2} = \pm 1 &\Rightarrow y_t = A_1(1)^t + A_2(-1)^t \equiv y_t = c + a_0(-1)^t
\end{aligned}
$$

d)

$$
\begin{aligned}
y_t - y_{t-2} &= \varepsilon_t \\
y_t - y_{t-2} &= 0 \Rightarrow b_{1,2} = \pm 1 \\
y_t^c &= A_1 + A_2(-1)^t \\
y_t^p - L^2 y_t^p &= \varepsilon_t \Rightarrow \\
y_t^p &= \frac{\varepsilon_t}{1-L^2} = \frac{1}{(1-L)}\frac{\varepsilon_t}{(1+L)} = \sum_{i=0}^{\infty}(1)^i \sum_{i=0}^{\infty}(-1)^i \varepsilon_{t-i} \\
&= i\sum_{i=0}^{\infty}(-1)^i \varepsilon_{t-i} = \sum_{i=0}^{\infty}(-1)^i i \varepsilon_{t-i}
\end{aligned}
$$

$$
y_t = y_t^c + y_t^p = A_1 + A_2(-1)^t + \sum_{i=0}^{\infty}(-1)^i i \varepsilon_{t-i}
$$

---

9. a) i.

$$
\begin{aligned}
y_t - 1.2y_{t-1} + 0.2t_{t-2} = 0 \\
\Rightarrow b_1 &= \frac{1.2 + \sqrt{1.2^2 - 4(0.2)}}{2} = 1 \\
b_2 &= \frac{1.2 - \sqrt{1.2^2 - 4(0.2)}}{2} = 0.2
\end{aligned}
$$

A sequência $\{y_t\}$ não é estável pois possui uma raiz unitária. As raízes características são reais e positivas.

ii.

$$
\begin{aligned}
y_t - 1.2Ly_t + 0.2L^2 y_t &= 0 \\
y_t(1 - 1.2L + 0.2L^2) &= 0 \\
\Rightarrow L^2 - 6L + 5 &= 0 \\
L_1 = 3 + \frac{\sqrt{6^2 - 4(5)}}{2} &= 5 \\
L_2 = 3 - \frac{\sqrt{6^2 - 4(5)}}{2} &= 1
\end{aligned}
$$

b) i.

$$
\begin{aligned}
y_t - 1.2y_{t-1} + 0.4y_{t-2} = 0 \\
\Rightarrow b_1 &= \frac{1.2 + \sqrt{1.2^2 - 4(0.4)}}{2} = 0.6 + i\,0.2 \\
b_2 &= \frac{1.2 - \sqrt{1.2^2 - 4(0.4)}}{2} = 0.6 - i\,0.2
\end{aligned}
$$

As raízes são imaginárias e a sequência $\{y_t\}$ possui padrão flutuante de natureza periódica. Como $R = \sqrt{a_2} = \sqrt{1.2} = 1.095 > 1$ a sequência não é estável. As partes reais são positivas.

ii.

$$\begin{aligned}
y_t - 1.2Ly_t + 0.4L^2 y_t &= 0 \\
y_t(1 - 1.2L + 0.4L^2) &= 0 \\
\Rightarrow 2.5 - 2L + L^2 &= 0 \\
L_1 = 1 + \frac{\sqrt{2^2 - 4(2.5)}}{2} &= 1 + i\sqrt{\frac{3}{2}} \\
L_2 = 1 - \frac{\sqrt{2^2 - 4(2.5)}}{2} &= 1 - i\sqrt{\frac{3}{2}}
\end{aligned}$$

c) i.

$$y_t - 1.2y_{t-1} + 1.2t_{t-2} = 0$$

$$\begin{aligned}
b_1 = 0.6 + \frac{\sqrt{1.2^2 + 4(1.2)}}{2} &= 1.85 \\
b_2 = 0.6 - \frac{\sqrt{1.2^2 + 4(1.2)}}{2} &= -0.65
\end{aligned}$$

As raízes são reais, uma sendo positiva e a outra negativa. Como uma das raízes é maior que um em valor absoluto, a sequência $\{y_t\}$ não é estável.

ii.

