TIME SERIES PREDICTION USING LYAPUNOV EXPONENTS IN EMBEDDING PHASE SPACE

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ABSTRACT

In this paper, a multi-dimension chaotic time series prediction method using Lyapunov Exponents in phase space is proposed. The method is based on the fundament characteristic of chaotic behaviour that sensitive dependence upon initial conditions (SDUIC) and Lyapunov Exponents (LEs) are the measure of the SDUIC in chaotic system. The fundamental characteristic of chaotic time series is analyzed, the proposal method is deduced. Numerical experimental results show that the proposed method is simple, practical and effective.

1. INTRODUCTION

Nonlinear and chaos time series analysis and prediction is becoming a more and more reliable tool for the study of complicated dynamics from measurements. And analysis and prediction of chaotic time series take a very important role in chaos research including chaos modeling, control and other applications.

There have been extensive research works been done on the chaotic time series prediction. In this paper, a multi-dimension prediction method is proposed. The innovation of this method consists in the following aspects:

1) One of the fundamental characteristic of chaos, which is that a chaotic system is sensitive to its the initial state, is explored. The Lyapunov Exponents which quantify the sensitivity are used to predict the unknown points based on a known time series in the embedding phase space.
2) A multi-dimension embedding phase space method is proposed to improve the accuracy of prediction.

This paper is organized as follows: *Section 2* introduces the fundamental characteristics of chaotic system and Lyapunov Exponents. In *Section 3* the embedding phase space reconstruction method is presented, including the multi-dimension method. *Section 4* gives the algorithms to implement the proposed prediction method. A numerical experiment is done to verify the effectiveness of the method. *Section 5* draws the conclusion.

2. CHAOS FUNDAMENTAL CHARACTERISTIC AND LYAPUNOV EXPONENTS

There are three fundamental characteristics of chaos: (1) an essentially continuous and possibly banded frequency spectrum that resembles random noise;(2) the sensitivity to initial conditions, that is, nearby orbits diverge very rapidly, which result in long-term unpredictabilility; and (3) an ergodicity and mixing of the dynamical orbits which in essence implies the wholesale visit of the entire phase space by the chaotic behavior and a loss of information.

Lyapunov Exponents describe, in a logarithmic scale, the growth or shrinkage rate of small perturbations in different directions in the phase space of the orbits. When at least one Lyapunov Exponent is positive, the attractor is extremely sensitive to its initial conditions and hence becomes chaotic after a transient period of time. For the formal definition of Lyapunov Exponents, consider a specific one-dimensional map given by $x_{n+1} = f(x_n)$. The difference between two initially nearby states after n^{th} steps is written as

$$f^n(x+\varepsilon) - f^n(x) \approx \varepsilon \cdot 2^{n \cdot \lambda} \tag{1}$$

For small ε, this expression becomes

$$\lambda \approx \frac{1}{n} \log_2 \left| \frac{df^n}{dx} \right| \tag{2}$$

The λ is the Lyapunov Exponent.

3. EMBEDDING PHASE SPACE RECONSTRUCTION AND MULTI-DIMENSION METHOD

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To study chaotic time series, the conventional research method is to reconstruct the embedding phase space using the given chaotic time series data. The embedding method was developed by Takens[5] [6].

From an observed time series $x(t)$, a data vector $Y(t) = F(x(t), x(t-T), \ldots, x(t-(D-1)T))$ is generated, where T is the time delay, and D is the embedding dimension. This vector indicates one point of a D-dimensional reconstructed phase space R^D. Different trajectories can be drawn given different embedding dimension. Assume that the target system is a deterministic dynamical system and that the observed time series is obtained through an observation system corresponding to a continuous mapping C^1 from the state space of dynamical system to the 1-dimension Euclidean space R. Then, the reconstructed trajectory is an embedding of the original trajectory when D value is sufficiently large. If any attractor has appeared in the original dynamical system, another attractor, which retains the phase structure of the origin attractor, will appear in the reconstructed state space. In order that such reconstruction achieves embedding, it has been proven that the dimension D should satisfy the condition:

$$D > 2m + 1 \tag{3}$$

where m is the state space dimension of the original dynamical system. However, this is a sufficient condition. Depending on the characteristics of the data, embedding can be established even when D is less than $2m+1$.

It is shown from the results of other researchers that there is not an exact embedding dimension which is the best value to the reconstruct the phase space and the chaotic attractor. A chaotic time series data can be embedded in phase space of different dimension while preserving the chaotic attractor behavior.

In this paper, instead of exploring the area of "how to select an optimal embedding parameters", an effective multi-dimension EPS prediction method is developed, which not only eliminates the effort of selecting optimal embedding parameters but achieves a higher accuracy of prediction.

