Logistics

- TA Office Hours:
  - Mon & Wed 18:30-19:30pm (by Zoom).
- Section's material https://github.com/luciacasal/-Cornell_ECON6100.

1 General Equilibrium Under Uncertainty [1]

1.1 Arrow-Debreu Equilibrium

The set up of the exchange economy we have seen is the following. There are L goods and N individuals. Consumption of individual i is denoted by $x^i \in \mathbb{R}^L_+$. Endowment is denoted by $\omega^i \in \mathbb{R}^L_+$. Each individual has a preference $\succeq_i$ over the goods represented by $u^i : \mathbb{R}^L_+ \longrightarrow \mathbb{R}$. Equilibrium happens when all individuals solve their UMP's and markets clear.

We now introduce uncertainty into the model. This can be done by having S states of nature, only one of which will occur. We start with the Arrow-Debreu model. The key to Arrow-Debreu is that we can think of good l in state s as a different good from good l in state s'. This is the notion of *state-contingent commodities*. All decisions in the model happen before the state of the world is realized, so formally, the model can be thought of as having two time periods $t = 0$, where individuals solve their UMP's, and $t = 1$, where the state is realized and trade happens.

With this set up, individuals are not thinking about how much of the L goods to consume, but rather how much of the $L + LS = L(S + 1)$ contingent commodities to consume.[2] The consumption and endowment vectors for individual i are denoted by $x^i \in \mathbb{R}^{L(S+1)}_+$ and $\omega^i \in \mathbb{R}^{L(S+1)}_+$. Prices are denoted by $p \in \mathbb{R}^{L(S+1)}_{++}$.

What about utility? Individuals still have utility function u^i on the L physical goods as before, but they have subjective beliefs on which state will occur π^i on S. However, they may deal with uncertainty in period 1 in different ways. That is, each i takes u^i and π^i , and marry them in some way. For example if i is an expected utility maximizer, then i's utility function over the contingent commodities is: $V_i = u^i(x^i_{10}, ..., x^i_{L0}) + \sum_s \pi^i u^i(x^i_{1s}, ...x^i_{Ls})$. The individual may be pessimistic and only value future consumption in worst state: $min_s u^i(x^i_{1s}, ...x^i_{Ls})$. We can also have the case where i has multiple π^i's in $\prod^i$ and i is a maximin expected utility maximizer over period 1 contingent commodities (Gilboa-Schmeidler): $min_{\pi^i \in \prod^i} \sum_s \pi^i_s u^i(x^i_{1s}, ...x^i_{Ls})$.

[1] Borrowed from Abhi Ananth notes from 2021 and Jaden Chen's notes from 2020.

[2] Some books, for instance MWG, do not allow individuals to consume in period 0, though this is usually not explicitly mentioned. This is a special case of our set up with everyone having zero endowment in period 0.

To summarize, the ingredients of the Arrow-Debreu model are:

- L physical goods, S states of the world, N individuals.
- $x^i \in \mathbb{R}_+^{L(S+1)}$, $\omega^i \in \mathbb{R}_+^{L(S+1)}$,$p \in \mathbb{R}_{++}^{L(S+1)}$.
- $u^i : \mathbb{R}_+^L \to \mathbb{R}$ and $\prod^i = \left\{\pi^i = (\pi_1^i, ..., \pi_S^i) \gg 0, \sum_s \pi_s^i = 1\right\}$, and some way of combining them.

Definition 1: [Arrow-Debreu Equilibrium] An Arrow-Debreu equilibrium denotes an allocation $(x^{i*})_{i=1}^N \in \mathbb{R}_+^{L(S+1)N}$ and a price vector $p^* \in \mathbb{R}_{++}^{L(S+1)}$ such that:

- (Utility Maximization): all i choose x to max $V^i(x)$ subject to $x \in \mathcal{B}_{AD}^i(p, \omega^i)$, with

$$\mathcal{B}_{AD}^i(p, \omega^i) = \left\{x \in \mathbb{R}_+^{L(S+1)} : p \cdot x \leq p \cdot \omega^i\right\}.$$

- (Market clearing): $\sum_i x_i^* = \sum_i \omega^i$.

All trade takes place in period 0, before the state is realized, so the equilibrium is determined in period 0. Welfare theorems in Walrasian equilibrium can be applied here without modification.

