# MICRO II PS # 2 (Q2 & Q3)

2. Consider the linear program

$$v_P(b) = \max c_x x + c_y y$$
$$\text{subject to} \quad x \le 1$$
$$x \ge 0$$

   (a) Rewrite this in the canonical form.

   (b) Rewrite this in the standard form.

(a)  Since $y$ doesn't have the nonnegativity constraint,
     we can express $y$ as the difference of two nonnegative
     variables.

$$V_P(b) = \text{MAX} \ (C_x \quad C_y \quad -C_y) \begin{bmatrix} x \\ y_1 \\ y_2 \end{bmatrix}$$

$$\text{S.T.} \quad (1 \quad 0 \quad 0) \begin{bmatrix} x \\ y_1 \\ y_2 \end{bmatrix} \le 1 \quad \checkmark$$

$$\begin{bmatrix} x \\ y_1 \\ y_2 \end{bmatrix} \ge 0$$

(b)  $V_P(b) = \text{MAX} \ C_x x + C_y y_1 - C_y y_2 + 0 \cdot S$

$$\text{S.T.} \quad x + S = 1$$

$$x, y_1, y_2, S \ge 0.$$

$$\Rightarrow \quad V_P(b) = \text{MAX} \ (C_x \quad C_y \quad -C_y \quad 0) \begin{bmatrix} x \\ y_1 \\ y_2 \\ S \end{bmatrix}$$

$$\text{S.T.} \quad [1 \quad 0 \quad 0 \quad 1] \begin{bmatrix} x \\ y_1 \\ y_2 \\ S \end{bmatrix} = 1 \quad \checkmark$$

$$\begin{bmatrix} x \\ y_1 \\ y_2 \\ S \end{bmatrix} \ge 0$$

3. Consider the primal problem

$$v_p(b) = \max x_1 + 2x_2$$
$$\text{s.t. } x_1 + x_2 \le 4$$
$$x_1 + 3x_2 \le b$$

(a) Write down the dual.

(b) For $b = 1$, plot the constraint sets for both problems, and solve them.

(c) Describe $v_P(b)$, and compute $\partial v_P(b)$ on the range $0 \le b \le 14$.

(a) We want to first rewrite the problem with nonnegativity constraints.

$$V_P(b) = MAX \quad \underbrace{(X_1 - X_2)}_{X_1} + 2\underbrace{(X_3 - X_4)}_{X_2}$$

Note: $X_1, X_2$ here are not the same as the ones in the problem.

$$S.T. \quad (X_1 - X_2) + (X_3 - X_4) \le 4$$
$$(X_1 - X_2) + 3(X_3 - X_4) \le b$$
$$X_1, X_2, X_3, X_4 \ge 0.$$

$$\Rightarrow V_P(b) = MAX \quad \underbrace{\begin{bmatrix} 1 & -1 & 2 & -2 \end{bmatrix}}_{C} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

$$S.T. \quad \underbrace{\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 3 & -3 \end{pmatrix}}_{A} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \le \underbrace{\begin{bmatrix} 4 \\ b \end{bmatrix}}_{b.}$$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \ge 0$$

Hence, the dual problem is

$$V_D(c) = MIN \quad \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 4 \\ b \end{bmatrix}$$

$$S.T. \quad \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 3 & -3 \end{bmatrix} \ge \begin{bmatrix} 1 & -1 & 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \ge 0.$$

(b) First consider the primal problem

$$v_p(b) = \max x_1 + 2x_2$$
$$\text{s.t. } x_1 + x_2 \le 4$$
$$x_1 + 3x_2 \le 1$$