The basic steps are:

1. Use traditional method to select a minimum embedding dimension D.
2. Predict the unknown phase space points on different unfold attractor trajectory in different phase space (e.g. $D-1$, D, $D+1$).
3. Average the predicting chaotic time series data using the error calculations theory [Hans-Jochen Bartsch,1974] which will be shown briefly in the following. The prediction result is relatively more accurate than the conventional prediction based on only a single dimension phase space.

Consider the prediction is done in n different dimensions, the predicted value obtained from n different dimension prediction is respectively $Y_1, Y_2, \ldots, Y_n$. The average value is X, and the errors for different dimension are $m_1, m_2, \ldots, m_n$. The apparent error is $v_i = Y_i - X, (i = 1,2,..,n)$. Then, the probability of the apparent error is [8]:

$$P(v_i) = \frac{1}{m_i\sqrt{2\pi}} e^{-\frac{v_i^2}{2m_i^2}dv_i} \tag{4}$$

As the predicted values from different dimension EPS are independent from each other, the unit probability of all predicted values is $\prod_{i=1}^{n} P(v_i)$. The larger the probability P of the predicted value is, the more reliable the prediction is. It can be induced that when $\sum_{i=1}^{n} v_i^2$ is the minimum value, the most reliable prediction can be obtained.

Therefore, the multi-dimension prediction value is calculated as :

$$Y_{Multi-\text{dimension}} = \frac{\sum_{i=1}^{n} p_i Y_i}{\sum_{i=1}^{n} p_i} \tag{5}$$

where n is the number of multi-dimension, Y_i is the predicted value and p_i is the prediction probability in i dimension (generally $p_i=1$ for independent process).

4. CHAOTIC TIME SERIES PREDICTION

4.1 The algorithm implementation

The proposed chaotic time series prediction method is consisted in three steps. The first step is to embed the known time series data from time domain to the phase space. Next predict the unknown phase space points by chaos behavior. Thirdly recover the predicted phase space points to time domain and thus the unknown time series values are obtained.

Given a chaotic time series data set $x_1, x_2, \ldots, x_N$, where N is the length of the known data set. Set the embedding dimension as D and the time delay as T, a set of phase space points $Y(I)$,

$I = 1,2,\ldots,N-(D-1)T$ is generated in the EPS, which are :

$$Y(1) = [x(1), x(2), \ldots, x(1+(D-1)T)],$$
$$Y(2) = [x(2), x(2+T), \ldots, x(2+(D-1)T)],$$
$$\ldots$$
$$Y(I) = [x(N-(D-1)T, x(N-(D-1)T+T), \ldots, x(N)],$$

Then we get N-(D-1)T embedding phase space points. Based on those points, we can find a point which is the nearest to the Y(N-(D-1)T) point. Mark the point as Y(min_dist). We can calculate the distance between Y(N-(D-1)T) and Y(min_dist) and name it the initial distance Dist_0. If Dist_1/Dist_0 has a little changed in every evolution step, from the Lyapunov Exponent estimation theory, we know that we can estimate the distance Dist_1 between Y(N-(D-1)T+1) and Y(min_dist+1). Which follow the equation:

$$Dist\_1 = Dist\_0 \cdot 2^{K \cdot \lambda} \tag{6}$$

where λ is the Lyapunov Exponent and K is the number of steps from Dist_0 to Dist_1. Then we can get Dist_1. As we know the position of Y(min_dist+1) point, we can calculate the position of Y(N-(D-1)T+1) point. Y(N-(D-1)T+1) is composed as [x(N-(D-1)T+1) x(N-(D-1)T+T+1) …x(N+1)]. Now we can predict the value of x(N+1). By the same methods, we can get more long term prediction (the length of prediction depends on the chaos system behavior and value of Lyapunov exponent which described in section 2).

This can be shown in phase space map Fig1. Given N chaotic time series data, we can embedding (N-(D-1)T) phase space points. If we want to prediction No.(N+1) time series data, i.e. prediction (N-(D-1)T+1) phase space point.

Based No.(N-(D-1)T) phase space point ①, we can find the nearest point of No.(N-(D-1)T) phase space point ③ from all known phase space point. After calculate the distance of point ① and point ③ , we can get the initial distance of this iteration Dist_0. Follow the trajectory of phase space, we can calculate the distance Dist_1 between ② and ④ point by chaotic behavior. Then we can calculate the ④ point position and get the time series predicting data which we want to predict.