1.2 Radner Equilibrium

The idea of a Radner economy, or a financial economy as it is often called, is that in addition to physical goods and states, you also have tradable financial assets. Suppose there are J financial assets with asset j paying out $a_{sj} \geq 0$ in state s. We can write the $S \times J$ asset return matrix A as

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1J} \\ \vdots & \ddots & \vdots \\ a_{S1} & \cdots & a_{SJ} \end{bmatrix}$$

Let $z^i \in \mathbb{R}^J$ denotes the portfolio of individual i, i.e., i holds z_j^i units of asset j (note that this may be negative), then the payout in state s is $\left(Az^i\right)_s$, the s-th row of Az^i . We say that the market is complete if A has full row rank, i.e., $rank(A) = S$. For example, for two assets and two states we might have

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

Let $z^i = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$,then $Az^i = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$. If state 1 occurs, i gets return 5, while if state 2 occurs, i gets return 0. The market is complete in this case.

We denote the prices of assets by r and to distinguish from the Arrow-Debreu prices, denote the prices of physical state-contingent commodities by q. To summarize, the ingredients of the Radner (financial economy) model are:

- L physical goods, S states of the world, N individuals.
- $x^i \in \mathbb{R}_+^{L(S+1)}, \omega^i \in \mathbb{R}_+^{L(S+1)}, q \in \mathbb{R}_{++}^{L(S+1)}$.
- $z^i \in \mathbb{R}^J$, $A_{S\times J}$ with $a_{sj} \geq 0$, and $r \in \mathbb{R}_{++}^J$.
- $u^i : \mathbb{R}_+^L \to \mathbb{R}$ and $\prod^i = \left\{\pi^i = (\pi_1^i, ..., \pi_S^i) \gg 0, \sum_s \pi_s^i = 1\right\}$, and some way of combining them.

In addition to assets, the timing of trade is the key feature of the Radner model. There are still two time periods. In period 0, physical commodities and assets are traded. The budget constraint in period 0 is

$$q_0 \cdot x_0^i + r \cdot z^i \leq q_0 \cdot \omega_0^i.$$

In period 1, which depends on the realized state, you can trade physical commodities. Suppose state s is realized, individual i's income in period 1 comes from selling i's endowment in state s plus the return from portfolio z^i. The budget constraint in state s is

$$q_s \cdot x_s^i \leq q_s \cdot \omega_s^i + (Az^i)_s.$$

As we have defined, the return of the asset is in monetary unit, for this to make sense in each market we need to have a numeraire good. The convention is to take the first good in each state as the numeraire. In other words, we take $q_{s1} = 1$.

Definition 3: [Radner Equilibrium] A Radner equilibrium is defined by (x^*, z^*, q^*, r^*) such that

- (Utility maximization): all i choose x, z to max $V^i(x)$ subject to $x \in \mathcal{B}_R^i(q, r, \omega^i)$, with

$$\mathcal{B}_R^i(q, r, \omega^i) = \left\{(x, z) \in \mathbb{R}_+^{L(S+1)} \times \mathbb{R}^J : q_0 \cdot x_0^i + r \cdot z^i \leq q_0 \cdot \omega_0^i \text{ and } q_s \cdot x_s^i \leq q_s \cdot \omega_s^i + (Az^i)_s, \forall s \in S\right\}.$$

- (Goods market clearing): $\sum_i x_i^* = \sum_i \omega^i$.
- (Financial market clearing): $\sum_i z^{i*} = 0$.

1.3 Questions

1. Suppose consumers are trading contingent claims on a single commodity, say sheep. States are in $S = a, b$. The aggregate endowment of sheep is independent of the state, but each individual's endowment of sheep is state-dependent. Suppose that individual 1 has preference of Gilboa-Schmeidler form. Individual 1 has a closed set of probability distributions P, and for any contract $c \in \mathbb{R}_+^{|S|}$,

$$V_1(c) = min_{p \in P} \sum_s u_1(c(s))p(s)$$

where Individual 2 is expected utility maximizer with Bernoulli utility function $u_2(.)$ and belief q. Assume that $u_1(.)$ and $u_2(.)$ are strictly concave, differentiable, strictly increasing and satisfying Inada conditions.

   (a) Let P contain two probability distributions $p_1 = (\frac{1}{4}, \frac{3}{4})$ and $p_2 = (\frac{3}{4}, \frac{1}{4})$, and let $q = (\frac{1}{2}, \frac{1}{2})$. Draw an indifference curve for individual 1 and individual 2.

   (b) Describe the set of Pareto optimal allocations.

