# 1 Problem 1

*Use the strong separating hyper-plane theorem to show that if $C$ is a closed convex set, then $C$ is the intersection of the half-spaces containing it.*

Here, we need to show that the closed convex set, $C$, is the intersection of the half spaces containing it. In order to do that, it suffices to prove two things, (a) that $C$ is a subset of the intersection of all half-spaces containing it, i.e. that $C \subseteq \bigcap\{s : s \in S\}$, where $S$ is the set of all half-spaces containing $C$ and (b) that the intersection of all half-spaces is a subset of set $C$, i.e. that $\bigcap\{s : s \in S\} \subseteq C$.

(a) By definition of the set $S$ each end every element $s_k$ of set $S$ is a half-space containing $C$. Hence, if we choose a point $x \in C$, then, by definition, we know that $x \in s_n$ and, by extension it is also contained in the intersection of all $s_k$'s, i.e. $x \in \bigcap\{s : s \in S\}$. Therefore, we conclude that $C \subseteq \bigcap\{s : s \in S\}$

(b) We aim to prove the backward direction through the use of a contra-positive and the Strong Separating Hyper-plane Theorem. We begin by picking a point $x$ such that $x \notin C$. We notice that the point $x$ itself constitutes a closed and convex set $\{x\}$ and so is set C, by definition. Therefore, we know that there is a separating hyper-plane, $\{p = a\}$ that strictly separates the two (disjoint) sets. This separating hyper-plane defines two half-spaces, one, $s$, containing set $C$ and one, $h$, containing set $\{x\}$. Further, we note that half-space $s$ is an element of set $S$, i.e. of all the half-spaces containing set $C$. Therefore, if $x \in h$, then it follows that $x \notin s$ and, subsequently, it is not in the set of all dub-spaces containing $C$ either, i.e. $x \notin S$. Finally, it follows that $x$ is not contained in the intersection of all the half-spaces containing $C$, i.e. $x \notin \bigcap\{s : s \in S\}$. Therefore, by use of a contra-positive, we have proved that $\bigcap\{s : s \in S\} \subseteq C$.

Combining proofs (a) and (b) above, we conclude that $C = \bigcap\{s : s \in S\}$, which is the desired conclusion, i.e. that the closed and convex set $C$, it is equal to the intersection of all half-spaces containing it.

## 2 Problem 2

*For $p \in \mathbb{R}^n$, define $e_C(p) = \inf\{p \cdot x, x \in C\}$. $e_C(p)$ is called the concave support function of $C$.*

Here, we consider that $e(p)$ below is essentially the same as $e_C(p)$.

*(a) Show that $e(p)$ is concave.*

We begin with the definition of $e_C(p) = \inf\{p \cdot x, x \in C\}$. We pick two prices $q$ and $z$ and we consider the expenditure function of a convex combination of those two prices, as follows for $\alpha \in [0, 1]$:

$$e(\alpha q + (1-\alpha)z) = \inf\{(\alpha q + (1-\alpha)z) \cdot x\}$$

Manipulating the above expression, we obtain:

$$e(\alpha q + (1-\alpha)z) = \inf\{\alpha q \cdot x + (1-\alpha)z \cdot x\}$$

Now, by the triangle inequality, we have:

$$\inf\{\alpha q \cdot x + (1-\alpha)z \cdot x\} \geq \inf\{\alpha q \cdot x\} + \inf\{(1-\alpha)z \cdot x\}$$

To see that the above, in fact, holds it is easier to consider that the infimum is, in fact, attained for some value of $\hat{x}$ and remove the infimum function, however the result is essentially the same. The right-hand side can be re-written as follows:

$$\inf\{\alpha q \cdot x\} + \inf\{(1-\alpha)z \cdot x\} = \alpha \inf\{q \cdot x\} + (1-\alpha)\inf\{z \cdot x\}$$

Hence, we conclude that:

$$e(\alpha q + (1-\alpha)z) \geq \alpha e(q) + (1-\alpha)e(z)$$

which is the definition of concavity for a function. Therefore, we have proved that $e(p)$ is concave in $p$.

