Logistics

- TA Office Hours:
  - Mon & Wed 18:30-19:30pm
  - In person (Uris 465) or by Zoom (same link as before).
- All problem sets are due before Friday section.
- Section's material https://github.com/luciacasal/-Cornell_ECON6100.

1 Hecksher-Ohlin-Vanek Model [1]

1.1 Questions

1. Suppose in a small open economy, world output prices for good 1 and 2 are p and 1, respectively. The production functions are:

$$q_1 = k^{1/2} l^{1/2}$$
$$q_2 = k^{3/4} l^{1/4}$$

   (a) Suppose both goods are produced with positive quantity, compute the equilibrium factor prices.

   (b) Compute $\frac{\partial w}{\partial p}$ when both goods are produced.

   (c) Suppose $p = 1$. If the endowment of capital and labor are both 100, do both firms operate? Compute factor prices.

   (d) Suppose $p = 1$. If the endowment of capital and labor are 100 and 400 respectively, do both firms operate? Compute factor prices.

2. A small open economy provides 2 goods, A and B, using 2 inputs, capital k and labor l. The production function for the two goods are:

$$f_A(l_A, k_A) = min\{\alpha_A l_A, \beta_A k_A\}$$
$$f_A(l_A, k_A) = min\{\alpha_B l_B, \beta_B k_B\}$$

   Let p_A and p_B be the world output prices of good A and B respectively. Also, let the country's stock of capital and labor be $(K, L) \gg 0$. Denote the prices of capital and labor by r and w respectively. Suppose $\frac{K}{L} \in (1/4, 1/2)$.

[1] These notes borrow form Jaden Chen's notes from 2020.

(a) Let $p_A = \alpha_A = \alpha_B = 1$ and $\beta_A = 4$, $\beta_B = 2$. Suppose we are in the situation where both A and B are produced in positive quantities. Solve for the competitive equilibrium when both wage rate and rental rate are positive.

(b) Suppose the economy specified above is in a diversified competitive equilibrium with $w, r > 0$. What can you say regarding the factor intensity of industry A compared to industry B? What is the effect of an increase in p_B on the equilibrium input prices?

3. For a country with an endowment in the interior of the cone of diversification, conjecture and prove a result on the effects of a small increase in the quantity of a factor on output.

HOV

①
- 2×2 production model
- 2 factors: K, L
- 2 industries: f_A, f_B (CRS)
- Exog V: (p_A, p_B, K, L)
- Endog V: $(w, r, x_A, x_B, K_A, L_A, K_B, L_B)$

②

Determination of eq. (factor) prices and output:

1) π-max: $p_i = c_i(w,r)$
2) Market clearing: $\nabla c_A x_A + \nabla c_B x_B = \begin{pmatrix} L \\ K \end{pmatrix}$

$$\underbrace{\frac{\partial c_A}{\partial w}}_{L_A} x_A + \underbrace{\frac{\partial c_B}{\partial w}}_{L_B} x_B = L$$

(Sheppard's) lemma

$$\frac{\partial c_A}{\partial r} x_A + \frac{\partial c_B}{\partial r} x_B = K$$

HOV Question.

Suppose small open economy
world output price for good 1 and good 2 are p and 1.

$$q_1 = k_1^{1/2}\, l_1^{1/2}$$
$$q_2 = k_2^{3/4}\, l_2^{1/4}$$

a) suppose $\underbrace{\text{diversification}}$ both goods are produced. compute the eq factor prices.

$$P_1 = p \qquad P_2 = 1$$

$x_1 > 0 \quad x_2 > 0$ diversification $\rightarrow$ interior solution

we know in equilibrium $P_1 = c_1(w, r)$
$P_2 = c_2(w, r)$

STEP 1: Solve for $c_i(r, w)$.