$$\begin{aligned}
y_t(1 - 1.2L + 1.2L2) &= 0 \\
\Rightarrow \frac{10}{12} - L + L^2 &= 0 \\
L_1 = 0.5 + \frac{\sqrt{1^2 - 4(\frac{10}{12})}}{2} &= 0.5 + i\sqrt{\frac{28}{3}} \\
L_2 = 0.5 - \frac{\sqrt{1^2 - 4(\frac{10}{12})}}{2} &= 0.5 - i\sqrt{\frac{28}{3}}
\end{aligned}$$

d) i.

$$\begin{aligned}
y_t + 1.2y_{t-1} &= 0 \\
\Rightarrow b + 1 = 0,\ \ b &= -1
\end{aligned}$$

A sequência possui raiz unitária, real e negativa, portanto não é estável.

ii.

$$\begin{aligned}
y_t(1 + 1.2L^2) &= 0 \\
\Rightarrow \frac{10}{12} + L^2 &= 0 \\
L_1 &= \sqrt{\frac{5}{6}} \\
L_2 &= -\sqrt{\frac{5}{6}}
\end{aligned}$$

e) i.

$$\begin{aligned}
y_t - 0.7y_{t-1} - 0.25y_{t-2} + 0.175y_{t-3} &= 0 \\
\Rightarrow b^3 - 0.7b^2 - 0.25b + 0.175 &= 0 \\
(b - 0.5)(b + 0.5)(b - 0.7) &= 0 \\
b_1 = 0.5,\ b_2 = -0.5,\ b_3 &= 0.7
\end{aligned}$$

A sequência é estável, com três raízes reais sendo duas positivas e uma negativa.

ii.

$$
\begin{aligned}
y_t(1 - 0.7L - 0.25L^2 + 0.175L^3) &= 0\\
\Rightarrow 1 - \frac{7}{10}L - \frac{1}{4}L^2 + \frac{7}{40}L^3 &= 0\\
\frac{1}{40}(7L-10)(L-2)(L+2) &= 0\\
(7L-10)(L-2)(L+2) &= 0\\
L_1 = \frac{10}{7},\ L_2 = 2,\ L_3 &= -2
\end{aligned}
$$

10.

$$y_t = 0.8y_{t-1} + \varepsilon_t - 0.5\varepsilon_{t-1}$$

a)

$$
\begin{aligned}
&y_1 = 1\\
&y_2 = 0.8y_1 + \varepsilon_2 - 0.5\varepsilon_1 = \varepsilon_2 + 0.3\\
&y_3 = 0.8y_2 + \varepsilon_3 - 0.5\varepsilon_2 = 0.8(\varepsilon_2 + 0.3) + \varepsilon_3 - 0.5\varepsilon_2 = \varepsilon_3 + 0.3\varepsilon_2 + 0.24\\
&y_4 = 0.8y_3 + \varepsilon_4 - 0.5\varepsilon_3 = 0.8(\varepsilon_3 + 0.3\varepsilon_2 + 0.24) + \varepsilon_4 - 0.5\varepsilon_3 = \varepsilon_4 + 0.3\varepsilon_3 + 0.24\varepsilon_2 + 0.192\\
&y_5 = 0.8y_4 + \varepsilon_5 - 0.5\varepsilon_4 = 0.8(\varepsilon_4 + 0.3\varepsilon_3 + 0.24\varepsilon_2 + 0.192) + \varepsilon_5 - 0.5\varepsilon_4\\
&\quad = \varepsilon_5 + 0.3\varepsilon_4 + 0.24\varepsilon_3 + 0.192\varepsilon_2 + 0.1536
\end{aligned}
$$

b)

