Logistics

- TA Office Hours:
  - Mon & Wed 18:30-19:30pm (in person (Uris 465) or by Zoom).
- All problem sets are due before Friday section.
- Section's material https://github.com/luciacasal/-Cornell_ECON6100.

1 Questions:

1. In a two-person, two good exchange economy, individuals A and B have the following preferences:

$$u^A(x, y) = min\{x, 2y\}$$
$$u^B(x, y) = min\{3x, y\}$$

   where x and y are two goods. The aggregate endowment is $(12, 12)$. Answer the following using an Edgeworth box:

   (a) Find the set of Pareto optima.

   (b) Assume A has all of good x and B all of good y. Describe the competitive equilibria.

   (c) Which endowment allocations have competitive equilibria with positive prices for both goods?

2. An exchange economy has two people and three goods. Let x_n denote person 1's consumption of good n, and let y_n denote person 2's consumption of good n. Utility for each consumer is quasi-linear. For person 1:

$$u_1(x, y) = x_1 - \frac{\alpha_1}{2}x_1^2 + x_2 - \frac{\alpha_2}{2}x_2^2 + x_3 - \lambda_1 y_1 - \lambda_2 y_2 - \lambda_3 y_3$$

   For person 2:

$$u_2(x, y) = y_1 - \frac{\beta_1}{2}y_1^2 + y_2 - \frac{\beta_2}{2}y_2^2 + y_3 - \gamma_1 x_1 - \gamma_2 x_2 - \gamma_3 x_3$$

   Notice that there are externalities in consumption. The endowment for consumer 1 is $(1, 0, e)$ and for consumer 2 is $(0, 1, f)$. There is a sales tax on goods 1 and 2. The seller of good n receives price p_n, while the buyer pays price $p_n + t_n$. Tax revenues split equally and handed back to the consumers as a lump-sum transfer. (That is, in deciding how much of a commodity to buy, the consumer does not account for the return of some share of the cost through the tax split.)

(a) What is each consumer's budget constraint?

(b) Define a sensible notion of competitive equilibrium for this economy.

(c) Compute the competitive equilibrium for arbitrary (small) taxes t_1 and t_2, assuming the allocation is interior.

(d) Compute the derivative of both individuals' utilities with respect to the tax rates.

(e) Are there small but non-zero taxes that would give an equilibrium allocation Paretobetter than that achieved at $t_1 = t_2 = 0$?

3. (Welfare PS Q2 continued from previous section) Fix utility levels u_1 and u_2 for individuals 1 and 2, respectively, and define $U(u_1, u_2)$ to be the set of all aggregate endowments that can be allocated so as to give individual 1 utility at least u_1 and person 2 utility at least u_2 . The lower boundary of this set is called the community indifference curve for the utility pair (u_1, u_2).

   (a) Let (x^*, p^*) be a competitive equilibrium for an exchange economy with aggregate endowment $e = (e_x, e_y)$. Let u_i denote the utility realized by person i at the competitive equilibrium. Show that if preferences are locally non-satiated, then at at a competitive equilibrium price vector, expenditure on $U(u_1, u_2)$ is minimized at the aggregate endowment.

   (b) Show that if the utility functions of the two individuals is strictly quasi-concave, $U(u_1, u_2)$ is a "strictly convex" set in the sense that the interior of the line segment connecting any two points in the set is in the interior of the set.

4. (For you to try at home) (2005 Aug III). There are three agents in the economy A, B and C. There are three goods in the economy (x_1, x_2, x_3). Agent A has 1 unit of good x_1, agent B has $b \in [1, 2)$ units of good x_2, and agent C has 1 unit of good x_3.The utility functions of the agents are:

$$u^A(x_1 x_2, x_3) = min\{x_1, x_2\}$$
$$u^B(x_1 x_2, x_3) = min\{x_2, x_3\}$$
$$u^C(x_1 x_2, x_3) = min\{x_1, x_3\}$$

   Let p_1, p_2, p_3 denote the prices of the goods.

   (a) In a CE, can all prices be positive? What happens when 2 or all prices are 0?

   (b) Write down the excess demand function of each good.

(c) If $p_3 = 1$, find the other prices.

(d) Suppose $p_3 = 1$, then how will each agent's utility change with a change in b?