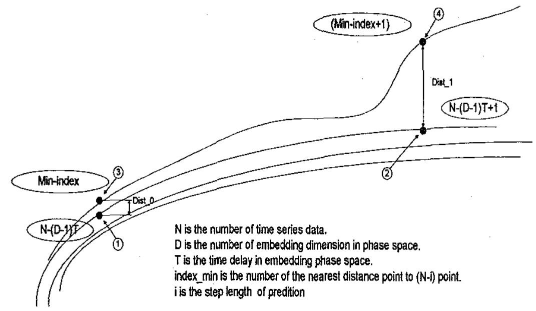

Fig. 1 Prediction method based on Lyapunov Exponent

The proposed algorithm introduced above is further presented in the following steps:

Step one *Load a known time series data set x(t) and the length of the data set is N;*

Step two *Choose the time delays, T ;*

Step three *Choose the dimension of reconstructed phase space,D;*

Step four *Calculate the Lyapunov Exponent denoted as "lyexps" based on know N length data set, i.e. finite-size Lyapunov Exponents .;*
If all lyexps< 0, then goto step ten;
If there is at least a lyexp>0 then next step;

Step five *Reconstruct phase space based on the chaotic time delay and embedding dimension;*

Step six *Find the nearest point min_dist to the last known phase space point Y(N-(D-1)T);*

Step seven *Calculate the distance between min_dist point and the last point Y(N-(D- 1)T);*

Step eight *Calculate the distance between (min_dist+1) point and Y(N-(D-1)T+1) point;*

Step nine *Get the coordinate of (min_dist+1) point in embedding phase space and predict the x(N+1) value in time domain using Lyapunov Exponent;*

Step ten *Use different embedding dimension to calculate from step four to step nine. and get more accurate prediction value using some calculate technique;*

Step eleven *Stop.*

4.2. Numerical Experiments:

In order to prove the validation of the proposed method, a known chaotic system is utilized as an example to test the method.

Example : Logistic equation which is in the form of

$$x(t+1) = 4.0 \cdot x(t) \cdot (1 - x(t)) \tag{7}$$

with an initial state of $x(0) = 0.36$ is used to generate a time series with length of 512. The data in time domain is shown in Fig. 2.

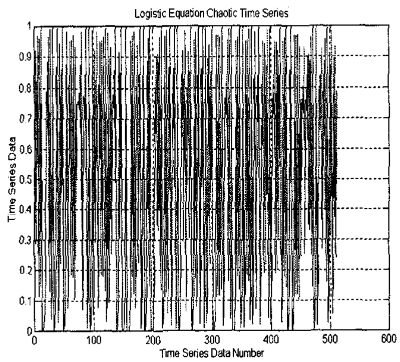

Fig. 2 A time series generated from Logistic Equation, x(0)=0.36

The prediction has been carried out in two dimension of the embedding phase space, i.e. D=2 and D=3 respectively. The time delay is chosen as 1. The reconstructed attractors in the two & three embedding phase space are shown in Fig. 3 and Fig. 4 respectively.

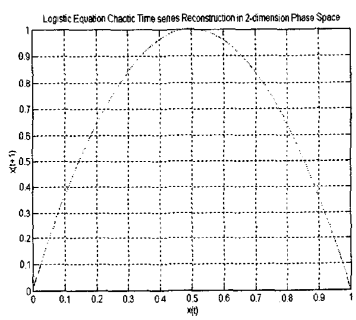

Fig. 3 Reconstructed attractor in 2-dimension EPS

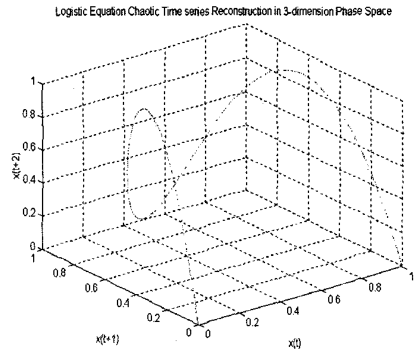

Fig. 4 Reconstructed attractor in 3-dimension EPS

The prediction accuracy is evaluated by a number of criteria, namely, the absolute error summation (AES), the relative error summation (RES), the average absolute error (AAE), the average relative error (ARE), and the mean square error (MSE). The calculation equations are given as follows:

Absolute Error Sum:

$$AES = \sum_{i=1}^{K} (prediction\_value - real\_value)$$

Relative Error Sum:

$$RES = \sum_{i=1}^{K} \frac{prediction\_value - real\_value}{real\_value}$$

Average Absolute Error:

$$AAE = \frac{AES}{K}$$

Average Relative Error:

$$ARE = \frac{RES}{K}$$

Mean Square Error:

$$MSE = \sum_{i=1}^{N} (prediction\_value(i) - real\_value(i))^2 \cdot p_i$$

Table 1 gives the results of these criteria of the 2-dimension, 3-dimension and the multi-dimension:

