1 Problem 7

There are two countries, and a continuum of goods indexed by $z \in [0, 1]$. There is one input, labor. Production of each good is CRS, and $a(z)$ and $b(z)$ are the unit labor requirements in countries A and B, respectively. Without loss of generality, goods are ordered so that $\alpha(z) = a(z)/b(z)$ is decreasing. Assume it is strictly decreasing. Let $p(z)$ denote the free-trade world price of good z. Let w_a and w_b denote the wage rate in countries A and B, respectively, and let L_A and L_B denote their respective labor endowments.

(a) Describe the implications of profit maximization that must hold for all goods z in countries A and B.

Here we begin by writing the maximization problem for the proceeds made by the two countries, subject to the constraints stemming from labor availability. The problem, then, looks as follows:

$$\max_{q_A, q_B} \int_0^1 [p(z)q_A(z) + p(z)q_B(z)]dz = \max_{q_A, q_B} \int_0^1 p(z)[q_A(z) + q_B(z)]dz$$

$$s.t. \int_0^1 a(z)q_A(z)dz \leq L_A$$

$$\int_0^1 b(z)q_B(z)dz \leq L_B$$

$$q_A(z), q_B(z) \geq 0$$

where $q_A(z), q_B(z)$ stands for the quantity produced by each country. The complementary slackness conditions are then given by:

$$[\int_0^1 a(z)q_A(z)dz - L_A]w_a = 0 \tag{1}$$

$$[\int_0^1 b(z)q_B(z)dz - L_B]w_b = 0 \tag{2}$$

The dual for this problem is, then, given by:

$$\min_{w_a, w_b} w_a L_A + w_b L_B$$

$$s.t. a(z) w_a \geq p(z), \forall z$$

$$b(z) w_b \geq p(z), \forall z$$

$$w_a, w_b \geq 0$$

The complementary slackness conditions are given by:

$$q_A(z)(p(z) - w_a a(z)) = 0 \tag{3}$$

$$q_B(z)(p(z) - w_b b(z)) = 0 \tag{4}$$

where we have equations 1-4 being the profit maximization conditions.

(b) What additional relations does profit maximization imply for goods produced in A and in B?

In the spacial case that we have $q_A(z), q_B(z) > 0$, then we obtain the following system of equations:

$$a(z) w_a = p(z)$$

$$b(z) w_b = p(z)$$

Combining the two yields:

$$a(z) w_a = b(z) w_b \Rightarrow \frac{b(z)}{a(z)} = \frac{w_a}{w_b} \Rightarrow$$

$$\alpha(z) = \frac{w_b}{w_a} \tag{5}$$

(c) Proposition: In any equilibrium, there exists $z^* \in [0,1]$ such that A produces all goods $z < z^*$ and B produces all goods $z > z^*$. Prove it.

For a good that is produced in both countries, we know that:

$$\alpha(z^*) = \frac{w_b}{w_a}$$

Now, picking $z \leq z^*$, and taking into account that $\alpha(z)$ is strictly decreasing we have:

$$\alpha(z) \geq \alpha(z^*) \Rightarrow \frac{a(z)}{b(z)} \geq \frac{w_b}{w_a}$$

Now, if we have $q_a(z) > 0$,it follows that $q_b(z) = 0$, and from complementary slackness conditions 3 and 4, we have:

$$w_a a(z) = p(z)$$
$$p(z) \leq w_b b(z)$$

Therefore, it follows that:

$$w_a a(z) \leq w_b b(z)$$

which contradicts what was found above. Hence, it follows that in the case that we have $z \leq z^*$, we have $q_b(z) > 0$ and $q_a(z) = 0$.

Equivalently, by symmetry, we can obtain, by picking $z \geq z^*$, that in the case that we have $z \geq z^*$, we have $q_b(z) = 0$ and $q_a(z) > 0$.

(d) Suppose that α is continuous. Show that profit maximization requires $\alpha(z^*) = w_B/w_A$.

This is exactly what was shown in equation 5 above.

Suppose that consumers have identical Cobb-Douglas preferences, and let $s(z)$ denote the share of expenditures on good z. So $s(z) = p(z)c_i(z)/w_i L_i$ for $i = A, B$.

(e) Why must $\int_0^1 s(z)dz = 1$?