2 Notes[1]

Enjoy your break!...but (for you information) the following are GE type questions in recent Q's. Note that some are GE with uncertainty (asset pricing GE) questions, which we will not cover until later in the semester:

- 2016 June IV
- 2015 June IV
- 2015 Aug IV
- 2013 June 2
- 2013 June 5
- 2013 Aug 4
- 2012 June IV
- 2012 June V
- 2012 Aug I
- 2012 Aug V
- 2011 June III
- 2011 June IV
- 2011 Aug II
- 2011 Aug III
- 2011 Aug IV
- 2010 June IV
- 2010 June V
- 2010 Aug II
- 2010 Aug V
- 2009 June II
- 2009 Aug III
- 2009 Aug V

[1] Borrowed from Fikri Pitsuwan section notes from 2017.

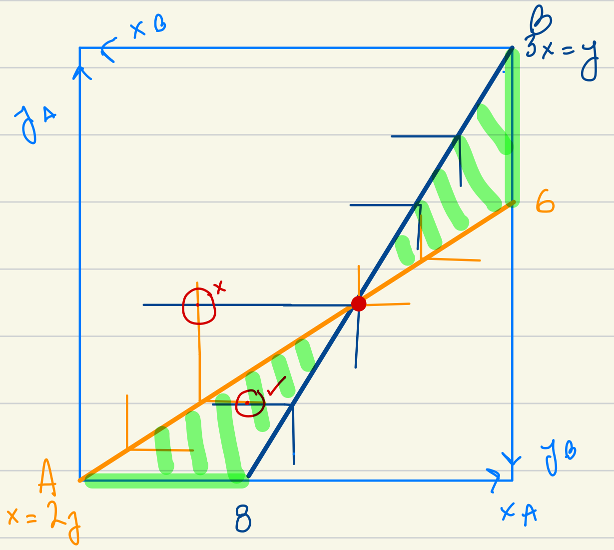
Q1)

$U^A = \min\{x, 2y\}$ $\quad e = (e_x, e_y) = (12, 12)$

$U^B = \min\{3x, y\}$

careful: p_x, p_y may be 0 since preferences are not strongly monotone.

a) Find set of Pareto optimal allocations.



b) $e^A = (12, 0)$ $\quad e^B = (0, 12)$

For consumer (A):

$$\begin{cases} \max \ \min\{x, 2y\} \\ \text{st} \quad p_x \cdot x^A + p_y \cdot y^A \leq 12 p_x \\ \qquad x^A, y^A \geq 0 \end{cases}$$

For consumer (B):

$$\begin{cases} \max \ \min\{3x, y\} \\ \text{st} \quad p_x \cdot x^B + p_y \, y^B \leq 12 p_y \\ \qquad x^B, y^B \geq 0 \end{cases}$$

CASE 1: $p_x > 0$; $p_y > 0$

normalize $p_x = p$; $p_y = 1$.

(A) $x^A = 2y^A \xrightarrow{BC} p\,2y^A + y^A = 12p \longrightarrow y^A = 12p/(2p+1) \longrightarrow x_A = 24p/(2p+1)$

(B) $3x^B = y^B \xrightarrow{BC} p\,x^B + 3x^B = 12 \longrightarrow x^B = 12/(p+3) \longrightarrow y_B = 36/(p+3)$

From market clearing: $x^A + x^B = 12 \longrightarrow \dfrac{24p}{2p+1} + \dfrac{12}{p+3} = 12 \longrightarrow p = 2$

CASE 2: $p_x > 0$; $p_y = 0$

(A) $x_A = 12$ $\quad y_A \geq 6$

(B) $x_B = 0$ $\quad y_B \geq 0$

In equilibrium: $x_A = 12$ $\quad y_A \in [6, 12]$

$x_B = 0$ $\quad y_B = 12 - y_A$

CASE 3: $p_y > 0$ $\quad p_x = 0$

(A) $x_A \geq 0$ $\quad y_A = 0$

(B) $x_B \geq 4$ $\quad y_B = 12$

In equilibrium: $x_A = 12 - x_B$ $\quad y_A = 0$

$x_B = [4, 12]$ $\quad y_B = 12$

CASE 4: $p_x = p_y = 0$

markets don't clear! impossible

unbounded demand

$\because$ $\succsim$ are monotone we know $p^* > 0$

$(p^* \geq 0 \,;\, p^* \neq 0)$.

