Logistics

- TA Office Hours: Mon & Wed 18:30-19:30pm.
- Same link for office hours and sections.
- All Problem Sets are due before Friday section.
- Please send the Problem Set by email to lc944@cornell.edu

1 Linear programming (continued)[1]

- **Fundamental Theorem of LP:** If a linear program in standard form has a feasible solution, then it has a basic feasible solution; if it has an optimal solution, then it has a basic optimal solution.

2 Duality of Linear Programming

2.1 Important concepts:

- **Definition 1: Duality.** Given a linear program in canonical form, we write the *primal problem* and the *dual problem*:

$$\begin{array}{cc} Primal & Dual \\ V_P(b) = max\, c^T x & V_D(c) = min\, y^T b \\ s.t.\ Ax \leq b & s.t.\ y^T A \geq c^T \\ x \geq 0 & y \geq 0 \end{array}$$

- **Theorem 1: Duality.** For a primal problem and a dual problem:
  1. Both are feasible and have optimal solutions, and $V_P(b) = V_D(c)$.
  2. One is unbounded, one is infeasible.
  3. Neither is feasible.

[1]These notes are based on Jaden Chen's notes from 2020 and Abhi Ananth notes from 2021.

- **Theorem 2: Complementary slackness.** If x and y are feasible solutions for the primal and the dual respectively, then they are optimal if and only if $y^T(b - Ax) = 0$ and $(y^T A - c)x = 0$.

  - Example: Consider the following primal problem:

$$\begin{array}{lll} max & 2x_1 + x_2 & \\ s.t. & x_1 + x_2 & \leq \; 1 \\ & x_1 \geq 0 & \\ & x_2 \geq 0 & \end{array}$$

The corresponding dual problem is:

$$\begin{array}{llll} min & y_1 & & \\ s.t. & y_1 & \geq & 2 \\ & y_1 & \geq & 1 \\ & y_1 & \geq & 0 \end{array}$$

Clearly, the solution to the dual is $y_1^* = 2$. According to CS, if y_1^* is an optimal solution of the dual, then an optimal solution of the primal must then satisfy:

$$\begin{aligned} y^*(b - Ax^*) &= 0 \\ y_1^*(1 - x_1^* - x_2^*) &= 0 \\ x_1^* + x_2^* &= 1 \qquad (1) \end{aligned}$$

Also, CS states that if x_1^*, x_2^* is an optimal solution of the primal, then an optimal solution of the dual must satisfy:

$$\begin{aligned} (y^{*T} A - c)x^* &= 0 \qquad (2) \\ (y_1^* - 2)x_1^* + (y_1^* - 1)x_2^* &= 0 \end{aligned}$$

Substituing $y_1^* = 2$, and combining 1 and 2 we obtain the optimal solution to the primal: $(x_1^*, x_2^*) = (1, 0)$.

2.2 Questions:

1. (From Section 1). Consider the primal problem:

$$\begin{array}{lrcl} max & x_1 + x_2 & & \\ s.t. & x_1 + 2x_2 & \leq & 6 \\ & x_1 - x_2 & \leq & 3 \\ & x_1 \geq 0 & & \\ & x_2 \geq 0 & & \end{array}$$

   (a) Draw the constraint set and solve graphically.

   (b) State and solve the dual problem.

   (c) Verify that the values coincide and that the complementary slackness conditions hold.

2. Consider the primal problem:

$$\begin{array}{lrcl} max & x_1 + 2x_2 & & \\ s.t. & x_1 + x_2 & \leq & 4 \\ & x_1 + 3x_2 & \leq & b \\ & x_1 \geq 0 & & \\ & x_2 \geq 0 & & \end{array}$$

   (a) Draw the constraint set.

   (b) Solve the problem and plot $V_P(b)$.

   (c) State and solve the dual problem. How does the solution of the dual problem depend on b?

   (d) Let $b = 6$, verify the complementary slackness conditions.

3. Solve the following linear program:

$$\begin{array}{lrcl} min & 4y_1 + 12y_2 + y_3 & & \\ s.t. & y_1 + 4y_2 - y_3 & \geq & 1 \\ & 2y_1 + 2y_2 + y_3 & \geq & 1 \\ & y_1 \geq 0 & & \\ & y_2 \geq 0 & & \\ & y_3 \geq 0 & & \end{array}$$