

Solution for Applied Econometric Time Series (Enders, 2014)

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Capítulo 1

Equações em diferença

1. a) Encontrando a solução de Jill:

$$\begin{aligned}
 y_t &= a_0 + a_1 y_{t-1} \\
 &= a_0 + a_1(a_0 + a_1 y_{t-2}) \\
 &= a_0 + a_0 a_1 + a_0 a_1^2 + \cdots + a_0 a_1^{t-1} + a_1^t y_0 \\
 &= a_0 \sum_{i=0}^{t-1} a_1^i + a_1^t y_0.
 \end{aligned}$$

O primeiro termo da soma é uma progressão geométrica finita, então

$$a_0 \sum_{i=0}^{t-1} a_1^i = a_0 \frac{(1 - a_1^t)}{(1 - a_1)} = \frac{a_0}{(1 - a_1)} - \frac{a_0 a_1^t}{(1 - a_1)}.$$

Substituindo a P.G. finita na solução de Jill, temos

$$\begin{aligned}
 y_t &= \frac{a_0}{(1 - a_1)} - \frac{a_0 a_1^t}{(1 - a_1)} + a_1^t y_0 \\
 &= \frac{a_0}{(1 - a_1)} - a_1^t \left[y_0 - \frac{a_0}{(1 - a_1)} \right].
 \end{aligned}$$

Solução para Bill:

A função complementar é dada pela solução homogênea de $y_t = a_0 + a_1 y_{t-1}$.

$$\begin{aligned}
 y_t - a_1 y_{t-1} &= 0 \\
 \text{se } y_t &= Ab^t, \quad y_{t-1} = a_1 Ab^{t-1} \\
 \text{então } Ab^t - a_1 Ab^{t-1} &= 0 \\
 Ab^t(1 - a_1 b^{-1}) &= 0 \Rightarrow b = a_1
 \end{aligned}$$

Com isso a função complementar se torna $y_t^c = Aa_1^t$,

A solução particular pode ser $y_t^p = k$, então

$$\begin{aligned}
 k &= a_0 + a_1 k \\
 k &= \frac{a_0}{1 - a_1} \Rightarrow y_t^p = \frac{a_0}{1 - a_1}
 \end{aligned}$$

então a solução geral é

$$y_t = y_t^p + y_t^c = \frac{a_0}{1 - a_1} + Aa_1^t$$

Para encontrarmos a solução definida usamos a condição inicial y_0 em $t = 0$ na solução geral.

$$\begin{aligned} y_0 &= \frac{a_0}{1 - a_1} + Aa_1^0 \\ &= \frac{a_0}{1 - a_1} + A \Rightarrow A = y_0 - \frac{a_0}{1 - a_1} \end{aligned}$$

Substituindo a constante arbitrária A na solução geral,

$$y_t = \frac{a_0}{1 - a_1} + a_1^t \left[y_0 - \frac{a_0}{1 - a_1} \right],$$

assim temos a igualdade entre as soluções de Jill e Bill.

b) Para Jill, se $a_1 = 1$, temos

$$\begin{aligned} y_t &= a_0 + y_{t-1} \\ &= a_0 + a_0 + y_{t-2} \\ &= a_0 + a_0 + \cdots + a_0 + y_0 \\ &= a_0 t + y_0 \end{aligned}$$

Método de Bill:

$$\begin{aligned} y_t^c &= Ab^t \Rightarrow Ab^t = Ab^{t-1} \\ b &= 1 \\ \Rightarrow y_t^c &= A \end{aligned}$$

$$\begin{aligned} y_t^p &= kt \Rightarrow kt = a_0 + k(t - 1) \\ \Rightarrow a_0 &= k \\ \Rightarrow y_t^p &= k + k(t - 1) = kt \end{aligned}$$

$$y_t = y_t^c + y_t^p = A + kt \text{ (solução geral)}$$

em $t = 0$, $y_t = y_0 \Rightarrow y_0 = A$, então

$$y_t = y_0 + kt$$

para $t = 1$, $y_1 = a_0 + y_0 \Rightarrow y_0 = y_1 - a_0$

substituindo em $y_t = y_0 + kt$ avaliada em $t = 1$

$$y_1 = y_1 - a_0 + k \Rightarrow k = a_0$$

$$y_t = a_0 t + y_0 \text{ (solução definida)}$$

2. a) solução homogênea

$$\begin{aligned} p_t^* - (1 - \alpha)p_{t-1}^* &= 0 \\ p_t^* &= Ab^t \Rightarrow Ab^t = (1 - \alpha)Ab^{t-1} \\ \Rightarrow b &= (1 - \alpha) \text{ e } p_t^* = A(1 - \alpha)^t \end{aligned}$$

b) solução particular

$$\begin{aligned} p_t^* &= \alpha L p_t + (1 - \alpha)L p_t^* \\ p_t^* &= \frac{\alpha L p_t}{1 - (1 - \alpha)L} \equiv \alpha p_t \sum_{i=0}^{\infty} (1 - \alpha)^i L^{i+1} \end{aligned}$$

Somando a solução homogênea e particular

$$p_t^* = A(1 - \alpha)^t + \alpha p_t \sum_{i=0}^{\infty} (1 - \alpha)^i L^{i+1}$$

No período $t = 0$, $p_t^* = p_0^*$, então

$$\begin{aligned} p_0^* &= A + \alpha p_0 \sum_{i=0}^{\infty} (1 - \alpha)^i L^{i+1} \\ A &= p_0^* - \alpha p_0 \sum_{i=0}^{\infty} (1 - \alpha)^i L^{i+1} \end{aligned}$$

No período inicial $p_0^* = p_0$, então a solução definida fica

$$\begin{aligned} p_t^* &= \left[p_0 - \alpha p_0 \sum_{i=0}^{\infty} (1 - \alpha)^i L^{i+1} \right] (1 - \alpha)^t + \alpha p_t \sum_{i=0}^{\infty} (1 - \alpha)^i L^{i+1} \\ p_t^* &= p_0 (1 - \alpha)^t + \alpha p_t \sum_{i=0}^{t-1} (1 - \alpha)^i L^{i+1} \end{aligned}$$

Substituindo na equação em diferença original:

Antes temos,

$$p_{t-1}^* = p_0 (1 - \alpha)^{t-1} + \alpha p_t \sum_{i=0}^{t-1} (1 - \alpha)^i L^{i+2}$$

Então

$$\begin{aligned} p_0(1-\alpha)^t + \alpha p_t \sum_{i=0}^{t-1} (1-\alpha)^i L^{i+1} &= \alpha p_{t-1} + (1-\alpha) \left[p_0(1-\alpha)^{t-1} + \alpha p_t \sum_{i=0}^{t-1} (1-\alpha)^i L^{i+2} \right] \\ &= \alpha p_{t-1} + (1-\alpha) p_0(1-\alpha)^{t-1} + \alpha p_t \sum_{i=0}^{t-1} (1-\alpha)^{i+1} L^{i+2} \end{aligned}$$

tratando apenas o primeiro e o terceiro termo da expressão acima

$$\begin{aligned} \Leftrightarrow \alpha p_{t-1} + \alpha p_t \sum_{i=0}^{t-1} (1-\alpha)^{i+1} L^{i+2} &= \alpha p_t L(1-\alpha)^0 + \alpha p_t \sum_{i=0}^{t-1} (1-\alpha)^{i+1} L^{i+2} \\ &\equiv \alpha p_t \sum_{i=0}^{t-1} (1-\alpha)^i L^{i+1} \end{aligned}$$

$$\Rightarrow p_0(1-\alpha)^t + \alpha p_t \sum_{i=0}^{t-1} (1-\alpha)^i L^{i+1} = p_0(1-\alpha)^t + \alpha p_t \sum_{i=0}^{t-1} (1-\alpha)^i L^{i+1}$$

Com isso os dois lados se tornam iguais, o que prova a veracidade da solução.

