ECON 6100: Problem Set 4 - Linear Production Models

Due Date: February 18th, 2021

Question 1.

Are the following matrices productive? Prove your answers.

$$A_1 = \begin{bmatrix} 0.6 & 0.2 & 0.1 \\ 0.3 & 0.2 & 0.4 \\ 0.2 & 0.4 & 0.3 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0.6 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}$$

Solution 1.

First recall the definition of a productive matrix. For a matrix A to be productive, we need to have $x \gg 0$ such that $x \gg Ax$. That is, in other words, $x - Ax \gg 0$.

Now, let's focus on A_1. Let $x = (10, 10, 10)$. If follows that:

$$x - A_1 x = \begin{bmatrix} 0.4 & -0.2 & -0.1 \\ -0.3 & 0.8 & -0.4 \\ -0.2 & -0.4 & 0.7 \end{bmatrix} x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \gg 0$$

Hence, we can conclude that A_1 is productive.

Now, let's turn to A_2 and pick $x = (15, 10)$. Then,

$$x - A_2 x = (I - A_2) x = \begin{bmatrix} 0.4 & -0.5 \\ -0.1 & 0.5 \end{bmatrix} x = \begin{bmatrix} 14 \\ 6.5 \end{bmatrix} \gg 0$$

Thus, A_2 is productive.

Question 2.

Prove that A is productive if and only if $(I - A)^{-1}$ exists and is non-negative.

Solution 2.

First, let's recall two theorems that will be crucial in proving the iff statement:

1. <u>Thrm 1</u>: If A is productive, then $(I - A)$ has full rank.

2. <u>Thrm 2</u>: If A is productive, for any $z \geq 0$, then the system $(I - A)x = z$ has a unique, non-negative solution.

Now, with these two results in our hand, let's turn to proving the iff statement. First, we prove the if direction $\implies$.

Let A be productive, then it follows by Thrm 1 that $(I - A)$ has full rank. Hence, $(I - A)^{-1}$ exists. Now, notice that $(I - A)(I - A)^{-1} = I$. Hence, it follows that: $(I - A)k^i = e^i$, where k^i is the $i - th$ column of $(I - A)^{-1}$ and e^i is the standard orthonormal basis. Finally, we can utilise Thrm 2, which states that for any $z \geq 0$, the system $(I - A)x = z$ has a unique, non-negative solution. Notice that this is exactly our case, since $e^i \geq 0$, we have $k^i \geq 0$ and then $(I - A)^{-1}$ exists and is non-negative.

Now turn to the other direction of the iff statement $\impliedby$.

Let assume $(I - A)^{-1}$ exists, and $(I - A)^{-1} \geq 0$. Since the inverse exists by assumption, we can construct a $z = (I - A)^{-1}e$.

Moreover, notice that the existence of the inverse implies that $(I - A)^{-1}$ does not have a column of all zeroes. It then follows that since $(I - A)^{-1}$ is non-negative, what we are left with is: $z = (I - A)^{-1}e \gg 0$. Now we can pre-multiply by $(I - A)$ on both sides of the equality $z = (I - A)^{-1}e$, which leaves us with $z - Az >> 0 \implies z >> Az$, which is the definition of A being productive. $\square$

Question 3.

Prove that if A is productive, then there is at least one row sum of A which is less than 1.

Solution 3.

Let's assume, towards a contradiction that A is productive, but all rows sum up to a value greater than 1.

Since A is productive, this means that there exists an $x \gg 0$ such that $x - Ax \gg 0$.. This, expressed as a system of linear inequality gives us:

$$x_1 - \sum_{i=1}^{n} a_{1i}x_i > 0$$

$$\vdots$$

$$x_n - \sum_{i=1}^{n} a_{ni}x_i > 0.$$

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Now, we can combine these linear inequality, that is:

$$\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ji} \right) x_i > 0.$$

However, one can easily notice that by the fact that A is productive, we have that $\sum_{j=1}^{n} a_{ij} \geq 1$ $\forall j$. This, in turn implies that $\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ji} \right) x_i < 0$, which draws the contradiction we were seeking for.