Here, we have identical preferences of the consumers, yielding the following maximization problem:

$$\max_{c_i(z)} \int_0^1 U_i(c_i(z))dz$$
$$s.t. \int_0^1 p(z)c_i(z)dz \leq w_i L_i$$

for each of the two countries. Given monotonicity of the utility function, we have that the above constraint has to be satisfied with equality, as follows:

$$\int_0^1 p(z)c_i(z)dz = w_i L_i \Rightarrow$$
$$\int_0^1 \frac{p(z)c_i(z)}{w_i L_i} dz = 1 \Rightarrow$$
$$\int_0^1 s(z)dz = 1 \tag{6}$$

Let $\theta(z^*) = \int_0^{z*} s(z)dz$ denote the fraction of income spent by both countries on a good produced in country A.

(f) What are the equilibrium budget constraints for both countries on expenditures at home and abroad?

Here we will have $\theta_A(z^*) = \int_{z^*}^1 s(z)dz$ being the fraction of income spent on Country A, and the fraction of income spent on country B will be $\theta_B(z^*) = \int_0^{z^*} s(z)dz$.

Therefore, we will have the following budget constraints:

$$w_a L_a = \theta_A(z^*)[w_a L_a + w_b L_b] \Rightarrow (1 - \theta_A(z^*))w_a L_a = \theta_A(z^*)w_b L_b$$

Equivalently, for country B we have:

$$w_b L_b = \theta_B(z^*)[w_a L_a + w_b L_b] \Rightarrow (1 - \theta_B(z^*))w_b L_b = \theta_B(z^*)w_a L_a$$

Now, it only remains to notice that $\theta_A(z^*) = 1 - \theta_B(z^*)$ to conclude that the two budget constraints coincide.

(g) Use these relations to derive a relationship between z^* and the equilibrium ration w_B/w_A. Denote this relationship as $w_B/w_A = \beta(z^*)$.

Utilizing the first of the two constraints, we have:

$$(1 - \theta_A(z^*))w_a L_a = \theta_A(z^*)w_b L_b \Rightarrow \frac{w_b}{w_a} = \frac{(1 - \theta_A(z^*))L_a}{\theta_A(z^*)L_b} \Rightarrow$$

$$\beta(z^*) = \frac{w_b}{w_a} = \frac{\theta_B(z^*)L_a}{\theta_A(z^*)L_b}$$

(h) What is an increase of L_A on the equilibrium z^*?

Here, we basically need to carry out comparative statics. We begin by utilizing the above result and differentiating with respect to L_a, as follows:

$$\frac{\partial}{\partial L_a}\beta(z^*) = \beta'(z^*)\frac{\partial z^*}{\partial L_a} = \frac{\theta_B(z^*)}{\theta_A(z^*)L_b} - L_a\frac{\theta_B(z^*) + \theta_A(z^*)}{\theta_A^2(z^*)L_b}\frac{\partial\theta(z^*)}{\partial z^*}\frac{\partial z^*}{\partial L_a} \Rightarrow$$

$$\beta'(z^*)\frac{\partial z^*}{\partial L_a} = \frac{\theta_B(z^*)}{\theta_A(z^*)L_b} - \frac{L_a}{\theta_A^2(z^*)L_b}\frac{\partial\theta(z^*)}{\partial z^*}\frac{\partial z^*}{\partial L_a} \Rightarrow$$

$$\beta'(z^*)\frac{\partial z^*}{\partial L_a} = \frac{\theta_B(z^*)}{\theta_A(z^*)L_b} + \frac{L_a s(z^*)}{\theta_A^2(z^*)L_b}\frac{\partial z^*}{\partial L_a} \Rightarrow$$

$$\frac{\partial z^*}{\partial L_a}[\beta'(z^*) - \frac{L_a s(z^*)}{\theta_A^2(z^*)L_b}] = \frac{\theta_B(z^*)}{\theta_A(z^*)L_b} \Rightarrow$$

$$\frac{\partial z^*}{\partial L_a}[\beta'(z^*)\theta_A(z^*)L_b - \frac{L_a s(z^*)}{\theta_A(z^*)}] = \theta_B(z^*) \Rightarrow$$

$$\frac{\partial z^*}{\partial L_a} = \frac{\theta_B(z^*)}{\beta'(z^*)\theta_A(z^*)L_b - \frac{L_a s(z^*)}{\theta_A(z^*)}}$$

Now, we know that the numerator is a positive number, while the denominator is a negative one, concluding that the derivative is negative. Hence, any increase in L_a will yield a decrease in z^*.

