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TIME SERIES PREDICTION USING LYAPUNOV EXPONENTS IN EMBEDDING PHASE SPACE

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ABSTRACT

In this paper, a chaotic time series prediction method was proposed. This method was based on the fundament characteristic of chaotic behaviour that sensitive dependence upon initial conditions(SDUIC) and Lyapunov Exponents(LEs) is a measure of the SDUIC in chaotic system. Because LEs of chaotic time series data provided a quantitative analysis of system dynamics in different embedding dimension after embedding a chaotic time series in different embedding dimension phase spaces, a multi-dimension chaotic time series prediction using LEs was proposed in this paper. This is done by first reconstructing a phase space using chaotic time series and then using LEs as quantitative parameter to predict unknown phase space point, then transfer the phase space point to time domain, we can get the predicted chaotic time series data. We analysis of the fundament characteristic of chaotic time series and LEs, deduce the proposal method. Then a computer simulation was done. The result of simulation shown that the proposal method is simple, practical and effective.

1. Introduction

General, what we observed chaos phenomenon in real world is in time series data format. i.e. The Chaos which we meet in real world is in the form of a time series data. So analysis and prediction chaotic time series take a very important role in chaos research include chaos control and chaos application.

State-space reconstruction method is by the analysis of the geometry of the embedded data to discovering the relationship between past and future points in a time series. Famer and Sidorowich[2] report on a direct implementation of this idea, and discuss the scaling of prediction error for several artificial and experimental time series. Their idea is to recognize that any manifold is locally linear. After embedding a time series in a state space using delay co-ordinates, they "learn" the induced nonlinear mapping using a local approximation. This method allows make a short-term prediction of the future behaviour of a time series, using information based only past values.

In this paper, we propose a new method to predict the chaotic time series. The main difference between this paper and [2][3] consists in following aspects:

1) The estimate method is different. In our method, we use the fundamental characteristic of chaos, a chaotic system is its sensitivity to the initial state, to estimate the unknown points in embedded phase space.

2) We use multi-dimension embedding phase space method to improve the accurate of prediction.

2. Chaos Fundamental Characteristic and Lyapunov Exponents

There are three fundamental characteristics of chaos: (1) an essentially continuous and possibly banded frequency spectrum that resembles random noise;(2) the sensitivity to initial conditions, that is, nearby orbits diverge very rapidly, which result in long-term unpredictabilility; and (3) an ergodicity and mixing of the dynamical orbits which in essence implies the wholesale visit of the entire phase space by the chaotic behavior and a loss of information.

Lyapunov Exponents describe, in a logarithmic scale, the growth or shrinkage rate of small perturbations in different directions in the phase space of the orbits. When at least one Lyapunov Exponent is positive, the attractor is extremely sensitive to its initial conditions and hence becomes chaotic after a transient period of time. For the formal definition of Lyapunov Exponents, consider a specific one-dimensional map given by $x_{n+1} = f(x_n)$. The difference between two initially nearby states after n^{th} steps is written as

$$f^n(x+\varepsilon) - f^n(x) \approx \varepsilon \cdot 2^{n \cdot \lambda} \quad (1)$$

For small ε, this expression becomes

$$\lambda \approx \frac{1}{n} \log_2 \left| \frac{df^n}{dx} \right| \quad (2)$$

The λ is the Lyapunov Exponent.

3. Embedding phase space reconstruction and multi-dimension method

To study chaotic time series, the conventional research method is to reconstruct the embedding phase space using the given chaotic time series data. The embedding method was developed by Takens[5] [6].

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From an observed time series ***x(t)***, a data vector $Y(t) = F(x(t), x(t-T), \ldots, x(t-(D-1)T))$ is generated, where $\boldsymbol{T}$ is the time delay, and $\boldsymbol{D}$ is the embedding dimension. This vector indicates one point of a ***D***-dimensional reconstructed phase space $\boldsymbol{R^D}$. In order that such reconstruction achieves embedding, it has been proven that the dimension $\boldsymbol{D}$ should satisfy the condition:

$$D > 2m + 1 \tag{3}$$

where m is the state space dimension of the original dynamical system. However, this is a sufficient condition. Depending on the characteristics of the data, embedding can be established even when $\boldsymbol{D}$ is less than ***2m+1***.

