Logistics

- TA Office Hours:
  - Mon & Wed 18:30-19:30pm (in person (Uris 465) or by Zoom).
- All problem sets are due before Friday section.
- Section's material https://github.com/luciacasal/-Cornell_ECON6100.

1 Problems on matching [1]

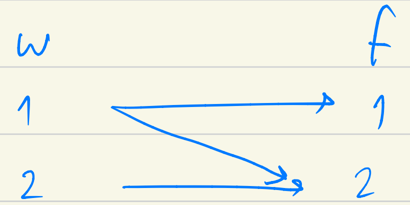
1. (Problem 4 of the matching problem set)

2. Suppose that a TU matching problem is given with I individuals and J firms, and match valuations $v_{ij} > 0$. Another matching problem has match valuations $v_{ij} + u_i + p_j$,with all the $u_i, p_j \geq 0$. Suppose that $I = J$. How do optimal matchings differ in these two problems?

3. (Roommate problem) Consider a "roommate problem", wherein individuals in a single population match with each other. Suppose there are three individuals, a, b and c. The value of an $a \leftrightarrow b$ match is $v_{ab}; v_{ac}$ and v_{bc} are defined similarly. The return to being unmatched is 0. Assume that $v_{ab} > v_{ac} > v_{bc}$. Let y_a, y_b and y_c denote the payoffs to the three individuals in a stable matching.
   (a) Suppose that $v_{ab} = 4$,$v_{ac} = 2$ and $v_{bc} = 1$. Does a stable matching exist? If so, what are the payoffs to a, b and c.
   (b) Suppose that $v_{ab} = 4$, $v_{ac} = 3$ and $v_{bc} = 2$. Does a stable matching exist? If so, what are the payoffs to a, b and c.
   (c) Show that for arbitrary values of the v_{ij}, any stable matching requires that the match be efficient, that is, that a matches with b and c is unmatched.

[1] Borrowed from Jaden Chen's Section Notes from 2020.

1. 2 workers, 2 firms



a) Set of feasible matching $(x_{11}, x_{12}, x_{21}, x_{22})$ where $x_{21} = 0$

$$\begin{cases} x_{11} + x_{12} \leq 1 & \text{worker 1} & \\ x_{22} \leq 1 & \text{worker 2} & x_{22} + x_{21} \\ x_{11} \leq 1 & \text{firm 1} & x_{11} + x_{21} \\ x_{12} + x_{22} \leq 1 & \text{firm 2} & \\ x_{11}, x_{12}, x_{22} \in \underbrace{\{0, 1\}}_{x_{ij} \geq 0} ; & x_{21} = 0 & \end{cases}$$

b) Assume Birckoff V-N true, then the primal is:

PRIMAL

$$\begin{cases} \max & v_{11} x_{11} + v_{12} x_{12} + v_{22} x_{22} \\ \text{st} & \begin{cases} x_{11} + x_{12} \leq 1 & [w_1] \\ x_{22} \leq 1 & [w_2] \\ x_{11} \leq 1 & [\pi_1] \\ x_{12} + x_{22} \leq 1 & [\pi_2] \\ x_{ij} \geq 0 & \end{cases} \end{cases}$$

In matrix form:

$$\begin{cases} \max \begin{pmatrix} v_{11} & v_{12} & v_{22} \end{pmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{22} \end{bmatrix} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{22} \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} ; \quad x_{ij} \geq 0 \end{cases}$$

DUAL

$$\min \begin{pmatrix} w_1 & w_2 & \pi_1 & \pi_2 \end{pmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{st} \quad \begin{pmatrix} w_1 & w_2 & \pi_1 & \pi_2 \end{pmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \geq \begin{bmatrix} v_{11} & v_{12} & v_{22} \end{bmatrix}$$

$$w, \pi \geq 0$$

$\longrightarrow \min \; w_1 + w_2 + \pi_1 + \pi_2$

$$\text{st} \begin{cases} w_1 + \pi_1 \geq v_{11} \\ w_1 + \pi_2 \geq v_{12} \\ w_2 + \pi_2 \geq v_{22} \\ w, \pi \geq 0 \end{cases}$$

MISS $w_2 + \pi_1 \geq v_{21}$

(It is possible that $w_2 + \pi_1 < v_{21}$.) !!

