Braid Group Cryptography

May 5, 2009

Practical Public-key Cryptosystems

- Diffie-Hellman
- RSA
- ElGamal
- Elliptic Curve Cryptosystems, etc

Quantum computers

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What kinds of group-theoretic problems can be used that the current classical computers and the future quantum computers can not solve?

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It is unclear when these or other algebraic problems will be well enough understood to produce practical public key cryptographic primitives with reliable security estimates.

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- For each $a \in G$, there exists an element $b \in G$ such that $a * b = b * a = e$, where e is the identity element.

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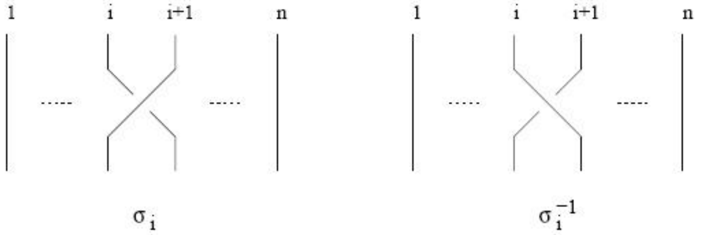
- For $n > 2$, the group B_n is not commutative and it contains an infinite cyclic subgroup.

Geometric presentation



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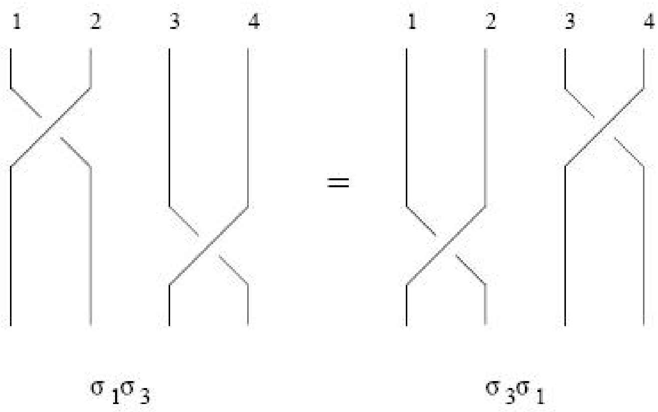

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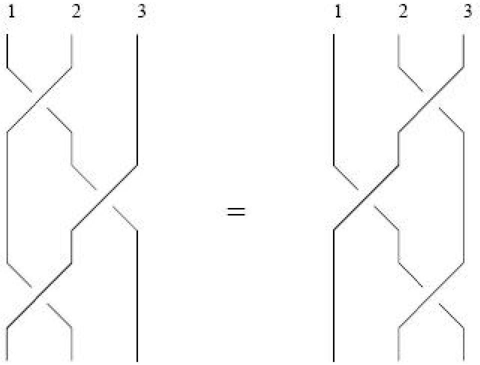

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Reduced Burau representation

$$\sigma_i \mapsto \begin{pmatrix} I_{i-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & t & -t & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-i-2} \end{pmatrix}$$

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4. **Decomposition Problem:** Assuming that for the braids u and w there are braids a_1 and a_2 such that $w = a_1 u a_2$ find witnesses, i.e., find braids x and y satisfying $w = xuy$.

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$$\text{KEY} : K_1 = K_2$$

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