*(b) Show that $e(p)$ is homogeneous of degree 1.*

To show this we begin by taking a constant $\alpha$ and manipulating the expression $e(p)$, as follows:

$$e(\alpha p) = \inf\{\alpha p \cdot x, x \in C\} = \alpha \inf\{p \cdot x, x \in C\} = \alpha e(p)$$

proving that $e(p)$ is, in fact, homogeneous of degree 1.

*(c) What does it mean if $e(p) = -\infty$? You can explain with a picture.*

On the figure below there is a representation of multiple expenditure functions for different $x$'s, $x \in C$. As the numbers of expenditure functions we draw grows larger, the support function begins to resemble a logarithmic function (shown in green):

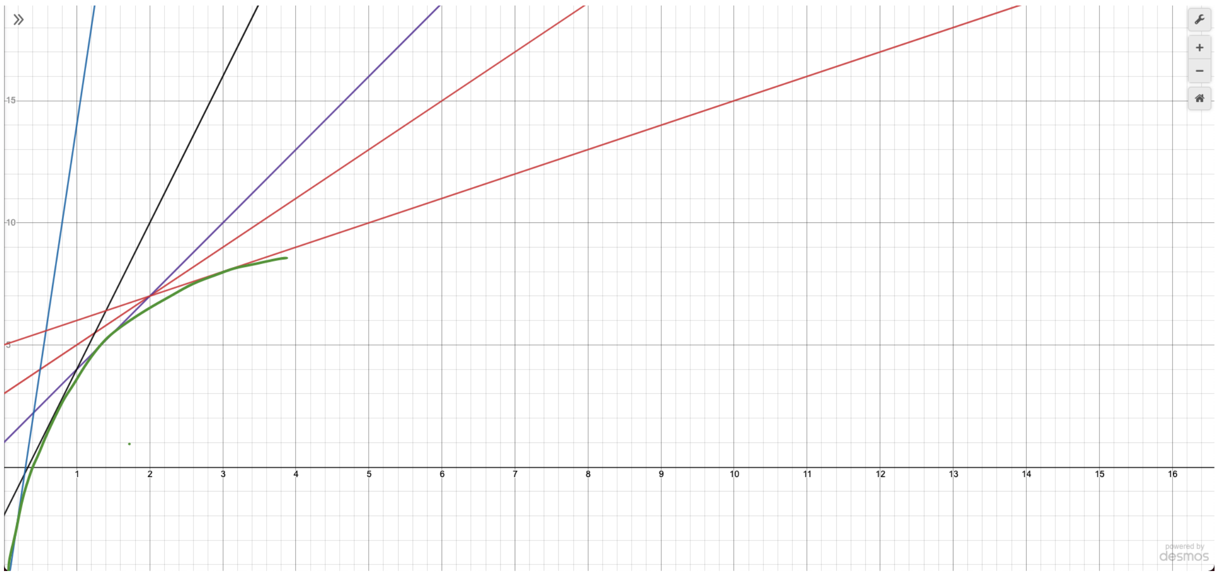


Figure 1: Graphical Representation of Support Function of $e(p)$ (Using Desmos)

Therefore, in the case that the value of the support function of the expenditure function becomes $-\infty$ it essentially means that there are elements of $C$ which yield a value for $p \cdot x$ which is negative without a bound from below, as $p$ goes to zero.

*(d) Show that $[p \geq \alpha]$ ($[-p \leq -\alpha]$) contains $C$ iff $\alpha \leq e(p)$.*

Here, we begin by noting the following equivalency: $[p \geq \alpha] = \{x : p \cdot x \geq \alpha\}$. Next, we note that the above is an if and only if statement, therefore we need to prove the forward direction (a) that if $C \subseteq \{x : p \cdot x \geq \alpha\}$, then we have $\alpha \leq e(p)$ and (b) if $\alpha \leq e(p)$ then we have $C \subseteq \{x : p \cdot x \geq \alpha\}$.