2 Problem 12

Consider a generalized Leontief model with B and A such that:

$$B - A = \begin{bmatrix} 1 & -1/2 & 1/2 \\ -1/2 & 1 & 1/4 \end{bmatrix}$$

with labor requirement vector $a_0 = (1, 1, 1/2)$. Is there a technology τ which is efficient for all net outputs $(y_1, y_2) \geq 0$?

Here, we begin by writing out the cost minimization problem, as follows:

$$\min_{x_1,x_2,x_3} \begin{bmatrix} 1 & 1 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$s.t. \begin{bmatrix} 1 & -1/2 & 1/2 \\ -1/2 & 1 & 1/4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq 0$$

Now, we continue by writing out the dual of this problem, as follows:

$$\max_{p_1,p_2} \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$s.t. \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} 1 & -1/2 & 1/2 \\ -1/2 & 1 & 1/4 \end{bmatrix} \leq \begin{bmatrix} 1 & 1 & 1/2 \end{bmatrix}$$

$$\begin{bmatrix} p_1 & p_2 \end{bmatrix} \geq 0$$

Now, the complementary slackness conditions are given by:

$$x_1[1 - p_1 + \frac{1}{2}p_2] = 0$$

$$x_2[1 + \frac{1}{2}p_1 - p_2] = 0$$

$$x_3[\frac{1}{2} - \frac{1}{2}p_1 - \frac{1}{4}p_2] = 0$$

Now, the constraint set of the dual problem are shown in Figure 1 below, enclosed in the four points:

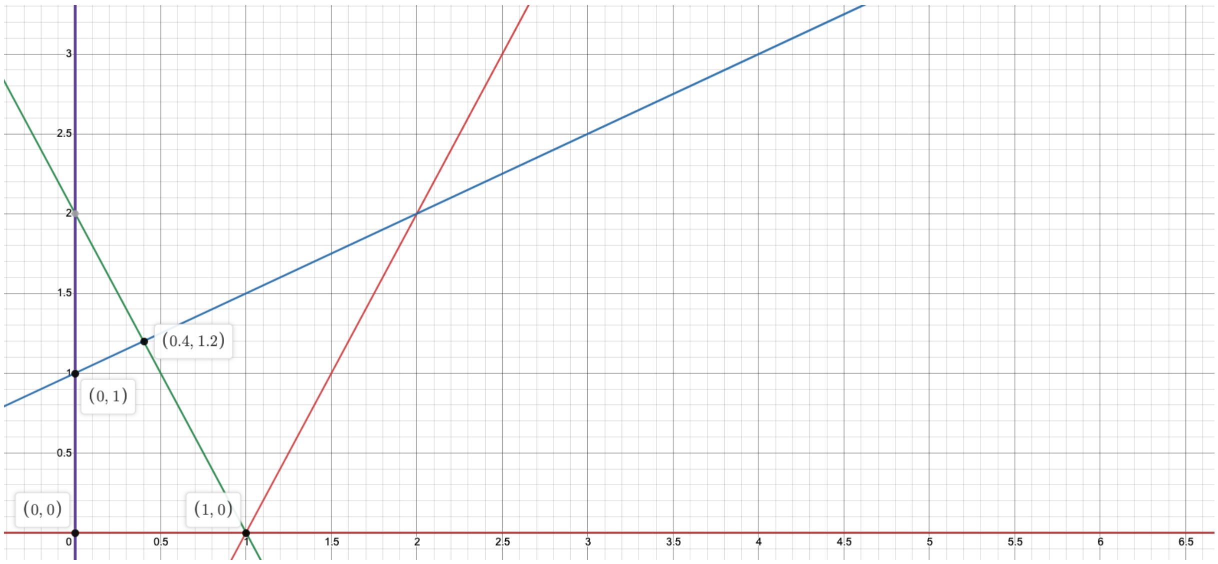

Figure 1: Constraint set of the dual problem

For the case in which we have $(y_1, y_2) = (1, 0)$, the solutions from vertex theorem are $(p_1, p_2) = (1, 0)$. Then, from complementary slackness above, we obtain $x_1 > 0$, $x_2 = 0$, $x_3 > 0$.

