Logistics

- TA Office Hours: Mon & Wed 18:30-19:30pm.
- Same link for office hours and sections.
- All Problem Sets are due before Friday section.
- Please send the Problem Set by email to lc944@cornell.edu

1 Convex sets[1]

1.1 Important concepts:

- **Separating Hyperplane Theorem:** Let C and D be two convex sets in $\mathbb{R}^n$ that do not intersect (i.e., $C \bigcap D = \emptyset$). Then, there exists $p \in \mathbb{R}^n, p \neq 0$ and $\alpha \in \mathbb{R}$ such that $p^T x \leq \alpha$ for all $x \in C$ and $p^T y \geq \alpha$ for all $y \in D$.

- **Farkas' Lemma:** one and only one of the following alternatives is true:
  1. The system $Ax = b, x \geq 0$ has a solution.
  2. The system $y^T A \geq 0, y^T b < 0$ has a solution.

1.2 Questions

1. *(From Convex Sets PS, Q5).* Use Farkas' Lemma to prove Gordans' Lemma: only one of this alternatives is true
   (a) $Ax = 0, x > 0$ has a solution.
   (b) $y^T A \gg 0$ has a solution.

[1] These notes borrow from Jaden Chen's notes from 2020.

2 Linear programming

2.1 Important concepts:

- **Canonical and Standard form:** a linear program can be written in *canonical* form or in *standard* form.

Canonical	*Standard*
$v_P(b) = max\ c \cdot x$	$v_P(b) = max\ c \cdot x$
$s.t.\ Ax \leq b$	$s.t.\ Ax = b$
$x \geq 0$	$x \geq 0$

- **Definitions:**

  1. *Solution:* any $x \in \mathbb{R}^n$ is called a solution.
  2. *Constraint (or Feasible) set:*
     (a) canonical form: $C = [x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$
     (b) standard form: $C = [x \in \mathbb{R}^n : Ax = b, x \geq 0\}$
  3. *Feasible solution:* any $x \in C$, where C is the constraint set, is called a feasible solution.
  4. *Optimal solution:* a vector x that solves the linear program is called an optimal solution, i.e., $x \in C$ such that $c \cdot x \geq c \cdot x'$, for all $x' \in C$.
  5. *Vertex:* a vector $x \in C$ is a vertex of C if and only if there is no $y \neq 0$ such that $x + y$ and $x - y$ are both in C.
  6. *Basic solution:* a solution to a linear program in standard form is a basic solution iff column vectors a_j corresponding to $x_j > 0$ are linearly independent.

- **Theorem 1.** A solution x is basic if and only if it is a vertex.

- **Theorem 2.** *Vertex Theorem.* For a linear program in standard form with feasible solutions:

  1. A vertex exists.
  2. If $v_P(b) < \infty$ and $x \in C$, then there is a vertex x' such that $c \cdot x' \geq c \cdot x$.

2.2 Questions

1. Consider the following linear program

$$\begin{array}{lrcl} max & x_1 + x_2 & & \\ s.t. & x_1 + 2x_2 & \leq & 6 \\ & x_1 - x_2 & \leq & 3 \\ & x_1 \geq 0 & & \\ & x_2 \geq 0 & & \end{array}$$

   (a) Express the linear program in canonical form and draw the constraint set and solve the problem graphically.

   (b) Express the linear program in standard form.

   (c) Use Simplex Algorithm to solve this problem.