1. Find cost functions for the following production functions;

(a) Cobb-Douglas:

Consider the following Cobb-Douglas production function:

$$f(K,L) = K^{\alpha}L^{\beta}$$

Then, the minimization cost problem is given by

$$C = wL + rK$$

s.t.

$$K^{\alpha}L^{\beta} \geq Y$$

We have no corner solutions because constraints bind due to monotonicity, then we can work with interior solutions, which means that the constraint is going to bind and the problem to solve is given by the following lagrangian:

$$\mathcal{L} = wL + rK + \lambda(Y - K^{\alpha}L^{\beta})$$

Then, the FOC are

$$w - \lambda\beta K^{\alpha}L^{\beta-1} = 0$$

$$r - \lambda\alpha K^{\alpha-1}L^{\beta} = 0$$

Then,

$$\frac{w}{r} = \frac{\beta K}{\alpha L} \rightarrow K = \frac{w\alpha}{r\beta}L$$

Then,

$$Y = \left(\frac{w\alpha}{r\beta}\right)^{\alpha} L^{\alpha+\beta}$$

$$L^{*} = Y^{\frac{1}{\alpha+\beta}}\left(\frac{r\beta}{w\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}$$

Then,

$$K^{*} = Y^{\frac{1}{\alpha+\beta}}\left(\frac{w\alpha}{r\beta}\right)^{\frac{\beta}{\alpha+\beta}}$$

Then the cost function is given by

$$C^*(w,r) = Y^{\frac{1}{\alpha+\beta}}\left[w\left(\frac{r\beta}{w\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} + r\left(\frac{w\alpha}{r\beta}\right)^{\frac{\beta}{\alpha+\beta}}\right]$$

(b) CES: $\boldsymbol{f(L,K) = (aK^r + bL^r)^{1/r}}$

Again, the problem is given by

$$\mathcal{L} = wL + rK + \lambda\left(Y - (aK^r + bL^r)^{1/r}\right)$$

$$w - \frac{\lambda}{r} rbL^{r-1}(aK^r + bL^r)^{1-r/r} = 0$$

$$r - \frac{\lambda}{r} raK^{r-1}(aK^r + bL^r)^{1-r/r} = 0$$

$$\frac{w}{r} = \frac{bL^{r-1}}{aK^{r-1}} \rightarrow L = \left(\frac{wa}{br}\right)^{\frac{1}{r-1}} K$$

$$Y = \left(aK^r + b\left(\left(\frac{wa}{br}\right)^{\frac{1}{r-1}} K\right)^r\right)^{1/r} = \left(aK^r + b\left(\frac{wa}{br}\right)^{\frac{r}{r-1}} K^r\right)^{1/r} = Y$$

$$K\left(a + b\left(\frac{wa}{br}\right)^{\frac{r}{r-1}}\right)^{1/r} = Y$$

$$K = \frac{Y}{\left(a + b\left(\frac{wa}{br}\right)^{\frac{r}{r-1}}\right)^{1/r}}$$

And

$$L = \frac{\left(\frac{wa}{br}\right)^{\frac{1}{r-1}} Y}{\left(a + b\left(\frac{wa}{br}\right)^{\frac{r}{r-1}}\right)^{1/r}}$$

Then, the cost function is given by

$$C^*(w,r) = Y\left[\frac{r}{\left(a + b\left(\frac{wa}{br}\right)^{\frac{r}{r-1}}\right)^{1/r}} + \frac{w\left(\frac{wa}{br}\right)^{\frac{1}{r-1}} Y}{\left(a + b\left(\frac{wa}{br}\right)^{\frac{r}{r-1}}\right)^{1/r}}\right]$$

(C) Linear function: $\boldsymbol{f(K,L) = aK + bL}$

Then, we have to minimize the following cost function

$$C = wL + rK$$

Subject to

$$Y = aK + bL$$

Given that both are linear, we are going to have to compare the slopes of both functions, we can see that the slope of the isocosts are given by $-\frac{w}{r}$ because if we write $K$ as a function of $L$ we have that

$$K = \frac{C}{r} - \frac{w}{r}L$$

In the case of the isoquants, analogously we have that the slope is given by $-\frac{b}{a}$. Then, we have to compare the slopes of both functions.