   (c) If (p, x_1, x_2) is a competitive equilibrium, what can you say about p?

2. Consider an exchange economy with two consumers. In period 0 consumers consume nothing and trade only financial assets. In period 1 there are two possible states, H and T. A single good is available in each state. The aggregate endowment is 3 in each state. Consumer A is endowed with 2 units in state H and 1 unit in state T. Consumer B is endowed with the rest. Both consumers believe each state is equally likely. Utility over second period consumption for A and B are

$$u^A(x_H, x_T) = \frac{1}{2}v^A(x_H) + \frac{1}{2}v^A(x_T)$$
$$u^B(x_H, x_T) = \frac{1}{2}v^B(x_H) + \frac{1}{2}v^B(x_T)$$

The function v^A and v^B are strictly concave, increasing, differentiable and satisfy Inada conditions. There are two assets available for trading in period 0. The first asset pays 1 unit in each state, the second asset pays 1 unit in H and $a \neq 1$ unit of good in state T.

   (a) For all $a \neq 1$and $a > 0$, compute the Radner equilibrium.

   (b) what happens to equilibrium if $a = 1$?

ECONOMIC GROWTH
From Prof. Caunedo Assignment for ECON6140:

Assignment 2

Macro PhD Core

Spring 2021

The due date for this assignment is Friday February 26.

Feel free to make additional assumptions that you think are critical to solve a problem. These assumptions need to be clearly stated in your solution.

Capital Utilization

Consider an economy in which each individual ranks consumption sequences according to the functional

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{with } \beta \in (0,1)$$

Assume that u is strictly concave, increasing in both arguments, and twice differentiable. This economy imports investment goods. Assume that the international price of the investment good is $p_k \geq 1$.Thus, the feasibility constraint faced by the planner is

$$c_t + p_{kt}x_{kt} \leq f(k_t^e) \qquad k_0^e > 0$$

where k_t^e denotes effective units of capital per worker (to be defined). The planner chooses that intensity of use of the capital stocks. Let intensity of use be denoted υ_t, where $\upsilon_t \in (0,1)$ all t. If the economy has available k units of capital and it is used at intensity υ_t, the effective supply of capital is $k_t^e = \upsilon_t k_t$. In this case, the depreciation rate is $\delta(\upsilon_t) = \upsilon_t^{1+\lambda}$, where $\lambda > 0$, and $\upsilon_t \in (0,1)$ [1].

In this economy there are installation costs. If x units of investment goods are allocated to the production of new capital, they produce $G(x_t, a)$ units of new capital goods. Thus, the aggregate law of motion of capital is

$$k_{t+1} \leq (1 - \delta_k(\upsilon_t))k_t + G(x_t, a)$$

[1] In general you could choose the timing for the selection of the intensity of use that you would like. That is, you may want to choose it in period $t-1$ jointly with the stock of capital for next period. Or you could choose it in the current period t. Whatever way you choose to solve for it should be fine in an economy without uncertainty like this one.

where a is a factor in fixed supply. The function G is assumed increasing in each argument, differentiable and concave. One interpretation of G is that it represents installation costs that are necessary to make capital goods productive.
Assume that the production function is given by

$$f(k_t^e) = (k_t^e)^\alpha \quad \text{with } \alpha \in (0,1)$$

Assume that the economy is at the steady state.

1. Assume that $G(x,a) = x$, and that λ is sufficiently high so that the solution is interior. Argue that the steady state value of capital (and you can also show it for others like consumption and utilization) is independent of the price of investment goods.

2. Let $G(x,a) = x^{1-\theta}a^\theta$, with $0 < \theta < 1$. Go as far as you can describing how the price of investment goods (p_k) affects capital utilization. Describe how changes in this price affect the capital-output ratio and the investment outputratio. Be explicit about the assumptions that guarantee an interior solution.

3. Let $G(x,a) = x^{1-\theta}a^\theta$, with $0 < \theta < 1$. Given that the economy is at the steady state, is it possible to describe the dynamics associated with a once and for all permanent increase in the international price of the investment good? If it is possible, describe the dynamics. If it is not, explain why.

Problems

1. Suppose consumers are trading contingent claims on a single commodity, say sheep. States are in $S = \{a, b\}$. The aggregate endowment of sheep is independent of the state, but each individual's endowment of sheep is state-dependent. Suppose that individual 1 has preference of Gilboa-Schmeidler form. Individual 1 has a closed set of probability distributions P, and for any contract $c \in \mathbb{R}_+^{|S|}$,

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(b) Describe the set of Pareto optimal allocations.