Q2) Exchange economy. 2 people, 3 goods. Let x_n denote person 1's consumption of good n, y_n " person 2 of good n.
Utility quasi-linear:

Person 1: $$u_1(x,y) = x_1 - \frac{\alpha_1}{2}x_1^2 + x_2 - \frac{\alpha_2}{2}x_2^2 + x_3 - \lambda_1 y_1 - \lambda_2 y_2 - \lambda_3 y_3$$

Person 2: $$u_2(x,y) = y_1 - \frac{\beta_1}{2}y_1^2 + y_2 - \frac{\beta_2}{2}y_2^2 + y_3 - \delta_1 x_1 - \delta_2 x_2 - \delta_3 x_3$$

Externalities in consumption!

Endowments: Consumer 1: $(1, 0, e)$
Consumer 2: $(0, 1, f)$

$\exists$ sales tax on goods 1 and 2. Seller receives p_n while buyer pays $p_n + t_n$.

Tax revenues split equally and handed back to consumers as a lump-sum transfer

a) Each consumer's budget constraint?

Consumer 1:

$$(p_1 + t_1)x_1 + (p_2 + t_2)x_2 + p_3 x_3 \leq p_1 \cdot 1 + p_2 0 + p_3 e + \overset{\text{constant}}{T}$$

Consumer 2:

$$(p_1 + t_1)y_1 + (p_2 + t_2)y_2 + p_3 y_3 \leq p_1 0 + p_2 \cdot 1 + p_3 f + T$$

b) Notion of CE for this economy.

A CE in this economy is a bundle of allocations $\{x_1^*, x_2^*, x_3^*\}$ and $\{y_1^*, y_2^*, y_3^*\}$ and a system of prices $\{p_1^*; p_2^*; p_3^*\}$ (one for each commodity) such that, given that prices:

a) (x_1^*, x_2^*, x_3^*) max consumer's 1 utility, given consumer's 2 optimal bundles

b) (y_1^*, y_2^*, y_3^*) max consumer's 2 utility, given consumer's 1 optimal bundles.

c) markets clear: good 1 $x_1^* + y_1^* = 1$

good 2 $x_2^* + y_2^* = 1$

good 3 $x_3^* + y_3^* = e + f$

ex good 2 pollution $\to p_2 = 0$ nobody wants it

if I check $\to p > 0 \to$ MC ✓ $\to dd = SS$

c) Compute the CE for arbitrarily small t_1, t_2, assuming interior solution.

For simplicity we will normalize price system as: $(p_1, p_2, 1)$

Consumer 1 $\begin{cases} \text{UMP} \quad \max_{x_1, x_2, x_3} u_1(x, y) = x_1 - \frac{\alpha_1}{2} x_1^2 + x_2 - \frac{\alpha_2}{2} x_2^2 + x_3 - \lambda_1 y_1 - \lambda_2 y_2 - \lambda_3 y_3 \\ \text{st } (p_1 + t_1) x_1 + (p_2 + t_2) x_2 + \overset{=1}{p_3} x_3 \leq p_1 \cdot 1 + p_2 0 + \overset{=1}{p_3} e + T \end{cases}$

$$\mathcal{L}(x_1, x_2, x_3; \mu_1) = x_1 - \frac{\alpha_1}{2} x_1^2 + x_2 - \frac{\alpha_2}{2} x_2^2 + x_3 - \lambda_1 y_1 - \lambda_2 y_2 - \lambda_3 y_3$$

$$+ \mu_1 \left[p_1 + 1.e + T - (p_1 + t_1) x_1 - (p_2 + t_2) x_2 - 1.x_3 \right]$$

FOC:

$$[x_1] \quad 1 - \frac{\alpha_1}{\cancel{2}} \cancel{2} x_1 = \mu_1 \left[(p_1 + t_1) \right]$$

$$[x_2] \quad 1 - \frac{\alpha_2}{\cancel{2}} \cancel{2} x_2 = \mu_1 \left[(p_2 + t_2) \right]$$

$$[x_3] \quad 1 = \mu_1$$

$$[\mu_1] \quad (p_1 + t_1) x_1 + (p_2 + t_2) x_2 + \overset{=1}{p_3} x_3 = p_1 \cdot 1 + p_2 0 + \overset{=1}{p_3} e + T_1$$

(given $\mu_1 \neq 0$) by CS!