Hence, we proved that if A is productive, then there is at least one row sum of A which is less than 1. $\qquad \square$

Question 4.

Let

$$A = \begin{bmatrix} 0.1 & 0.4 & 0.3 \\ 0.2 & 0.7 & 0.0 \\ 0.1 & 0.1 & 0.5 \end{bmatrix}$$

Suppose too that the vector of labor requirements is $(1, 1, 1)$. Compute competitive equilibrium prices for this economy.

Solution 4.

First, we let x be a vector s.t. :

$$x = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}.$$

You can also prove that is productive by proving that $(I-A)^{-1}$ exists and is non-negative

Notice that, this implies:

$$x >> \begin{bmatrix} 8 \\ 9 \\ 7 \end{bmatrix} = Ax,$$

Hence, by our well known definition, A is productive.

Now, notice that the vector of labor requirements is strictly positive. This implies that there exists a competitive equilibrium where the profits earned from each commodity are zero.

Now define per-unit profits as $\pi = p(I - A) - a_0$. It then follows that prices in equilibrium are given by: $p^* = a_0(I - A)^{-1}$.
Then by substituting A into our expression, and then solving for p we get:

$$p^* = \begin{bmatrix} 3.75 & 9.75 & 4.25 \end{bmatrix}.$$

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Question 5.

Associated with any $n \times n$ non-negative matrix A is a graph G where there is a directed edge from i to j iff $a_{ij} > 0$. A matrix is said to be irreducible iff the graph is strongly connected; that is iff there is a path from any i to any j.

Part A)

Show that a square matrix A is irreducible iff for each pair i and j there is an m such that the i, j 'th element of A^m is positive. [Hint: What does it mean for an element of A^2 to be positive, in terms of the graph?]

Solution to Part A)

First let's prove the if direction of the statement $\implies$.

If A is irreducible, then the graph associated with A is strongly connected. Hence, there exists a directed path p from any vertex i to vertex j of length $k \leq n - 1$. Thus, this means that $A_{ij}^k \geq 1 > 0$.

Now, let's turn to the other direction $\impliedby$.

If for each pair i and j there is an m such that the $i, j - th$ element of A^m is positive, then for any vertices i, j, there is some $k \leq n - 1$ such that $A_{ij}^k > 0$. However, by construction, it is impossible for the entries of A^l to be negative entries, which implies that $\sum_{k=1}^{n-1} A^k >> 0$, which implies that A must be irreducible. $\qquad \square$

Part B)

Use the preceding fact to show that the assumption $a_0 \gg 0$ can be replaced with the assumptions $a_0 > 0$ and A is irreducible in the characterization of equilibrium in the simple Leontief model.

Solution to Part B)

First, it is useful to recall a result from class. Namely, recall that in a zero-profit price competitive equilibrium, the price vector p^* must satisfy: $p^* = a_0 (I - A)^{-1}$.

It follows that, if one wants all prices to be strictly positive (and thus the quantities produced as well), under the second assumption that $a_0 > 0$, it must be that $(I - A)^{-1} \gg 0$.

Here is where the previous fact come into play. In fact, if one can also assume that A is irreducible. It follows that, since $\sum_{k=1}^{n-1} A^k >> 0$ and $A^l \geq 0$ for any $l \geq 0$, what we get is: $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k >> 0$.

Hence, one can replace the hypothesis that $a_0 \gg 0$ with the proposed hypothesis that $a_0 > 0$ and the fact that A is irreducible to reach the same result. □

Question 6.