We begin with the forward direction, taking as given that $C \subseteq \{x : p \cdot x \geq \alpha\}$. Subsequently, we note that $\forall x \in C$, we have $p \cdot x \geq \alpha$. Then, if this is true for all $x$, it is also true for its minimum value and we conclude that:

$$\inf\{p \cdot x, x \in C\} \geq \alpha \Rightarrow e(p) \geq \alpha$$

which proves the forward direction.

We now proceed with the backward direction, taking as given that $\alpha \leq e(p)$. If there is a unique value $\hat{x}$ that minimizes the function, we know that for all other values of $x' \in C$, we have:

$$\alpha \leq \inf\{p \cdot x, x \in C\} = p \cdot \hat{x} \leq p \cdot x'$$

Hence, we conclude that $\forall x \in C$, we have that $p \cdot x \geq \alpha$. Therefore, it follows that:

$$C \subseteq \{x : p \cdot x \geq \alpha\} = [p \geq \alpha]$$

which concludes the backward part of the proof. The combination of the above two proofs concludes the proof of the if and only if statement of the question.

# 3 Problem 3

*Show that a function $f : \mathbb{R}^n \to \mathbb{R}$ is concave iff $\{(x, y) \in \mathbb{R}^n \times \mathbb{R} : y \leq f(x)\}$ is convex. This set is called the sub-graph or hypo-graph of f.*

This is an if an only if statement, hence we need to prove both the forward and the backward direction. The forward direction (a) entails proving that if $f(x)$ is a concave function then the sub-graph is convex, while the backward direction (b) entails proving that if the sub-graph is convex, then $f(x)$ is concave.

(a) Beginning with the forward direciton, we take as given that the function $f(x)$ is a concave function. By applying the definition we can write down the following expression for $\alpha \in [0, 1]$:

$$\alpha f(x_1) + (1 - \alpha) f(x_2) \leq f(\alpha x_1 + (1 - \alpha) x_2)$$

Now, we pick two points in the sub-graph, namely $(x_1, A)$ and $(x_2, B)$. By the definition of the sub-graph, we have that:

$$A \leq f(x_1)$$

$$B \leq f(x_2)$$

Now, multiplying the first by $\alpha$ and the latter by $(1 - \alpha)$ and adding the two, we obtain:

$$A\alpha + B(1 - \alpha) \leq \alpha f(x_1) + (1 - \alpha) f(x_2) \leq f(\alpha x_1 + (1 - \alpha) x_2)$$

Hence, we have managed to prove that after picking two points in the graph, their convex combination is still below the line of the function, i.e. within the sub-graph, concluding that the set is, indeed, convex.

(b) For the backward direction, we need to prove, given that the sub-graph is a convex set, that the function is, in fact, concave. Towards this end, we pick two points in the sub-graph, namely $(x, f(x))$ and $(y, f(y))$. Given that we know the sub-graph is a convex set, any convex combination of the two points we picked, is going to be contained in the set as well. Therefore, the following convex combination will also be in the sub-graph:

$$(\alpha x + (1 - \alpha) y, \alpha f(x) + (1 - \alpha) f(y))$$

Now, by definition of sub-graph, we know that for any point contained in the sub-graph the following must be true: the y-coordinate must be less than or equal to the function of the x-coordinate, or $y \leq f(x)$. For the point that we obtained above, this is expressed as follows:

$$\alpha f(x) + (1-\alpha)f(y) \leq f(\alpha x + (1-\alpha)y)$$

However, this is the very definition of concavity. Hence, we have proved that for a convex sub-graph, the corresponding function ought to be concave.

Therefore, we have proved both the forward and the backward directions, essentially confirming that a function $f$ is concave if and only if its sub-graph is convex.