$$y_t^c = A(0.8)^t$$

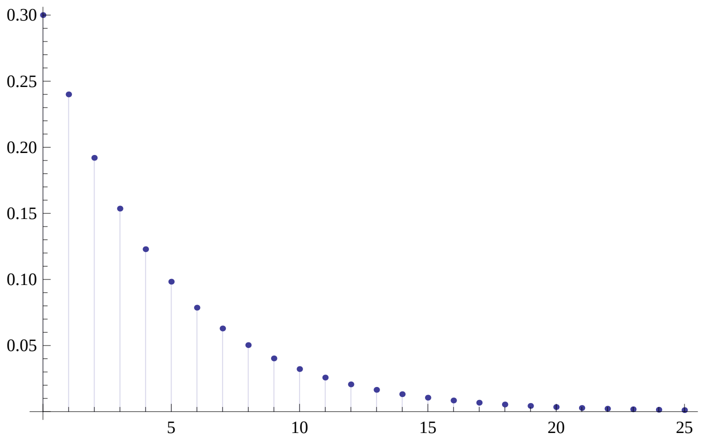
$$
\begin{aligned}
y_t^p \Rightarrow y_t &= 0.8Ly_t + \varepsilon_t - 0.5L\varepsilon_t\\
y_t^p &= \frac{\varepsilon_t - 0.5L\varepsilon_t}{(1-0.8L)} = \sum_{i=0}^{\infty}(0.8)^i(\varepsilon_{t-i}) - \sum_{i=0}^{\infty}(0.8)^i(0.5\varepsilon_{t-1-i}) = \varepsilon_t + \sum_{i=0}^{\infty}(0.8)^i(0.3\varepsilon_{t-(i+1)})\\
y_t^* &= A(0.8)^t + \varepsilon_t + \sum_{i=0}^{\infty}(0.8)^i(0.3\varepsilon_{t-(i+1)})
\end{aligned}
$$

c)

$$
\begin{aligned}
&y_0 = 0 \Rightarrow 0 = A \quad \therefore\\
&y_t^* = \varepsilon_t + \sum_{i=0}^{\infty}(0.8)^i(0.3\varepsilon_{t-(i+1)})
\end{aligned}
$$

d)

$$
\begin{aligned}
\frac{\partial y_t}{\partial \varepsilon_t} &= 1\\
\frac{\partial y_t}{\partial \varepsilon_{t-i}} &= (0.8)^i 0.3
\end{aligned}
$$



11. $0 < \alpha < 1$ e $\beta > 0$

$$\begin{aligned}
&\text{caso } d > 0\\
a_1 + a_2 < 1 \Rightarrow & \alpha(1+\beta) - \alpha\beta < 1\\
& \therefore \alpha < 1
\end{aligned}$$

$$\begin{aligned}
a_2 < 1 + a_1 \Rightarrow & -\alpha\beta < 1 + \alpha(1+\beta)\\
& 0 < 1 + \alpha(1+\beta) + \alpha\beta
\end{aligned}$$

$$\begin{aligned}
&\text{caso } d = 0\\
&|a_1| = 2 \Rightarrow |\alpha(1+\beta)| < 2, \alpha < \frac{2}{1+\beta}
\end{aligned}$$

$$\begin{aligned}
&\text{caso } d < 0\\
&-a_2 < 1 \Rightarrow \alpha\beta < 1, \quad \beta < \frac{1}{\alpha}
\end{aligned}$$

---

12. a) i.

$$\begin{aligned}
y_t &= 3 + 0.75y_{t-1} - 0.125y_{t-2} + \varepsilon_t\\
\Rightarrow y_t^p &= b_0 + b_1 t + b_2 t^2 + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i} \ \therefore
\end{aligned}$$

$$\begin{aligned}
b_0 + b_1 t + b_2 t^2 + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i} = 3 + 0.75\left[b_0 + b_1(t-1) + b_2(t-1)^2 + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-1-i}\right]\\
- 0.125\left[b_0 + b_1(t-2) + b_2(t-2)^2 + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-2-i}\right] + \varepsilon_t
\end{aligned}$$

Tratando a parte determinística da solução particular:

$$b_0 + b_1 t + b_2 t^2 = 3 + 0.75[b_0 + b_1(t-1) + b_2(t-1)^2] - 0.125[b_0 + b_1(t-2) + b_2(t-2)^2]$$