Equivalently, we have for the case in which we have $(y_1, y_2) = (0, 1)$, the solutions from vertex theorem are $(p_1, p_2) = (0.4, 1.2)$. Then, from complementary slackness above, we obtain $x_1 = 0$, $x_2 > 0$, $x_3 > 0$.

Given that the set of technologies for y_1 and y_2 do not coincide, we do not have a technology that is efficient for all $y_1, y_2 \geq 0$.

Problem 13

- For $y_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ the cost function:

$$\lambda(y_1) = \min\ x_1 + x_2$$

$$\text{st} \begin{pmatrix} 3/4 & r-1/4 \\ -1/4 & 3/4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$x_1, x_2 \geq 0$$

Draw the constraint set and you'll notice you can divide in two cases: $\begin{cases} r \leq 1 \rightarrow \text{interior solution } (x_1, x_2 > 0) \\ r > 1 \rightarrow \text{corner solution } (x_1 = 0) \end{cases}$

we then obtain:

$$c(r, (1,0)) = \begin{cases} 4/(2+r) & \text{if } r \leq 1 \\ 4/(4r-1) & \text{if } r > 1 \end{cases} \quad x(r) = \begin{cases} 3/(2+r) & \text{if } r \leq 1 \\ 0 & \text{if } r > 1 \end{cases} \quad y(r) = \begin{cases} 1/(2+r) & \text{if } r \leq 1 \\ 4/(4r-1) & \text{if } r > 1 \end{cases}$$

- For $y_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ the cost function:

$$\lambda(y_2) = \min\ x_1 + x_2$$

$$\text{st} \begin{pmatrix} 3/4 & r-1/4 \\ -1/4 & 3/4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x_1, x_2 \geq 0$$

Draw constraint set and you will find 2 cases: $\begin{cases} r \leq 1/4 \rightarrow \text{interior solution } x_1, x_2 > 0 \\ r > 1/4 \rightarrow \text{corner solution } (x_1 = 0) \end{cases}$

$$c(r, (0,1)) = \begin{cases} \frac{4-4r}{(2+r)} & \text{if } r \leq 1/4 \\ 4/3 & \text{if } r \geq 1/4 \end{cases} \quad x(r) = \begin{cases} \frac{1-4r}{(2+r)} & \text{if } r \leq 1/4 \\ 0 & \text{if } r \geq 1/4 \end{cases} \quad y(r) = \begin{cases} 3/(2+r) & \text{if } r \leq 1/4 \\ 4/3 & \text{if } r \geq 1/4 \end{cases}$$

4 Problem 14

This problem is set is entirely about the theoretical analysis of an extension of the basic Ricardian trade model. The point of the extension is to introduce consideration of transportation costs. This problem was suggested to me by a comment someone made in class. Thank you

Transportation is a productive activity that turns apples in country i into apples in country j. In the competitive model, this technology will be owned by a firm just as the technology for making apples and the technology for making beer are owned by firms. The technology is known as the iceberg technology: For every unit shipped from country $i, 0 < k < 1$ units arrive at country j. Otherwise, everything else is as usual: Output of good A in country i is $q_i^A = \alpha_i l_i^A$, $q_i^B = \alpha_i l_i^B$. The output of each good in country i is divided between home consumption and exports. The consumption of each good in country i is the sum of home production and imports. Both countries have identical utility, which is Cobb-Douglas, in log form $u(c^A, c^B) = \delta \log c^A + \gamma \log c^B$, with $\gamma = 1 - \delta$. Parts 1 through 7 are questions having only to do with the technology and the fact that each country must consume some of both goods in equilibrium. Parts 8 and 9 require facts from the demand side of the model.

Remember that you have a limited number of relations that can be applied to answer these questions: All profits must be non-positive. A produced good must earn 0 profits. A production activity which earns negative profits will not be used. And finally, supply equals demand. I'll publish some hints, but if you really want to have the research experience, you should try to avoid them.

(a) How many goods does this model have

This model has 6 goods in total.