3. a)

$$\begin{aligned} m_t &= m + \rho m_{t-1} + \varepsilon_t \\ \Rightarrow m_{t+1} &= m + \rho m_t + \varepsilon_{t+1} \\ m_{t+2} &= m + \rho m_{t+1} + \varepsilon_{t+2} \\ &\vdots \\ m_{t+n} &= m + \rho m_{t+n-1} + \varepsilon_{t+n} \end{aligned}$$

substituindo recursivamente o termo defasado

$$\begin{aligned} m_{t+1} &= m + \rho m_t + \varepsilon_{t+1} \\ m_{t+2} &= m + \rho(m + \rho m_t + \varepsilon_{t+1}) + \varepsilon_{t+2} = m + \rho m + \rho^2 m_t + \rho \varepsilon_{t+1} + \varepsilon_{t+2} \\ m_{t+3} &= m + \rho(m + \rho m + \rho^2 m_t + \rho \varepsilon_{t+1} + \varepsilon_{t+2}) + \varepsilon_{t+3} \\ &\equiv m + \rho m + \rho^2 m + \rho^3 m_t + \rho^2 \varepsilon_{t+1} + \rho \varepsilon_{t+2} + \varepsilon_{t+3} \\ &\vdots \\ m_{t+n} &= \sum_{j=0}^{n-1} \rho^j m + \rho^n m_t + \sum_{i=0}^n \rho^{n-i} \varepsilon_{t+i} \end{aligned}$$

b)

$$\begin{aligned} E[m_{t+n}] &= E \left[\sum_{j=0}^{n-1} \rho^j m + \rho^n m_t + \sum_{i=0}^n \rho^{n-i} \varepsilon_{t+i} \right] \\ &= \sum_{j=0}^{n-1} \rho^j m + \rho^n m_t + E \left[\sum_{i=0}^n \rho^{n-i} \varepsilon_{t+i} \right] \\ &= \sum_{j=0}^{n-1} \rho^j m + \rho^n m_t + \sum_{i=0}^n \rho^{n-i} E[\varepsilon_{t+i}] \\ E[m_{t+n}] &= \sum_{j=0}^{n-1} \rho^j m + \rho^n m_t = \frac{1-\rho^{n-1}}{(1-\rho)} m + \rho^n m_t \end{aligned}$$

Como m_{t+n} depende somente de uma variável conhecida m_t e uma sequência de termos de erro $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{t+n}\}$ de média zero, um modelo univariado pode ser útil para prever a oferta monetária n períodos no futuro. Isto é possível estimando ρ por meio de técnicas univariadas de séries temporais.

4. a) i.

$$y_t - 1.5y_{t-1} + 0.5y_{t-2} = 0$$

$$y_t = Ab^t \Rightarrow y_{t-1} = Ab^{t-1} \text{ e } y_{t-2} = Ab^{t-2}$$

$$Ab^t - 1.5Ab^{t-1} + 0.5Ab^{t-2} = 0$$

$$b^2 - 1.5b + 0.5 = 0$$

$$b_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}, \quad a_1 = -1.5, \quad a_2 = 0.5$$

$$\Rightarrow b_1 = \frac{1.5 + \sqrt{(-1.5)^2 - 4 \times 0.5}}{2} = 1$$

$$b_2 = \frac{1.5 - \sqrt{(-1.5)^2 - 4 \times 0.5}}{2} = 0.5$$

A solução homogênea fica $A_1 + A_2(0.5)^t$.

ii.

$$y_t - y_{t-2} = 0$$

$$y_t = Ab^t \Rightarrow y_{t-2} = Ab^{t-2}$$

$$Ab^t - Ab^{t-2} = 0, \Rightarrow b^2 - 1 = 0,$$

$$b_1 = 1 \text{ ou } b_2 = -1$$

iii.

$$y_t - 2y_{t-1} + y_{t-2} = 0$$

$$\Rightarrow Ab^t - 2Ab^{t-1} + Ab^{t-2} = 0$$

$$b^2 - 2b + 1 = 0$$

$$b_1 = b_2 = 1$$

como são raízes repetidas, a solução homogênea fica $A_1 + A_2t$.

iv.

$$y_t - y_{t-1} - 0.25y_{t-2} + 0.25y_{t-3} = 0$$

$$\Rightarrow Ab^t - Ab^{t-1} - 0.25Ab^{t-2} + 0.25Ab^{t-3} = 0$$

$$[b^3 - b^2 - 0.25b + 0.25 = 0] \times 4$$

$$(2b)^2b - (2b)^2 - 1b + 1 = 0$$

$$\equiv (b-1)(2b+1)(2b-1) = 0$$

$$\Rightarrow b_1 = 1, \quad b_2 = 0.5 \quad b_3 = -0.5$$

A solução homogênea fica $A_1 + A_2(0.5)^t + A_3(-0.5)^t$.

b) i.

$$\begin{aligned}
 y_t &= 1.5y_{t-1} - 0.5y_{t-2} + \varepsilon_t \\
 \Rightarrow y_t &= 1.5Ly_t - 0.5L^2y_t + \varepsilon_t \\
 y_t - 1.5Ly_t + 0.5L^2y_t &= \varepsilon_t \\
 (1 - L)(1 - 0.5L)y_t &= \varepsilon_t \\
 y_t &= \frac{\varepsilon_t}{(1 - L)(1 - 0.5L)}
 \end{aligned}$$

Embora a expressão $\varepsilon_t/(1 - 0.5L)$ seja convergente, a expressão $\varepsilon_t/(1 - L)$ não é, portanto a solução retrospectiva é não convergente.

ii.

$$\begin{aligned}
 y_t &= y_{t-2} + \varepsilon_t \\
 \Rightarrow y_t - L^2y_t &= \varepsilon_t \\
 (1 - L^2)y_t &= \varepsilon_t \\
 (1 - L)(1 + L)y_t &= \varepsilon_t \\
 y_t &= \frac{\varepsilon_t}{(1 - L)(1 + L)}
 \end{aligned}$$

A expressão $\varepsilon_t/(1 - L)$ não converge, portanto a solução retrospectiva é não convergente.

iii.

$$\begin{aligned}
 y_t &= 2y_{t-1} - y_{t-2} + \varepsilon_t \\
 \Rightarrow y_t - 2Ly_t + L^2y_t &= \varepsilon_t \\
 (1 - 2L + L^2)y_t &= \varepsilon_t \\
 (1 - L)(1 - L)y_t &= \varepsilon_t \\
 y_t &= \frac{\varepsilon_t}{(1 - L)(1 - L)}
 \end{aligned}$$

Portanto a solução não converge.

iv.

$$\begin{aligned}
 y_t &= y_{t-1} + 0.25y_{t-2} - 0.25y_{t-3} + \varepsilon_t \\
 y_t - Ly_t - 0.25L^2y_t + 0.25L^3y_t &= \varepsilon_t \\
 (1 - L - 0.25L^2 + 0.25L^3)y_t &= \varepsilon_t \\
 (1 - L)(1 + 0.5L)(1 - 0.5L)y_t &= \varepsilon_t \\
 y_t &= \frac{\varepsilon_t}{(1 - L)(1 + 0.5L)(1 - 0.5L)}
 \end{aligned}$$

que não converge devido à expressão $\varepsilon_t/(1 - L)$.

c)

$$y_t = 1.5y_{t-1} - 0.5y_{t-2} + \varepsilon_t$$

$$y_t - y_{t-1} = 1.5y_{t-1} - y_{t-1} - 0.5y_{t-2} + \varepsilon_t$$

$$y_t - y_{t-1} = 0.5y_{t-1} - 0.5y_{t-2} + \varepsilon_t$$

$$\Delta y_t = 0.5\Delta y_{t-1} + \varepsilon_t$$

$$\Delta y_t = 0.5L\Delta y_t + \varepsilon_t$$

$$\Delta y_t - 0.5L\Delta y_t = \varepsilon_t$$

$$(1 - 0.5L)\Delta y_t = \varepsilon_t$$

$$\Delta y_t = \frac{\varepsilon_t}{(1 - 0.5L)} \equiv \sum_{i=0}^{\infty} (0.5)^i \varepsilon_{t-i}$$

d) ii.

$$y_t = y_{t-2} + \varepsilon_t$$

$$y_t - y_{t-1} = -y_{t-1} + y_{t-2} + \varepsilon_t$$

$$\Delta y_t = -\Delta y_{t-1} + \varepsilon_t$$

$$\Delta y_t = -\Delta L y_t + \varepsilon_t$$

$$(1 + L)\Delta y_t = \varepsilon_t$$

$$\Delta y_t = \frac{\varepsilon_t}{(1 + L)} \equiv \sum_{i=0}^{\infty} (-1)^i \varepsilon_{t-i}$$

não converge, portanto não há solução.

iii.

$$y_t = 2y_{t-1} - y_{t-2} + \varepsilon_t$$

$$y_t - y_{t-1} = 2y_{t-1} - y_{t-1} - y_{t-2} + \varepsilon_t$$

$$\Delta y_t = \Delta y_{t-1} + \varepsilon_t$$

$$\Delta y_t = \Delta L y_t + \varepsilon_t$$

$$(1 - L)\Delta y_t = \varepsilon_t$$

$$\Delta y_t = \frac{\varepsilon_t}{(1 - L)} \equiv \sum_{i=0}^{\infty} (1)^i \varepsilon_{t-i}$$

não converge, portanto não há solução.

iv.