Como a sequência $\{y_t\}$ não possui raiz unitária, a solução não possui tendência e portanto $b_1 = b_2 = 0$. Então:

$$\begin{aligned} b_0 &= 3 + 0.75b_0 - 0.125b_0 \\ b_0 &= 8 \end{aligned}$$

Tratando a parte estocástica da solução particular:

$$\begin{aligned} \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i} = 0.75\left[\sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-1-i}\right] &- 0.125\left[\sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-2-i}\right] + \varepsilon_t \\ \Rightarrow \alpha_0 &= 1 \\ \alpha_1 &= 0.75\alpha_0 \\ \alpha_2 &= 0.75\alpha_1 - 0.125\alpha_0 \\ \alpha_3 &= 0.75\alpha_2 - 0.125\alpha_1 \\ &\vdots \\ \alpha_i &= 0.75\alpha_{i-1} - 0.125\alpha_{i-2} \\ \Rightarrow \alpha_i &= A_1(0.5)^i + A_2(0.25)^i \\ \alpha_0 = 1 \Rightarrow A_1 &= 1 - A_2 \\ \alpha_i &= (1 - A_2)(0.5)^i + A_2(0.25)^i \\ \alpha_1 &= 0.75 \Rightarrow 0.75 = (1 - A_2)0.5 + A_2 0.25, \quad A_2 = -1, \quad A_1 = 2 \\ \alpha_i &= 2(0.5)^i - (0.25)^i \end{aligned}$$

$$y_t^p = 8 + \sum_{i=0}^{\infty}\left[2(0.5)^i - (0.25)^i\right]\varepsilon_{t-i}$$

ii.

$$y_t = 3 + 0.25y_{t-1} + 0.375y_{t-2} + \varepsilon_t$$

parte determinística:

$$\begin{aligned} y_t &= b_0 \\ \Rightarrow b_0 &= 3 + 0.25b_0 + 0.375b_0 \\ b_0 &= \frac{3}{(0.375)} = 8 \end{aligned}$$

parte estocástica:

$$\begin{aligned} \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i} = 0.25\left[\sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-1-i}\right] &+ 0.375\left[\sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-2-i}\right] + \varepsilon_t \\ \alpha_0 &= 1 \\ \alpha_1 &= 0.25\alpha_0 \\ \alpha_2 &= 0.25\alpha_1 + 0.375\alpha_1 \\ \alpha_3 &= 0.25\alpha_2 + 0.375\alpha_2 \\ &\vdots \\ \alpha_i &= 0.25\alpha_{i-1} + 0.375\alpha_{i-2} \end{aligned}$$

$$\Rightarrow \alpha_i = A_1(-0.5)^i + A_2(0.75)^i$$
$$\alpha_0 = 1 \Rightarrow A_1 = 1 - A_2$$
$$\alpha_i = (1 - A_2)(-0.5)^i + A_2(0.75)^i$$
$$\alpha_1 = 0.25 \Rightarrow 0.25 = (1 - A_2)(-0.5) + A_2(0.75)$$
$$A_2 = 0.6, \quad A_1 = 0.4$$

$$\alpha_i = 0.4(-0.5)^i + 0.6(0.75)^i$$

$$y_t^p = 8 + \sum_{i=0}^{\infty} \left[0.4(-0.5)^i + 0.6(0.75)^i\right] \varepsilon_{t-i}$$

b) i.

$$y_t^h = A_1(0.5)^t + A_2(0.25)^t$$

ii.

$$y_t^h = A_1(-0.5)^t + A_2(0.75)^t$$

c) i.

$$y_t = A_1(0.5)^t + A_2(0.25)^t + 8 + \sum_{i=0}^{\infty} \left[2(0.5)^i - (0.25)^i\right] \varepsilon_{t-i}$$

$$y_0 = 8 \Rightarrow 8 = A_1 + A_2 + 8, \quad A_1 = -A_2$$
$$y_1 = 8 \Rightarrow 8 = -A_2 0.5 + A_2 0.25 + 8, \quad A_2 = 0, \quad A_1 = 0$$

ii.

$$y_t = A_1(-0.5)^t + A_2(0.75)^t + 8 + \sum_{i=0}^{\infty} \left[0.4(-0.5)^i + 0.6(0.75)^i\right] \varepsilon_{t-i}$$
$$y_0 = 8 \Rightarrow 8 = A_1 + A_2 + 8, \quad A_1 = -A_2$$
$$y_1 = 8 \Rightarrow 8 = -A_2(-0.5) + A_2 0.75 + 8, \quad A_2 = 0, \quad A_1 = 0$$