If $\frac{w}{r} > \frac{b}{a}$ then we are going to demand 0 of $L$ and $\frac{Y}{a}$ of $K$ because the relative cost of labor is greater than its relative marginal productivity compared to capital.

If $\frac{w}{r} < \frac{b}{a}$ then we are going to demand 0 of $K$ and $\frac{Y}{b}$ of $L$ because the relative cost of capital is greater than its relative marginal productivity compared to labor.

If $\frac{w}{r} = \frac{b}{a}$ then we have that both functions are going to overlap and we have infinity possible combinations of $L$ and $K$ that solve our problem.

Then, the cost function is given by

$$C(w,r) = \begin{cases} \frac{rY}{a} \; if \; \frac{w}{r} \geq \frac{b}{a} \\ \frac{wY}{b} \; if \; \frac{w}{r} < \frac{b}{a} \end{cases}$$

**(d) Leontief, $\boldsymbol{f(K,L) = \min\{aK, bL\}}$**

We know that

$$Y = \min\{aK, bL\}$$

Which means that the demand of the factors is going to be given by

$$K^* = \frac{Y}{a}$$

$$L^* = \frac{Y}{b}$$

Then, the optimal cost function is given by

$$C^*(w,r) = \left(\frac{r}{a} + \frac{w}{b}\right)Y$$

(e) von Thünen's production function

$$f(K,L) = (1-e^{-aK})\left(1-e^{-bL}\right) = 1-e^{-bL}-e^{-aK}+e^{-aK-bL}$$

$$f_L = be^{-bL}(1-e^{-aK}) = 0$$

$$f_K = ae^{-aK}\left(1-e^{-bL}\right) = 0$$

Then,

$$\frac{f_L}{f_K} = \frac{be^{-bL}(1-e^{-aK})}{ae^{-aK}(1-e^{-bL})} = \frac{w}{r}$$

Given this conditions we solve the system of equations of two unknowns $(L,K)$ and two equations and find $L^*$ and $K^*$ and replace them in the cost function:

$$C^*(w,r) = wL^* + rK^*$$

**2. Suppose in a two-country world, countries $\boldsymbol{A}$ and $\boldsymbol{B}$, that $\boldsymbol{f_A}$ and $\boldsymbol{f_B}$ are the usual neoclassical production functions of capital and labor, with Inada conditions.**

**(a) Show that all revenues (national product) are distributed to the factors.**

If the production functions meet with CRS then we have that by Euler's theorem

$$\frac{\partial f_i}{\partial L}L + \frac{\partial f_i}{\partial K}K = f_i(L,K) \quad ,i = A,B$$

Then, multiplying both sides by the price of the good produced in country $A$

$$p_i\frac{\partial f_i}{\partial L}L + p_i\frac{\partial f_i}{\partial K}K = p_i f_i(L,K) \quad ,i = A,B$$

And we know that, in the optimum

$$p_i\frac{\partial f_i}{\partial L} = w \quad \wedge p_i\frac{\partial f_i}{\partial K} = r$$

It is, that the value of the marginal products of the factors have to equal the marginal cost of them. We also know that $p_i f_i(L,K)$ is the total income of country $i$. Let's define the total income of country $i$ as $Y_i$, then

$$Y_i = wL + rK$$

Which means that all the income of the country is distributed between the two factors $L$ and $K$.