# 4 Problem 4

*Give an example of two closed convex sets that cannot be strongly separated.*

The two, closed, convex, sets we could use to demonstrate the above are as follows:

$$A = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y \leq 0, x \in \mathbb{R}\}$$

and

$$B = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y \geq \frac{1}{x}, x > 0\}$$

The two sets are both convex (as is evident by the schematic, and closed as they contain all their endpoints. These two sets are graphically demonstrated as follows:

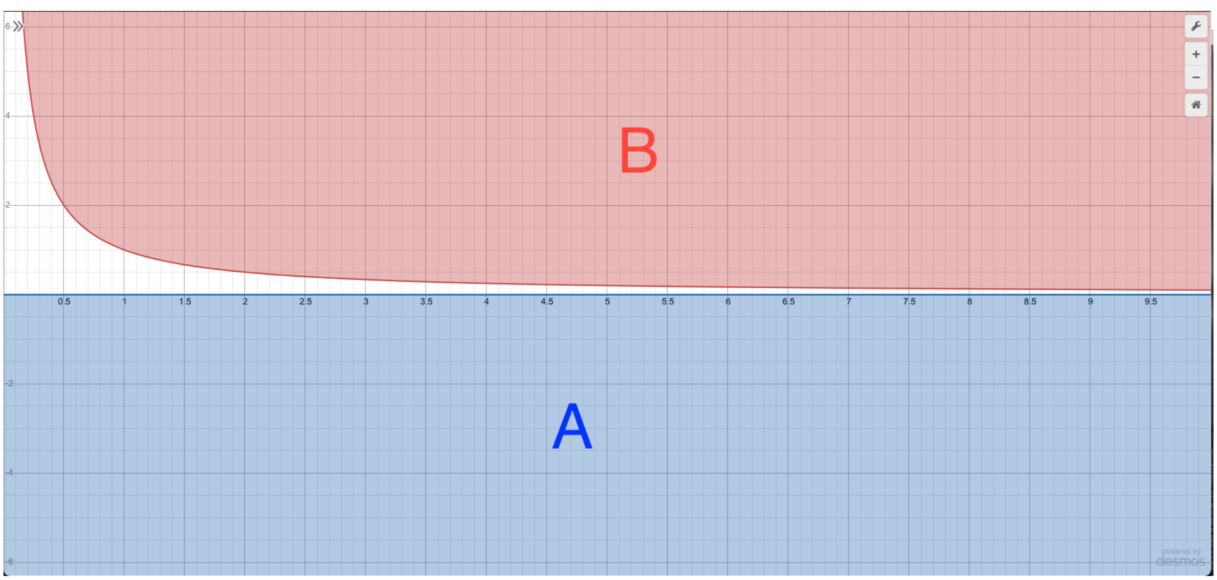


Figure 2: Graphical Representation of Sets A and B (Using Desmos)

Given that the line $y = 0$ is included in set $A$, there is no hyper-plane separating the two sets.

# 5 Problem 5

*Prove Gordan's Lemma: Either $\boldsymbol{Ax} = \boldsymbol{0}$, $\boldsymbol{x} > \boldsymbol{0}$ has a solution or $\boldsymbol{y'A} >> \boldsymbol{0}$ has a solution. Hint: This follows from Farkas' lemma.*

In order to solve this problem we will be utilizing Farkas' Lemma, beginning by bringing the second part of Gordan's Lemma in the form of Farkas' Lemma.

According to Farkas' Lemma, exactly one of the two statements have a solution: either (a) $\mathbf{Ax{=}b}$, $\mathbf{x} \geq \mathbf{0}$, or (b) $\mathbf{yA} \geq \mathbf{0}$, $\mathbf{yb} < \mathbf{0}$.

Now, we begin from the second part of Gordan's Lemma and assume that $\mathbf{y'A} >> \mathbf{0}$ has a solution. If this is the case, we know that $\exists c > 0$ such that $[\mathbf{y'A} - c\mathbf{e}] \geq \mathbf{0}$, where $\mathbf{e}$ is a vector whose columns are all equal to 1.