In this question we develop the Ricardian trade model. There are two countries, England and Spain. (Portugal is traditional here, but then we have too many p 's floating around.) Each country can make both wine (v) and mutton (m). England requires a_{ve} units of English labor to produce a unit of wine, and a_{me} units of English labor to produce a unit of mutton. Spain requires a_{vs} units of Spanish labor to produce a unit of wine, and a_{ms} units of Spanish labor to produce a unit of mutton. Neither good is used as an intermediate product in the production of any good; itself or the other. England and Spain have l_e and l_s units of labor, respectively, and because of BREXIT, no laborer may cross the Channel to work in the other country.

Part A)

Describe the production of wine and mutton as a general Leontief model. How many primary factors are there? What are A and B ?

Solution to Part A)

For the general Leontief model, firstly we can define the activities level matrix, x as:

$$x = \begin{bmatrix} x_{ve} \\ x_{me} \\ x_{vs} \\ x_{ms} \end{bmatrix}$$

Notice that, of course, each entry identify a country and activity. Then we define the labor input requirements matrix, A_0 as:

$$A_0 = \begin{bmatrix} a_{ve} & a_{me} & 0 & 0 \\ 0 & 0 & a_{vs} & a_{ms} \end{bmatrix}$$

and finally the labor endowment vector as:

$$L = \begin{bmatrix} l_e \\ l_s \end{bmatrix}$$

in England and Spain.

Since labor is the only input used in production, then the input requirements matrix A is simply

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and of course, since the level output of each good per unit of each activity is one, it follows that B is given by:

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Finally, y is the matrix of output goods, denoted by:

$$y = \begin{bmatrix} y_v \\ y_m \end{bmatrix}$$

Part B)

Describe the production possibility set as a convex polyhedron, that is, as a solution set to a system of linear inequalities. Show that it is closed and convex.

Solution to Part B)

The production possibility set is:

$$Y = \{ y \geq 0 : Bx \geq y, A_0 x \leq L, x \geq 0. \}$$

Since Y is the solution to a set of three matrix inequalities, specifically weak inequalities. Hence, this implies that is a closed polyhedron.

Now, we want to show convexity. First, let's consider $y_1, y_2 \in Y$ and let y be a convex combination of $y_1, y_2 \in Y$. Namely,

$$y = \alpha y_1 + (1 - \alpha) y_2$$

for some $\alpha \in [0, 1]$. Moreover, let x_1 and x_2 be the solution of the system for y_1 and y_2 and finally we let

$$x = \alpha x_1 + (1 - \alpha) x_2$$

Now, one can note that:

$$Bx = B\alpha x_1 + (1 - \alpha) Bx_2 \geq \alpha y_1 + (1 - \alpha) y_2 = y$$

Furthermore,

$$A_0 x = A_0 \alpha x_1 + (1 - \alpha) A_0 x_2 \leq \alpha L + (1 - \alpha) L = L$$

This means that, since $x_1, x_2 \geq 0$, their convex combination must satisfy: $\alpha x_1 + (1-\alpha) x_2 \geq 0$, which implies $x \geq 0$.

Therefore, y must be in the production set of Y, and Y hence must be convex.

Part C)

The production possibility set is convex. The convex support function is one way of characterizing a closed convex set C. It is defined as

$$h(p) = \max\{ p \cdot x : x \in C \}.$$

The convex support function is one way of giving the dual description of C, the set of all half spaces containing it. (The concave support function does this with min's.) Write down a linear programming problem that gives the convex support function. Hint: It begins, $h(\alpha, \beta) = \max \alpha v + \beta m$ over some variables including, obviously, v and m, subject to some constraints.

Solution to Part C)

Let's define the convex support function of Y by the means of the following linear program:

$$h(\alpha, \beta) = \max_{x,y} \begin{bmatrix} \alpha & \beta \end{bmatrix} y$$

$$\text{s.t.} \quad y - Bx \leq 0$$
$$A_0 x \leq L$$
$$x, y \geq 0.$$

Note that the constraint $y - Bx \leq 0$ will always bind in equilibrium (market clearing condition). Hence, one can simplify the program to:

$$h(\alpha, \beta) = \max_x \begin{bmatrix} \alpha & \beta \end{bmatrix} Bx$$

$$\text{s.t.} \quad A_0 x \leq L$$
$$x \geq 0.$$

Part D)

What is the dual of your program?