**(b) Suppose that $\boldsymbol{p_A}$ increases. Under what conditions on $\boldsymbol{f_A}$ and $\boldsymbol{f_B}$ will the capital share of national product increase?**

We have that when the rental price increase, then the share of capital in the national product will increase as well. First, I am going to show this. Suppose $r' > r$

Then we know that

$$\frac{r'K}{rK} > \frac{wL}{wL}$$

$$r'KwL > rKwL \rightarrow r'KwL + r'KrK > rKwL + r'KrK$$

$$r'K(wL + rK) > rK(wL + r'K)$$

$$\frac{r'K}{wL + r'K} > \frac{rK}{wL + rK}$$

Then, it is enough to find under what conditions on $f_A$ and $f_B$ will the rental price $r$ increase when $p_A$ increases. In order to prove this, we can work with the Stolper-Samuelson theorem which states that an increase in the price of a good is going to increase the price of the factor in which the production of that good is intensive. In this case, what we hope is that if $p_A$ increases the rental rate $r$ and therefore the share of capital in the national product is going to increase, then the condition that must be hold is that $A$ is intensive in capital.

Let's prove Stolper-Samuelson theorem. The optimality conditions indicate that the price equals to marginal cost in each country, then

$$C_A(w, r) = p_A$$

$$C_B(w, r) = p_B$$

Which means that

$$F_1(w, r, p_A, p_B) = C_A(w, r) - p_A$$

$$F_2(w, r, p_A, p_B) = C_B(w, r) - p_B$$

Then, by taking derivatives relative to $p_A$ we have that

$$\frac{\partial C_A}{\partial w}\frac{\partial w}{\partial p_A} + \frac{\partial C_A}{\partial r}\frac{\partial r}{\partial p_A} = 1$$

$$\frac{\partial C_B}{\partial w}\frac{\partial w}{\partial p_A} + \frac{\partial C_B}{\partial r}\frac{\partial r}{\partial p_A} = 0$$

$$\begin{bmatrix} \frac{\partial C_A}{\partial w} & \frac{\partial C_A}{\partial r} \\ \frac{\partial C_B}{\partial w} & \frac{\partial C_B}{\partial r} \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial p_A} \\ \frac{\partial r}{\partial p_A} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We know that by Shepard's Lemma and market clearing, it is true that in the optimum the derivative of the cost function relative to the prices of the factors are going to give the optimal demand of the factors in each country, then

$$\frac{\partial C_A}{\partial w} = L_A, \frac{\partial C_B}{\partial w} = L_B, \frac{\partial C_A}{\partial r} = K_A, \frac{\partial C_B}{\partial r} = K_B$$

$$\begin{bmatrix} L_A & K_A \\ L_B & K_B \end{bmatrix} \begin{bmatrix} \dfrac{\partial w}{\partial p_A} \\ \dfrac{\partial r}{\partial p_A} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We have that

$$\frac{\partial r}{\partial p_A} = \frac{1}{L_A K_B - K_A L_B} \begin{vmatrix} L_A & 1 \\ L_B & 0 \end{vmatrix} = \frac{-L_B}{L_A K_B - K_A L_B}$$

We assumed that $A$ is intensive in capital, then it is true that

$$\frac{K_A}{K_B} > \frac{L_A}{L_B} \rightarrow L_A K_B < K_A L_B$$

Then,

$$\frac{\partial r}{\partial p_A} = \frac{\overbrace{-L_B}^{<0}}{\underbrace{L_A K_B - K_A L_B}_{<0}} > 0$$

Then, an increase in $p_A$ increases the price of the factor in which $A$ is intensive, it is, increases $r$.

**3. For a country with an endowment in the interior of the cone of diversification, derive and prove a result on the effects of a small increase in the quantity of a factor on output**

We know that by Shepard's Lemma and Market clearing then

$$X_A \frac{\partial C_A(w,r)}{\partial w} + X_B \frac{\partial C_B(w,r)}{\partial w} = L$$

$$X_A \frac{\partial C_A(w,r)}{\partial r} + X_B \frac{\partial C_B(w,r)}{\partial r} = K$$

Then

$$X_A L_A + X_B L_B = L$$

$$X_A K_A + X_B K_B = K$$

$$\begin{bmatrix} C_{Aw} & C_{BW} \\ C_{Ar} & C_{Br} \end{bmatrix} \begin{bmatrix} \dfrac{\partial X_A}{\partial L} & \dfrac{\partial X_A}{\partial K} \\ \dfrac{\partial X_B}{\partial L} & \dfrac{\partial X_B}{\partial K} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then,

$$\begin{bmatrix} \dfrac{\partial X_A}{\partial L} & \dfrac{\partial X_A}{\partial K} \\ \dfrac{\partial X_B}{\partial L} & \dfrac{\partial X_B}{\partial K} \end{bmatrix} = \frac{1}{C_{Aw} C_{Br} - C_{BW} C_{Ar}} \begin{bmatrix} C_{Br} & -C_{BW} \\ -C_{Ar} & C_{Aw} \end{bmatrix}$$

Then, we are going to have that an increase in the quantity of a factor is going to increase the output of the good that is intensive on that factor.

