Section 8

Logistics

- TA Office Hours:
  - Mon & Wed 18:30-19:30pm
  - In person (Uris 465) or by Zoom (same link as before).
- All problem sets are due before Friday section.
- Section's material https://github.com/luciacasal/-Cornell_ECON6100.

1 Walrasian equilibrium: pure exchange economy [1]

1.1 Important concepts:

1. **Setup.** Consider an exchange economy:
   (a) L goods, I consumers
   (b) endowment $e^i = (e^i_1, ..., e^i_L) \in \mathbb{R}^L_+$ and aggregate endowents $e = \sum_i e^i$
   (c) prices $p = (p_1, ..., p_L) \in \mathbb{R}^L_+ / \{0\}$
   (d) demand function of consumer i is $d_i(p, e^i) \in \mathbb{R}^L_+$, excess demand is $z_i(p, e) = d_i(p, e^i) - e^i \in \mathbb{R}^L$, with aggregate excess demand $z(p, e) = \sum_i z_i(p, e^i) \in \mathbb{R}^L$.
2. **Definition 1. Walrasian equilibrium:** (or competitive equilibrium) is a price vector p^* such that (1) $z(p^*, e) \leq 0$; (2) $p^* z(p^*, e) = 0$.
3. **Theorem 1. Existence of Walrasian Equilibrium.** If excess demand function $z(p, e)$ satisfies:
   (a) A1: [HOD0] $z(p, e)$ is HOD in p.
   (b) A2: [Warlas' Law] $\sum_l p_l z_l(p, e) = 0$
   (c) A3: [Continuity] $z(p, e)$ is continuous in p.

   then there is a Walrasian equilibrium price p^*.

[1] These notes borrow from Jaden Chen's notes from 2020.

1.2 Questions

1. (Varian 17.4) There are two consumers A and B with the following utility functions and endowments

$$u^A(x,y) = \alpha logx + (1-\alpha)logy \qquad e^A = (0,1)$$
$$u^B(x,y) = min\{x,y\} \qquad e^B = (1,0)$$

   (a) Illustrate this situation in and Edgeworth box diagrame for $\alpha = 1/2$. What are the market clearing prices and equilibrium allocation?

   (b) For $\alpha \in (0,1)$, calculate the market clearing prices and equilibrium allocations. How do things change with α?

2. (Modified from Q2009 June) Consider an exchange economy with L goods and N consumers. Each consumer's utility function is of the form $u_n(x_1,...,x_L) = \sum_l v_n(x_l)$, where each v_n is strictly concave, strictly increasing, differentiable and satisfies an Inada condition at the origin. Suppose that each consumer has a strictly positive endowment $\omega_n = (\omega_{n1},...\omega_{nL}) \gg 0$.

   (a) If $\sum_n \omega_{n1} = \sum_n \omega_{n2} = ... = \sum_n \omega_{nL}$, what are equilibrium prices?

   (b) Show that if $\sum_n \omega_{n1} > \sum_n \omega_{n2} > ... > \sum_n \omega_{nL}$ then for the competitive equilibrium price vector p^*, we have $p_1^* < p_2^* < ... < p_L^*$.

2 General Equilibrium

2.1 Important concepts:

1. Definition 1: Private-ownership economy. We move from pure exchange economy to a private-ownership economy by introducing a set of producers.

   (a) L commodities, aggreagte endowments $\bar{\omega} \in \mathbb{R}^L$.

   (b) Firms: $j = 1, ..., J$ with nonempty and closed production set $Y_j \in \mathbb{R}^L$

   (c) Consumers: $i = 1, ..., I$ with consumption set $X_i \subset \mathbb{R}^L$, preferences $\succeq_i$ and endowment $\omega_i \in X_i$. Consumer $i's$ ownership of firm j is $\theta_{ij} \in [0,1]$ and $\sum_i \theta_{ij} = 1$.