Prod function $\quad q = k^\alpha l^{1-\alpha}$

$$\begin{cases} c(r,w) = \min\limits_{l,k} \; wl + rk \\ \text{st} \quad k^\alpha l^{1-\alpha} = 1 \end{cases}$$

$$\mathcal{L} = wl + rk + \lambda\left[1 - k^\alpha l^{1-\alpha}\right]$$

$[l] \quad w = k^\alpha (1-\alpha) l^{-\alpha} \cdot \lambda$

$[r] \quad r = \alpha k^{\alpha-1} l^{1-\alpha} \lambda$

$[\lambda] \quad k^\alpha l^{1-\alpha} = 1$

$$\rightarrow \frac{w}{r} = \frac{(1-\alpha)}{\alpha}\frac{k}{l} \rightarrow \boxed{k = \frac{\alpha}{(1-\alpha)} \cdot \frac{w}{r} \cdot l} \quad (1)$$

plug into $[\lambda]$:

$$\left[\frac{\alpha}{1-\alpha}\frac{w}{r} l\right]^\alpha l^{1-\alpha} = 1$$

$$\left(\frac{\alpha}{1-\alpha}\right)^\alpha \left(\frac{w}{r}\right)^\alpha l = 1$$

$$\rightarrow \boxed{l = \left(\frac{1-\alpha}{\alpha}\right)^\alpha \cdot \left(\frac{r}{w}\right)^\alpha} \quad (2)$$

plug 2 into 1 to obtain k in terms of parameters:

$$\boxed{k = \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} \left(\frac{w}{r}\right)^{1-\alpha}} \quad (3)$$

Now substitute 2 and 3 into cost function $C(w,r) = wl + rk$

$$wl + rk = w \left(\frac{1-\alpha}{\alpha}\right)^{\alpha} \left(\frac{r}{w}\right)^{\alpha} + r \cdot \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} \left(\frac{w}{r}\right)^{1-\alpha}$$

$$= w^{1-\alpha} r^{\alpha} \left[(1-\alpha)^{\alpha} \alpha^{-\alpha} + \alpha^{1-\alpha} (1-\alpha)^{-(1-\alpha)}\right]$$

$$= w^{1-\alpha} r^{\alpha} \cdot \alpha^{-\alpha} (1-\alpha)^{\alpha} \left[1 + \alpha (1-\alpha)^{-1}\right]$$

$$= w^{1-\alpha} r^{\alpha} \quad \alpha^{-\alpha} (1-\alpha)^{\alpha} \left[\frac{1-\cancel{\alpha}+\cancel{\alpha}}{1-\alpha}\right]$$

$$C(w,r) = w^{1-\alpha} r^{\alpha} \cdot \frac{1}{(1-\alpha)^{1-\alpha} \cdot \alpha^{\alpha}}$$

$$\boxed{C(w,r) = \left[\frac{w}{(1-\alpha)}\right]^{1-\alpha} \left[\frac{r}{\alpha}\right]^{\alpha}} \quad (4)$$

$2^{1/2} \cdot 2^{1/2}$

good 1 : $\alpha = 1/2 \longrightarrow C_1(w,r) = \left(\frac{w}{1/2}\right)^{1/2} \cdot \left(\frac{r}{1/2}\right)^{1/2}$

$$\boxed{C_1(w,r) = 2\sqrt{wr}} \quad (5)$$

good 2: $\alpha = 3/4 \longrightarrow C_2(w,r) = \left(\frac{w}{1/4}\right)^{1/4} \cdot \left(\frac{r}{3/4}\right)^{3/4}$

$$\boxed{C_2(w,r) = \frac{4}{3^{3/4}} \cdot w^{1/4} r^{3/4}} \quad (6)$$

In equilibrium:
$$\begin{cases} P_1 = p = C_1(w,r) \\ P_2 = 1 = C_2(w,r) \end{cases}$$

1) $p = 2\sqrt{wr} \longrightarrow \boxed{w = \left(\frac{p}{2}\right)^2 \frac{1}{r}}$ (7)

2) $1 = \frac{4}{3^{3/4}} w^{1/4} r^{3/4}$

substitute (7) to get:

$$1 = \frac{4}{3^{3/4}} \cdot \left(\frac{p}{2}\right)^{2/4} \left(\frac{1}{r}\right)^{1/4} r^{3/4} \longrightarrow \boxed{r = \left(\frac{3}{4}\right)^{3/2} \frac{1}{p}} \quad (8)$$

plug (8) in (7): $\longrightarrow \boxed{w = \sqrt{\frac{4}{27}} \cdot p^3}$ (9)

