2. Consider the linear program

$$
\begin{aligned}
v_P(b) = {} & \max c_x x + c_y y \\
\text{subject to} \quad & x \le 1 \\
& x \ge 0
\end{aligned}
$$

(a) Rewrite this in the canonical form.

(b) Rewrite this in the standard form.

(a) Since y doesn't have the nonnegativity constraint, we can express y as the difference of two nonnegative variables.

$$V_P(b) = \text{MAX} \ (C_x \ \ C_y \ \ -C_y) \begin{bmatrix} x \\ y_1 \\ y_2 \end{bmatrix}$$

$$\text{S.T.} \quad [1 \ \ 0 \ \ 0] \begin{bmatrix} x \\ y_1 \\ y_2 \end{bmatrix} \le 1$$

$$\begin{bmatrix} x \\ y_1 \\ y_2 \end{bmatrix} \geq 0$$

(b) $V_P(b) = \text{MAX} \ C_x x + C_y y_1 - C_y y_2 + 0 \cdot S$

S.T. $x + S = 1$

$x, y_1, y_2, S \geq 0.$

$$\Rightarrow V_P(b) = \text{MAX} \ (C_x \ \ C_y \ \ -C_y \ \ 0) \begin{bmatrix} x \\ y_1 \\ y_2 \\ S \end{bmatrix}$$

$$\text{S.T.} \quad [1 \ \ 0 \ \ 0 \ \ 1] \begin{bmatrix} x \\ y_1 \\ y_2 \\ S \end{bmatrix} = 1$$

$$\begin{bmatrix} x \\ y_1 \\ y_2 \\ S \end{bmatrix} \geq 0$$

3. Consider the primal problem

$$\begin{aligned} v_p(b) = \max\ & x_1 + 2x_2 \\ \text{s.t. } & x_1 + x_2 \le 4 \\ & x_1 + 3x_2 \le b \end{aligned}$$

(a) Write down the dual.

(b) For $b = 1$, plot the constraint sets for both problems, and solve them.

(c) Describe $v_P(b)$, and compute $\partial v_P(b)$ on the range $0 \le b \le 14$.

(a) We want to first rewrite the problem with nonnegativity constraints.

$$V_P(b) = \text{MAX } \overbrace{(x_1 - x_2)}^{x_1} + 2\overbrace{(x_3 - x_4)}^{x_2}$$

Note: x_1, x_2 here are not the same as the ones in the problem.

$$\text{S.T. } (x_1 - x_2) + (x_3 - x_4) \le 4$$

$$(x_1 - x_2) + 3(x_3 - x_4) \le b$$

$$x_1, x_2, x_3, x_4 \ge 0.$$

$$\Rightarrow V_P(b) = \text{MAX } \underbrace{\begin{bmatrix} 1 & -1 & 2 & -2 \end{bmatrix}}_{c} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\text{S.T. } \underbrace{\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 3 & -3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \le \underbrace{\begin{bmatrix} 4 \\ b \end{bmatrix}}_{b.}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \geq 0$$

Hence, the dual problem is

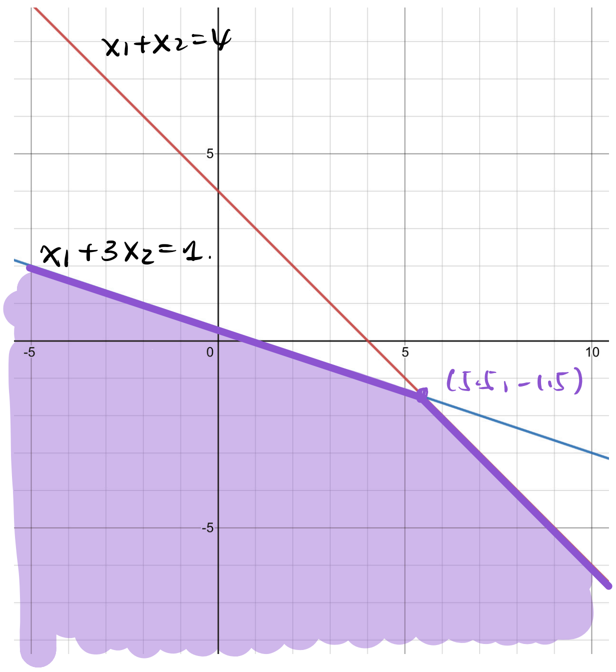
$$V_D(c) = \text{MIN } \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 4 \\ b \end{bmatrix}$$

