18 марта

Интегрирование сложной ф-ии:

1 $f(x) \in I_R[a;b] \Rightarrow m, M;$

2 $\varphi(y)$ опр-на на $[m; M]$

$\exists L > 0: |\varphi(\bar{y}) - \varphi(\bar{\bar{y}})| < L|\bar{y} - \bar{\bar{y}}|$ — условие Липшица. Тогда: $\underbrace{\varphi(f(x)) \in I_R[a;b]}_{h(x)}$

Доказательство.

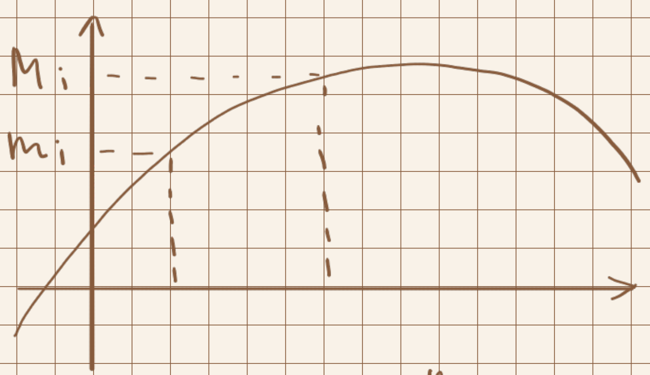
$f(x) \in I_R[a;b] \Rightarrow \exists \int_a^b f(x)\,dx \Leftrightarrow$

$\Leftrightarrow \lim\limits_{\lambda \to 0} \underline{S} = I_* = I^* = \lim\limits_{\lambda \to 0} \bar{S} \Rightarrow$

$\Rightarrow \lim\limits_{\lambda \to 0} (\bar{S} - \underline{S}) = 0 \Leftrightarrow$

$\Leftrightarrow \forall \varepsilon > 0 \ \exists \delta > 0: \forall p, \xi_i \ \ \lambda > \delta \Rightarrow \bar{S} - \underline{S} < \varepsilon,$

т.е. $\exists \int_a^b f(x)\,dx = I \Leftrightarrow \bar{S} - \underline{S} < \varepsilon$



$$\bar{S} - \underline{S} = \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta x_i$$

$$= \sum_{i=1}^{n} (M_i - m_i) \Delta x_i$$

$$= \sum_{i=1}^{n} \omega_i \Delta x_i < \varepsilon$$

$$\exists I \Leftrightarrow \sum_{i=1}^{n} \omega_i \Delta x_i \xrightarrow[\lambda \to 0]{} 0$$

Для $[\Delta x_i] = [x_{i-1}; x_i]$

$$|\bar{y}_i - \bar{\bar{y}}_i| = |f(\bar{x}_i) - f(\bar{\bar{x}}_i)| \le M_i^f - m_i^f$$
$$= \omega_i^f$$

$$\overline{S}_f - \underline{S}_f = \sum_{i=1}^{n} \omega_i \Delta x_i < \varepsilon \quad (\lambda < \delta)$$

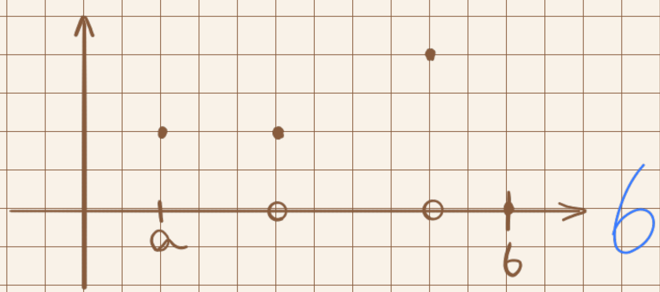
$$\omega_i^h = |h(\bar{x}_i) - h(\bar{\bar{x}}_i)| = |\varphi(\overbrace{f(\bar{x}_i)}^{\bar{y}}) - \varphi(\overbrace{f(\bar{\bar{x}}_i)}^{\bar{\bar{y}}})| \le \quad (*)$$
$$\le L|f(\bar{x}_i) - f(\bar{\bar{x}}_i)| \le L^f \omega_i^f = \omega_i^h$$

$$\overline{S}_h - \underline{S}_h = \sum_{i=1}^{n} \omega_i^h \Delta x_i = \sum_{i=1}^{n} L \omega_i^f \Delta x_i < \varepsilon \cdot L.$$

Доказано!