$$\begin{aligned}
y_t &= y_{t-1} + 0.25y_{t-2} - 0.25y_{t-3} + \varepsilon_t \\
y_t - y_{t-1} &= 0.25y_{t-2} - 0.25y_{t-3} + \varepsilon_t \\
\Delta y_t &= 0.25\Delta y_{t-2} + \varepsilon_t \\
\Delta y_t &= 0.25\Delta L^2 y_t + \varepsilon_t \\
(1 - 0.25L^2)\Delta y_t &= \varepsilon_t \\
\Delta y_t &= \frac{\varepsilon_t}{(1 + 0.5L)(1 - 0.5L)} = \frac{\varepsilon_t}{(1 + 0.5L)} \frac{1}{(1 - 0.5L)} \\
&\equiv \left[\sum_{i=0}^{\infty} (-0.5)^i \varepsilon_{t-i} \right] \left[\sum_{i=0}^{\infty} (0.5)^i (1) \right] = 2 \sum_{i=0}^{\infty} (-0.5)^i \varepsilon_{t-i}
\end{aligned}$$

que converge, portanto esta solução existe.

e) Já foi realizado anteriormente.

f)

$$\begin{aligned}
y_t &= a_0 - y_{t-1} + \varepsilon_t \\
y_0 &= y_0 \\
y_1 &= a_0 - y_0 + \varepsilon_1 \\
y_2 &= a_0 - y_1 + \varepsilon_2 = a_0 - (a_0 - y_0 + \varepsilon_1) + \varepsilon_2 = y_0 + \varepsilon_2 - \varepsilon_1 \\
y_3 &= a_0 - y_2 + \varepsilon_3 = a_0 - (y_0 + \varepsilon_2 - \varepsilon_1) + \varepsilon_3 = a_0 - y_0 + \varepsilon_3 - \varepsilon_2 + \varepsilon_1 \\
\Rightarrow y_t &= \frac{a_0 + \varepsilon_t}{1 + L} \equiv \sum_{i=0}^{t-1} (-1)^i a_0 + \sum_{i=0}^{t-1} (-1)^i \varepsilon_{t-i}
\end{aligned}$$

não converge.

5. a) i.

$$\begin{aligned}
y_t &= 0.75y_{t-1} - 0.125y_{t-2} \\
\Rightarrow b^2 - 0.75b + 0.125 &= 0 \\
b_{1,2} &= \frac{-a_1 \pm \sqrt{d}}{2}, \quad d = (a_1)^2 - 4a_2, \quad a_1 = -0.75, \quad a_2 = 0.125 \\
d &= (-0.75)^2 - 4 \times 0.125 = 0.0625 \\
b_1 &= \frac{0.75 + \sqrt{0.0625}}{2} = 0.5 \\
b_2 &= \frac{0.75 - \sqrt{0.0625}}{2} = 0.25 \\
y_t &= A_1 0.5^t + A_2 0.25^t
\end{aligned}$$

ii.

$$\begin{aligned}
y_t &= 1.5y_{t-1} - 0.75y_{t-2} \\
\Rightarrow b^2 - 1.5b + 0.75 &= 0 \\
b_{1,2} &= \frac{1.5 \pm \sqrt{(-1.5)^2 - 4 \times 0.75}}{2} \\
b_1 &= \frac{1.5 + \sqrt{-0.75}}{2} = 0.75 + i\sqrt{\frac{0.75}{4}} = 0.75 + i\sqrt{0.1875} \\
b_2 &= \frac{1.5 - \sqrt{-0.75}}{2} = 0.75 - i\sqrt{0.1875} \\
y_t &= A_1(0.75 + i\sqrt{0.1875})^t + A_2(0.75 - i\sqrt{0.1875})^t
\end{aligned}$$

iii.

$$\begin{aligned}
y_t &= 1.8y_{t-1} - 0.81y_{t-2} \\
\Rightarrow b^2 - 1.8b + 0.81 &= 0 \\
b_{1,2} &= \frac{1.8 \pm \sqrt{(-1.8)^2 - 4 \times 0.81}}{2} \\
d &= 0 \\
b_1 &= b_2 = 0.9 \\
y_t &= A_1(0.9)^t + A_2t(0.9)^t
\end{aligned}$$

iv.

$$\begin{aligned}
y_t &= 1.5y_{t-1} - 0.5625y_{t-2} \\
\Rightarrow b^2 - 1.5b + 0.5625 &= 0 \\
b_{1,2} &= \frac{1.5 \pm \sqrt{(-1.5)^2 - 4 \times 0.5625}}{2} \\
d &= 0 \\
b_1 &= b_2 = 0.75 \\
y_t &= A_1(0.75)^t + A_2t(0.75)^t
\end{aligned}$$

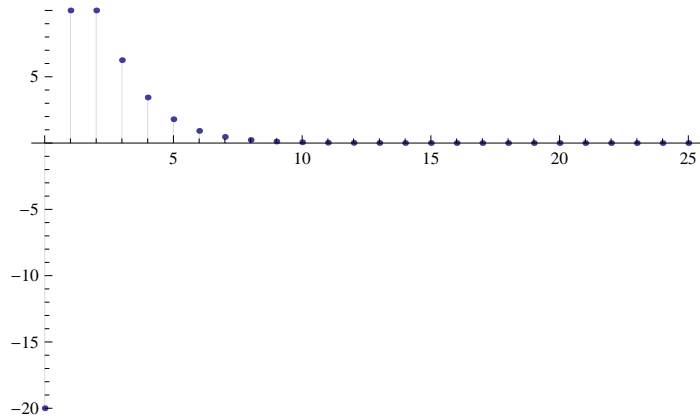
b) i.

$$\begin{aligned}
 y_t &= A_1 0.5^t + A_2 0.25^t, \quad y_1 = y_2 = 10 \\
 \Rightarrow y_1 &= 10 = A_1 0.5^1 + A_2 0.25^1 \\
 A_1 &= 20 - A_2 0.5
 \end{aligned}$$

$$\begin{aligned}
 y_2 &= 10 = (20 - A_2 0.5)(0.5)^2 + A_2 (0.25)^2 \\
 10 &= (20 - A_2 0.5)(0.25) + A_2 (0.0625) \\
 10 &= (5 - A_2 0.125) + A_2 (0.0625) \\
 5 &= -A_2 0.0625 \\
 A_2 &= -80
 \end{aligned}$$

$$\begin{aligned}
 A_1 &= 20 - (-80)0.5 \\
 A_1 &= 60
 \end{aligned}$$

$$y_t = 60(0.5)^t - 80(0.25)^t$$



ii.

$$y_t = A_1(0.75 + i\sqrt{0.1875})^t + A_2(0.75 - i\sqrt{0.1875})^t$$

$$(h \pm iv)^t = R^t(\cos(\theta t) \pm i \sin(\theta t))$$

$$R = \sqrt{0.75}, \quad \cos(\theta) = \frac{1.5}{2\sqrt{0.75}} = \sqrt{0.75}, \quad \sin(\theta) = \sqrt{1 - \frac{(-1.5)^2}{4(0.75)}} = 0.5 \rightarrow \theta = \frac{\pi}{6}$$

$$\begin{aligned}
 \Rightarrow y_t &= \sqrt{0.75}^t [A_1 \{\cos(\theta t) + i \sin(\theta t)\} + A_2 \{\cos(\theta t) - i \sin(\theta t)\}] \\
 &= \sqrt{0.75}^t \{[A_1 + A_2] \cos(\theta t) + [A_1 - A_2] i \sin(\theta t)\} \\
 y_t &= \sqrt{0.75}^t \{A_5 \cos(\frac{\pi}{6} t) + A_6 \sin(\frac{\pi}{6} t)\} \quad (\text{forma polar})
 \end{aligned}$$

$$y_1 = y_2 = 10 \Rightarrow$$

$$10 = \sqrt{0.75} \{A_5 \cos(\frac{\pi}{6}) + A_6 \sin(\frac{\pi}{6})\}$$

$$\frac{10}{\sqrt{0.75}} = A_5 \sqrt{0.75} + A_6 0.5$$

$$A_6 = \frac{20}{\sqrt{0.75}} - A_5 2\sqrt{0.75}$$

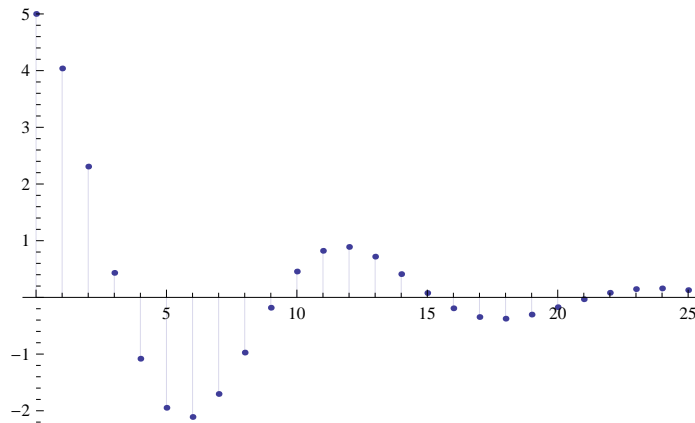
$$10 = \sqrt{0.75}^2 \{A_5 \cos(2\frac{\pi}{6}) + A_6 \sin(2\frac{\pi}{6})\}$$