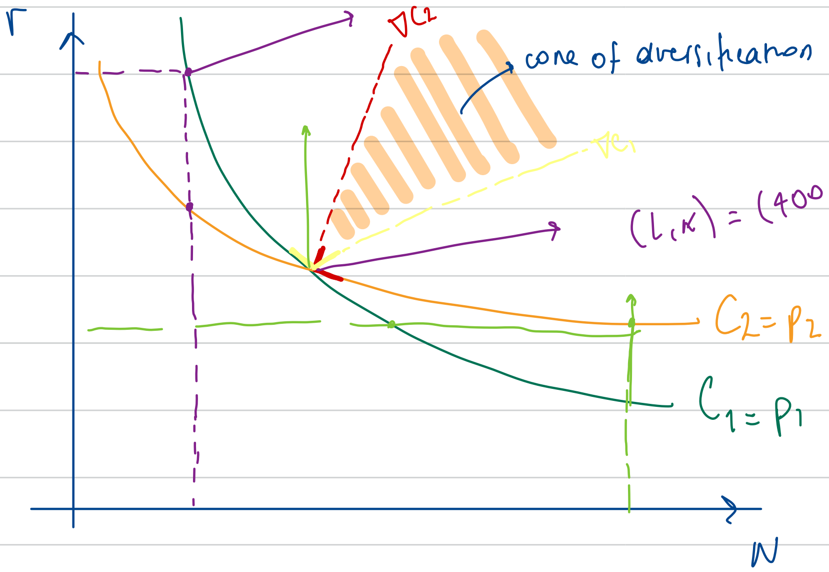
b) Compute $\frac{dw}{dp}$ when diversification:

$$\frac{dw}{dp} = \sqrt{\frac{4}{\underbrace{27}_{3^3}}}\, 3p^2 = \frac{2}{3} p^2$$

c) Suppose $p = 1$. If $K = 100$ and $L = 100$, do both firms operate? compute factor prices.

$P_1 = 1$; $P_2 = 1$ $\quad K, L = (100, 100)$

In this case $\boxed{(w^*, r^*) = \left(\frac{2}{\sqrt{27}}, \frac{\sqrt{27}}{8}\right)}$ (10)



we need to find the slopes of $C_1(w,r) = \partial C_1/\partial r \,/\, \partial C_1/\partial w$

$C_2(w,r) = \partial C_2/\partial r \,/\, \partial C_2/\partial w$

$$\nabla C_1 = \left[\frac{\partial C_1}{\partial r} ; \frac{\partial C_1}{\partial w}\right]$$

$$\frac{\partial C_1}{\partial r} = \frac{\partial\, 2\sqrt{wr}}{\partial r} = \sqrt{\frac{w}{r}} \qquad \frac{\partial C_1}{\partial w} = \sqrt{\frac{r}{w}}$$

In the equilibrium (int. solution): (substitute (10)):

$$\frac{\partial C_1}{\partial r} = \frac{\left(2/\sqrt{27}\right)^{1/2}}{\left(\sqrt{27}/8\right)^{1/2}} = \frac{4}{\sqrt{27}} \quad ; \quad \frac{\partial C_1}{\partial w} = \frac{\sqrt{27}}{4}$$

$$\rightarrow \boxed{\frac{\partial C_1/\partial r}{\partial C_1/\partial w} = \frac{\frac{4}{\sqrt{27}}}{\frac{\sqrt{27}}{4}} = \frac{16}{27}} \qquad (11)$$

$$\nabla C_2 = \left(\frac{\partial C_2}{\partial r} ; \frac{\partial C_2}{\partial w} \right)$$

$$\frac{\partial C_2}{\partial r} = 3^{1/4} \left(\frac{w}{r}\right)^{1/4} \qquad \frac{\partial C_2}{\partial w} = \frac{1}{3^{3/4}} \left(\frac{r}{w}\right)^{3/4}$$

In equilibrium (substitute (10)) interior solution:

$$\frac{\partial C_2}{\partial r} = \left(\frac{3}{27}\right)^{1/4} \cdot 2 \qquad \frac{\partial C_2}{\partial w} = \frac{1}{3^{3/4}} \cdot \frac{27^{3/4}}{2^3}$$

$$\longrightarrow \boxed{\frac{\partial C_2}{\partial r} \Big/ \frac{\partial C_2}{\partial w} = \frac{16}{9}} \quad (12)$$

$$\text{cone of diversification} \in \left[\frac{16}{27} ; \frac{16}{9} \right]$$

with $K = 100$; $L = 100$

$$\text{slope} = \frac{K}{L} = 1 \in \left[\frac{16}{27} ; \frac{16}{9} \right]$$

$\longrightarrow$ we have diversification (both goods are produced)

Factor prices are given by (10). [Inside the cone of diversification, factor prices do not change when K and L change].