2. Definition 2. Walrasian equilibrium for a private-ownership economy: denotes an allocation (x^*, y^*) and a price vector p^* such that:

   (a) (Profit maximization): For all j, $p^* \cdot y_j^* \geq p^* \cdot y_j$ for all $y_j \in Y_j$

   (b) (Utility maximization): For all i, x_i^* is $\succeq_i -maximal$ in the budget set:

$$\left\{ x_i \in X_i : p^* \cdot x_i \leq p^* \cdot \omega_i + \sum_i \theta_{ij} p^* \cdot y_j^* \right\}$$

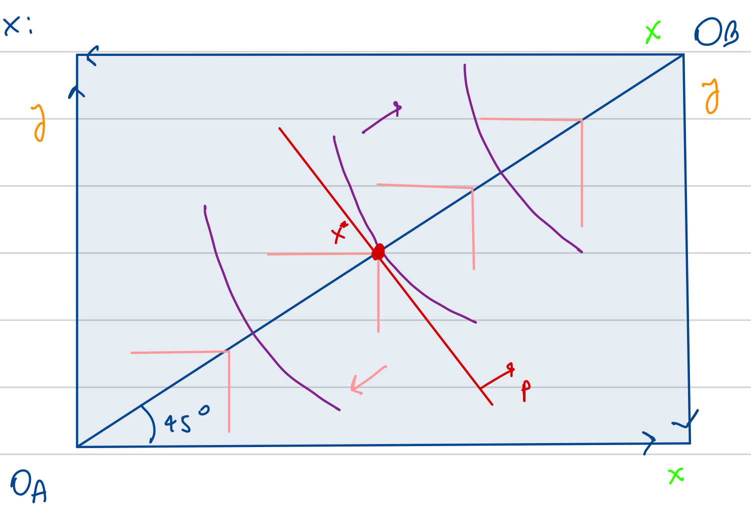
   (c) (Markets clear): $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$

Q1) 2 consumers; A and B. with utility functions:

$$U^A(x,y) = \alpha \log x + (1-\alpha)\log y \qquad e^A = (0,1)$$
$$U^B(x,y) = \min\{x,y\} \qquad e^B = (1,0)$$

a) $\alpha = 1/2$. Edgeworth box. Mkt clearing prices and eq. allocations.

Edgeworth box:



normalize $p_x = p \quad p_y = 1$

consumer A

$$\begin{cases} \max & \alpha \log x_A + (1-\alpha)\log y_A \\ \text{st} & p \cdot x_A + 1 \cdot y_A \leq p \cdot 0 + 1 \cdot 1 \\ & x_A, y_A \geq 0 \end{cases}$$

$\because$ Inada cond $\longrightarrow$ interior solution ✓ $\longrightarrow$ ignore non-neg

$\because$ concave $\longrightarrow$ SOC ✓ (FOC sufficient)

$\because$ monotone preferences $\longrightarrow$ walras law $\longrightarrow$ $p_x x_A + p_y y_A = 1$

We know in a competitive equilibrium $\dfrac{MgU_{x_A}}{MgU_{y_A}} = \dfrac{p_x}{p_y} = p$

$$\longrightarrow \quad \frac{\frac{\alpha}{x_A}}{\frac{1-\alpha}{y_A}} = p$$

$$\frac{y_A}{x_A} \cdot \frac{\alpha}{1-\alpha} = p$$

$$\boxed{y_A = p\, x_A \cdot \frac{1-\alpha}{\alpha}}$$

From budget constraint: $p \cdot x_A + y_A = 1$

$$p\, x_A + p\, x_A \cdot \frac{1-\alpha}{\alpha} = 1$$

$$p\, x_A \left(\frac{1}{\alpha}\right) = 1$$

① $x_A^* = \dfrac{\alpha}{p}$ $\qquad$ ② $y_A = 1-\alpha$

For $\alpha = 1/2$ $\qquad x_A = \dfrac{1}{2p} \qquad y_A = 1/2$

- consumer B

$$\max \; \min\{x_B; y_B\}$$
$$\text{st} \quad p x_B + y_B \leq 1 \cdot p$$
$$x_B, y_B \geq 0$$

we know $p_x, p_y > 0$ because:

Claim: if $\exists$ i such that $\succsim_i$ is strongly monotone, then equilibrium price $p^* >> 0$. (If not, the agent would like to consume ∞ amounts of the good when price is 0, and market clearing condition would be violated).