$$10 = 0.75 \{A_5 \cos(\frac{\pi}{3}) + (\frac{20}{\sqrt{0.75}} - A_5 2\sqrt{0.75}) \sin(\frac{\pi}{3})\}$$

$$10 = 0.75 \{A_5 0.5 + (\frac{20}{\sqrt{0.75}} - A_5 2\sqrt{0.75}) \sqrt{0.75}\}$$

$$15 = A_5 0.5 + 20 - A_5 1.5 \Rightarrow A_5 = 5 \text{ e } A_6 = \frac{2\sqrt{3}}{3}$$

$$y_t = \sqrt{0.75}^t \{5 \cos(\frac{\pi}{6}t) + \frac{2\sqrt{3}}{3} \sin(\frac{\pi}{6}t)\}$$



iii.

$$y_t = A_1(0.9)^t + A_2 t(0.9)^t$$

$$y_1 \Rightarrow 10 = A_1(0.9) + A_2(0.9)$$

$$A_1 = \frac{10}{0.9} - A_2$$

$$y_2 \Rightarrow 10 = A_1(0.9)^2 + A_2 2(0.9)^2$$

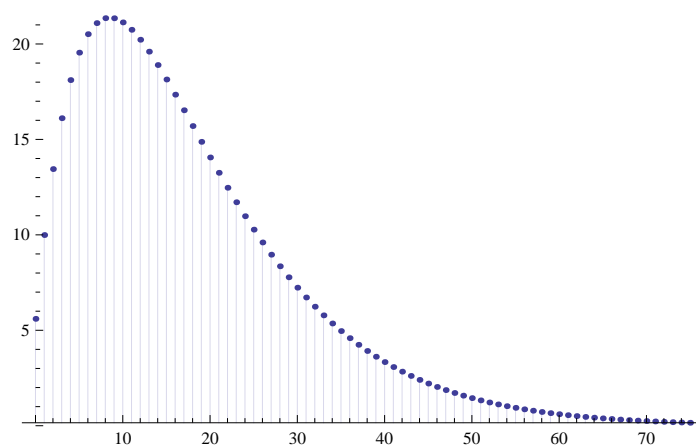
$$10 = (\frac{10}{0.9} - A_2)^2 + A_2 1.62 = \frac{100}{0.81} - 2A_2 \frac{10}{0.9} + A_2 1.62$$

$$10 - \frac{100}{0.81} = -A_2 \frac{20}{0.9} + A_2 1.62$$

$$-113.45679 = -20.60222A_2$$

$$\Rightarrow A_2 \approx 5.5 \text{ e } A_1 \approx 5.6$$

$$y_t = 5.6(0.9)^t + 5.5t(0.9)^t$$



iv.

$$y_t = A_1(0.75)^t + A_2 t(0.75)^t$$

$$y_1 \Rightarrow 10 = A_1(0.75) + A_2(0.75)$$

$$A_1 = \frac{40}{3} - A_2$$

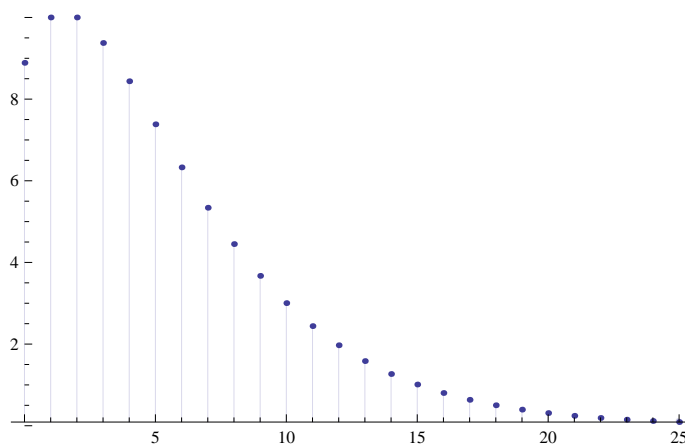
$$y_t = \left(\frac{40}{3} - A_2\right)(0.75)^t + A_2 t(0.75)^t$$

$$y_2 \Rightarrow 10 = \left(\frac{40}{3} - A_2\right)(0.75)^2 + A_2 2(0.75)^2$$

$$10 = \frac{15}{2} - A_2 \frac{9}{16} + A_2 \frac{18}{16}$$

$$A_2 = \frac{5}{2} \frac{16}{9} = \frac{40}{9} \text{ e } A_1 = \frac{80}{9}$$

$$y_t = \frac{80}{9}(0.75)^t + \frac{40}{9}t(0.75)^t$$



6. a)

$$\begin{aligned}
y_t &= 1 + 0.7y_{t-1} - 0.1y_{t-2} + \varepsilon_t \\
y_t - 0.7y_{t-1} + 0.1y_{t-2} &= 0 \Rightarrow b_{1,2} = \frac{0.7 \pm \sqrt{0.7^2 - 4(0.1)}}{2} \\
b_1 &= 0.5 \\
b_2 &= 0.2 \\
y_t^c &= A_1^t 0.5 + A_2 0.2^t
\end{aligned}$$

A solução particular teste neste caso é

$$\begin{aligned}
y_t^p &= b_0 + b_1 t + b_2 t^2 + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i} \\
\Rightarrow b_0 + b_1 t + b_2 t^2 + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i} &= 1 + 0.7[b_0 + b_1(t-1) + b_2(t-1)^2 + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-1-i}] \\
&\quad - 0.1[b_0 + b_1(t-2) + b_2(t-2)^2 + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-2-i}] + \varepsilon_t
\end{aligned}$$

Tratando primeiramente da parte estocástica da solução particular

$$\begin{aligned}
\Rightarrow \alpha_0 &= 1 \\
\alpha_1 &= 0.7\alpha_0 \\
\alpha_2 &= 0.7\alpha_1 - 0.1\alpha_0 \\
\alpha_3 &= 0.7\alpha_2 - 0.1\alpha_1 \\
\alpha_4 &= 0.7\alpha_3 - 0.1\alpha_2 \\
&\vdots \\
\alpha_i &= 0.7\alpha_{i-1} - 0.1\alpha_{i-2} \\
\Rightarrow \alpha_i - 0.7\alpha_{i-1} + 0.1\alpha_{i-2} &= 0 \\
\alpha_i &= A_1(0.2)^i + A_2(0.5)^i \\
\alpha_0 = 1 &\Rightarrow A_1 = 1 - A_2
\end{aligned}$$

$$\begin{aligned}
\alpha_i &= (1 - A_2)(0.2)^i + A_2(0.5)^i \\
\alpha_1 = 0.7 &\Rightarrow 0.7 = (1 - A_2)(0.2) + A_2(0.5) \\
0.7 &= 0.2 - A_2(0.2) + A_2(0.5) \\
0.5 &= A_2(0.3) \Rightarrow A_2 = \frac{5}{3} \text{ e } A_1 = -\frac{2}{3}
\end{aligned}$$

$$\alpha_i = -\frac{2}{3}(0.2)^i + \frac{5}{3}(0.5)^i$$

Como neste problema não existe raiz unitária, não existe tendência na solução particular, então $b_1 = b_2 = 0$. Se $y_t^p = b_0$ e tratando $\varepsilon_{t-i} = 0$, podemos encontrar a parte

determinística da solução particular.

$$\begin{aligned} b_0 &= 1 + 0.7b_0 - 0.1b_0 \\ b_0 &= \frac{1}{(1 - 0.7 + 0.1)} = \frac{5}{2} \\ \Rightarrow y_t^p &= \frac{5}{2} + \sum_{i=0}^{\infty} \left[\frac{5}{3}(0.5)^i - \frac{2}{3}(0.2)^i \right] \varepsilon_{t-i} \end{aligned}$$