BC holds with equality

Subsituting $[x_3]$ into $[x_1]$ and $[x_2]$:

$$1 - \alpha_1 x_1 = p_1 + t_1 \longrightarrow \boxed{x_1^\circ = \left(1 - p_1 - t_1\right) \cdot \frac{1}{\alpha_1}}$$

$$1 - \alpha_2 x_2 = p_2 + t_2 \longrightarrow \boxed{x_2^\circ = \left(1 - p_2 - t_2\right) \cdot \frac{1}{\alpha_2}}$$

We subsitute x_1° and x_2° into $[\mu_1]$ to get an expression for x_3°:

$$\boxed{-\left[(p_1 + t_1)\left(1 - p_1 - t_1\right)\frac{1}{\alpha_1}\right] - \left[(p_2 + t_2)\left(1 - p_2 - t_2\right)\frac{1}{\alpha_2}\right] + p_1 + e + T = x_3^\circ}$$

Consumer 2

$$\begin{cases} \text{UMP} \\ \max \; u_1(x,y) = y_1 - \frac{k_1}{2} y_1^2 + y_2 - \frac{k_2}{2} y_2^2 + y_3 - \gamma_1 x_1 - \gamma_2 x_2 - \gamma_3 x_3 \\ \text{st} \quad (p_1+t_1) y_1 + (p_2+t_2) y_2 + \overbrace{p_3}^{=1} y_3 \leq p_1 0 + p_2 \cdot 1 + \overbrace{p_3}^{=1} F + T \end{cases}$$

The UMP is analogous to the one of Consumer 1, so we have:

$$\boxed{y_1^* = \left(1 - p_1 - t_1\right) \cdot \frac{1}{k_1}}$$

$$\boxed{y_2^* = \left(1 - p_2 - t_2\right) \cdot \frac{1}{k_2}}$$

and in the BC condition [μ_2], we get an expression for y_3^*, using also the optimal values x_1^*, x_2^*:

$$\boxed{y_3^* = p_2 + F - (p_1+t_1)\left[\left(1 - p_1 - t_1\right)\frac{1}{k_1}\right] - (p_2+t_2)\left[\left(1 - p_2 - t_2\right)\frac{1}{k_2}\right] + T}$$

- Now we have to find the eq. prices: p_1^*, p_2^*, p_3^*

We normalized $p_3^*=1$; so we concentrate on p_2^* and p_1^*
To do that we use the market clearing conditions:

markets clear: $\begin{cases} \text{good 1} & x_1^* + y_1^* = 1 \\ \text{good 2} & x_2^* + y_2^* = 1 \\ \text{good 3} & x_3^* + y_3^* = e+f \end{cases}$

- $x_1^* + y_1^* = \left(1 - p_1 - t_1\right)\left(\frac{1}{\alpha_1} + \frac{1}{\beta_1}\right) = 1$

$$1 - p_1 - t_1 = \left(\frac{\alpha_1 \beta_1}{\beta_1 + \alpha_1}\right)$$

$$\boxed{p_1^* = 1 - t_1 - \left(\frac{\alpha_1 \beta_1}{\beta_1 + \alpha_1}\right)}$$

- $x_2^* + y_2^* = \left(1 - p_2 - t_2\right)\left(\frac{1}{\alpha_2} + \frac{1}{\beta_2}\right) = 1$

Analogously to p_1^*:

$$\boxed{p_2^* = 1 - t_2 - \left(\frac{\alpha_2 \beta_2}{\alpha_2 + \beta_2}\right)}$$

- We can now substitute p_1°, p_2° and p_3° into $x_1^\circ, x_2^\circ, x_3^\circ, y_1^\circ, y_2^\circ, y_3^\circ$:

$$x_1^\circ = \left(1 - p_1 - t_1\right) \cdot \frac{1}{\alpha_1} = \left(\cancel{1} - \left(\cancel{1} - t_1 - \frac{\alpha_1 \beta_1}{\alpha_1 + \beta_1}\right) - t_1\right) \cdot \frac{1}{\alpha_1}$$

$$x_1^\circ = \frac{\cancel{\alpha_1} \beta_1}{\alpha_1 + \beta_1} \cdot \frac{1}{\cancel{\alpha_1}} \rightarrow \boxed{x_1^\circ = \frac{\beta_1}{\alpha_1 + \beta_1}}$$

$$x_2^\circ = \left(1 - p_2 - t_2\right) \cdot \frac{1}{\alpha_2} = \left(\cancel{1} - \left(\cancel{1} - t_2 - \frac{\alpha_2 \beta_2}{\alpha_2 + \beta_2}\right) - t_2\right) \frac{1}{\alpha_2}$$

$$= \frac{\cancel{\alpha_2} \beta_2}{\alpha_2 + \beta_2} \frac{1}{\cancel{\alpha_2}} \rightarrow \boxed{x_2^\circ = \frac{\beta_2}{\alpha_2 + \beta_2}}$$