Comments: compare this set up with the old setup:

① Is the surplus higher, lower, equal? weakly lower

you want to max the same obj function, but with a constraint that is a subset of the old one.

smaller constraint set, so you will get a smaller no.

⟹ Value for primal problem is less, because it adds more constraints, so the resulting matching is less "optimal" (ie; achieves less social surplus)

(2) Does this mean that it Pareto less efficient?
↳ the new one

(Use def of Pareto Efficiency).

It does not mean that because to compare PO btw the two, we have to compare what each manual gets: w_1, w_2, π_1, π_2, are they $w_i < w_i'$ or $\pi_i < \pi_i'$; $w_2' < w_2'$; $\pi_2 < \pi_2'$?
In this setting we can not conclude like that.

→ for any solution (w, π) in the new problem, $\exists$ (w', π') when solves the old problem (uncont). and $w_i' \geq w_i$; $\pi_i' \geq \pi_i$, so by imposing +1 constraint we get that the answer is less Pareto efficient.

2.

2. Suppose that a TU matching problem is given with I individuals and J firms, and match valuations $v_{ij} > 0$. Another matching problem has match valuations $v_{ij}+u_i+p_j$, with all the $u_i, p_j \geq 0$. Suppose that $I = J$. How do optimal matchings differ in these two problems?

I workers, J firms $v_{ij} > 0$ Prelim 1 old question!

New matching: if i and j get matched, the valuation is $v_{ij} + u_i + p_j$ $u_i; p_j \geq 0$

How do optimal matching differ?

Ans:

$I = J$ **key**

- older PRIMAL:

$$\max \sum_{i,j} x_{ij} v_{ij}$$
$$\text{st} \quad \sum_i x_{ij} \le 1 \quad \forall j$$
$$\sum_j x_{ij} \le 1 \quad \forall i$$
$$x_{ij} \geq 0 \quad (\text{BvN thm})$$

- new PRIMAL

$$\max \sum_{i,j} x_{ij} (v_{ij} + u_i + p_j)$$
$$\text{st} \quad \sum_i x_{ij} \le 1 \quad \forall j$$
$$\sum_j x_{ij} \le 1 \quad \forall i$$
$$x_{ij} \geq 0 \quad (\text{BvN thm})$$

; u_i ; $p_j \geq 0$

objective

$$\overbrace{\sum_{ij} x_{ij} v_{ij}}^{\text{same as old problem!}} + \underbrace{\sum_{ij} x_{ij} u_i}_{1} + \underbrace{\sum_{ij} x_{ij} p_j}_{2}$$

1
$$\sum_{ij} x_{ij} u_i = \sum_i \sum_j x_{ij} u_i = \sum_i \left(\sum_j x_{ij} \right) u_i$$
$$= \sum_i u_i \underbrace{\left(\sum_j x_{ij} \right)}_{= 1}$$

for an optimal match

Because $I = J$ every worker can get matched with each firm. and $v_{ij} > 0$. You have incentive to make the match because they will get sposue payoff

If they match. $\boxed{\therefore \sum_j x_{ij} = 1 \; ; \; \sum_i x_{ij} = 1}$

(2) $\sum_j p_j x_{ij} = \sum_j p_j$

$\Longrightarrow \sum_{ij} x_{ij} \left(V_{ij} + U_i + p_j \right)$

$= \sum_{ij} x_{ij} V_{ij} + \underbrace{\sum_i U_i + \sum_j p_j}_{\text{Constant!}}$

$\Longrightarrow$ Value Old Primal = Value New Primal

- How would the dual problem change? think

Value Dual' = Value Dual

but optimal w and π will differ check

TRY

3. (Roommate problem) Consider a "roommate problem", wherein individuals in a single population match with each other. Suppose there are three individuals, a, b and c. The value of an $a \leftrightarrow b$ match is v_{ab}; v_{ac} and v_{bc} are defined similarly. The return to being unmatched is 0. Assume that $v_{ab} > v_{ac} > v_{bc}$. Let y_a, y_b and y_c denote the payoffs to the three individuals in a stable matching.

(a) suppose that $v_{ab} = 4$, $v_{ac} = 2$ and $v_{bc} = 1$. Does a stable matching exist? If so, what are the payoffs to a, b and c.

(b) Suppose that$v_{ab} = 4$, $v_{ac} = 3$ and $v_{bc} = 2$. Does a stable matching exist? If so, what are the payoffs to a, b and c.