Dim	***RES***	***AAE***	***ARE***	***MSE***
2	2586.6	0.025	5.173	0.0144
3	3200.2	0.042	6.400	0.0387
Multi-Dim	-	0.009	1.261	0.0003

Table 1 Results of the evaluation criteria

Prediction results are also given in time series, absolute error curve, relative error curve respectively in 2-dimension, 3-dimension from Fig.5~Fig.8

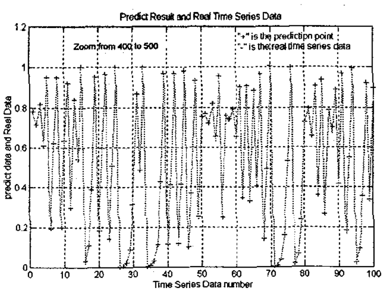

Fig. 5 Prediction result of time series from 2-dimensin prediction

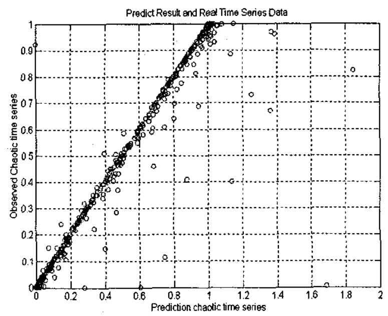

Fig. 6 Prediction result of relative error curve from in 2-demension EPS prediction

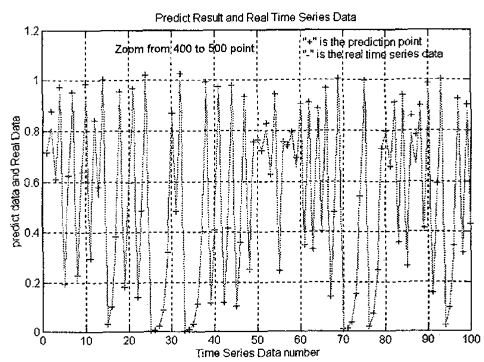

Fig. 7 Prediction result of time series from 3-dimensin prediction

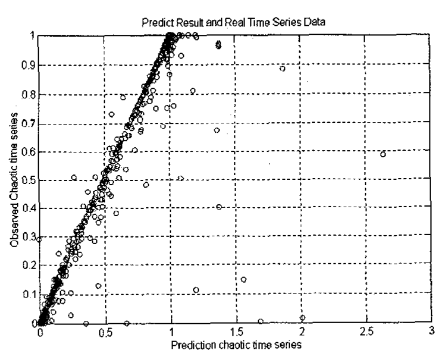

Fig. 8 Prediction result of relative error curve from in 3-demension EPS prediction

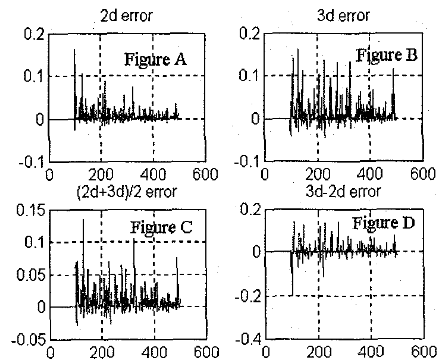

Fig. 9 Prediction result of absolute error and relative error curve from 2-demension EPS prediction

Fig. 9A is the prediction error which is from 2-dimension EPS.
Fig. 9B is the prediction error which is from 3-dimension EPS.
Fig. 9C is the prediction error which is from multi-dimension method.
Fig. 9D is the different between Fig. 9A and Fig. 9B.

From the above figures, it is obvious that the multi-dimension method results in the minimum prediction error, proving that the multi-dimension prediction method demonstrates a higher prediction accuracy.

5. CONCLUSION

The proposed method for predicting chaotic time series in embedding phase space is based on one of the fundamental characteristic of chaotic behaviour which is the sensitivity dependence upon initial conditions(SDUIC) and the fact that Lyapunov Exponents(LEs) are measures of the SDUIC in a chaotic system. A few conclusions can be drawn which are:

1) Chaotic time series displays some stochastic behavior in time domain and display some deterministic behavior in the embedding phase space. This property is exploited in the proposed method.
2) It is feasible to predict chaotic time series based on the fundamental characteristic of a chaotic system which is its sensitivity to the initial state.
3) Multi-dimension embedding phase space method provides a higher prediction accuracy.
4) More available data will give better prediction because the points in reconstructed phase space will be denser and

searching for the closest point will be more accurate.

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