---

13. a)

$$\begin{aligned} y_{t-1} &= 0.75 y_{t-2} + \varepsilon_{t-1} \\ &= 0.75[0.75 y_{t-3} + \varepsilon_{t-2}] + \varepsilon_{t-1} = (0.75)^2 y_{t-3} + 0.75\varepsilon_{t-2} + \varepsilon_{t-1} \\ &= 0.75[0.75[0.75 y_{t-4} + \varepsilon_{t-3}] + \varepsilon_{t-2}] + \varepsilon_{t-1} = (0.75)^3 y_{t-4} + (0.75)^2\varepsilon_{t-3} + 0.75\varepsilon_{t-2} + \varepsilon_{t-1} \\ y_{t-1} &= (0.75)^{t-1} y_0 + (0.75)^{t-2}\varepsilon_1 + (0.75)^{t-3}\varepsilon_2 + ... + (0.75)^2\varepsilon_{t-3} + 0.75\varepsilon_{t-2} + \varepsilon_{t-1} \end{aligned}$$

b)

$$\alpha_0^t y_0 + \sum_{i=1}^{\infty} \alpha_i \varepsilon_{t-i} + \varepsilon_t = 0.75\left[\alpha_0^{t-1} y_0 + \varepsilon_{t-1} + \sum_{i=1}^{\infty} \alpha_i \varepsilon_{t-1-i}\right] + \varepsilon_t$$

$$\begin{aligned}
\alpha_0^t &= 0.75\alpha_0^{t-1}, \quad \alpha_0 = 0.75 \\
\alpha_1 &= 0.75 \\
\alpha_2 &= 0.75\alpha_1 \\
\alpha_3 &= 0.75\alpha_2 \\
&\vdots \\
\alpha_i &= 0.75\alpha_{i-1} \\
\alpha_i &= (0.75)^i, \quad i = 1, 2, ...
\end{aligned}$$

c)

$$\begin{aligned}
b_0 + b_1 t + b_2 t^2 + \sum_{i=1}^{\infty} \alpha_i \varepsilon_{t-i} + \varepsilon_t = 0.75&\left[b_0 + b_1(t-1) + b_2(t-1)^2 + \varepsilon_{t-1} + \sum_{i=1}^{\infty} \alpha_i \varepsilon_{t-1-i}\right] \\
&- 0.125\left[b_0 + b_1(t-2) + b_2(t-2)^2 + \varepsilon_{t-2} + \sum_{i=1}^{\infty} \alpha_i \varepsilon_{t-2-i}\right] + \varepsilon_t
\end{aligned}$$

Tratando a parte determinística da solução particular:

$b_0 + b_1 t + b_2 t^2 = 3 + 0.75[b_0 + b_1(t-1) + b_2(t-1)^2] - 0.125[b_0 + b_1(t-2) + b_2(t-2)^2]$

Como a sequência $\{y_t\}$ não possui raiz unitária, a solução não possui tendência e portanto $b_1 = b_2 = 0$. Então:

$$\begin{aligned}
b_0 &= 0.75b_0 - 0.125b_0 \\
b_0 &= 0
\end{aligned}$$

Tratando a parte estocástica da solução particular:

$$
\begin{aligned}
\sum_{i=1}^{\infty}\alpha_i\varepsilon_{t-i}+\varepsilon_t = 0.75\left[\varepsilon_{t-1}+\sum_{i=1}^{\infty}\alpha_i\varepsilon_{t-1-i}\right] &- 0.125\left[\varepsilon_{t-2}+\sum_{i=1}^{\infty}\alpha_i\varepsilon_{t-2-i}\right]+\varepsilon_t \\
\Rightarrow \alpha_1 &= 0.75 \\
\alpha_2 &= 0.75\alpha_1 - 0.125 \\
\alpha_3 &= 0.75\alpha_2 - 0.125\alpha_1 \\
&\vdots \\
\alpha_i &= 0.75\alpha_{i-1} - 0.125\alpha_{i-2} \\
\Rightarrow \alpha_i &= A_1(0.5)^i + A_2(0.25)^i \\
\alpha_1 = 0.75 \Rightarrow 0.75 &= A_1(0.5) + A_2(0.25) \\
A_2 &= 3 - 2A_1 \\
\alpha_i &= (A_1)(0.5)^i + (3-2A_1)(0.25)^i \\
\alpha_2 = 0.4375 \Rightarrow 0.4375 &= (A_1)(0.5)^2 + (3-2A_1)(0.25)^2 \\
0.4375 &= (A_1)(0.25) + (3-2A_1)(0.0625) \\
0.4375 &= (A_1)(0.25) + (0.1875 - 0.125A_1) \\
0.25 &= 0.125A_1, \quad A_1 = 2, \quad A_2 = -1
\end{aligned}
$$