(b) Will any country both import and export the same good?

In the case that country 1 exports good A, we have:

$$kp_2^A - p_1^A = 0$$

In the case that country 1 imports good A, we have:

$$kp_1^A - p_2^A = 0$$

Now, solving the irst equation and substituting into the second one, we have:

$$k^2 p_2^A - p_2^A = 0 \Rightarrow p_2^A(k^2 - 1) = 0$$

We know that the above cannot be true as $0 < k < 1$ and $p_2^A > 0$ as we have production of both goods. Therefore, we reach a contradiction and we conclude that we cannot have country 1 both being exporting and importing good A.

By symmetry, the same is true for good B, and then for country 2 and both goods.

(c) Will a country export the good in which it has a comparative disadvantage? One hint just to get you started: Suppose without loss of generality that $\alpha_1/\beta_1 > \alpha_2/\beta_2$, so that country 1 has a comparative advantage in A and country 2 in B. Suppose country 1 exports B. The previous part proved that country 2 cannot then export B, and it must export something or there can be no trade, so it must export A. Go from there.

The profit maximization problem for country 1, if it producing good A is:

$$\max p_1^A \alpha_1 l_1^A - w_1 l_1^A \text{ s.t. } p_1^A \alpha_1 \leq w_1$$

Now, if country 1 produces good B, we have:

$$\max p_1^B \beta_1 l_1^B - w_1 l_1^B \text{ s.t. } p_1^B \beta_1 = w_1$$

And from the zero profit condition for country 1 exporting B, we have:

$$kp_2^B - p_1^B = 0 \Rightarrow kp_2^B = p_1^B$$

Now, from the above relationships, we obtain:

$$kp_2^B \beta_1 = w_1 \geq p_1^A \alpha_1 \tag{7}$$

We now move to country 2, if it producing good B is:

$$\max p_2^B \beta_2 l_1^B - w_2 l_2^B \text{ s.t. } p_2^B \beta_2 \leq w_2$$

Now, if country 1 produces good B, we have:

$$\max p_2^A \alpha_2 l_1^A - w_2 l_2^A \text{ s.t. } p_2^A \alpha_2 = w_2$$

And from the zero profit condition for country 1 exporting A, we have:

$$kp_1^A - p_2^A = 0 \Rightarrow kp_1^A = p_2^A$$

Now, from the above relationships, we obtain:

$$kp_1^A \alpha_2 = w_2 \geq p_2^B \beta_2 \tag{8}$$

Now, combining 7 and 8, we have:

$$\frac{\alpha_1}{k\beta_1} \leq k\frac{\alpha_2}{\beta_2} \Rightarrow \frac{\alpha_1}{\beta_1} \leq k^2 \frac{\alpha_2}{\beta_2}$$

which given the range of values of k cannot hold, as we must have:

$$\frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2}$$

Therefore, we conclude that we cannot have a country exporting goods over which it has a comparative disadvantage against the other country.

(d) Can any country totally specialize in production (of the good in which it has a comparative advantage) but not trade?

This is not possible, as the nature of the utility function dictates that both goods have to be produced.

(e) Under what conditions can there be a no-trade equilibrium? How can this be strengthened to a sufficient condition for a no-trade equilibrium?

For the absence of trade, we must have both countries producing both goods at home with $p_1^A \alpha_1 = w_1$ and $p_1^B \beta_1 = w_1$, yielding:

$$\frac{\alpha_1}{\beta_1} = \frac{p_1^B}{p_1^A}$$

Equivalently, the zero profit condition for country 2 is $p_2^A \alpha_2 = w_2$ and $p_2^B \beta_2 = w_2$, yielding:

$$\frac{\alpha_2}{\beta_2} = \frac{p_2^B}{p_2^A}$$

Now, the condition from negative profits from trade for country 1 is $kp_2^A < p_1^A$ and for country 2 it is $kp_1^B < p_2^B$. Therefore, combining the two we obtain:

$$\frac{p_1^B}{p_1^A} < \frac{\alpha_2}{k^2 \beta_2}$$

Now, combining the three equations above, we have:

$$\frac{\alpha_1}{\beta_1} < \frac{\alpha_2}{k^2 \beta_2}$$

(f) Is it possible to have a trade equilibrium in which neither country is specialized in production? Disprove or give a necessary condition.