In this paper, instead of exploring the area of "how to select an optimal embedding parameters", an effective multi-dimension EPS prediction method is developed, which not only eliminates the effort of selecting optimal embedding parameters but achieves a higher accuracy of prediction.

Consider the prediction is done in ***n*** different dimensions, the predicted value obtained from n different dimension prediction is respectively $Y_1, Y_2, \ldots, Y_n$. The average value is X, and the errors for different dimension are $m_1, m_2, \ldots, m_n$. The apparent error is $v_i = Y_i - X, (i = 1,2,..,n)$. Then, the probability of the apparent error is [8]:

$$P(v_i) = \frac{1}{m_i\sqrt{2\pi}} e^{-\frac{v_i^2}{2m_i^2} dv_i} \tag{4}$$

As the predicted values from different dimension EPS are independent from each other, the unit probability of all predicted values is $\prod_{i=1}^{n} P(v_i)$. The larger the probability $\boldsymbol{P}$ of the predicted value is, the more reliable the prediction is. It can be induced that when $\sum_{i=1}^{n} v_i^2$ is the minimum value, the most reliable prediction can be obtained.

4. Chaotic time series prediction

4.1 The algorithm implementation

The proposed chaotic time series prediction method is consisted in three steps. The first step is to embed the known time series data from time domain to the phase space. Next predict the unknown phase space points by chaos behavior. Thirdly recover the predicted phase space points to time domain and thus the unknown time series values are obtained.

Given a chaotic time series data set $x_1, x_2, \ldots, x_N$, where N is the length of the known data set. Set the embedding dimension as D and the time delay as T, a set of phase space points $Y(I), Y(I)$, $I = 1,2,\ldots, N-(D-1)T$ is generated in the EPS, which are :

$$Y(1) = [x(1), x(2), \ldots, x(1+(D-1)T)],$$

$$Y(2) = [x(2), x(2+T), \ldots, x(2+(D-1)T)],$$

$$\ldots$$

$$Y(I) = [x(N-(D-1)T, x(N-(D-1)T+T), \ldots, x(N)],$$

Then we get N-(D-1)T embedding phase space points. Based on those points, we can find a point which is the nearest to the Y(N-(D-1)T) point. Mark the point as Y(min_dist). We can calculate the distance between Y(N-(D-1)T) and Y(min_dist) and name it the initial distance Dist_0. If Dist_1/Dist_0 has a little changed in every evolution step, from the Lyapunov Exponent estimation theory, we know that we can estimate the distance Dist_1 between Y(N-(D-1)T+1) and Y(min_dist+1). Which follow the equation:

$$Dist\_1 = Dist\_0 \cdot 2^{K \cdot \lambda} \tag{5}$$

where λ is the Lyapunov Exponent and K is the number of steps from Dist_0 to Dist_1. Then we can get Dist_1. As we know the position of Y(min_dist+1) point, we can calculate the position of Y(N-(D-1)T+1) point. Y(N-(D-1)T+1) is composed as [x(N-(D-1)T+1) x(N-(D-1)T+T+1) ...x(N+1)]. Now we can predict the value of x(N+1). By the same methods, we can get more long term prediction (the length of prediction depends on the chaos system behavior and value of Lyapunov exponent which described in section 2).

This can be shown in phase space map Fig1. Given N chaotic time series data, we can embedding (N-(D-1)T) phase space points. If we want to prediction No.(N+1) time series data, i.e. prediction (N-(D-1)T+1) phase space point.

Based No.(N-(D-1)T) phase space point ①, we can find the nearest point of No.(N-(D-1)T) phase space point ③ from all known phase space point. After calculate the distance of point ① and point ③, we can get the initial distance of this iteration Dist_0. Follow the trajectory of phase space, we can calculate the distance Dist_1 between ② and ④ point by chaotic behavior. Then we can calculate the ④ point position and get the time series predicting data which we want to predict.