Então a solução geral fica:

$$y_t = y_t^c + y_t^p = A_1 0.5^t + A_2 0.2^t + \frac{5}{2} + \sum_{i=0}^{\infty} \left[\frac{5}{3}(0.5)^i - \frac{2}{3}(0.2)^i \right] \varepsilon_{t-i}$$

b) Função complementar

$$\begin{aligned} y_t &= 1 - 0.3y_{t-1} + 0.1y_{t-2} + \varepsilon_t \\ b_1 &= \frac{-0.3 + \sqrt{(0.3)^2 + 4(0.1)}}{2} = 0.2 \\ b_2 &= \frac{-0.3 - \sqrt{(0.3)^2 + 4(0.1)}}{2} = -0.5 \end{aligned}$$

$$y_t^c = A_1(0.2)^t + A_2(-0.5)^t$$

solução particular determinística

$$\begin{aligned} y_t^{p,d} = b_0 &\Rightarrow b_0 = 1 - 0.3b_0 + 0.1b_0 \\ y_t^{p,d} = b_0 &= \frac{1}{1 + 0.3 - 0.1} = 0.8333 \end{aligned}$$

solução particular estocástica

$$y_t^{p,e} = \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i}$$

substituindo na equação em diferença

$$\sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i} = 1 - 0.3 \left[\sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-1-i} \right] + 0.1 \left[\sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-2-i} \right] + \varepsilon_t$$

$$\begin{aligned} \Rightarrow \alpha_0 &= 1 \\ \alpha_1 &= -0.3\alpha_0 = -0.3 \\ \alpha_2 &= -0.3\alpha_1 + 0.1\alpha_0 \\ &\vdots \\ \alpha_i &= -0.3\alpha_{i-1} + 0.1\alpha_{i-2} \end{aligned}$$

$$\begin{aligned}
&\Rightarrow \alpha_i = A_1(0.2)^i + A_2(-0.5)^i \\
&\alpha_0 = 1 \Rightarrow A_1 = 1 - A_2 \\
&\alpha_i = (1 - A_2)(0.2)^i + A_2(-0.5)^i \\
&\alpha_1 = -0.3 \Rightarrow -0.3 = (1 - A_2)(0.2) + A_2(-0.5) \\
&\quad -0.3 = 0.2 - A_2(0.2) + A_2(-0.5) \Rightarrow A_2 = \frac{5}{7} \text{ e } A_1 = \frac{2}{7} \\
&\alpha_i = \left(\frac{2}{7}\right)(0.2)^i + \left(\frac{5}{7}\right)(-0.5)^i \\
&y_t^{p,e} = \sum_{i=0}^{\infty} \left[\left(\frac{2}{7}\right)(0.2)^i + \left(\frac{5}{7}\right)(-0.5)^i \right] \varepsilon_{t-i}
\end{aligned}$$

A solução geral fica

$$y_t = y_t^c + y_t^{p,d} + y_t^{p,e} = A_1(0.2)^t + A_2(-0.5)^t + \frac{12}{10} + \sum_{i=0}^{\infty} \left[\left(\frac{2}{7}\right)(0.2)^i + \left(\frac{5}{7}\right)(-0.5)^i \right] \varepsilon_{t-i}$$

7. $y_t = a_0 + a_2 y_{t-2} + \varepsilon_t$

a)

$$\begin{aligned}
&y_t - a_2 y_{t-2} = 0 \\
&\Rightarrow b^2 - a_2 = 0, \quad b_{1,2} = \pm \sqrt{a_2} \\
&y_t^c = A_1 \sqrt{a_2}^t + A_2 (-\sqrt{a_2})^t
\end{aligned}$$

b)

$$\begin{aligned}
&y_t^p = b_0 + b_1 t + b_2 t^2 + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i} \\
&\Rightarrow b_0 + b_1 t + b_2 t^2 + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i} = a_0 + a_2 \left[b_0 + b_1 t + b_2 t^2 + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-2-i} \right] + \varepsilon_t
\end{aligned}$$

$$\begin{aligned}
&\alpha_0 = 1 \text{ e } \alpha_1 = 0 \\
&\alpha_2 = a_2 \alpha_0 = a_2 \\
&\alpha_3 = a_2 \alpha_1 = 0 \\
&\alpha_4 = a_2 \alpha_2 = a_2^2 \\
&\alpha_5 = a_2 \alpha_3 = 0 \\
&\vdots \\
&\alpha_i = a_2 \alpha_{i-2} \\
&\Rightarrow \alpha_i = A_1 \sqrt{a_2}^i + A_2 (-\sqrt{a_2})^i \\
&\alpha_0 = 1 \Rightarrow A_1 = 1 - A_2 \\
&\alpha_i = [1 - A_2] \sqrt{a_2}^i + A_2 (-\sqrt{a_2})^i \\
&\alpha_1 = 0 \Rightarrow 0 = [1 - A_2] \sqrt{a_2} + A_2 (-\sqrt{a_2}) \\
&\quad A_2 = \frac{1}{2\sqrt{a_2}} \text{ e } A_1 = 1 - \frac{1}{2\sqrt{a_2}} \\
&\alpha_i = \left[1 - \frac{1}{2\sqrt{a_2}} \right] \sqrt{a_2}^i + \left[\frac{1}{2\sqrt{a_2}} \right] (-\sqrt{a_2})^i
\end{aligned}$$

$$\begin{aligned}
b_0 + b_1 t + b_2 t^2 &= a_0 + a_2 \left[b_0 + b_1 t + b_2 t^2 \right] \\
&\Rightarrow b_0(1 - a_2) + b_1(1 - a_2)t + b_2(1 - a_2)t^2 - a_0 = 0
\end{aligned}$$

$$\text{se } a_2 \neq 1 \Rightarrow b_1 = b_2 = 0 \text{ e } b_0 = \frac{a_0}{(1 - a_2)}$$

$$y_t^p = \frac{a_0}{(1 - a_2)} + \sum_{i=0}^{\infty} \left\{ \left[1 - \frac{1}{2\sqrt{a_2}} \right] \sqrt{a_2}^i + \left[\frac{1}{2\sqrt{a_2}} \right] (-\sqrt{a_2})^i \right\} \varepsilon_{t-i}$$

Se $a_2 = 1$ implica em $a_0 = 0$ e temos duas raízes unitárias 1 e -1 . Com isso, o efeito do termo de erro não diminui à medida que o tempo passa e a solução geral não converge.

c)

$$\begin{aligned}
y_t^p &= a_0 + a_2 L^2 y_t + \varepsilon_t \\
y_t^p &= \frac{a_0}{(1 - a_2 L^2)} + \frac{\varepsilon_t}{(1 - a_2 L^2)} \\
&\equiv \frac{a_0}{1 - a_2} + \frac{1}{(1 - \sqrt{a_2} L)(1 + \sqrt{a_2} L)} \varepsilon_t \\
y_t^p &= \frac{a_0}{1 - a_2} + \frac{1}{(1 - \sqrt{a_2})} \sum_{i=0}^{\infty} (-\sqrt{a_2})^i \varepsilon_{t-i}
\end{aligned}$$

8. a)

$$\begin{aligned}
y_t - y_{t-1} &= 0 \\
\Rightarrow A b^t - A b^{t-1} &= 0 \\
b &= 1 \Rightarrow y_t = A \equiv y_t = c
\end{aligned}$$

b)

$$\begin{aligned}
y_t - y_{t-1} &= a_0 \\
\Rightarrow y_t^c &= A \\
y_t^p &= k t \Rightarrow k t - k(t-1) = a_0 \\
k &= a_0 \Rightarrow y_t^p = a_0 t \\
y_t &= c + a_0 t
\end{aligned}$$

c)

$$\begin{aligned}
y_t - y_{t-2} &= 0 \\
\Rightarrow b^2 - 1 &= 0 \\
b_{1,2} = \pm 1 &\Rightarrow y_t = A_1(1)^t + A_2(-1)^t \equiv y_t = c + a_0(-1)^t
\end{aligned}$$

d)

$$y_t - y_{t-2} = \varepsilon_t$$

$$y_t - y_{t-2} = 0 \Rightarrow b_{1,2} = \pm 1$$

$$y_t^c = A_1 + A_2(-1)^t$$

$$y_t^p - L^2 y_t^p = \varepsilon_t \Rightarrow$$

$$\begin{aligned} y_t^p &= \frac{\varepsilon_t}{1 - L^2} = \frac{1}{(1 - L)(1 + L)} \varepsilon_t = \sum_{i=0}^{\infty} (1)^i \sum_{i=0}^{\infty} (-1)^i \varepsilon_{t-i} \\ &= i \sum_{i=0}^{\infty} (-1)^i \varepsilon_{t-i} = \sum_{i=0}^{\infty} (-1)^i i \varepsilon_{t-i} \end{aligned}$$

$$y_t = y_t^c + y_t^p = A_1 + A_2(-1)^t + \sum_{i=0}^{\infty} (-1)^i i \varepsilon_{t-i}$$

9. a) i.