Analogously, $\boxed{y_1^\circ = \frac{\alpha_1}{\alpha_1 + \beta_1}}$ $\boxed{y_2^\circ = \frac{\alpha_2}{\alpha_2 + \beta_2}}$

To get x_3° and y_3° we have to substitute $p_1^\circ, p_2^\circ, p_3^\circ$ into x_3° and y_3°

(Also need to check for SOC to check FOC are sufficient).

d) Derivative of both individual's utilities w.r.t tax rates

$$u_1(w_1, p, t) = x_1^\circ - \frac{\alpha_1}{2} x_1^{\circ 2} + x_2^\circ - \frac{\alpha_2}{2} x_2^{\circ 2} + x_3^\circ - \lambda_1 y_1^\circ - \lambda_2 y_2^\circ - \lambda_3 y_3^\circ$$

$$u_2(w_2, p, t) = y_1^\circ - \frac{\mu_1}{2} y_1^{\circ 2} + y_2^\circ - \frac{\mu_2}{2} y_2^{\circ 2} + y_3^\circ - \gamma_1 x_1^\circ - \gamma_2 x_2^\circ - \gamma_3 x_3^\circ$$

Plugging in the optimal bundles in eq. we get the indirect utility functions for consumer 1 and 2:

$$\frac{\partial u_1(w_1, p, t)}{\partial t_1} = \frac{\partial x_3^\circ}{\partial t_1} - \lambda_3 \frac{\partial y_3^\circ}{\partial t_1} = \frac{-1}{2} - \frac{\lambda_3}{2}$$

$$\frac{\partial u_1(w, p, t)}{\partial t_2} = \frac{\partial x_3^\circ}{\partial t_2} - \lambda_3 \frac{\partial y_3^\circ}{\partial t_2} = \frac{1}{2} + \frac{\lambda_3}{2}$$

Consumer's 2:

$$\frac{\partial u_2}{\partial t_1} = \frac{1}{2} + \frac{\gamma_3}{2}$$

$$\frac{\partial u_2}{\partial t_2} = \frac{-1}{2} - \frac{\gamma_3}{2}$$

e) Let Δ_1 be the change in t_1

Δ_2 be " " in t_2

Given that change the utility change for consumer's will be:

Consumer 1: Change in utility is given by $\Delta U_i = \Delta_1 \frac{\partial U_i}{\partial t_1} + \Delta_2 \frac{\partial U_i}{\partial t_2}$

$$\Delta_1 \left(\frac{-1}{2} - \frac{\lambda_3}{2} \right) + \Delta_2 \left(\frac{1}{2} + \frac{\lambda_3}{2} \right)$$

Consumer 2:

$$\Delta_1 \left(\frac{1}{2} + \frac{\gamma_3}{2} \right) + \Delta_2 \left(\frac{-1}{2} - \frac{\gamma_3}{2} \right)$$

In a Pareto improvement, after a tax change, implies the tax change harms no one and benefits at least one person.

To have a Pareto improvement we need both consumers to be at least better off after tax change:

Consumer 1 $\quad \Delta_2 \left(\frac{1}{2} + \frac{\lambda_3}{2} \right) > \Delta_1 \left(\frac{1}{2} + \frac{\lambda_3}{2} \right)$

Consumer 2 $\quad \Delta_1 \left(\frac{1}{2} + \frac{\gamma_3}{2} \right) > \Delta_2 \left(\frac{1}{2} + \frac{\gamma_3}{2} \right)$