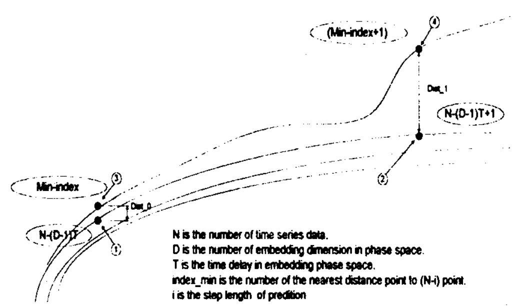

Fig. 1 Prediction method based on Lyapunov Exponent

The proposed algorithm introduced above is further presented in the following steps:

Step one *Load a known time series data set x(t) and the length of the data set is N;*

Step two *Choose the time delays, T ;*

Step three	*Choose the dimension of reconstructed phase space, D;*
Step four	*Calculate the Lyapunov Exponent denoted as "lyexps" based on know N length data set, i.e. finite-size Lyapunov Exponents .;*
	If all lyexps< 0, then goto step ten;
	If there is at least a lyexp>0 then next step;
Step five	*Reconstruct phase space based on the chaotic time delay and embedding dimension;*
Step six	*Find the nearest point min_dist to the last known phase space point Y(N-(D-1)T);*
Step seven	*Calculate the distance between min_dist point and the last point Y(N-(D- 1)T);*
Step eight	*Calculate the distance between (min_dist+1) point and Y(N-(D-1)T+1) point;*
Step nine	*Get the coordinate of (min_dist+1) point in embedding phase space and predict the x(N+1) value in time domain using Lyapunov Exponent;*
Step ten	*Use different embedding dimension to calculate from step four to step nine. and get more accurate prediction value using some calculate technique;*
Step eleven	*Stop.*

4.2. Numerical Experiments:

In order to prove the validation of the proposed method, a known chaotic system is utilized as an example to test the method.

Example : Logistic equation which is in the form of

$$x(t+1) = 4.0 \cdot x(t) \cdot (1 - x(t)) \qquad (6)$$

with an initial state of $x(0) = 0.36$ is used to generate a time series with length of 512. The data in time domain is shown in Fig. 2.

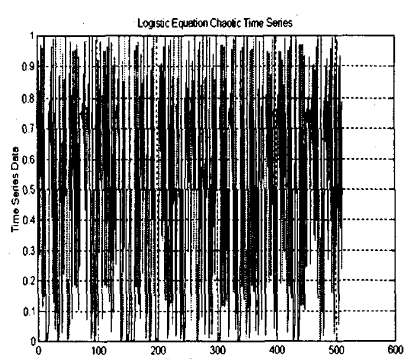

Fig. 2 A time series generated from Logistic Equation, x(0)=0.36

The prediction has been carried out in two dimension of the embedding phase space, i.e. D=2 and D=3 respectively. The time delay is chosen as 1. The reconstructed attractors in the two & three embedding phase space are shown in Fig. 3 and Fig. 4 respectively.

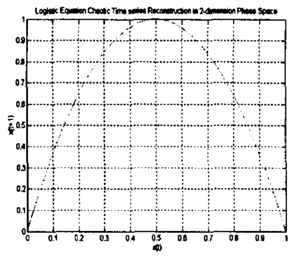

Fig. 3 Reconstructed attractor in 2-dimension EPS

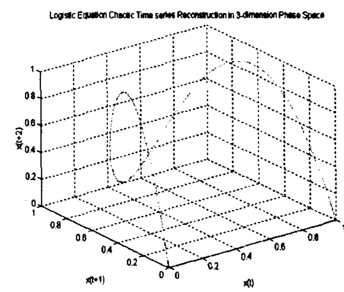

Fig. 4 Reconstructed attractor in 3-dimension EPS

The prediction accuracy is evaluated by a number of criteria, namely, the absolute error summation (AES), the relative error summation (RES), the average absolute error (AAE), the average relative error (ARE), and the mean square error (MSE). The calculation equations are given as follows:

