SEIR Model with Stochastic Uncertainties – A Polynomial Chaos Approach

Tillmann Mühlpfordt

March 26, 2020

Abstract

We show how a (variant of the classic) SEIR model can be overloaded with uncertainties.

1 Deterministic equations

The so-called susceptible, exposed, infected, recovered (SEIR) model is often used to model pandemics. Assuming no births or deaths, the dynamics can be modelled in the form of a system of ordinary differential equations [2, 1]

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\frac{R_0}{t_{\text{inf}}N} SI, \qquad S(0) = S_0 \geq 0, \tag{1a}$$
$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{R_0}{t_{\text{inf}}N} SI - \frac{E}{t_{\text{lat}}}, \qquad E(0) = E_0 \geq 0, \tag{1b}$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{E}{t_{\text{lat}}} - \frac{I}{t_{\text{inf}}}, \qquad I(0) = I_0 \geq 0, \tag{1c}$$
$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{I}{t_{\text{inf}}}, \qquad R(0) = R_0 \geq 0. \tag{1d}$$

The meaning of the parameters of the SEIR model is explained in Table 1.

Table 1: Parameters for SEIR model (1).

Name	Meaning
N	Total population
R_0	Basic reproduction number
t_{inf}	Infectious period
t_{lat}	Latent period

In addition to the SEIR model, we would like to model the number of people in need of intensive care ($IN(t)$), for which we introduce another two equations (which we call intensive care unit (ICU) system)

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{p}{t_{\text{lat}}} E - \frac{L}{t_{\text{lag}}}, \qquad L(0) = L_0 \geq 0, \tag{2a}$$

$$\frac{\mathrm{d}IN}{\mathrm{d}t} = \frac{L}{t_{\text{lag}}} - \frac{IN}{t_{\text{lay}}}, \qquad IN(0) = IN_0 \geq 0. \tag{2b}$$

Reading the system (2) bottom-up, we see that the number of people in intensive care decreases according to the average lay time t_{lay}, and increases according to the average lag time t_{lag}. The fictitious compartment $L(t)$ models the people who are transitioning from the infected compartment ($I(t)$) to the intensive-care compartment ($IN(t)$).

The meaning of the parameters of the ICU model is given in Table 2.

Table 2: Parameters for ICU model (2).

Name	Meaning
p	Fraction of infected people needing intensive care
t_{lag}	Lag period from infection to intensive care
t_{lay}	Period of hospitalization

2 Introducing uncertainty

A key issue with both the SEIR model and the ICU model is parametric uncertainty: the precise numerical values of the parameters from either Table 1 or Table 2 are not known. To account for this uncertainty explicitly, we can model the parameters as random variables. For instance,

$$\mathsf{R}_0 \sim \mathcal{N}(\mu_{R_0}, \sigma_{R_0}) \tag{3}$$

means to model the basic reproduction rate as a Gaussian random variable with mean μ_{R_0} and standard deviation σ_{R_0}.

What is the effect of this uncertainty on the solution of the SEIR-ICU model? One way to study this effect is to do Monte Carlo simulation:

1. draw samples of the random variable,
2. solve the respective system of differential equations (1) & (2),
3. and repeat.

Another way is to use polynomial chaos expansion (PCE), a method that is to a random variable what a Fourier series is to a periodic signal: an orthogonal decomposition. This method allows to propagate uncertainties through differential equations in a single shot. By that we mean the following

1. introduce PCE for all uncertain quantities,
2. derive a new set of equations via so-called Galerkin projection, and

3. solve this new system *once*.

The advantage of PCE over Monte-Carlo is that no sampling is required whatsoever; by solving the new set of equations, all statistical information are available such as expected value and/or standard deviations over time.