$$\alpha_i = 2(0.5)^i - (0.25)^i, \quad i = 1, 2, ...$$

$$y_t^p = \sum_{i=1}^{\infty}\left[2(0.5)^i - (0.25)^i\right]\varepsilon_{t-i} + \varepsilon_t$$

$$y_t^* = B_1(0.5)^t + B_2(0.25)^t + \sum_{i=1}^{\infty}\left[2(0.5)^i - (0.25)^i\right]\varepsilon_{t-i} + \varepsilon_t$$

Capítulo 2

Modelos de séries de tempo estacionárias

1. $w_t = \frac{1}{4}\varepsilon_t + \frac{1}{4}\varepsilon_{t-1} + \frac{1}{4}\varepsilon_{t-2} + \frac{1}{4}\varepsilon_{t-3}$

   a) i.

$$\begin{aligned} E(w_t) &= E(\frac{1}{4}\varepsilon_t + \frac{1}{4}\varepsilon_{t-1} + \frac{1}{4}\varepsilon_{t-2} + \frac{1}{4}\varepsilon_{t-3}) \\ &= \frac{1}{4}E(\varepsilon_t) + \frac{1}{4}E(\varepsilon_{t-1}) + \frac{1}{4}E(\varepsilon_{t-2}) + \frac{1}{4}E(\varepsilon_{t-3}) = 0 \end{aligned}$$

   ii.

$$\begin{aligned} E(w_t|\varepsilon_{t-3} = \varepsilon_{t-2} = 1) &= E(\frac{1}{4}\varepsilon_t + \frac{1}{4}\varepsilon_{t-1} + \frac{1}{4}(1) + \frac{1}{4}(1)) \\ &= \frac{1}{4}E(\varepsilon_t) + \frac{1}{4}E(\varepsilon_{t-1}) + \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \end{aligned}$$

   b) i.

$$\begin{aligned} \mathrm{var}(w_t) &= E(w_t^2) - E(w_t)^2, \quad E(w_t) = 0 \\ \Rightarrow E(w_t^2) &= E([\frac{1}{4}\varepsilon_t + \frac{1}{4}\varepsilon_{t-1} + \frac{1}{4}\varepsilon_{t-2} + \frac{1}{4}\varepsilon_{t-3}]^2) \\ &= \frac{1}{16}E(\varepsilon_t^2) + \frac{1}{16}E(\varepsilon_{t-1}^2) + \frac{1}{16}E(\varepsilon_{t-2}^2) + \frac{1}{16}E(\varepsilon_{t-3}^2), \quad \text{pois } E(\varepsilon_t, \varepsilon_{t-s}) = 0 \;\forall\; s > 0 \\ &= \frac{1}{16}\sigma^2 + \frac{1}{16}\sigma^2 + \frac{1}{16}\sigma^2 + \frac{1}{16}\sigma^2 = \frac{1}{4}\sigma^2 \end{aligned}$$

   ii.

$$\begin{aligned} \mathrm{var}(w_t|\varepsilon_{t-3} = \varepsilon_{t-2} = 1) &= E(w_t^2|\varepsilon_{t-3} = \varepsilon_{t-2} = 1) - E(w_t|\varepsilon_{t-3} = \varepsilon_{t-2} = 1)^2, \\ E(w_t|\varepsilon_{t-3} = \varepsilon_{t-2} = 1) &= \frac{1}{2} \\ \Rightarrow \mathrm{var}(w_t|\varepsilon_{t-3} = \varepsilon_{t-2} = 1) &= E(w_t^2|\varepsilon_{t-3} = \varepsilon_{t-2} = 1) - \frac{1}{4} \end{aligned}$$

$$\begin{aligned} E(w_t^2|\varepsilon_{t-3} = \varepsilon_{t-2} = 1) &= E([\frac{1}{4}\varepsilon_t + \frac{1}{4}\varepsilon_{t-1} + \frac{1}{2}]^2) \\ &= \frac{1}{16}E(\varepsilon_t^2) + \frac{1}{16}E(\varepsilon_{t-1}^2) + \frac{1}{4} \\ &= \frac{1}{4} + \frac{1}{8}\sigma^2 \\ \mathrm{var}(w_t|\varepsilon_{t-3} = \varepsilon_{t-2} = 1) &= \frac{1}{4} + \frac{1}{8}\sigma^2 - \frac{1}{4} = \frac{1}{8}\sigma^2 \end{aligned}$$

c) i.