First, we know that by Shephard's Lemma

$$C_{Aw} = L_A$$

$$C_{Br} = K_B$$

$$C_{BW} = L_B$$

$$C_{Ar} = K_A$$

Where $C_{ij}$ indicates the derivative of the cost function of country $i$ relative to factor $j$. Then,

$$\begin{bmatrix} \frac{\partial X_A}{\partial L} & \frac{\partial X_A}{\partial K} \\ \frac{\partial X_B}{\partial L} & \frac{\partial X_B}{\partial K} \end{bmatrix} = \frac{1}{L_A K_B - L_B K_A} \begin{bmatrix} K_B & -L_B \\ -K_A & L_A \end{bmatrix}$$

Then, suppose that $A$ is intensive in $L$ then we have that

$$\frac{\partial X_A}{\partial L} = \frac{K_B}{L_A K_B - L_B K_A}$$

We know that $\frac{L_A}{L_B} > \frac{K_A}{K_B} \rightarrow L_A K_B - L_B K_A > 0$

Then,

$$\frac{\partial X_A}{\partial L} > 0$$

As expected by Rybczynski Theorem.

4. Here is another two-sector model. Sector 1 produces investment goods (capital goods). Sector 2 produces consumption goods. Each sector is characterized by a neoclassical production function (strictly concave, $\boldsymbol{C^2}$, Inada conditions at 0) with constant returns to scale. Write $\boldsymbol{Y_i = F_i(K_i, L_i)}$ for output in sector $\boldsymbol{i}$ as a function $\boldsymbol{F_i}$ of capital $\boldsymbol{K_i}$ and labor $\boldsymbol{L_i}$ employed in sector $\boldsymbol{i}$.

(a) Rewrite these relationships in terms of the output/labor and capital/labor ratio: $\boldsymbol{y_i = f_i(k_i)}$ where $\boldsymbol{y_i = \frac{Y_i}{L_i}}$, etc. What properties do the foregoing assumptions imply for $\boldsymbol{f_i}$?

By definition we have that

$$y_i = \frac{Y_i}{L_i}$$

Then,

$$y_i = \frac{F_i(K_i, L_i)}{L_i}$$

$$y_i = F\left(\frac{K_i}{L_i}, 1\right)$$

By homogeneity of degree one we have that

$$y_i = f_i(k_i)$$

Where $k_i = K_i/L_i$

We know that $F_i$ is strictly concave in $K_i$ and $L_i$, then $f_i$ is concave in $k_i$. The Inada conditions still apply for $f_i$. We also know that $f_i$ is $C^2$. Now, we have that $f_i$ is not going to have constant returns to scale. Originally, we had that

$$F_i(\alpha L_i, \alpha K_i) = \alpha F(L_i, K_i)$$

However, if we consider that $f_i(k_i)$ we have that

$$f_i(\alpha k_i) = F\left(\frac{\alpha K_i}{L_i}, 1\right) = \alpha F\left(k_i, \frac{1}{\alpha}\right) \neq \alpha f_i(k_i)$$