$\longrightarrow \qquad x_B = y_B$ (see graph)

$\therefore$ from budget constraint $\qquad p x_B + x_B = p$

③ $y_B = x_B = \dfrac{p}{1+p}$

Now we use market clearing to find prices:

$$\begin{cases} x_A + x_B = 1 = e_{x_A} + e_{x_B} \\ y_A + y_B = 1 = e_{y_A} + e_{y_B} \end{cases}$$

Substitute ① ② ③ :

- $\frac{\alpha}{p} + \frac{p}{1+p} = 1$
- $(1-\alpha) + \frac{p}{1+p} = 1 \longrightarrow \frac{p}{1+p} = \alpha \longrightarrow p = \frac{\alpha}{1-\alpha}$

$*$

For $\alpha = 1/2 \longrightarrow p = 1 \longrightarrow$ $x_A = 1/2$, $x_B = 1/2$, $y_A = 1/2$, $y_B = 1/2$

b) Done in part a. If $\alpha \nearrow \rightarrow p \nearrow$ α is like the importance of the $\log x$...

claim:

$*$ If Walras law holds, only need to solve for $N-1$ equations in the market clearing.

Proof. take $N = 2$

If WL holds, $p(x^A + x^B) + 1(y^A + y^B) = pe_x + 1e_y$

If $\underbrace{x^A + x^B = e_x}_{\text{market clearing for } x}$, then we have $\underbrace{y^A + y^B = e_y}_{\text{mkt clearing for } y}$.

Q2) (Modified from Q 2009 June)

L goods

N consumers

each n:

$\max \quad \sum_\ell v_n(x_\ell)$

st $\quad \sum_\ell p_\ell x_\ell^n \leq \sum_\ell w_\ell^n$

for each consumer n:

$U_n(x_1, \dots x_L) = \sum_\ell v_n(x_\ell)$; with v_n strictly concave (SOC ✓)

" increasing (walras law)

differentiable (calculus)

Inada cond (interior sol)

suppose strictly positive endowment $w_n = (w_{n1}, \dots w_{nL}) \gg 0$

a) If $\sum_n w_{n1} = \sum_n w_{n2} = \dots = \sum_n w_{nL}$, what are eq. prices?

from FOC: for all n and ℓ, m:

$$\frac{v_n'(x_\ell^n)}{v_n'(x_m^n)} = \frac{p_\ell}{p_m}$$

If $\exists$ ℓ, m such that $p_\ell > p_m$, since v_n is strictly concave, we have $x_m^n > x_\ell^n$. $\forall n$

$\Longrightarrow$ ① $\sum_n x_m^n > \sum_n x_\ell^n$

$\because$ in equilibrium $\begin{cases} \sum_n x_\ell^n = \sum_n w_\ell^n \\ \sum_n x_m^n = \sum_n w_m^n \end{cases}$ [market clearing]

then from ①: $\sum_n w_m^n > \sum_n w_\ell^n$ contradiction! with

$\Longrightarrow$ $p_\ell^* = p_m^* \quad \forall\, m, \ell$

b) Show that if $\sum_n w_{n1} > \sum_n w_{n2} > \dots > \sum_n w_{nL}$, then for the competitive equilibrium price vector p^*, we have
$$p^*_1 < p^*_2 < \dots < p^*_L$$

notice that if there are m and l such that

$$\sum_n w_l > \sum_n w_m$$

but $p_l \geq p_m$

from a) we must have $x^n_l \leq x^n_m \;\forall n$

$$\underbrace{\sum_n x_l}_{=\sum_n w_l} \leq \underbrace{\sum_n x_m}_{=\sum_n w_m}$$

$$\sum_n w_l < \sum_n w_m$$

market clearing

contradiction!