

2. Consider the linear program

$$\begin{aligned} v_P(b) &= \max c_x x + c_y y \\ \text{subject to } & x \leq 1 \\ & x \geq 0 \end{aligned}$$

(a) Rewrite this in the canonical form.

(b) Rewrite this in the standard form.

(a) Since y doesn't have the nonnegativity constraint, we can express y as the difference of two nonnegative variables.

$$\begin{aligned} V_P(b) &= \text{MAX } (c_x \ c_y \ -c_y) \begin{pmatrix} x \\ y_1 \\ y_2 \end{pmatrix} \\ \text{s.t. } & \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y_1 \\ y_2 \end{pmatrix} \leq 1 \\ & \begin{pmatrix} x \\ y_1 \\ y_2 \end{pmatrix} \geq 0 \end{aligned}$$

$$(b) \quad V_P(b) = \text{MAX } c_x x + c_y y_1 - c_y y_2 + 0 \cdot s$$

$$\text{s.t. } \quad x + s = 1$$

$$x, y_1, y_2, s \geq 0.$$

$$\Rightarrow V_P(b) = \text{MAX } (c_x \ c_y \ -c_y \ 0) \begin{pmatrix} x \\ y_1 \\ y_2 \\ s \end{pmatrix}$$

$$\text{s.t. } \quad \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y_1 \\ y_2 \\ s \end{pmatrix} = 1$$

$$\begin{pmatrix} x \\ y_1 \\ y_2 \\ s \end{pmatrix} \geq 0$$

3. Consider the primal problem

$$\begin{aligned} v_p(b) = \max & x_1 + 2x_2 \\ \text{s.t. } & x_1 + x_2 \leq 4 \\ & x_1 + 3x_2 \leq b \end{aligned}$$

- (a) Write down the dual.
 (b) For $b = 1$, plot the constraint sets for both problems, and solve them.
 (c) Describe $v_p(b)$, and compute $\partial v_p(b)$ on the range $0 \leq b \leq 14$.

(a) We want to first rewrite the problem with nonnegativity constraints.

$$\begin{aligned} V_p(b) = \max & \overbrace{(x_1 - x_2)}^{x_1} + 2 \overbrace{(x_3 - x_4)}^{x_2} \quad \text{Note: } x_1, x_2 \text{ here are not} \\ \text{s.t. } & (x_1 - x_2) + (x_3 - x_4) \leq 4 \quad \text{the same as the ones in} \\ & (x_1 - x_2) + 3(x_3 - x_4) \leq b \quad \text{the problem.} \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

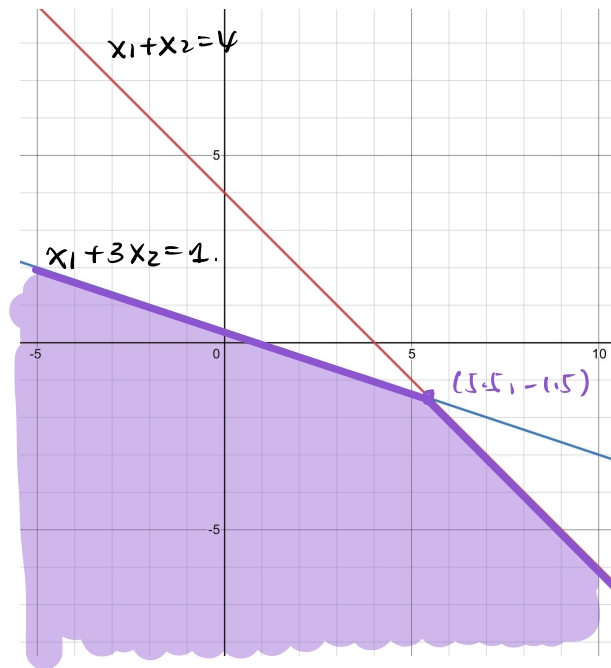
$$\begin{aligned} \Rightarrow V_p(b) = \max & \underbrace{[1 \ -1 \ 2 \ -2]}_C \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\ \text{s.t. } & \underbrace{\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 3 & -3 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \leq \underbrace{\begin{bmatrix} 4 \\ b \end{bmatrix}}_{b.} \\ & \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \geq 0 \end{aligned}$$