$$y_t - 1.2y_{t-1} + 0.2y_{t-2} = 0$$

$$\Rightarrow b_1 = \frac{1.2 + \sqrt{1.2^2 - 4(0.2)}}{2} = 1$$

$$b_2 = \frac{1.2 - \sqrt{1.2^2 - 4(0.2)}}{2} = 0.2$$

A sequência $\{y_t\}$ não é estável pois possui uma raiz unitária. As raízes características são reais e positivas.

ii.

$$y_t - 1.2Ly_t + 0.2L^2y_t = 0$$

$$y_t(1 - 1.2L + 0.2L^2) = 0$$

$$\Rightarrow L^2 - 6L + 5 = 0$$

$$L_1 = 3 + \frac{\sqrt{6^2 - 4(5)}}{2} = 5$$

$$L_2 = 3 - \frac{\sqrt{6^2 - 4(5)}}{2} = 1$$

b) i.

$$y_t - 1.2y_{t-1} + 0.4y_{t-2} = 0$$

$$\Rightarrow b_1 = \frac{1.2 + \sqrt{1.2^2 - 4(0.4)}}{2} = 0.6 + i 0.2$$

$$b_2 = \frac{1.2 - \sqrt{1.2^2 - 4(0.4)}}{2} = 0.6 - i 0.2$$

As raízes são imaginárias e a sequência $\{y_t\}$ possui padrão flutuante de natureza periódica. Como $R = \sqrt{a_2} = \sqrt{1.2} = 1.095 > 1$ a sequência não é estável. As partes reais são positivas.

ii.

$$y_t - 1.2Ly_t + 0.4L^2y_t = 0$$

$$y_t(1 - 1.2L + 0.4L^2) = 0$$

$$\Rightarrow 2.5 - 2L + L^2 = 0$$

$$L_1 = 1 + \frac{\sqrt{2^2 - 4(2.5)}}{2} = 1 + i\sqrt{\frac{3}{2}}$$

$$L_2 = 1 - \frac{\sqrt{2^2 - 4(2.5)}}{2} = 1 - i\sqrt{\frac{3}{2}}$$

c) i.

$$y_t - 1.2y_{t-1} + 1.2y_{t-2} = 0$$

$$b_1 = 0.6 + \frac{\sqrt{1.2^2 + 4(1.2)}}{2} = 1.85$$

$$b_2 = 0.6 - \frac{\sqrt{1.2^2 + 4(1.2)}}{2} = -0.65$$

As raízes são reais, uma sendo positiva e a outra negativa. Como uma das raízes é maior que um em valor absoluto, a sequência $\{y_t\}$ não é estável.

ii.

$$y_t(1 - 1.2L + 1.2L^2) = 0$$

$$\Rightarrow \frac{10}{12} - L + L^2 = 0$$

$$L_1 = 0.5 + \frac{\sqrt{1^2 - 4(\frac{10}{12})}}{2} = 0.5 + i\sqrt{\frac{28}{3}}$$

$$L_2 = 0.5 - \frac{\sqrt{1^2 - 4(\frac{10}{12})}}{2} = 0.5 - i\sqrt{\frac{28}{3}}$$

d) i.

$$y_t + 1.2y_{t-1} = 0$$

$$\Rightarrow b + 1 = 0, \quad b = -1$$

A sequência possui raiz unitária, real e negativa, portanto não é estável.

ii.

$$y_t(1 + 1.2L^2) = 0$$

$$\Rightarrow \frac{10}{12} + L^2 = 0$$

$$L_1 = \sqrt{\frac{5}{6}}$$

$$L_2 = -\sqrt{\frac{5}{6}}$$

e) i.

$$y_t - 0.7y_{t-1} - 0.25y_{t-2} + 0.175y_{t-3} = 0$$

$$\Rightarrow b^3 - 0.7b^2 - 0.25b + 0.175 = 0$$

$$(b - 0.5)(b + 0.5)(b - 0.7) = 0$$

$$b_1 = 0.5, \quad b_2 = -0.5, \quad b_3 = 0.7$$

A sequência é estável, com três raízes reais sendo duas positivas e uma negativa.

ii.

$$\begin{aligned}
y_t(1 - 0.7L - 0.25L^2 + 0.175L^3) &= 0 \\
\Rightarrow 1 - \frac{7}{10}L - \frac{1}{4}L^2 + \frac{7}{40}L^3 &= 0 \\
\frac{1}{40}(7L - 10)(L - 2)(L + 2) &= 0 \\
(7L - 10)(L - 2)(L + 2) &= 0 \\
L_1 = \frac{10}{7}, L_2 = 2, L_3 = -2
\end{aligned}$$

10.

$$y_t = 0.8y_{t-1} + \varepsilon_t - 0.5\varepsilon_{t-1}$$

a)

$$\begin{aligned}
y_1 &= 1 \\
y_2 &= 0.8y_1 + \varepsilon_2 - 0.5\varepsilon_1 = \varepsilon_2 + 0.3 \\
y_3 &= 0.8y_2 + \varepsilon_3 - 0.5\varepsilon_2 = 0.8(\varepsilon_2 + 0.3) + \varepsilon_3 - 0.5\varepsilon_2 = \varepsilon_3 + 0.3\varepsilon_2 + 0.24 \\
y_4 &= 0.8y_3 + \varepsilon_4 - 0.5\varepsilon_3 = 0.8(\varepsilon_3 + 0.3\varepsilon_2 + 0.24) + \varepsilon_4 - 0.5\varepsilon_3 = \varepsilon_4 + 0.3\varepsilon_3 + 0.24\varepsilon_2 + 0.192 \\
y_5 &= 0.8y_4 + \varepsilon_5 - 0.5\varepsilon_4 = 0.8(\varepsilon_4 + 0.3\varepsilon_3 + 0.24\varepsilon_2 + 0.192) + \varepsilon_5 - 0.5\varepsilon_4 \\
&= \varepsilon_5 + 0.3\varepsilon_4 + 0.24\varepsilon_3 + 0.192\varepsilon_2 + 0.1536
\end{aligned}$$

b)

$$y_t^c = A(0.8)^t$$

$$y_t^p \Rightarrow y_t = 0.8Ly_t + \varepsilon_t - 0.5L\varepsilon_t$$

$$y_t^p = \frac{\varepsilon_t - 0.5L\varepsilon_t}{(1 - 0.8L)} = \sum_{i=0}^{\infty} (0.8)^i (\varepsilon_{t-i}) - \sum_{i=0}^{\infty} (0.8)^i (0.5\varepsilon_{t-1-i}) = \varepsilon_t + \sum_{i=0}^{\infty} (0.8)^i (0.3\varepsilon_{t-(i+1)})$$

$$y_t^* = A(0.8)^t + \varepsilon_t + \sum_{i=0}^{\infty} (0.8)^i (0.3\varepsilon_{t-(i+1)})$$

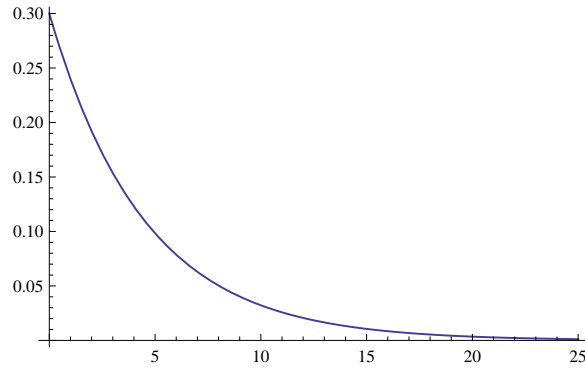
c)

$$y_0 = 0 \Rightarrow 0 = A \quad \therefore$$

$$y_t^* = \varepsilon_t + \sum_{i=0}^{\infty} (0.8)^i (0.3\varepsilon_{t-(i+1)})$$

d)

$$\begin{aligned}
\frac{\partial y_t}{\partial \varepsilon_t} &= 1 \\
\frac{\partial y_t}{\partial \varepsilon_{t-i}} &= (0.8)^i 0.3
\end{aligned}$$



11. $0 < \alpha < 1$ e $\beta > 0$

caso $d > 0$

$$a_1 + a_2 < 1 \Rightarrow \alpha(1 + \beta) - \alpha\beta < 1$$

$$\therefore \alpha < 1$$

$$a_2 < 1 + a_1 \Rightarrow -\alpha\beta < 1 + \alpha(1 + \beta)$$

$$0 < 1 + \alpha(1 + \beta) + \alpha\beta$$

caso $d = 0$

$$|a_1| = 2 \Rightarrow |\alpha(1 + \beta)| < 2, \alpha < \frac{2}{1 + \beta}$$

caso $d < 0$

$$-a_2 < 1 \Rightarrow \alpha\beta < 1, \quad \beta < \frac{1}{\alpha}$$

12. a) i.