FACTOR PRICE EQUALIZATION THEOREM.

d) Suppose $p = 1$, If $(K, L) = (100, 400)$, do both firms operate? compute factor prices.

now new slope $(100,400) = \frac{100}{400} = \frac{1}{4} \notin \left[\frac{16}{27}; \frac{16}{9}\right]$

sector 1 $\rightarrow$ $\pi = 0$

sector 2 $\rightarrow$ $\pi < 0$ (won't produce)

Factor prices? Eq conditions:

1 $\quad c_1(w,r) = p_1$

2 $\quad \begin{bmatrix} \partial c_1/\partial w \\ \partial c_1/\partial r \end{bmatrix} x_1 = \begin{bmatrix} L \\ K \end{bmatrix}$

1 $\quad c_1 = 2\sqrt{wr} = 1 = p_1$

2 $\quad \nabla c_1 = \left[\frac{\partial c_1}{\partial w} \quad \frac{\partial c_1}{\partial r}\right] = \left[\sqrt{\frac{r}{w}} ; \sqrt{\frac{w}{r}}\right]$

$(L, K) = (400, 100)$

$$\frac{\partial c_1/\partial r}{\partial c_1/\partial w} = \frac{\sqrt{w/r}}{\sqrt{r/w}} = \frac{w}{r} = \frac{1}{4} = \frac{K}{L}$$

$\begin{cases} w/r = 1/4 \\ 2\sqrt{wr} = 1 \end{cases} \longrightarrow \boxed{r = 1 ; \; w = 1/4}$

Q2. 2 goods $\to A$, $\to B$ 2 inputs $\to k$, $\to l$

$$F_A(k_A, l_A) = \min\{\alpha_A l_A ; \beta_A k_A\}$$
$$F_B(k_B, l_B) = \min\{\alpha_B l_B ; \beta_B k_B\}$$

p_A, p_B $(K, L) \gg 0$ r, w

$$\frac{K}{L} \in \left(\frac{1}{4}, \frac{1}{2}\right)$$

1. Suppose diversification (both goods are produced). Solve for CE.

$p_A = \alpha_A = \alpha_B = 1$; $\beta_A = 4$; $\beta_B = 2$.

- First, we have to find the cost function. Notice that to produce a level of output 1, we must have:

- $\alpha_i l_i = 1 \to l_i = 1/\alpha_i \to l_A = 1$, $l_B = 1$
- $\beta_i k_i = 1 \to k_i = 1/\beta_i \to k_A = 1/4$, $k_B = 1/2$

$$\Rightarrow c_i(r, w) = r \frac{1}{\beta_i} + w \cdot \frac{1}{\alpha_i}$$

- $C_A(r, w) = r \cdot \frac{1}{4} + w$.
- $C_B(r, w) = r \cdot \frac{1}{2} + w$

[interior sol]

$$\begin{cases} 1 = p_A = C_A(r,w) \\ p_B = C_B(r,w) \end{cases}$$

$$1 = \frac{r}{4} + w \longrightarrow r = 4(1-w)$$

$$p_B = \frac{r}{2} + w \longrightarrow p_B = \frac{4(1-w)}{2} + w$$

$$w = 2 - p_B$$
$$r = -4 + 4p_B$$

- We also know the system balance condition (market clearing):

$$\begin{cases} \frac{\partial C_A}{\partial w} x_A + \frac{\partial C_B}{\partial w} \cdot x_B = L \\ \frac{\partial C_A}{\partial r} x_A + \frac{\partial C_B}{\partial r} x_B = K \end{cases}$$

$$\frac{\partial C_A}{\partial w} = 1 \qquad \frac{\partial C_A}{\partial r} = \frac{1}{4}$$

$$\frac{\partial C_B}{\partial w} = 1 \qquad \frac{\partial C_B}{\partial r} = \frac{1}{2}$$

$$\begin{cases} x_A + x_B = L \longrightarrow x_A = L - x_B \\ \frac{1}{4} x_A + \frac{1}{2} x_B = K \end{cases}$$

$$\longrightarrow \frac{1}{4}(L - x_B) + \frac{1}{2} x_B = K$$

$$x_B^* = 4K - L$$
$$x_A^* = 2L - 4K$$

- $$\ell_A^* = \frac{x_A^*}{\alpha_A} = \frac{2L-4K}{1} \qquad k_A^* = \frac{x_A^*}{\beta_A} = \frac{2L-4K}{4}$$

$$\ell_B^* = \frac{x_B^*}{\alpha_B} = \frac{4K-L}{1} \qquad k_B^* = \frac{x_B^*}{\beta_B} = \frac{4K-L}{1}$$

