1 Problem 1

A two person zero-sum game is described by a single matrix M; m_{ij} is the payoff to the row player of choosing strategy i when the column player chooses strategy j. Since the game is zero-sum, the payoff to the column player of the i, j pair is $-m_{ij}$.

(a) Suppose row chooses mixed strategy p. What is the vector of payoffs against each pure strategy j that column could choose.

Here, the matrix of payoffs of the row player choosing a strategy i when column player chooses a strategy j, is given by:

$$\underbrace{\begin{bmatrix} m_{11} & \dots & m_{1j} \\ \vdots & \ddots & \vdots \\ m_{i1} & \dots & m_{ij} \end{bmatrix}}_{i \times j}$$

Then, we introduce a set of mixed strategies p for the row player, and the payoff vector for the row player against pure strategies of the column player becomes:

$$\underbrace{\begin{bmatrix} p_1 & \dots & p_i \end{bmatrix}}_{1 \times i} \underbrace{\begin{bmatrix} m_{11} & \dots & m_{1j} \\ \vdots & \ddots & \vdots \\ m_{i1} & \dots & m_{ij} \end{bmatrix}}_{i \times j} = \underbrace{\begin{bmatrix} \sum_i p_i m_{i1} & \dots & \sum_i p_i m_{ij} \end{bmatrix}}_{1 \times j}$$

(b) Suppose that column's goal is to give row as little utility as possible. Row's security level is the maximal amount of utility that row can guarantee himself by a suitable strategy choice no matter what column does after seeing row's strategy choice. Formulate the problem of finding row's security level as a linear program. The solution should give you both a value z^* and a strategy p^*.

Now we proceed to formulate this problem in a linear program. We need to maximize the row player's security level s, subject to all mixed strategies summing up to one, each of the strategies being non-negative, as well as the security level being less than or equal to the each column of the payoff vector calculated above. In other words, the linear program that is produced looks as follows:

$$\begin{gathered}\max_{p\in\mathbb{R}^i} s \\ \text{such that } \sum_i p_i = 1 \\ s \leq \sum_i p_i m_{ij}, \forall j \\ p_i \geq 0, \forall i\end{gathered}$$

However, we note that the above is not in primal form as it contains both equality and inequality constraints. Since in the problem below we will be asked to transform the above into its dual, it is beneficial to write it in primal form.

The first step towards this end is to transform the equality constraint into two inequality constraints, as follows:

$$\begin{gathered}\max_{p\in\mathbb{R}^i} s \\ \text{such that } \sum_i p_i \leq 1 \\ \text{such that } -\sum_i p_i \leq -1 \\ s - \sum_i p_i m_{ij} \leq 0, \forall j \\ p_i \geq 0, \forall i\end{gathered}$$

Now, we make the following manipulations in order to write the problem in a primal form using matrices that will yield the desired dual form in the next step:

$$\max \underbrace{\begin{bmatrix} 1 & -1 & 0 & \dots & 0 \end{bmatrix}}_{1\times(i+2)} \cdot \underbrace{\begin{bmatrix} s \\ 0 \\ p_1 \\ \vdots \\ p_i \end{bmatrix}}_{(i+2)\times 1}$$

$$\text{such that } \underbrace{\begin{bmatrix} 1 & -1 & -m_{1j} & \dots & -m_{ij} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & -m_{11} & \dots & -m_{i1} \\ 0 & 0 & 1 & \dots & 1 \\ 0 & 0 & -1 & \dots & -1 \end{bmatrix}}_{(j+2)\times(i+2)} \cdot \underbrace{\begin{bmatrix} s \\ 0 \\ p_1 \\ \vdots \\ p_i \end{bmatrix}}_{(i+2)\times 1} \leq \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ -1 \end{bmatrix}}_{(j+2)\times 1}$$

$$\underbrace{\begin{bmatrix} s \\ 0 \\ p_1 \\ \vdots \\ p_i \end{bmatrix}}_{(i+2)\times 1} \geq \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{(i+2)\times 1}$$

(c) Write down and interpret the dual of your linear program. The dual should give you both a value w^* and a strategy q^*.