$$\begin{aligned}
\text{cov}(w_t, w_{t-1}) &= E(w_t w_{t-1}) - E(w_t)E(w_{t-1}) \\
&= E(w_t w_{t-1}), \text{pois } E(w_{t-s}) = 0 \,\forall\, s \geq 0 \\
E(w_t w_{t-1}) &= E[(\frac{1}{4}\varepsilon_t + \frac{1}{4}\varepsilon_{t-1} + \frac{1}{4}\varepsilon_{t-2} + \frac{1}{4}\varepsilon_{t-3})(\frac{1}{4}\varepsilon_{t-1} + \frac{1}{4}\varepsilon_{t-2} + \frac{1}{4}\varepsilon_{t-3} + \frac{1}{4}\varepsilon_{t-4})] \\
&= \frac{3}{16}\sigma^2, \quad \text{pois } E(\varepsilon_t, \varepsilon_{t-s}) = 0 \,\forall\, s > 0
\end{aligned}$$

ii.

$$\begin{aligned}
\text{cov}(w_t, w_{t-2}) &= E(w_t w_{t-2}) - E(w_t)E(w_{t-2}) \\
&= E(w_t w_{t-2}) \\
E(w_t w_{t-2}) &= E[(\frac{1}{4}\varepsilon_t + \frac{1}{4}\varepsilon_{t-1} + \frac{1}{4}\varepsilon_{t-2} + \frac{1}{4}\varepsilon_{t-3})(\frac{1}{4}\varepsilon_{t-2} + \frac{1}{4}\varepsilon_{t-3} + \frac{1}{4}\varepsilon_{t-4} + \frac{1}{4}\varepsilon_{t-5})] \\
&= \frac{1}{8}\sigma^2
\end{aligned}$$

iii.

$$\begin{aligned}
\text{cov}(w_t, w_{t-5}) &= E(w_t w_{t-5}) - E(w_t)E(w_{t-5}) \\
&= E(w_t w_{t-5}) \\
E(w_t w_{t-5}) &= E[(\frac{1}{4}\varepsilon_t + \frac{1}{4}\varepsilon_{t-1} + \frac{1}{4}\varepsilon_{t-2} + \frac{1}{4}\varepsilon_{t-3})(\frac{1}{4}\varepsilon_{t-5} + \frac{1}{4}\varepsilon_{t-6} + \frac{1}{4}\varepsilon_{t-7} + \frac{1}{4}\varepsilon_{t-8})] \\
&= 0, \text{ pois só temos termos de erro } \textit{ruído branco} \text{ de períodos diferentes.}
\end{aligned}$$

Com este primeiro exercício podemos perceber que a variância condicional é menor do que a não condicional. Intuitivamente faz sentido, já que pelo fato de termos mais informações no exercício condicionado ($\varepsilon_{t-3} = \varepsilon_{t-2} = 1$), mais correta será nossa estimativa.

---

2 $y_t = a_0 + a_2 y_{t-2} + \varepsilon_t, \quad |a_2| < 1$

a) i. $E_{t-2} y_t = a_0 + a_2 y_{t-2}$

ii. $E_{t-1} y_t = a_0 + a_2 y_{t-2}$

iii. $E_t y_{t+2} = a_0 + a_2 y_t$

iv.

$$\begin{aligned}
y_t &= a_0 + a_2 y_{t-2} + \varepsilon_t \\
&= a_0 + a_2 a_0 + a_2^2 y_{t-4} + a_2 \varepsilon_{t-2} + \varepsilon_t \\
&= a_0 + a_2 a_0 + a_2^2 a_0 + a_2^3 y_{t-6} + a_2^2 \varepsilon_{t-4} + a_2 \varepsilon_{t-2} + \varepsilon_t \\
&\vdots \\
y_t &= \frac{a_0}{1 - a_2} + \sum_{\substack{i=0 \\ j=2i}}^{\infty} a_2^i \varepsilon_{t-j} + A a_2^t
\end{aligned}$$