**(b) Let $\boldsymbol{w}$ and $\boldsymbol{r}$ denote the equilibrium prices of capital, and $\boldsymbol{P_i}$ the price of output $\boldsymbol{i}$. Equilibrium in this model requires for $\boldsymbol{i = 1, 2}$. Interpret these equations, including the demand conditions (4).**

$$\textbf{(1)}\ \boldsymbol{Y_i = F_i(K_i L_i)}$$

$$\textbf{(2)}\ \boldsymbol{P_i \frac{\partial F_i}{\partial K_i} = r} \qquad \textbf{(3)}\ \boldsymbol{P_i \frac{\partial F_i}{\partial L_i} = w} \qquad \textbf{(4)}\ \boldsymbol{K_1 + K_2 = K}$$

$$\textbf{(5)}\ \boldsymbol{L_1 + L_2 = L} \qquad \textbf{(6)}\ \boldsymbol{P_1 Y_1 = rK} \qquad \textbf{(7)}\ \boldsymbol{P_2 Y_2 = wL}$$

First, we have that by condition (1), all the output is consumed in the economy. Then the conditions (2) and (3) states that the value of the marginal product of the factors (price times the marginal product) has to be equal to the marginal cost of each factor. The following two conditions (4) and (5) state that the factor markets clear, it is that all the factors are used either in the production of capital goods or in the production of consumption goods and finally the two last condition (6) and (7) state that the income from the capital goods sector $P_1 Y_1$ equals the income from the capital $rK$, and the income from labor in the economy $wL$ has to be equal to the production value of the consumption goods in the economy, $P_2 Y_2$.

**(c) Rewrite these conditions in terms of the aggregate capital/Labor ratio $\boldsymbol{k = K/L}$, the output/labor ratios $\boldsymbol{y_i}$, the labor shares $\boldsymbol{l_i = L_i/L}$ and the wage/rent ratio $\boldsymbol{\omega = w/r}$.**

We have that condition (1) is going to be

$$y_i = f(k_i)$$

We have that

$$y_i = \frac{F_i(K_i, L_i)}{L_i}$$

Then,

$$\frac{\partial y_i}{\partial L_i} = \frac{F_{iL}(K_i, L_i)L_i - F_i(K_i, L_i)}{L_i^2} = \frac{F_{iL}(K_i, L_i)}{L_i} - \frac{f_i(k_i)}{L_i}$$

Then,

$$\frac{\partial y_i}{\partial L_i} = \frac{F_{iL}(K_i, L_i)}{L_i} - \frac{f_i(k_i)}{L_i}$$

We also know that

$$\frac{\partial y_i}{\partial L_i} = -\frac{K_i}{L_i^2} f_i'(k_i)$$

From this we have that

$$\frac{F_{iL}(K_i, L_i)}{L_i} - \frac{f_i(k_i)}{L_i} = -\frac{K_i}{L_i^2} f_i'(k_i) \rightarrow \; F_{iL}(K_i, L_i) - f_i(k_i) = -\frac{K_i}{L_i} f_i'(k_i)$$

Then,

$$F_{i,L}(k_i, 1) = f_i(k_i) - k_i f_i'(k_i)$$

Then, we have that

$$\omega = \frac{w}{r} = \frac{P_i F_{i,L}(K_i, L_i)}{P_i F_{i,K}(K_i, L_i)} = \frac{F_{i,L}(k_i, 1)}{f_i'(k_i)} = \frac{f_i(k_i)}{f_i'(k_i)} - k_i$$

Then

$$\omega = \frac{f_i(k_i)}{f_i'(k_i)} - k_i$$

On the other hand, we know that

$$k = \frac{K}{L} = \frac{K_1 + K_2}{L} = \frac{K_1 L_1}{L_1 L} + \frac{K_2 L_2}{L_2 L} = l_1 k_1 + l_2 k_2$$

We also have that

$$\frac{L_1}{L} + \frac{L_2}{L} = l_1 + l_2 = \frac{L}{L} = 1$$

We also have that

$$P_1 Y_1 = rK$$

Then,

$$P_1 Y_1 = P_1 \frac{\partial F_1}{\partial K} K$$

Then

$$Y_1 = \frac{\partial F_1}{\partial K} K$$

Then,

$$\frac{Y_1}{L_1} = \frac{K}{L_1} F_{1K}(k_i, 1)$$

$$L_1 f_1(k_1) = K f_1'(k_1)$$

Dividing both sides by $L$ we have that

$$l_1 f_1(k_1) = k f_1{}'(k_1)$$

And for the second part, by a similar procedure we have that

$$l_1 f_2(k_2) = k_2 f_2'(k_2)$$

**(d) Compute $dk_i/d\omega$ and use this to prove that the capital/labor ratio in each sector is uniquely determined by the wage/rent ratio.**