(c) If (p, x_1, x_2) is a competitive equilibrium, what can you say about p?

- $S = \{a, b\}$ • $e_a = e_b$ (aggregate)
- $V_1(c) = \min \{ \mathbb{E}_p\, U_1(c) : p \in P \}$
- $V_2(c) = \mathbb{E}_q\, U_2(c)$

a) $P = \{p_1, p_2\}$ where $p_1 = \left(\frac{1}{4}, \frac{3}{4}\right)$ $p_2 = \left(\frac{3}{4}, \frac{1}{4}\right)$

$q = \left(\frac{1}{2}, \frac{1}{2}\right)$

$V_1(c) = \min \{ \mathbb{E}_{p_1} U_1(c) ; \mathbb{E}_{p_2} U_2(c) \} =$

$= \min \left\{ \frac{1}{4} U_1(c_a) + \frac{3}{4} U_1(c_b) ; \frac{3}{4} U_1(c_a) + \frac{1}{4} U_1(c_b) \right\}$

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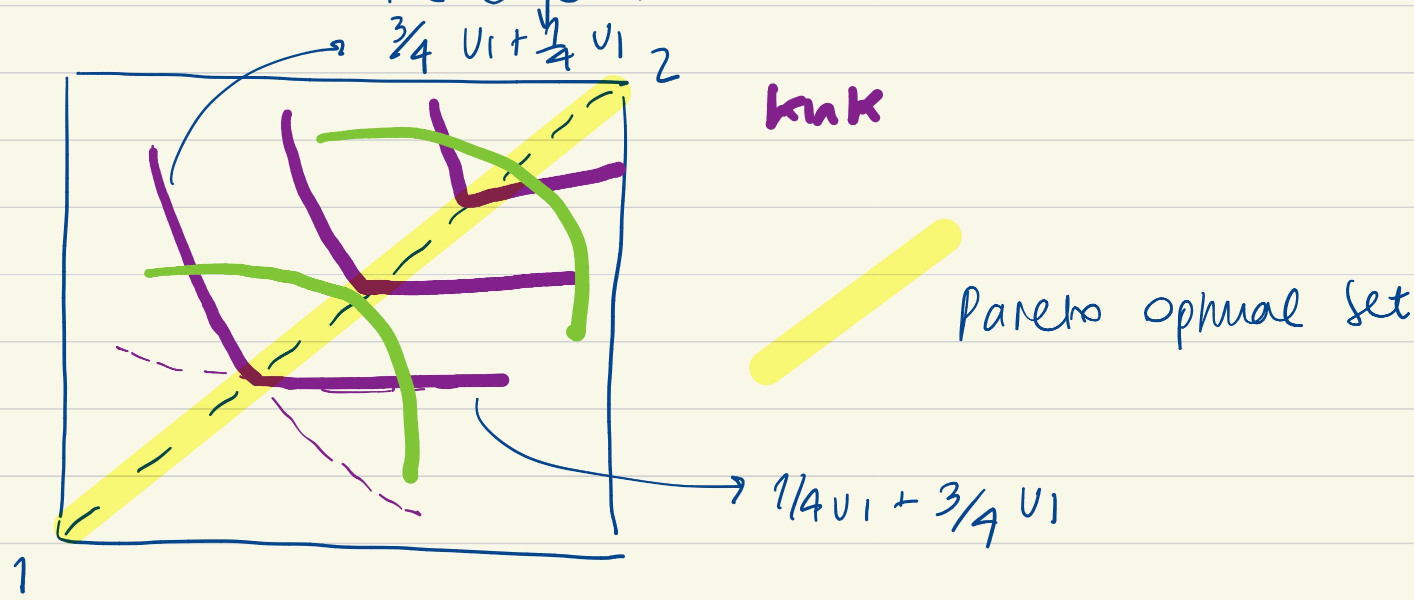
$$V_1(c) = \begin{cases} \frac{1}{4} u_1(ca) + \frac{3}{4} u_1(cb) & \text{if } cb < ca \\ \frac{3}{4} u_1(ca) + \frac{1}{4} u_1(cb) & \text{if } cb \geq ca \end{cases}$$

$$V_2(c) = \frac{1}{2} u_2(ca) + \frac{1}{2} u_2(cb)$$

Indifference curves in the edgeworth box



c) All allocations are PO (1WT). Bernoulli UF. nice property that you won't have corner sol.