Hence, the dual problem is

$$\begin{aligned} V_D(b) = \min & [y_1 \ y_2] \begin{bmatrix} 4 \\ b \end{bmatrix} \\ \text{s.t. } & [y_1 \ y_2] \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 3 & -3 \end{bmatrix} \geq [1 \ -1 \ 2 \ -2] \\ & [y_1 \ y_2] \geq 0. \end{aligned}$$

(b) First consider the primal problem

$$\begin{aligned} v_p(b) = \max \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 4 \\ & x_1 + 3x_2 \leq 1 \end{aligned}$$



The shaded region is the constraint set and by vertex theorem, the solution is $(5.5, -1.5)$, which gives $V_p(b) = 2.5$



Next consider the dual problem:

$$\text{MIN } 4y_1 + y_2$$

$$\text{s.t. } y_1 + y_2 \geq 1$$

$$-y_1 - y_2 \geq -1$$

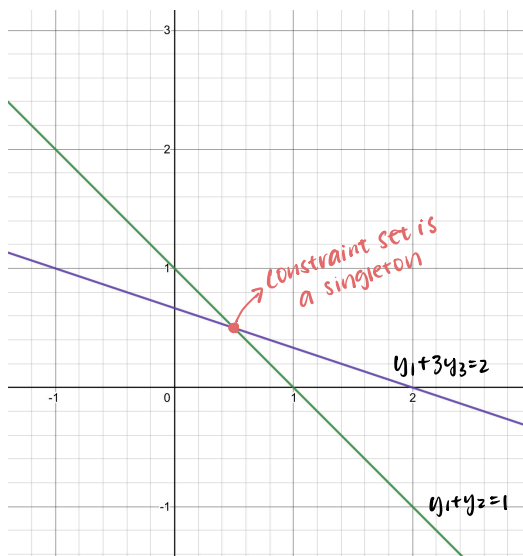
$$y_1 + 3y_2 \geq 2$$

$$-y_1 - 3y_2 \geq -2$$

$$y_1, y_2 \geq 0$$

$$\Rightarrow y_1 + y_2 = 1$$

$$\Rightarrow y_1 + 3y_2 = 2$$



Since the constraint set is a singleton, the solution to the dual problem is $(0.5, 0.5)$, which gives $V_d(b) = 2.5$.

$$\Rightarrow V_d(b) = V_p(b)$$



(c)

Shadow Prices

Theorem. If either $v_P(b)$ or $v_D(c)$ is finite, then

1. $v_P(b) = v_D(c)$,
2. both programs have optimal solutions, and
3. $\partial v_D(b)$ is the set of optimal solutions to the primal, and $\partial v_P(b)$ is the set of optimal solutions to the dual.

By computation above, we know that $V_P(b)$ and $V_D(c)$ are both finite when $b \in [0, 14]$ and $V_P(b) = V_D(c)$. The solution for the dual problem is $(0.5, 0.5)$.

Hence, $V_P(b) = V_D(c) = 4(0.5) + 0.5b = 2 + 0.5b$.

By the shadow price theorem, $\partial V_P\left(\begin{bmatrix} 4 \\ b \end{bmatrix}\right) = (0.5, 0.5)$ for $b \in [0, 14]$ ✓
and $V_P(b) = 0.5$ for $b \in [0, 14]$

You could also solve the primal for an arbitrary b , and then

$$\text{obtain } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{12-b}{2} \\ \frac{b-4}{2} \end{bmatrix}$$

and plug it back into

$$V_P(b) = x_1 + 2x_2 = 2 + \frac{b}{2}$$

$$\frac{dV_P(b)}{db} = \frac{1}{2}$$