$$y_t = 3 + 0.75y_{t-1} - 0.125y_{t-2} + \varepsilon_t$$

$$\Rightarrow y_t^p = b_0 + b_1t + b_2t^2 + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i} \quad \therefore$$

$$b_0 + b_1t + b_2t^2 + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i} = 3 + 0.75 \left[b_0 + b_1(t-1) + b_2(t-1)^2 + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-1-i} \right]$$

$$- 0.125 \left[b_0 + b_1(t-2) + b_2(t-2)^2 + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-2-i} \right] + \varepsilon_t$$

Tratando a parte determinística da solução particular:

$$b_0 + b_1t + b_2t^2 = 3 + 0.75[b_0 + b_1(t-1) + b_2(t-1)^2] - 0.125[b_0 + b_1(t-2) + b_2(t-2)^2]$$

Como a sequência $\{y_t\}$ não possui raiz unitária, a solução não possui tendência e portanto $b_1 = b_2 = 0$. Então:

$$b_0 = 3 + 0.75b_0 - 0.125b_0$$

$$b_0 = 8$$

Tratando a parte estocástica da solução particular:

$$\begin{aligned}
\sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i} &= 0.75 \left[\sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-1-i} \right] - 0.125 \left[\sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-2-i} \right] + \varepsilon_t \\
&\Rightarrow \alpha_0 = 1 \\
&\alpha_1 = 0.75\alpha_0 \\
&\alpha_2 = 0.75\alpha_1 - 0.125\alpha_0 \\
&\alpha_3 = 0.75\alpha_2 - 0.125\alpha_1 \\
&\vdots \\
&\alpha_i = 0.75\alpha_{i-1} - 0.125\alpha_{i-2} \\
&\Rightarrow \alpha_i = A_1(0.5)^i + A_2(0.25)^i \\
\alpha_0 = 1 &\Rightarrow A_1 = 1 - A_2 \\
\alpha_i &= (1 - A_2)(0.5)^i + A_2(0.25)^i \\
\alpha_1 = 0.75 &\Rightarrow 0.75 = (1 - A_2)0.5 + A_20.25, \quad A_2 = -1, \quad A_1 = 2 \\
\alpha_i &= 2(0.5)^i - (0.25)^i \\
y_t^p &= 8 + \sum_{i=0}^{\infty} \left[2(0.5)^i - (0.25)^i \right] \varepsilon_{t-i}
\end{aligned}$$

ii.

$$y_t = 3 + 0.25y_{t-1} + 0.375y_{t-2} + \varepsilon_t$$

parte determinística:

$$\begin{aligned}
y_t &= b_0 \\
\Rightarrow b_0 &= 3 + 0.25b_0 + 0.375b_0 \\
b_0 &= \frac{3}{(0.375)} = 8
\end{aligned}$$

parte estocástica:

$$\begin{aligned}
\sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i} &= 0.25 \left[\sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-1-i} \right] + 0.375 \left[\sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-2-i} \right] + \varepsilon_t \\
\alpha_0 &= 1 \\
\alpha_1 &= 0.25\alpha_0 \\
\alpha_2 &= 0.25\alpha_1 + 0.375\alpha_1 \\
\alpha_3 &= 0.25\alpha_2 + 0.375\alpha_2 \\
&\vdots \\
\alpha_i &= 0.25\alpha_{i-1} + 0.375\alpha_{i-2}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \alpha_i &= A_1(-0.5)^i + A_2(0.75)^i \\
\alpha_0 = 1 &\Rightarrow A_1 = 1 - A_2 \\
\alpha_i &= (1 - A_2)(-0.5)^i + A_2(0.75)^i \\
\alpha_1 = 0.25 &\Rightarrow 0.25 = (1 - A_2)(-0.5) + A_2(0.75) \\
A_2 &= 0.6, \quad A_1 = 0.4
\end{aligned}$$

$$\alpha_i = 0.4(-0.5)^i + 0.6(0.75)^i$$

$$y_t^p = 8 + \sum_{i=0}^{\infty} \left[0.4(-0.5)^i + 0.6(0.75)^i \right] \varepsilon_{t-i}$$

b) i.

$$y_t^h = A_1(0.5)^t + A_2(0.25)^t$$

ii.

$$y_t^h = A_1(-0.5)^t + A_2(0.75)^t$$

c) i.

$$y_t = A_1(0.5)^t + A_2(0.25)^t + 8 + \sum_{i=0}^{\infty} \left[2(0.5)^i - (0.25)^i \right] \varepsilon_{t-i}$$

$$\begin{aligned}
y_0 = 8 &\Rightarrow 8 = A_1 + A_2 + 8, \quad A_1 = -A_2 \\
y_1 = 8 &\Rightarrow 8 = -A_2(0.5) + A_2(0.25) + 8, \quad A_2 = 0, \quad A_1 = 0
\end{aligned}$$

ii.

$$y_t = A_1(-0.5)^t + A_2(0.75)^t + 8 + \sum_{i=0}^{\infty} \left[0.4(-0.5)^i + 0.6(0.75)^i \right] \varepsilon_{t-i}$$

$$\begin{aligned}
y_0 = 8 &\Rightarrow 8 = A_1 + A_2 + 8, \quad A_1 = -A_2 \\
y_1 = 8 &\Rightarrow 8 = -A_2(-0.5) + A_2(0.75) + 8, \quad A_2 = 0, \quad A_1 = 0
\end{aligned}$$

13. a)

$$\begin{aligned}
y_{t-1} &= 0.75y_{t-2} + \varepsilon_{t-1} \\
&= 0.75[0.75y_{t-3} + \varepsilon_{t-2}] + \varepsilon_{t-1} = (0.75)^2 y_{t-3} + 0.75\varepsilon_{t-2} + \varepsilon_{t-1} \\
&= 0.75[0.75[0.75y_{t-4} + \varepsilon_{t-3}] + \varepsilon_{t-2}] + \varepsilon_{t-1} = (0.75)^3 y_{t-4} + (0.75)^2 \varepsilon_{t-3} + 0.75\varepsilon_{t-2} + \varepsilon_{t-1} \\
y_{t-1} &= (0.75)^{t-1} y_0 + (0.75)^{t-2} \varepsilon_1 + (0.75)^{t-3} \varepsilon_2 + \dots + (0.75)^2 \varepsilon_{t-3} + 0.75\varepsilon_{t-2} + \varepsilon_{t-1}
\end{aligned}$$

b)

$$\alpha_0^t y_0 + \sum_{i=1}^{\infty} \alpha_i \varepsilon_{t-i} + \varepsilon_t = 0.75 \left[\alpha_0^{t-1} y_0 + \varepsilon_{t-1} + \sum_{i=1}^{\infty} \alpha_i \varepsilon_{t-1-i} \right] + \varepsilon_t$$

$$\alpha_0^t = 0.75 \alpha_0^{t-1}, \quad \alpha_0 = 0.75$$

$$\alpha_1 = 0.75$$

$$\alpha_2 = 0.75 \alpha_1$$

$$\alpha_3 = 0.75 \alpha_2$$

$$\vdots$$

$$\alpha_i = 0.75 \alpha_{i-1}$$

$$\alpha_i = (0.75)^i, \quad i = 1, 2, \dots$$

c)

$$\begin{aligned} b_0 + b_1 t + b_2 t^2 + \sum_{i=1}^{\infty} \alpha_i \varepsilon_{t-i} + \varepsilon_t &= 0.75 \left[b_0 + b_1(t-1) + b_2(t-1)^2 + \varepsilon_{t-1} + \sum_{i=1}^{\infty} \alpha_i \varepsilon_{t-1-i} \right] \\ &\quad - 0.125 \left[b_0 + b_1(t-2) + b_2(t-2)^2 + \varepsilon_{t-2} + \sum_{i=1}^{\infty} \alpha_i \varepsilon_{t-2-i} \right] + \varepsilon_t \end{aligned}$$

Tratando a parte determinística da solução particular:

$$b_0 + b_1 t + b_2 t^2 = 3 + 0.75[b_0 + b_1(t-1) + b_2(t-1)^2] - 0.125[b_0 + b_1(t-2) + b_2(t-2)^2]$$