$p^* = (1,1)$.

we can solve UMP and use MK

2. Consider an exchange economy with two consumers. In period 0 consumers consume nothing and trade only financial assets. In period 1 there are two possible states, H and T. A single good is available in each state. The aggregate endowment is 3 in each state. Consumer A is endowed with 2 units in state H and 1 unit in state T. Consumer B is endowed with the rest. Both consumers believe each state is equally likely. Utility over second period consumption for A and B are

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(a) For all $a \neq 1$ and $a > 0$, compute the Radner equilibrium.

(b) what happens to equilibrium if $a = 1$?

2 consumers: A and B.

A single good.

2 states: H and T

$e^A = (e^A_H, e^A_T) = (2, 1)$ and $e^B = (1, 2)$

- $u^i(x_H, x_T) = \frac{1}{2}v^i(x_H) + \frac{1}{2}v^i(x_T)$

- two assets

$$A = \begin{array}{cc} \text{asset 1} & \text{asset 2} \end{array}$$
$$\begin{array}{c} \\ \end{array} A = \begin{bmatrix} 1 & 1 \\ 1 & a \end{bmatrix} \begin{array}{c} H \\ T \end{array}$$

a) Compute Radner equilibrium.

A :

$$\max_{x, z} \quad \frac{1}{2}v^A(x_H^A) + \frac{1}{2}v^A(x_T^A)$$

s.t.
$$q_1 . z_1^A + q_2 z_2^A \leq 0$$

If they are endowing too put that here

$$x_H^A \leq 2 + z_1^A + z_2^A$$

$$\begin{cases} x_T^A \leq 1 + z_1^A + a \cdot z_2^A \\ x_H^A ; x_T^A \geq 0 \end{cases} \quad \text{(non-negativity)!}$$

Simplify:

$$z_2^A = -\frac{q_1}{q_2} z_1^A$$

$$\therefore \begin{cases} x_H^A = 2 + \dfrac{q_2 - q_1}{q_2} z_1^A \\ x_T^A = 1 + \dfrac{q_2 - a q_1}{q_2} z_1^A \end{cases}$$

$$\text{FOC}: \quad \frac{1}{2} v^{A\prime}(x_H^A) \frac{q_2 - q_1}{q_2} + \frac{1}{2} v^{A\prime}(x_T^A) \frac{q_2 - a q_1}{q_2} = 0$$

$$\therefore \frac{v^{A\prime}(x_H^A)}{v^{A\prime}(x_T^A)} = -\frac{q_2 - a q_1}{q_2 - q_1}$$

(B) symmetric

$$\begin{cases} x_H^B = 1 + \dfrac{q_2 - q_1}{q_2} z_1^B \\ x_T^B = 2 + \dfrac{q_2 - a q_1}{q_2} z_1^B \end{cases}$$

$MRS_1 = MRS_2$

$$\Longrightarrow \boxed{\frac{v^{B\prime}(x_H^B)}{v^{B\prime}(x_T^B)} = -\frac{q_2 - a q_1}{q_2 - q_1} = \frac{v^{A\prime}(x_H^A)}{v^{A\prime}(x_T^A)}}$$

η

If $\eta > 1 \rightarrow \dfrac{v''(x^i_H)}{v''(x^i_T)} > 1$

what do we know x^i_H and x^i_T we have a $\overset{\text{str.}}{\text{concave}}$

ut function $\rightarrow$ $\underbrace{x^i_H < x^i_T}_{\text{str. decreasing}}$ for both $i \in \{A, B\}$

$$\overbrace{x^A_H + x^B_H}^{e_H = 3} < \overbrace{x^A_T + x^B_T}^{e_T = 3} \quad \text{by MC}$$

contradiction because we assumed $e^H = e^T$ [mc] no aggregate uncertainty.

$$\therefore \quad -q_2 + a q_1 = q_2 - q_1$$

$$\therefore \quad \boxed{q_1 = \frac{2}{a+1} q_2} \qquad \therefore \boxed{MRS = 1}$$

$$z^A_2 = -\frac{q_1}{q_2} z^A_1 = -\frac{2}{a+1} z^A_1$$

$$x^H_A = x^A_T \quad \therefore \quad 2 + \left(1 - \overset{\frac{2}{a+1}}{\frac{q_1}{q_2}}\right) z^A_1 = 1 + \left(1 - \overset{\frac{2}{a+1}}{\frac{a q_1}{q_2}}\right) z^A_1$$

$$\Longrightarrow \quad 1 = \frac{2(1-a)}{a+1} z^A_1 \quad \text{or} \quad z^A_1 = \frac{a+1}{2-2a} \; ; \; z^A_2 = \frac{1}{a-1}$$