Writting the above in matrix form, we obtain:

$$\begin{bmatrix} y_1 & ... & y_k \end{bmatrix} \begin{bmatrix} A_{11} & ... & A_{1n} \\ & ... & \\ A_{k1} & ... & A_{kn} \end{bmatrix} >> \begin{bmatrix} 0 & ... & 0 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} y_1 & ... & y_k \end{bmatrix} \begin{bmatrix} A_{11} & ... & A_{1n} \\ & ... & \\ A_{k1} & ... & A_{kn} \end{bmatrix} - c \begin{bmatrix} 1 & ... & 1 \end{bmatrix} \geq \begin{bmatrix} 0 & ... & 0 \end{bmatrix}$$

Now, we define the following two new variables:

$$y^* = \begin{bmatrix} y_1 \\ ... \\ y_k \\ c \end{bmatrix}, A^* = \begin{bmatrix} A_{11} & ... & A_{1n} \\ & ... & \\ A_{k1} & ... & A_{kn} \\ -1 & ... & -1 \end{bmatrix}$$

Now, we have:

$$y^{*'} A^* = \begin{bmatrix} y_1 & ... & y_k & c \end{bmatrix} \begin{bmatrix} A_{11} & ... & A_{1n} \\ & ... & \\ A_{k1} & ... & A_{kn} \\ -1 & ... & -1 \end{bmatrix} \geq \begin{bmatrix} 0 & ... & 0 \end{bmatrix}$$

Which has a solution by the second part of Farkas' Lemma. Now, we consider a new term:

$$b^* = \begin{bmatrix} 0 \\ ... \\ 0 \\ -1 \end{bmatrix}$$

So that the equation $\mathbf{y}^{*'}\mathbf{b}^* < 0$ holds, as it yields $-c < 0$, which is true by definition of $c$. Therefore, we have established that $\mathbf{y}^{*'}\mathbf{A}^* \geq 0$, $\mathbf{y}^{*'}\mathbf{b}^* < 0$ has a solution by Farkas' Lemma.

The first term is, then:

$$\mathbf{y}^{*'}\mathbf{A}^* = \mathbf{y}\mathbf{A} - s\mathbf{e} \geq \mathbf{0} \Rightarrow \mathbf{y}^{'}\mathbf{A} >> \mathbf{0}$$

Then, also by Farka's Lemma, we have that, if the above is not true:

$$\mathbf{A}^*\mathbf{x} = \mathbf{b}^* \Rightarrow \begin{bmatrix} A_{11} & ... & A_{1n} \\ & ... & \\ A_{k1} & ... & A_{kn} \\ -1 & ... & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ ... \\ 0 \\ -1 \end{bmatrix}$$

with $\mathbf{x} \geq \mathbf{0}$. The above, then, yields two equations, as follows:

$$\begin{bmatrix} A_{11} & ... & A_{1n} \\ & ... & \\ A_{k1} & ... & A_{kn} \end{bmatrix} \begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ ... \\ 0 \end{bmatrix} \Rightarrow \mathbf{A}\mathbf{x} = \mathbf{0}$$

and, secondly:

$$\begin{bmatrix} -1 & ... & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix} = -1 \Rightarrow \mathbf{x} > \mathbf{0}$$

Therefore, we have managed to prove the first part of Gordan's Lemma, as well.

Therefore, transforming Gordan's Lemma into the equivalent of Farkas' Lemma, we have managed to prove that either $\mathbf{A}\mathbf{x} = \mathbf{0}$, $\mathbf{x} > \mathbf{0}$ will have a solution, or $\mathbf{y}'\mathbf{A} >> \mathbf{0}$ will have a solution.

As a last note, as according to Farkas' Lemma and the course slides, the two alternatives cannot hold at the same time, having constructed the equivalent of Farkas' Lemma here, the two alternatives of Gordan's Lemma cannot hold simultaneously either.