Solution to Part D)

Let $w = \begin{bmatrix} w_e & w_s \end{bmatrix}$. Then, the dual of the the previously stated program can be written as:

$$h'(\alpha, \beta) = \min_w wL$$

$$\text{s.t.} \quad wA_0 \geq \begin{bmatrix} \alpha & \beta \end{bmatrix} B$$
$$w \geq 0.$$

Part E)

Using complementary slackness, interpret solutions to the dual as competitive prices. There is no loss of generality in assuming $\alpha, \beta > 0$.

Solution to Part E)

First, one could restate the previous program:

$$h(\alpha, \beta) = \max_{x} \ \alpha x_{ve} + \beta x_{me} + \alpha x_{vs} + \beta x_{ms}$$
$$\text{s.t.} \quad a_{ve}x_{ve} + a_{me}x_{me} \leq l_e$$
$$a_{vs}x_{vs} + a_{ms}x_{ms} \leq l_s$$
$$x \geq 0.$$

similarly, for their dual, one could restate it as:

$$h'(\alpha, \beta) = \min_{w} w_e l_e + w_s l_s$$
$$\text{s.t.} \quad w_e a_{ve} \geq \alpha$$
$$w_e a_{me} \geq \beta$$
$$w_s a_{vs} \geq \alpha$$
$$w_s a_{ms} \geq \beta$$
$$w \geq 0.$$

Then, it follows by complementary slackness that the solution must satisfy for England and Spain respectively:

$$(w_e a_{ve} - \alpha)x_{ve} = 0$$
$$(w_e a_{me} - \beta)x_{me} = 0$$
$$w_e(a_{ve}x_{ve} + a_{me}x_{me} - l_e) = 0$$
$$(w_s a_{vs} - \alpha)x_{vs} = 0$$
$$(w_s a_{ms} - \beta)x_{ms} = 0$$
$$w_s(a_{vs}x_{vs} + a_{ms}x_{ms} - l_s) = 0,$$

One can notice that the solution to the dual (the wage) equalise the marginal cost of labour, with the marginal revenues for a good, produced in a country. Or in other words the solution wages, are the competitive price of labor.

Part F)

Under what conditions can all activities be used?

Solution to Part F)

If the following two conditions hold, then all activities can be used:

$$w_e a_{ve} = w_s a_{vs} = \alpha$$
$$w_e a_{me} = w_s a_{ms} = \beta$$

In other words, the previous condition states that the relative productivity in each country has to be the same, or formally we must have: $\frac{a_{ve}}{a_{me}} = \frac{a_{vs}}{a_{ms}}$.

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Part G)

Suppose that there is an optimal solution in which $x_{me} > 0$ and $x_{vs} > 0$. What can you infer about the ratios a_{me}/a_{ve} and a_{ms}/a_{vs}?

Solution to Part G)

Notice that, if we suppose what is state in Part G, then we must have that the wage in England must be such that the zero profit conditions hold in mutton production, while in the wine industry some non-positive profits are made; Conversely the wage in Spain must be such that the zero profit conditions holds in the wine industry and non-positive profits are attained in the mutton one.

Hence, by complementary slackness we have that for England:

$$\frac{\alpha}{\beta} \leq \frac{w_e a_{ve}}{w_e a_{me}} = \frac{a_{ve}}{a_{me}}$$

and for Spain:

$$\frac{\alpha}{\beta} \geq \frac{w_s a_{vs}}{w_s a_{ms}} = \frac{a_{vs}}{a_{ms}},$$

By combining the two previously stated condition, we can finally draw the relationship we where seeking in the part of the problem, that is:

$$\frac{a_{vs}}{a_{ms}} \leq \frac{a_{ve}}{a_{me}}$$

See also seehon notes 4
for reference
solution of problem
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