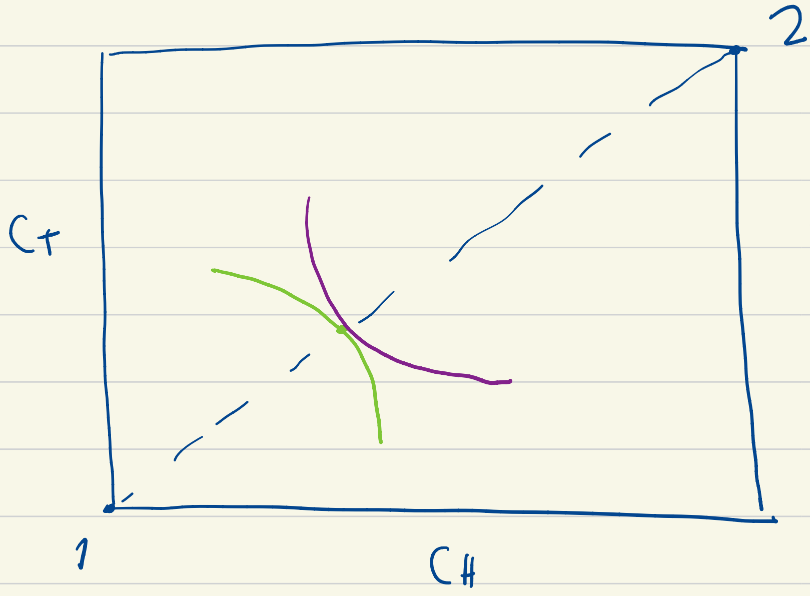
$x^A_H = x^A_T = 3/2$ similarly by mkt clearing:

$z_1^B = -z_1^A$ and $z^B_2 = -z_2^A$

$\underbrace{\qquad\qquad}$

no agg endowment in the mkt

So $x^B_H = x^B_T = 3/2$



By buying the asset all covers hedge the risk perfectly: and allocation is Pareto optimal. (1 WT implied).

on A: rank(A) = |S|

what happens if rank(A) ≠ S?

b) Suppose a = 1 Payoff Matrix becomes:

$$A = \begin{array}{cc} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ \begin{array}{cc} q_1 & q_2 \end{array} \end{array} \qquad \text{rank}(A) = 1 < |S| = 2$$

We will see if the Market Eq will give us a PO allocation!

What do we know about q_1 and q_2? they have to be equal if not the person can do ∞ wealth because buying q_1 or q_2

$q = q_1 = q_2$ ∵ no arbitrage

For (A) : $\max \ U_1$

$$\text{st} \quad q z_1^A + q z_2^A \leq 0$$

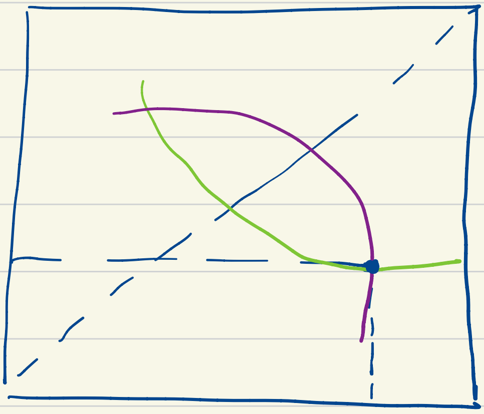
$$x_H^A \leq 2 + z_1^A + z_2^A \qquad (LNS) =$$
$$x_T^A \leq 1 + z_1^A + z_2^A$$

$$\downarrow$$

$$\text{st} \begin{cases} x_H^A = 2 + z_1^A + z_2^A \\ x_T^A = 1 + z_1^A + z_2^A \\ q z_1^A + q z_2^A = 0 \end{cases} \Longrightarrow z_1^A + z_2^A = 0$$

We must have $x_H^A = 2$; $x_T^A = 1$ autarchy corner 1 will only come by endowment

Similarly $x_H^B = 1$; $x_T^B = 2$



what could it not different because the asset market is incomplete!!