Оценки опр-ых интегралов.

1 $f(x) \ge 0 \ \ \forall x \in [a; b] \Rightarrow \int_a^b f(x)\,dx \ge 0$



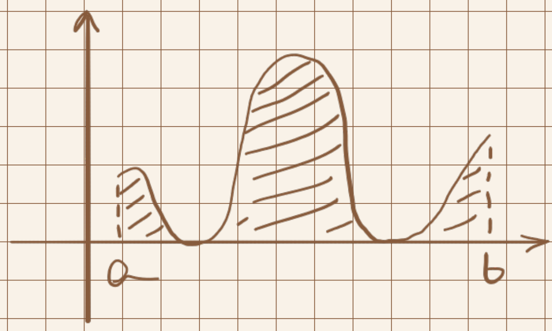
$$\int_a^b = 0$$

2 $f(x) \in C^0[a; b]$

$f(x) \ge 0$

$f(x) \not\equiv 0$

$$\Rightarrow \int f(x)\,dx > 0$$



3 $f(x) \geq g(x)$ $x \in [a; b]$, $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

$$f(x) - g(x) \geq 0 \Rightarrow \int_a^b (f(x) - g(x))\,dx =$$
$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \geq 0$$

4 $\exists \int_a^b f(x)\,dx \Rightarrow \exists \int_a^b |f(x)|\,dx \geq \left|\int_a^b f(x)\,dx\right|$

Док-во.

$$\left|\int_a^b f(x)\,dx\right| = \left|\lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i\right| \leq$$
$$\leq \lim_{\lambda \to 0} \sum_{i=1}^{n} |f(\xi_i)|\Delta x_i$$
$$= \int_a^b |f(x)|\,dx$$

5 $\int_a^b c \cdot dx = c(b-a)$

$$\int_a^b c\,dx = \lim_{\lambda \to 0} \sum_{i=1}^{n} c\,\Delta x_i = c \lim_{\lambda \to 0} \sum_{i=1}^{n} \Delta x_i = c \lim_{\lambda \to 0} (b-a)$$
$$= c(b-a)$$

6 $m \le f(x) \le M \Rightarrow m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

Первая теорема о среднем.

$f(x), g(x) \in I_R[a;b]$

Для определенности $g(x) \ge 0$.

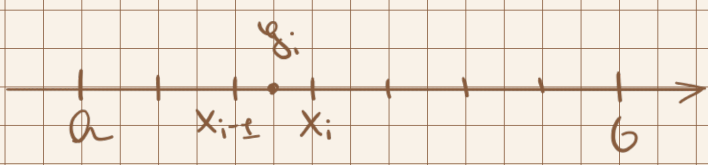
$\inf_{[a;b]} f(x) = m$

$\sup_{[a;b]} f(x) = M$

$$\Downarrow$$

$$\exists \mu \in [m;M]: \int_a^b f(x)g(x)dx = \mu \int_a^b g(x)dx$$

Док-во.