Here, we follow the exact same process as in (e), with the only difference that the zero profits from trade conditions hold with equality. Therefore, we obtain:

$$\frac{p_1^B}{p_1^A} = \frac{\alpha_2}{k^2 \beta_2}$$

Now, substituting as in (e) yields:

$$\frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{k^2 \beta_2}$$

Therefore, in this case the condition ought to hold with equality.

(g) When will there be an equilibrium in which at least one country is fully specialized? (Setting one marginal product to 0 is an obvious special case which provesit can happen. Now find good necessary condition.) Strengthen this to a sufficient condition.

As in the previous parts, for country 1 we have $p_1^A\alpha_1 = w_1$ and $p_1^B\beta_1 \leq w_1$, yielding:

$$p_1^B\beta_1 \leq p_1^A\alpha_1$$

For country 2 we have $p_2^B\beta_2 = w_2$ and $p_2^A\alpha_2 \leq w_2$, yielding:

$$p_2^A\alpha_2 \leq p_2^B\beta_2$$

Now, from the zero profit condition, we have $p_1^A = kp_2^A$ and $p_2^B = kp_1^B$, and by substituting above we have:

$$\frac{1}{k}p_2^B\beta_1 \leq kp_2^A\alpha_1 \Rightarrow \frac{p_2^B}{p_2^A} \leq k^2\frac{\alpha_1}{\beta_1}$$

And equivalently:

$$\frac{\alpha_2}{\beta_2} \leq \frac{p_2^B}{p_2^A}$$

Combining the above, we then have:

$$\frac{\alpha_2}{\beta_2} \leq k^2\frac{\alpha_1}{\beta_1}$$

(h) In a trade equilibrium, do both countries have the same relative consumption(c_A/c_B)? How does this compare to the standard Ricardian ($k = 1$) model?

Here we have the following maximization problem for each country:

$$\max \delta \log c_i^A + \gamma \log c_i^B$$

$$s.t. p_i^A c_i^A + p_i^B c_i^B$$

Taking the first order conditions we have $\frac{\delta}{c_i^A} = p_i^A$ and equivalently $\frac{1-\delta}{c_i^B} = p_i^B$. Now, combining the two, we have:

$$\frac{c_i^A}{c_i^B} = \frac{\delta p_i^B}{(1-\delta)p_i^A}$$

Now, in a trade equilibrium we wil have $\frac{p_2^B}{p_2^A} = k^2\frac{p_1^B}{p_1^A}$, with relative consumptions being different in each country. Now, for $k = 1$, we will have equal consumptions, as the following will hold:

$$\frac{p_2^B}{p_2^A} = \frac{p_1^B}{p_1^A}$$

(i) Can there be an equilibrium in which both countries are specialized? One obvious case is if in country 1 only A is produced while in country 2 only B is produced. So go farther and give a necessary and a sufficient condition for the existence of a equilibrium in which both countries are fully specialized. Your answer should depend upon the utility parameters and the labor endowments. Derive from this a necessary condition that depends only upon technology parameters. Interpret your answers.

In the case described above, we have $q_1^A = \alpha_1 L_1$ and $q_2^B = \beta_2 L_2$.

Then, demand of country 1 for good A is given by $c_1^A = \delta\alpha_1 L_1$, and equivalently for country 2 it is $c_2^A = k\delta\beta_2 \frac{p_2^B}{p_1^A} L_2$.

By the above we conclude that the demand of country 2 for the production of country 1's good A is $C_{2-1}^A = \delta\beta_2 \frac{p_2^B}{p_1^A} L_2$.

We know that the price ratio of the goods produced in each country will be equal to the ratio of total demand for one good to total supply. Combining this with the no-trade equilibrium condition obtained above, we have:

$$\frac{k\alpha_1}{\beta_1} \geq \frac{(1-\delta)\alpha_1 L_1}{\delta\beta_2 L_2} \geq \frac{\alpha_2}{k\beta_2}$$

(j) Give necessary and sufficient conditions for the existence of a trade equilibrium in which only one country is specialized.

This is the very result obtained in part (g) above.

(k) What other questions should we ask of this model?

Here we could potentially extend to having more than two products and see how this would affect the results.

See also section notes 5.