(c) Show that for arbitrary values of the v_{ij} , any stable matching requires that the match be efficient, that is, that a matches with b and c is unmatched.

Roomate Problem

{ TU — depending on
NTU — utility

① { 2 sided matching
e.g W ⟷ F ; M ⟷ F

② { one-side mateing
e.g roommate Problem

{ 3 individuals: a, b, c
V_{ab}, V_{ac}, V_{bc} ; if unmatched get 0
with $V_{ab} > V_{ac} > V_{bc}$
Payoffs Y_a, Y_b, Y_c.

a) $V_{ab} = 4$; $V_{ac} = 2$; $V_{bc} = 1$. Does a stable mateing exist?

Yes, there is a stable matching $a \longleftrightarrow b$ $\quad V_{ab} = 4$

$\because$

Set $\underbrace{y_a = 2{,}5}_{>2}$ $\quad \underbrace{y_b = 1{,}5}_{> v_{bc} = 1}$ $\quad y_a + y_b = 4$ $\quad y_c = 0$

so a ✓ $\quad$ so b ✓ $\quad y_a \geq 2$

$y_b \geq 1.$

nobody has incentive to deviate → stable matching.

b) $V_{ab} = 4$ $\quad V_{ac} = 3$ $\quad V_{bc} = 2$. Is there a stable matching?

No!

Method 1:

① $a \longleftrightarrow b$ so $y_a + y_b = V_{ab} = 4$; $y_c = 0$

To make it stable; $y_a \geq 3 = V_{ac}$ prevents A from breaking

$y_b \geq V_{bc} = 2$; $y_a + y_b \geq 5$

impossible. $a \longleftrightarrow b$ unstable

② $a \longleftrightarrow c$ $\quad y_a + y_c = V_{ac} = 3$ $\quad y_b = 0$

$y_a \geq V_{ab} = 4$ impossible $a \longleftrightarrow c$ unstable

③ $b \leftrightarrow c$ $\quad y_b + y_c = V_{bc} = 2$; $y_a = 0$ Need $y_b \geq V_{ab} = 4 > V_{bc}$

So $b \leftrightarrow c$ unstable.

Method 2

to make a matching stable we must have

$$\underbrace{y_i}_{w_i} + \underbrace{y_j}_{\pi_j} \geq v_{ij} \qquad \forall\, i, j$$

In DUAL constraint set!

$$\Longrightarrow \quad \begin{cases} y_a + y_b \geq v_{ab} \\ y_b + y_c \geq v_{bc} \\ y_a + y_c \geq v_{ac} \end{cases}$$

Sum up: $2y_a + 2y_b + 2y_c \geq v_{ab} + v_{bc} + v_{ac} = 9$ (1)

Suppose $i \leftrightarrow j$; $y_i + y_j = v_{ij}$; $y_k = 0$

(1) becomes $2\, v_{ij} + 0 \geq 9 \rightarrow v_{ij} \geq 4.5$

$\because$ $v_{ab} = 4$ $v_{ac} = 3$ $v_{bc} = 2$ $\therefore$ no such v_{ij}

$\Longrightarrow$ No stable matching

c) From b): stable matching can not be guaranteed
But optimal matching always exists, when is $a \leftrightarrow b$
$\therefore$ there is no such coincidence b/w optimality and stability as we saw in class.

So optimal $\not\rightarrow$ stability

But stability $\rightarrow$ optimality? we want to ask if stab $\rightarrow$ optimality?

Yes!

Suppose not. $i \leftrightarrow j$ stable, but not optimal. $\exists$ a unmatched k such that $v_{ik} > v_{ij}$

Note that $\because i \leftrightarrow j$; $y_i + z_j = v_{ij}$ during all steps!

In order to make $i \leftrightarrow j$ stable, we must have

$$y_i + y_k \geq v_{ik} \quad [\text{in (b) method 2}]$$

but $y_i + y_k = y_i + 0$ [k unmatched]
$\leq v_{ij} < v_{ik}$

So it is unstable, contradiction!!

$\therefore$ stability must imply optimality

stability $\longrightarrow$ optimality

$$\boxed{\begin{array}{c} OPT \longrightarrow STAB \\ OPT \longleftarrow STAB \end{array}}$$