2. Diversified CE $w, r > 0$

we have:

$$\frac{\partial C_A / \partial r}{\partial C_A / \partial w} = \frac{1}{4} < \frac{1}{2} = \frac{\partial C_B / \partial r}{\partial C_B / \partial w}$$

$\longrightarrow$ Industry B is relatively more intensive in capital.

Given $r^* = -4 + 4p_B$
$w^* = 2 - p_B$

If $p_B \uparrow \longrightarrow r^* \uparrow$ and $w^* \downarrow$

When the price of good B goes up, the price of the factor which is intensive in (capital) goes up and the price of the other factor goes down. Stolper-Samuelson theorem.

Q3. Investigate $(K, L) \longrightarrow (x_A, x_B)$

Inside the cone of diversification:

$$\begin{bmatrix} \frac{\partial c_A}{\partial w} & \frac{\partial c_B}{\partial w} \\ \frac{\partial c_A}{\partial r} & \frac{\partial c_B}{\partial r} \end{bmatrix} \cdot \begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} L \\ K \end{bmatrix}$$

Suppose there is an increase in L:

differentiate the system wrt L:

$$\underbrace{\begin{bmatrix} \frac{\partial c_A}{\partial w} & \frac{\partial c_B}{\partial w} \\ \frac{\partial c_A}{\partial r} & \frac{\partial c_B}{\partial r} \end{bmatrix}}_{D} \cdot \begin{bmatrix} \partial x_A / \partial L \\ \partial x_B / \partial L \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since diversification $\longrightarrow$ $\text{Det}(D) \neq 0$, then apply Cramer's rule:

$$\frac{\partial x_A}{\partial L} = \begin{vmatrix} 1 & \partial c_B / \partial w \\ 0 & \partial c_B / \partial r \end{vmatrix} \frac{1}{D} = \frac{\partial c_B / \partial r}{D} \qquad (1)$$

$$\frac{\partial x_B}{\partial L} = \begin{vmatrix} \partial c_A / \partial w & 1 \\ \partial c_A / \partial r & 0 \end{vmatrix} \frac{1}{D} = \frac{-\partial c_A / \partial r}{D} \qquad (2)$$

with $\det(D) = \partial c_A/\partial w \cdot \partial c_B / \partial r - \partial c_A / \partial r \cdot \partial c_B / \partial w$

Now assume that sector A is capital intensive; ie; : (B is L-intensive)

$$\frac{K_A}{L_A} > \frac{K_B}{L_B}$$

Therefore: $$\frac{\partial C_A/\partial r}{\partial C_A/\partial w} > \frac{\partial C_B/\partial r}{\partial C_B/\partial w}$$

which implies:

$$\partial C_A/\partial r \cdot \partial C_B/\partial w - \partial C_B/\partial r \cdot \partial C_A/\partial w > 0$$

$$\Longrightarrow \qquad D < 0$$

Since by Sheppards' lemma we know that $\frac{\partial C_i}{\partial r} = k_i \geq 0$ for $i = A, B$.

and $\frac{\partial C_i}{\partial w} = L_i \geq 0$

Back to (1) and (2):

- $$\frac{\partial x_A}{\partial L} = \frac{\overbrace{\partial C_B/\partial r}^{\geq 0}}{\underbrace{D}_{<0}} \leq 0$$

- $$\frac{\partial x_B}{\partial L} = -\frac{\overbrace{\partial C_A/\partial r}^{\geq 0}}{\underbrace{D}_{<0}} \geq 0$$

Rybcszynsk Theorem: In a 2x2 model, with no factor intensity (and with diversification (interior solution!)) reversion (only one intersection) if the endowment of a factor increases, then the production of the good that uses this factor relatively more intensive increases, and the production of the other good decreases.