with $\lambda_3, \gamma_3 > 0$ we need

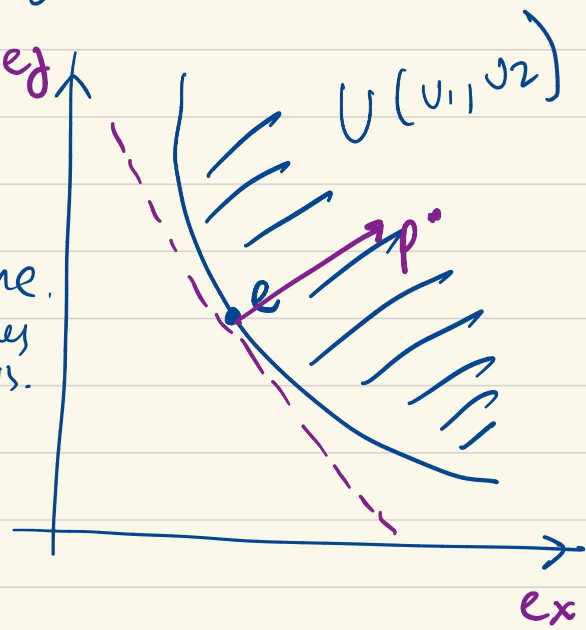
$$\left. \begin{array}{l} \Delta_2 > \Delta_1 \\ \Delta_1 > \Delta_2 \end{array} \right\} \text{contradiction!}$$

$\nexists$ a Pareto improvement after tax changes.

Q3) [Be explicit about your assumptions]

Problem 2

(a) If $\succsim$ are LNS, at a CE price vector, expenditure on $U(u_1, u_2)$ is minimized at agg endowment.



2 commodities: x, y
2 consumers: $1, 2$

I will assume $\succsim$ are LNS and monotone.

monotone implies LNS.

If $\exists\, e'$ st $e' \in U(u_1, u_2)$ and

$p^* e' < p^* e$. $\because e' \in U(u_1, u_2)$,

By definition there are c_1', c_2' such that:

[some write =] only need LNS

$$\begin{cases} c_1' + c_2' \leq e' \\ u_1(c_1') \geq u_1(c_1^*) \\ u_2(c_2') \geq u_2(c_2^*) \end{cases}$$

$\because \succsim$ is monotone, we have $p^* \geq 0$.

is not that we endow the economy with new endowment

So $p^* c_1 + p^* c_2 \leq p^* e'$
$< p^* e$

prices are the same p^*

[monotone]

$\therefore p^* c_1 + p^* c_2 < p^* e_1 + p^* e_2$

So there is some $i \in \{1, 2\}$ such that $p^* c_i' < p^* e_i$

But notice that $u_i(c_i') \geq u_i(c_i^\circ)$, violating LNS.

(b) utility strictly quasi-concave ; $U(v_1,v_2)$ strictly convex set

Assume that $\succsim$ is lower-semi continuous. ($\succsim$ is cont)

If $e', e'' \in U(v_1, v_2)$. WTS $\alpha e' + (1-\alpha) e'' \in \text{int}\, U(v_1, v_2)$

for $\alpha \in (0,1)$ (ie $\exists$ open set $O \subset U(v_1, v_2)$ st $\alpha e' + (1-\alpha) e'' \in O$)

Suppose $c_i' \succsim c_i^\circ$ and $c_i'' \succsim c_i^\circ$ where c_i' and c_i'' are allocation of e' and e''.

$\because$ $\succsim$ is strictly convex ($u(\cdot)$ is strictly quasiconcave)

$\therefore$ $\alpha c_i' + (1-\alpha) c_i'' \succsim c_i^*$. Define $P(c_i^\circ)$ as the strict upper contour set of c_i°

$$P(c_i^\circ) = \{ c_i : c_i \succsim_i c_i^\circ \}$$

$\therefore$ $\alpha c_i' + (1-\alpha) c_i'' \in P(c_i^\circ)$

Define $P \equiv P(c_1^\circ) + P(c_2^\circ) = \{ c : c = c_1 + c_2 ;\ c_1 \in P(c_1^*),\ c_2 \in P(c_2^*) \}$

$\because$ $\succsim$ is lower semi continuous.

$\therefore$ $P(c_i^\circ)$ is open (since strict lower semi-continuity)

$\therefore$ P is open $\subset U(v_1, v_2)$ (sum of two open sets)

Since $\alpha c_i' + (1-\alpha) c_i'' \in P(c_i^*)$ $\forall i$, then

$$\alpha (c_1' + c_2') + (1-\alpha)(c_1'' + c_2'') \in P$$

$\therefore \quad \alpha e' + (1-\alpha) e'' \in P$

$\therefore \quad \alpha e' + (1-\alpha) e'' \in \text{int } U(u1, u2)$