3 Simulation study

The following study is based on the SEIR-SEIRicu model from [2]. We introduce uncertainty for the basic reproduction number as a Gaussian random variable,

$$\mathsf{R}_0 \sim \mathcal{N}(\mu_{R_0}, \sigma_{R_0}). \tag{4a}$$

In addition, we introduce uncertainty for the percentage of people in need of intensive care as a uniform random variable with support $[\underline{p}, \overline{p}]$

$$\mathsf{R}_0 \sim \mathcal{N}(\mu_{R_0}, \sigma_{R_0}), \tag{4b}$$

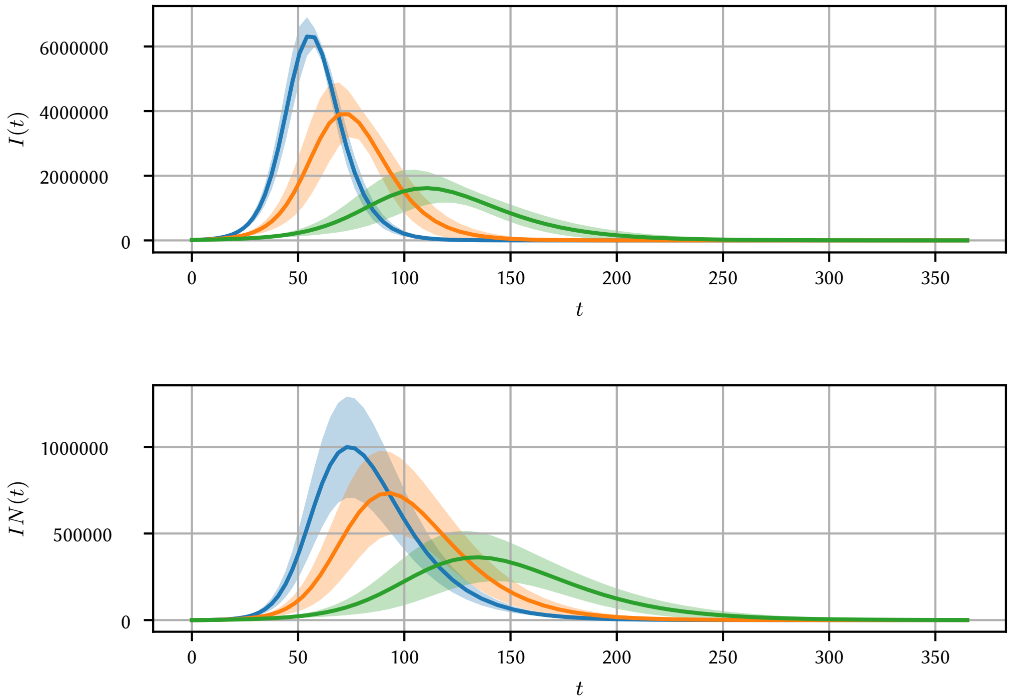
$$\mathsf{p} \sim \mathcal{U}(\underline{p}, \overline{p}). \tag{4c}$$

The results are shown in Figure 1. The mean values replicate the known behavior from [2]. Accounting specifically for the uncertainties allows to draw intervals that visualize the standard deviation. Not surprisingly, the peak values of the uncertainty are reached at the peak of the epidemic, with quite severe deviations.

References

[1] An der Heiden M. and Buchholz U. *Modellierung von Beispielszenarien der SARS-CoV-2-Epidemie 2020 in Deutschland.* Tech. rep. Robert Koch Institut, Mar. 2020. URL: `https://www.rki.de/DE/Content/InfAZ/N/Neuartiges_Coronavirus/Modellierung_Deutschland.pdf?__blob=publicationFile`.

[2] *Stellungnahme der Deutschen Gesellschaft für Epidemiologie (DGEpi) zur Verbreitung des neuen Coronavirus (SARS-CoV-2).* Tech. rep. Deutsche Gesellschaft für Epidemiologie, Mar. 2020. URL: `https://www.dgepi.de/assets/Stellungnahmen/Stellungnahme2020Corona_DGEpi-21032020.pdf`.



μ_{R_0}	σ_{R_0}	$\underline{p}$	$\overline{p}$	Color
2.5	0.10	0.02	0.06	blue
2.0	0.15	0.02	0.06	orange
1.5	0.10	0.02	0.06	green

Figure 1: Number of infected people $I(t)$ and ICU patients $IN(t)$ for different parameter values. The shaded area denotes the interval of $\pm$ one standard deviation; solid lines are mean values. Time axis represents days.