$$\text{S.T. } \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 3 & -3 \end{bmatrix} \geq \begin{bmatrix} 1 & -1 & 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \geq 0.$$

(b) First consider the primal problem

$$v_p(b) = \max x_1 + 2x_2$$
$$\text{s.t. } x_1 + x_2 \leq 4$$
$$x_1 + 3x_2 \leq 1$$



The shaded region is the constraint set and by vertex theorem, the solution is $(5.5, -1.5)$, which gives $V_P(b) = 2.5$

Next consider the dual problem:

MIN $4y_1 + y_2$

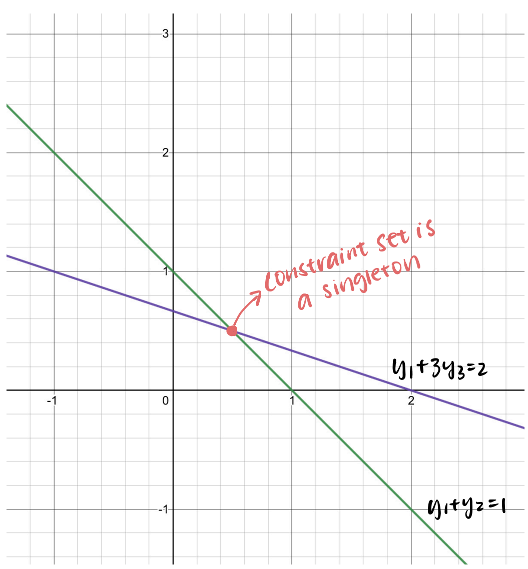
s.t. $y_1 + y_2 \geq 1$ $\Rightarrow y_1 + y_2 = 1$

$-y_1 - y_2 \geq -1$

$y_1 + 3y_2 \geq 2$ $\Rightarrow y_1 + 3y_3 = 2$

$-y_1 - 3y_2 \geq -2$

$y_1, y_2 \geq 0$



Since the constraint set is a singleton, the solution to the dual problem is $(0.5, 0.5)$, which gives $V_D(b) = 2.5$.

$\Rightarrow V_D(c) = V_P(b)$

(c)

Shadow Prices

Theorem. If either $v_P(b)$ or $v_D(c)$ is finite, then

1. $v_P(b) = v_D(c)$,
2. both programs have optimal solutions, and
3. $\partial v_D(b)$ is the set of optimal solutions to the primal, and $\partial v_P(b)$ is the set of optimal solutions to the dual.

By computation above, we know that $V_P(b)$ and $V_D(c)$ are both finite when $b \in [0,14]$ and $V_P(b) = V_D(c)$. The solution for the dual problem is $(0.5, 0.5)$.

Hence, $V_P(b) = V_D(c) = 4(0.5) + 0.5b = 2 + 0.5b$.

By the shadow price theorem, $\partial V_P\left(\begin{bmatrix} 4 \\ b \end{bmatrix}\right) = (0.5, 0.5)$ for $b \in [0, 14]$

and $V_P(b) = 0.5$ for $b \in [0, 14]$

You could also solve the primal for an arbitrary b, and then obtain $\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} \frac{12-b}{2} \\ \frac{b-4}{2} \end{bmatrix}$

and plug it back into

$V_P(b) = x_1^* + 2x_2^* = 2 + \frac{b}{2}$

$\frac{d V_P(b)}{d b} = \frac{1}{2}$