$x_1 + x_2 = 4$

$x_1 + 3x_2 = 1.$
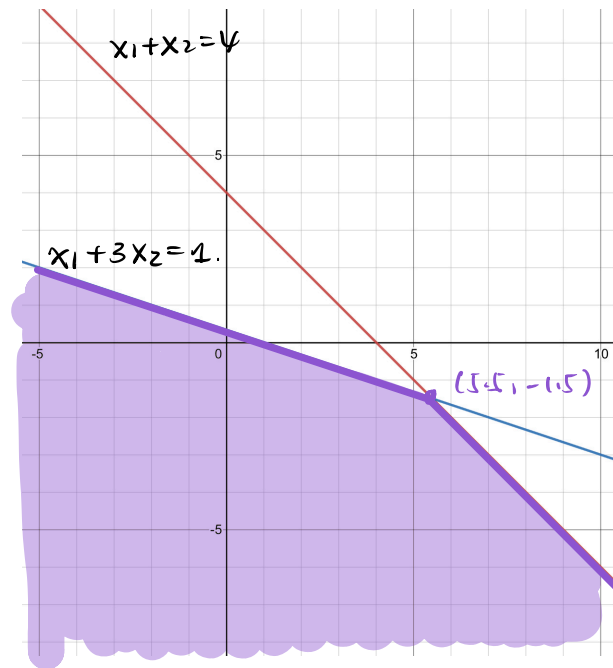
$(5.5, -1.5)$

The shaded region is the constraint set and by vertex theorem, the solution is $(5.5, -1.5)$, which gives $V_p(b) = 2.5$

Next consider the dual problem:

$$\text{MIN } 4y_1 + y_2$$
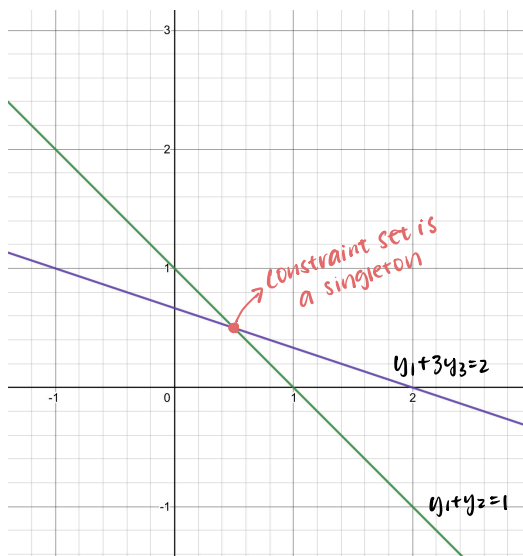$$\text{S.T. } y_1 + y_2 \ge 1 \qquad \Rightarrow y_1 + y_2 = 1$$
$$-y_1 - y_2 \ge -1$$
$$y_1 + 3y_2 \ge 2 \qquad \Rightarrow y_1 + 3y_3 = 2$$
$$-y_1 - 3y_2 \ge -2$$
$$y_1, y_2 \ge 0$$

constraint set is a singleton

$y_1 + 3y_3 = 2$

$y_1 + y_2 = 1$

Since the constraint set is a singleton, the solution to the dual problem is $(0.5, 0.5)$, which gives $V_D(b) = 2.5.$

$$\Rightarrow V_D(c) = V_p(b)$$

(C)

By computation above, we know that $V_P(b)$ and $V_D(c)$ are both finite when $b \in [0,14]$.

and $V_P(b) = V_D(c)$. The solution for the dual problem is $(0.5, 0.5)$.

Hence, $V_P(b) = V_D(c) = 4(0.5) + 0.5b = 2 + 0.5b$.

By the shadow price theorem, $\partial V_P\left(\begin{bmatrix} 4 \\ b \end{bmatrix}\right) = (0.5, 0.5)$ for $b \in [0,14]$ ✓

and $V_P(b) = 0.5$ for $b \in [0,14]$

You could also solve the primal for an arbitrary $b$, and then

obtain $\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} \frac{12-b}{2} \\ \frac{b-4}{2} \end{bmatrix}$

and plug it back into

$V_P(b) = \dot{x_1} + 2\dot{x_2} = 2 + \frac{b}{2}$

$\dfrac{d V_P(b)}{db} = \dfrac{1}{2}$