Como a sequência $\{y_t\}$ não possui raiz unitária, a solução não possui tendência e portanto $b_1 = b_2 = 0$. Então:

$$b_0 = 0.75 b_0 - 0.125 b_0$$

$$b_0 = 0$$

Tratando a parte estocástica da solução particular:

$$\begin{aligned}
 \sum_{i=1}^{\infty} \alpha_i \varepsilon_{t-i} + \varepsilon_t &= 0.75 \left[\varepsilon_{t-1} + \sum_{i=1}^{\infty} \alpha_i \varepsilon_{t-1-i} \right] - 0.125 \left[\varepsilon_{t-2} + \sum_{i=1}^{\infty} \alpha_i \varepsilon_{t-2-i} \right] + \varepsilon_t \\
 \Rightarrow \alpha_1 &= 0.75 \\
 \alpha_2 &= 0.75\alpha_1 - 0.125 \\
 \alpha_3 &= 0.75\alpha_2 - 0.125\alpha_1 \\
 &\vdots \\
 \alpha_i &= 0.75\alpha_{i-1} - 0.125\alpha_{i-2} \\
 \Rightarrow \alpha_i &= A_1(0.5)^i + A_2(0.25)^i \\
 \alpha_1 = 0.75 &\Rightarrow 0.75 = A_1(0.5) + A_2(0.25) \\
 A_2 &= 3 - 2A_1 \\
 \alpha_i &= (A_1)(0.5)^i + (3 - 2A_1)(0.25)^i \\
 \alpha_2 = 0.4375 &\Rightarrow 0.4375 = (A_1)(0.5)^2 + (3 - 2A_1)(0.25)^2 \\
 0.4375 &= (A_1)(0.25) + (3 - 2A_1)(0.0625) \\
 0.4375 &= (A_1)(0.25) + (0.1875 - 0.125A_1) \\
 0.25 &= 0.125A_1, \quad A_1 = 2, \quad A_2 = -1
 \end{aligned}$$

$$\alpha_i = 2(0.5)^i - (0.25)^i, \quad i = 1, 2, \dots$$

$$y_t^p = \sum_{i=1}^{\infty} \left[2(0.5)^i - (0.25)^i \right] \varepsilon_{t-i} + \varepsilon_t$$

$$y_t^* = B_1(0.5)^t + B_2(0.25)^t + \sum_{i=1}^{\infty} \left[2(0.5)^i - (0.25)^i \right] \varepsilon_{t-i} + \varepsilon_t$$

Capítulo 2

Modelos de séries de tempo estacionárias

1. $w_t = \frac{1}{4}\varepsilon_t + \frac{1}{4}\varepsilon_{t-1} + \frac{1}{4}\varepsilon_{t-2} + \frac{1}{4}\varepsilon_{t-3}$

a) i.

$$\begin{aligned} E(w_t) &= E\left(\frac{1}{4}\varepsilon_t + \frac{1}{4}\varepsilon_{t-1} + \frac{1}{4}\varepsilon_{t-2} + \frac{1}{4}\varepsilon_{t-3}\right) \\ &= \frac{1}{4}E(\varepsilon_t) + \frac{1}{4}E(\varepsilon_{t-1}) + \frac{1}{4}E(\varepsilon_{t-2}) + \frac{1}{4}E(\varepsilon_{t-3}) = 0 \end{aligned}$$

ii.

$$\begin{aligned} E(w_t | \varepsilon_{t-3} = \varepsilon_{t-2} = 1) &= E\left(\frac{1}{4}\varepsilon_t + \frac{1}{4}\varepsilon_{t-1} + \frac{1}{4}(1) + \frac{1}{4}(1)\right) \\ &= \frac{1}{4}E(\varepsilon_t) + \frac{1}{4}E(\varepsilon_{t-1}) + \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \end{aligned}$$

b) i.

$$\begin{aligned} \text{var}(w_t) &= E(w_t^2) - E(w_t)^2, \quad E(w_t) = 0 \\ \Rightarrow E(w_t^2) &= E\left(\left[\frac{1}{4}\varepsilon_t + \frac{1}{4}\varepsilon_{t-1} + \frac{1}{4}\varepsilon_{t-2} + \frac{1}{4}\varepsilon_{t-3}\right]^2\right) \\ &= \frac{1}{16}E(\varepsilon_t^2) + \frac{1}{16}E(\varepsilon_{t-1}^2) + \frac{1}{16}E(\varepsilon_{t-2}^2) + \frac{1}{16}E(\varepsilon_{t-3}^2), \quad \text{pois } E(\varepsilon_t, \varepsilon_{t-s}) = 0 \quad \forall s > 0 \\ &= \frac{1}{16}\sigma^2 + \frac{1}{16}\sigma^2 + \frac{1}{16}\sigma^2 + \frac{1}{16}\sigma^2 = \frac{1}{4}\sigma^2 \end{aligned}$$

ii.

$$\begin{aligned} \text{var}(w_t | \varepsilon_{t-3} = \varepsilon_{t-2} = 1) &= E(w_t^2 | \varepsilon_{t-3} = \varepsilon_{t-2} = 1) - E(w_t | \varepsilon_{t-3} = \varepsilon_{t-2} = 1)^2, \\ E(w_t | \varepsilon_{t-3} = \varepsilon_{t-2} = 1) &= \frac{1}{2} \\ \Rightarrow \text{var}(w_t | \varepsilon_{t-3} = \varepsilon_{t-2} = 1) &= E(w_t^2 | \varepsilon_{t-3} = \varepsilon_{t-2} = 1) - \frac{1}{4} \end{aligned}$$

$$\begin{aligned} E(w_t^2 | \varepsilon_{t-3} = \varepsilon_{t-2} = 1) &= E\left(\left[\frac{1}{4}\varepsilon_t + \frac{1}{4}\varepsilon_{t-1} + \frac{1}{2}\right]^2\right) \\ &= \frac{1}{16}E(\varepsilon_t^2) + \frac{1}{16}E(\varepsilon_{t-1}^2) + \frac{1}{4} \\ &= \frac{1}{4} + \frac{1}{8}\sigma^2 \\ \text{var}(w_t | \varepsilon_{t-3} = \varepsilon_{t-2} = 1) &= \frac{1}{4} + \frac{1}{8}\sigma^2 - \frac{1}{4} = \frac{1}{8}\sigma^2 \end{aligned}$$

c) i.

$$\begin{aligned}
\text{cov}(w_t, w_{t-1}) &= E(w_t w_{t-1}) - E(w_t)E(w_{t-1}) \\
&= E(w_t w_{t-1}), \text{ pois } E(w_{t-s}) = 0 \forall s \geq 0 \\
E(w_t w_{t-1}) &= E\left[\left(\frac{1}{4}\varepsilon_t + \frac{1}{4}\varepsilon_{t-1} + \frac{1}{4}\varepsilon_{t-2} + \frac{1}{4}\varepsilon_{t-3}\right)\left(\frac{1}{4}\varepsilon_{t-1} + \frac{1}{4}\varepsilon_{t-2} + \frac{1}{4}\varepsilon_{t-3} + \frac{1}{4}\varepsilon_{t-4}\right)\right] \\
&= \frac{3}{16}\sigma^2, \quad \text{pois } E(\varepsilon_t, \varepsilon_{t-s}) = 0 \forall s > 0
\end{aligned}$$

ii.

$$\begin{aligned}
\text{cov}(w_t, w_{t-2}) &= E(w_t w_{t-2}) - E(w_t)E(w_{t-2}) \\
&= E(w_t w_{t-2}) \\
E(w_t w_{t-2}) &= E\left[\left(\frac{1}{4}\varepsilon_t + \frac{1}{4}\varepsilon_{t-1} + \frac{1}{4}\varepsilon_{t-2} + \frac{1}{4}\varepsilon_{t-3}\right)\left(\frac{1}{4}\varepsilon_{t-2} + \frac{1}{4}\varepsilon_{t-3} + \frac{1}{4}\varepsilon_{t-4} + \frac{1}{4}\varepsilon_{t-5}\right)\right] \\
&= \frac{1}{8}\sigma^2
\end{aligned}$$

iii.

$$\begin{aligned}
\text{cov}(w_t, w_{t-5}) &= E(w_t w_{t-5}) - E(w_t)E(w_{t-5}) \\
&= E(w_t w_{t-5}) \\
E(w_t w_{t-5}) &= E\left[\left(\frac{1}{4}\varepsilon_t + \frac{1}{4}\varepsilon_{t-1} + \frac{1}{4}\varepsilon_{t-2} + \frac{1}{4}\varepsilon_{t-3}\right)\left(\frac{1}{4}\varepsilon_{t-5} + \frac{1}{4}\varepsilon_{t-6} + \frac{1}{4}\varepsilon_{t-7} + \frac{1}{4}\varepsilon_{t-8}\right)\right] \\
&= 0, \text{ pois só temos termos de erro } \textit{ruído branco} \text{ de períodos diferentes.}
\end{aligned}$$

Com este primeiro exercício podemos perceber que a variância condicional é menor do que a não condicional. Intuitivamente faz sentido, já que pelo fato de termos mais informações no exercício condicionado ($\varepsilon_{t-3} = \varepsilon_{t-2} = 1$), mais correta será nossa estimativa.