Absolute Error Sum:

$$AES = \sum_{i=1}^{K} (prediction\_value - real\_value)$$

Relative Error Sum:

$$RES = \sum_{i=1}^{K} \frac{prediction\_value - real\_value}{real\_value}$$

Mean Square Error:

$$MSE = \sum_{i=1}^{N} (prediction\_value(i) - real\_value(i))^2 \cdot p_i$$

Table 1 gives the results of these criteria of the 2-dimension, 3-dimension and the multi-dimension:

Dimension	***RES***	***MSE***
2	2586.6	0.0144
3	3200.2	0.0387
Multi-Dimension		0.0003

Table 1 Results of the evaluation criteria

Prediction results are also given in time series, absolute error curve, relative error curve respectively in 2-dimension, 3-dimension from Fig.5~Fig.6

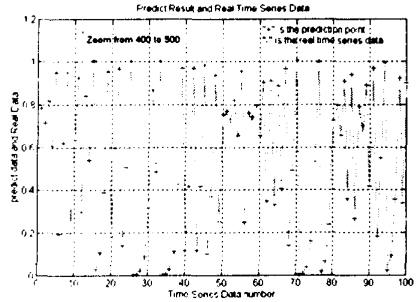

Fig. 5 Prediction result of time series from 2-dimensin prediction

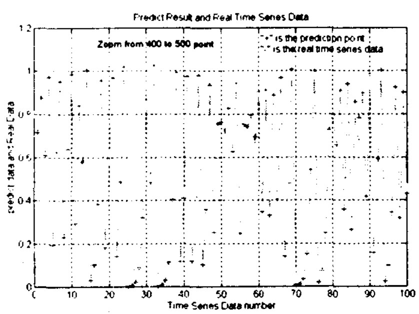

Fig. 6 Prediction result of time series from 3-dimensin prediction

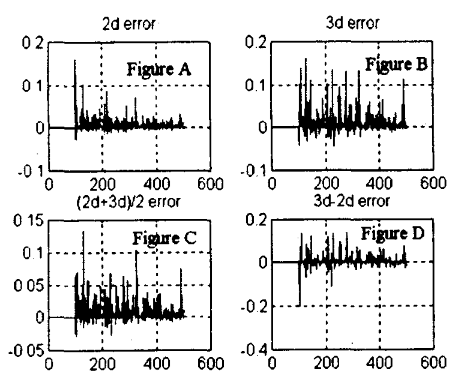

Fig. 7 Prediction result of absolute error and relative error curve from 2-demension EPS prediction

Fig. 7A is the prediction error which is from 2- dimension EPS.

Fig. 7B is the prediction error which is from 3- dimension EPS.

Fig. 7C is the prediction error which is from multi-dimension method.

Fig. 7D is the different between Fig. 9A and Fig. 9B.

From the above figures, it is obvious that the multi-dimension method results in the minimum prediction error, proving that the multi-dimension prediction method demonstrates a higher prediction accuracy.

5. Conclusion

The proposed method for predicting chaotic time series in embedding phase space is based on one of the fundamental characteristic of chaotic behaviour which is the sensitivity dependence upon initial conditions(SDUIC) and the fact that Lyapunov Exponents(LEs) are measures of the SDUIC in a chaotic system. A few conclusions can be drawn which are:

1) Chaotic time series displays some stochastic behavior in time domain and display some deterministic behavior in the embedding phase space. This property is exploited in the proposed method.

2) It is feasible to predict chaotic time series based on the fundamental characteristic of a chaotic system which is its sensitivity to the initial state.

3) Multi-dimension embedding phase space method provides a higher prediction accuracy.

4) More available data will give better prediction because the points in reconstructed phase space will be denser and searching for the closest point will be more accurate.

6. References

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