Based on the last form of the primal that we have, we write the dual as follows:

$$\min \underbrace{\begin{bmatrix} q_1 & \dots & q_j & w & 0 \end{bmatrix}}_{1\times(j+2)} \cdot \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{bmatrix}}_{(j+2)\times 1}$$

$$\text{such that } \underbrace{\begin{bmatrix} q_1 & \dots & q_j & w & 0 \end{bmatrix}}_{1\times(j+2)} \cdot \underbrace{\begin{bmatrix} 1 & -1 & -m_{1j} & \dots & -m_{ij} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & -m_{11} & \dots & -m_{i1} \\ 0 & 0 & 1 & \dots & 1 \\ 0 & 0 & -1 & \dots & -1 \end{bmatrix}}_{(j+2)\times(i+2)} \geq \underbrace{\begin{bmatrix} 1 & -1 & 0 & \dots & 0 \end{bmatrix}}_{1\times(i+2)}$$

$$\underbrace{\begin{bmatrix} q_1 & \dots & q_j & w & 0 \end{bmatrix}}_{1\times(j+2)} \geq \underbrace{\begin{bmatrix} 0 & \dots & 0 & 0 & 0 \end{bmatrix}}_{1\times(j+2)}$$

Re-writting the dual in simpler form, it states the following:

$$\min w$$
$$\text{such that } \sum_j q_j \geq 1$$
$$\text{such that } -\sum_j q_j \geq -1$$
$$w - \sum_j q_j m_{ij} \geq 0, \forall i$$
$$q_j \geq 0, \forall j$$

Condensing the two initial inequality constraints into an equality constraint, we obtain:

$$\min_{q \in \mathbb{R}^j} w$$

$$\text{such that } \sum_j q_j = 1$$

$$w \geq \sum_j q_j m_{ij}, \forall i$$

$$q_j \geq 0, \forall j$$

Hence, we can see that for any concave maximization problem of the row player's security level subject to the row player's randomized strategy, there is an equivalent minimization problem, where the column player is looking to minimize w, i.e. the value to the row player, subject to their own randomized strategy.

(d) The fundamental theorem of two-person zero sum games, due to von Neumann in1927,

or the Minimax Theorem, says that for a zero-sum game played by two persons, where the payoff matrix is given by $M \in \mathbb{R}^{i \times j}$, the following must hold:

$$\max_{p \in \mathbb{R}^i_+ : \sum_i p_i = 1} \min_{q \in \mathbb{R}^j_+ : \sum_j p_j = 1} p^T M q = \min_{q \in \mathbb{R}^j_+ : \sum_j p_j = 1} \max_{p \in \mathbb{R}^i_+ : \sum_i p_i = 1} p^T M q$$

(e) Suppose we think of the (p^*, q^*) pair as a solution concept for this class of games. How is this solution related to Nash equilibrium?

Here, we begin by assuming a pair $(\hat{p}, \hat{q})$ that constitute a Nash Equilibrium solution. Hence, in such a case, each player's action is the best response to the other player's action, which implies that:

$$\max_{p \in \mathbb{R}^i_+ : \sum_i p_i = 1} p^T M \hat{q} = \hat{p}^T M \hat{q} = \max_{q \in \mathbb{R}^j_+ : \sum_j p_j = 1} (-\hat{p}^T M q) = \min_{q \in \mathbb{R}^j_+ : \sum_j p_j = 1} \hat{p}^T M q$$

Next, we consider a set of optimal strategies that are solutions to the linear programming problem, namely (p^*, q^*). From their property as solutions to the linear programming problem, we have:

$$p^{*T} M q^* \geq \min_{q \in \mathbb{R}^j_+ : \sum_j p_j = 1} p^{*T} M q = \max_{p \in \mathbb{R}^i_+ : \sum_i p_i = 1} \min_{q \in \mathbb{R}^j_+ : \sum_j p_j = 1} p^T M q =$$

$$= \min_{q \in \mathbb{R}^j_+ : \sum_j p_j = 1} \max_{p \in \mathbb{R}^i_+ : \sum_i p_i = 1} p^T M q = \max_{p \in \mathbb{R}^i_+ : \sum_i p_i = 1} p^T M q^* \geq p^{*T} M q^*$$

Therefore, we conclude that:

$$p^{*T} M q^* = \max_{p \in \mathbb{R}^i_+ : \sum_i p_i = 1} p^T M q^* = \min_{q \in \mathbb{R}^j_+ : \sum_j p_j = 1} p^{*T} M q$$

This yields the conclusion that (p^*, q^*) coincides with $(\hat{p}, \hat{q})$ and the former is, therefore, a Nash Equilibrium.

(f) Describe some properties of the solution set to a given game. Compare to Nash equilibrium in other kinds of finite games.

One property of the solution of this specific game was that the solution to the problem was in fact a Nash Equilibrium. However, it should be noted that this is only due to the fact that this is a zero-sum game. More specifically, the following step, used above to deduce the solution, was based on the property of zero-sum:

$$\hat{p}^T M \hat{q} = \max_{q \in \mathbb{R}^j_+ : \sum_j p_j = 1} (-\hat{p}^T M q)$$

This is due to the fact that the payoffs for each player are the same in absolute value and opposite in sign. This might not be the case on a non-zero-sum game. Hence, in other finite games such as non-zero-sum games, the optimal solution does not necessarily coincide with the Nash Equilibrium.