Consider what we found before

$$\omega = \frac{f_i(k_i)}{f_i{}'(k_i)} - k_i$$

By totally differentiating we have that

$$d\omega = \frac{f_i'(k_i) f_i'(k_i) dk_i - f_i(k_i) f_i''(k_i) dk_i}{\left[f_i'(k_i)\right]^2} - dk_i$$

$$d\omega = \left[\frac{f_i'(k_i) f_i'(k_i) - f_i(k_i) f_i''(k_i)}{\left[f_i'(k_i)\right]^2} - 1\right] dk_i$$

$$\frac{dk_i}{d\omega} = \frac{1}{\left[\frac{f_i'(k_i) f_i'(k_i) - f_i(k_i) f_i''(k_i) - \left[f_i'(k_i)\right]^2}{\left[f_i'(k_i)\right]^2}\right]} = \frac{[f_i'(k_i)]^2}{\left[f_i'(k_i)\right]^2 - f_i(k_i) f_i''(k_i) - \left[f_i'(k_i)\right]^2}$$

$$\frac{dk_i}{d\omega} = \frac{\overbrace{\left[f_i'(k_i)\right]^2}^{>0}}{\underbrace{-f_i(k_i)\, \underbrace{f_i''(k_i)}_{<0}}_{>0}}$$

Then, we have that $\frac{dk_i}{d\omega} > 0$. Which means that the capital labor ratio is a strictly increasing function of the wage-rent ratio.

**(e) Find an equation that implicitly defines the equilibrium wage-rent ratio in terms of the capital-labor ratio.**

We know that

$$\omega = \frac{f_i(k_i)}{f_i{}'(k_i)} - k_i$$

And that

$$l_1 f_1(k_1) = k f_1{}'(k_1)$$

Define the capital-labor ratio in each industry depending on $\omega$ as follows: $k_1(\omega)$ and $k_2(\omega)$, then

$$\omega = \frac{f_1(k_1)}{f_1{}'(k_1)} - k_1 \rightarrow \frac{f_1(k_1)}{f_1'(k_1)} = \omega + k_1 \rightarrow \frac{f_1'(k_1)}{f_1(k_1)} = \frac{1}{\omega + k_1}$$

Then

$$l_1 = k\frac{f_1'(k_1)}{f_1(k_1)} \rightarrow l_1 = \frac{k}{\omega + k_1(\omega)}$$

Then we have that

$$l_1 + l_2 = 1$$

$$l_2 = 1 - \frac{k}{\omega + k_1(\omega)} = \frac{\omega + k_1(\omega) - k}{\omega + k_1(\omega)}$$

Then we had that

$$k = l_1 k_1 + l_2 k_2 \rightarrow k = \frac{k}{\omega + k_1(\omega)} k_1(\omega) + \frac{\omega + k_1(\omega) - k}{\omega + k_1(\omega)} k_2(\omega)$$

Then, we have that

$$k = \frac{k k_1(\omega) + \omega k_2(\omega) + k_1(\omega) k_2(\omega) - k k_2(\omega)}{\omega + k_1(\omega)}$$

$$k\big(\omega + k_1(\omega)\big) = k k_1(\omega) + \omega k_2(\omega) + k_1(\omega) k_2(\omega) - k k_2(\omega)$$

$$k\big[\big(\omega + k_1(\omega)\big) - k_1(\omega) + k_2(\omega)\big] = \omega k_2(\omega) + k_1(\omega) k_2(\omega)$$

$$k = \frac{k_2(\omega)\big(\omega + k_1(\omega)\big)}{\omega + k_2(\omega)}$$

According to which we have that the equilibrium wage-rent ratio $\omega$ is implicitly defined by the capital-labor ratio.