$P = \{a = x_0 < x_1 < \dots < x_{i-1} < x_i < \dots < x_n = b\}$

$m \le f(\xi_i) \le M$

$m\sum_{i=1}^n g(\xi_i)\Delta x_i \le \sum_{i=1}^n f(\xi_i)g(\xi_i)\Delta x_i \le M\sum_{i=1}^n g(\xi_i)\Delta x_i$

$\Downarrow(\lambda \to 0)\Downarrow$

$m\int_a^b f(x)dx \le \int_b^a f(x)g(x)dx \le M\int_a^b g(x)dx \,\Big|\, : \int_a^b g(x)dx > 0$

$$m \le \frac{\int_a^b f(x)\,g(x)\,dx}{\int_a^b g(x)\,dx} \le M$$

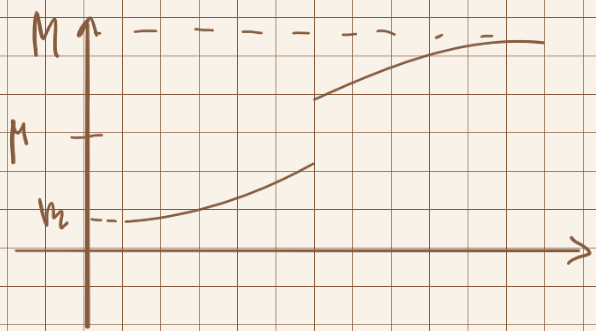
$$\parallel$$

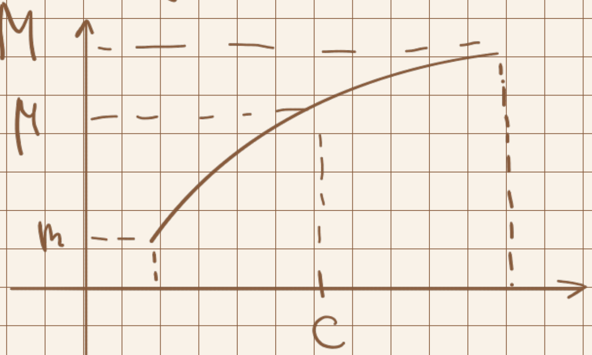
$$\mu \in [m; M]$$

$$\int_a^b f(x)\,g(x)\,dx = \mu \int_a^b g(x)\,dx.$$

Доказано!

Следствие. $f(x) \in C^{\circ}[a;b] \Rightarrow \exists c \in [a,b]:$
$f(c) = \mu$; $\int_a^b fg\,dx = f(c)\cdot\int_a^b g\,dx$





Вторая теорема о среднем (ф-ла Бонне).

$\overline{\underline{I}}$ $f(x) \in I_R[a;b]$, $g\downarrow$, $g \ge 0$ $\Rightarrow \int_a^b fg\,dx = \int_a^{\xi} f\,dx$, $\xi \in [a;b]$

$\overline{\underline{II}}$ $f(x) \in I_R[a;b]$, $g\uparrow$, $g \ge 0$ $\Rightarrow \int_a^b fg\,dx = g\int_{\xi}^b f(x)\,dx$

$\overline{\underline{III}}$ $f(x) \in I_R[a;b] \Rightarrow \int_a^b fg\,dx = g(a)\int_a^{\xi} f\,dx + g(b)\int_{\xi}^b f\,dx$

g монотонна

$g \in C^{\circ}[a;b]$

Инт-л как ф-ия верхнего предела.

Размышление 1. Неважность переменной интегрирования

$$I = \int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \dots,$$

интеграл посчитали ⇒ переменная исчезает.

Размышление 2. Заставим верхний предел меняться

$$I(x) = \int_a^x f(t)\,dt = \int_a^x f(x)\,dx$$

Теорема. Пусть $f(x) \in I_R[a;b]$. Тогда $f(x) \in C^0[a;b]$.

Док-во.

$$\forall x \in [a;b] \quad \exists\, I(x)$$

$$\forall x + \Delta x \in [a;b] \quad \exists\, I(x+\Delta x) = I(x) + \Delta I$$

$$\Delta x \qquad \Delta I = I(x+\Delta x) - I(x)$$

Докажем $\lim\limits_{\Delta x \to 0} I(x) = 0$

$$\Delta I = \int_a^{\Delta x + x} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+\Delta x} f(t)\,dt =$$

$$= \mu(x + \Delta x - x) = \mu\,\Delta x \xrightarrow[\Delta x \to 0]{} 0$$