Portanto a solução particular para $\{y_t\}$ fica:

$$\begin{aligned}
y_t^p &= \frac{a_0}{1 - a_2} + \sum_{\substack{i=0 \\ j=2i}}^{\infty} a_2^i \varepsilon_{t-j} \\
&= \frac{a_0}{1 - a_2} + \varepsilon_t + a_2 \varepsilon_{t-2} + a_2^2 \varepsilon_{t-4} + a_2^3 \varepsilon_{t-6} + \cdots
\end{aligned}$$

$$\text{cov}(y_t, y_{t-1}) = E(y_t - Ey_t)(y_{t-1} - Ey_{t-1})$$

$$\begin{aligned}
Ey_t = Ey_{t-1} &= \frac{a_0}{1-a_2} \therefore \\
\text{cov}(y_t, y_{t-1}) &= E[(\varepsilon_t + a_2\varepsilon_{t-2} + a_2^2\varepsilon_{t-4} + a_2^3\varepsilon_{t-6} + \cdots) \\
&\quad \times (\varepsilon_{-1} + a_2\varepsilon_{t-3} + a_2^2\varepsilon_{t-5} + a_2^3\varepsilon_{t-7} + \cdots)] \\
&= 0 \\
\Rightarrow \rho_1 &= 0 \therefore \phi_{11} = 0
\end{aligned}$$

Para encontrar ϕ_{22} precisamos de ρ_2, portanto de γ_0 e γ_2.

$$\begin{aligned}
\gamma_0 = E(y_t - Ey_t)^2 &= E[(\varepsilon_t + a_2\varepsilon_{t-2} + a_2^2\varepsilon_{t-4} + a_2^3\varepsilon_{t-6} + \cdots) \\
&\quad \times (\varepsilon_t + a_2\varepsilon_{t-2} + a_2^2\varepsilon_{t-4} + a_2^3\varepsilon_{t-6} + \cdots)] \\
&= \frac{\sigma^2}{1-a_2^2}
\end{aligned}$$

$$\begin{aligned}
\gamma_2 = E(y_t - Ey_t)(y_{t-2} - Ey_{t-2}) &= E[(+\varepsilon_t + a_2\varepsilon_{t-2} + a_2^2\varepsilon_{t-4} + a_2^3\varepsilon_{t-6} + \cdots) \\
&\quad \times (+\varepsilon_{t-2} + a_2\varepsilon_{t-4} + a_2^2\varepsilon_{t-6} + a_2^3\varepsilon_{t-8} + \cdots)] \\
&= \frac{a_2\sigma^2}{1-a_2^2} \therefore \\
\rho_2 &= \frac{\gamma_2}{\gamma_0} = a_2
\end{aligned}$$

$$\phi_{22} = \frac{\rho_2 - \rho_1^2}{1-\rho_1^2} = a_2$$

b)

$$y_t = \frac{a_0}{1-a_2} + \sum_{\substack{i=0 \\ j=2i}}^{\infty} a_2^i \varepsilon_{t-j} + Aa_2^t$$

Função impulso resposta:

$$\begin{aligned}
\frac{\partial y_t}{\partial \varepsilon_{t-1}} &= 0 \\
\frac{\partial y_t}{\partial \varepsilon_{t-2}} &= a_2 \\
\frac{\partial y_t}{\partial \varepsilon_{t-3}} &= 0 \\
\frac{\partial y_t}{\partial \varepsilon_{t-4}} &= a_2^2 \\
&\vdots \\
\frac{\partial y_t}{\partial \varepsilon_{t-j}} &= a_2^i, \quad i = 1, 2, ..., \quad j = 2i.
\end{aligned}$$

Efeito de ε_t sobre $\{y_t\}$:

$$\frac{\partial y_t}{\partial \varepsilon_t} = 1$$

$$\vdots$$

$$\frac{\partial y_{t+j}}{\partial \varepsilon_t} = a_2^i, \quad i = 1, 2, ..., \quad j = 2i.$$