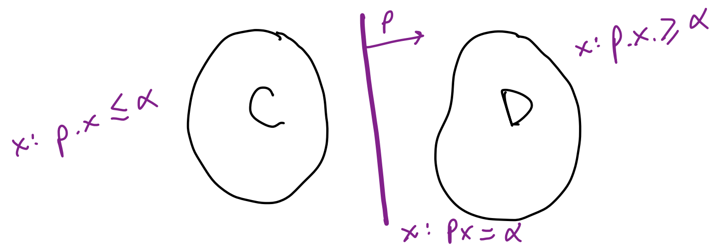
Logistics

- TA Office Hours: Mon & Wed 18:30-19:30pm.
- Same link for office hours and sections.
- All Problem Sets are due before Friday section.
- Please send the Problem Set by email to lc944@cornell.edu

1 Convex sets[1]

1.1 Important concepts:

- **Separating Hyperplane Theorem:** Let C and D be two convex sets in $\mathbb{R}^n$ that do not intersect (i.e., $C \bigcap D = \emptyset$). Then, there exists $p \in \mathbb{R}^n, p \neq 0$ and $\alpha \in \mathbb{R}$ such that $p^T x \leq \alpha$ for all $x \in C$ and $p^T y \geq \alpha$ for all $y \in D$.

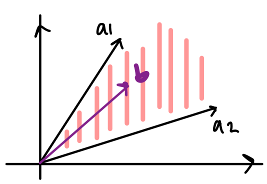


- **Farkas' Lemma:** one and only one of the following alternatives is true:

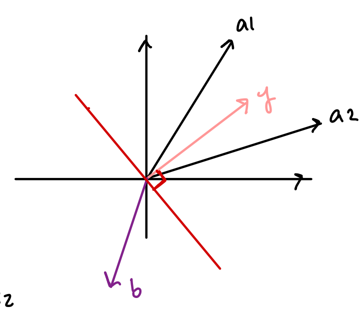
  1. The system $Ax = b, x \geq 0$ has a solution.
  2. The system $y^T A \geq 0, y^T b < 0$ has a solution.

For $yA \geq 0 \longrightarrow$ the angle btw y and A is $< 90°$

1.



2.



For $yb < 0 \longrightarrow$ the angle btw y and $b > 90°$

a_1, a_2 columns of A_{2x2}. vector b is an element of the convex cone spanned by a_1 and a_2

but notice that this case that the angle b can't be a positive LC of a_1, a_2.

1.2 Questions

1. *(From Convex Sets PS, Q5).* Use Farkas' Lemma to prove Gordans' Lemma: only one of this alternatives is true

   (a) $Ax = 0, x > 0$ has a solution.

   (b) $y^T A \gg 0$ has a solution.

[1] These notes borrow from Jaden Chen's notes from 2020.

Question 1

We will try to transform this problem so that it fits in Farkas lemma.

We want to show that if $yA \gg 0$ has a solution $\rightarrow$ $Ax = 0$; $x > 0$ has no solution.

Notice that

$y^T A \gg 0$ iff $\exists\, s > 0$ such that $y^T A - se \geq 0$; with $e = (1 \cdots 1)'$

Let $\hat{y} = \begin{pmatrix} y \\ s \end{pmatrix}$ and $\hat{A} = \begin{pmatrix} A \\ -e \end{pmatrix}$.

Then $\hat{y}^T A = (y \;\; s) \begin{pmatrix} A \\ -e \end{pmatrix} = y^T A - se \geq 0$

Now take $\hat{b} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -1 \end{pmatrix}$ so that $\hat{y}^T \cdot \hat{b} = (y^T \; s) \cdot \begin{pmatrix} 0 \\ \vdots \\ -1 \end{pmatrix} = -s < 0$

Therefore we have: $\hat{y}^T \hat{A} \geq 0$ and $\hat{y}^T b < 0$

From Farkas' lemma, we know that either:

1) $\hat{A}x = \hat{b}$; $x \geq 0$ has a solution.
2) $\hat{y}^T \hat{A} \geq 0$; $\hat{y}^T \hat{b} < 0$ has a solution.

$\rightarrow$ If $\hat{y}^T \hat{A} \geq 0$, $\hat{y}^T \hat{b} < 0$ has a solution $\rightarrow$ $\underbrace{\hat{A}x = \hat{b};\ x \geq 0 \text{ has no solution.}}$

$\hat{A} = \begin{pmatrix} A \\ -e \end{pmatrix} \cdot x = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

no solution.

notice that if $x = 0$ $\rightarrow$ $Ax = 0$ has a solution, but since in Gordan's lemma we impose $x > 0$, then the system $Ax = 0$ has no solution.

2 Linear programming

2.1 Important concepts:

- **Canonical and Standard form:** a linear program can be written in *canonical* form or in *standard* form.

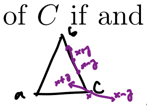
slack variables

Canonical	*Standard*
$v_P(b) = max\ c \cdot x$	$v_P(b) = max\ c \cdot x$
$s.t.\ Ax \leq b$	$s.t.\ Ax = b$
$x \geq 0$	$x \geq 0$

$Ax \geq b$
$-Ax \leq -b$

- **Definitions:**

  1. *Solution:* any $x \in \mathbb{R}^n$ is called a solution.
  2. *Constraint (or Feasible) set:*
     (a) canonical form: $C = [x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$
     (b) standard form: $C = [x \in \mathbb{R}^n : Ax = b, x \geq 0\}$
  3. *Feasible solution:* any $x \in C$, where C is the constraint set, is called a feasible solution.
  4. *Optimal solution:* a vector x that solves the linear program is called an optimal solution, i.e., $x \in C$ such that $c \cdot x \geq c \cdot x'$, for all $x' \in C$.
  5. *Vertex:* a vector $x \in C$ is a vertex of C if and only if there is no $y \neq 0$ such that $x + y$ and $x - y$ are both in C.



  6. *Basic solution:* a solution to a linear program in standard form is a basic solution iff column vectors a_j corresponding to $x_j > 0$ are linearly independent.

$\underset{2\times 4}{A_{m\times n}} \cdot \underset{4\times 1}{x_{n\times 1}} = \underset{2\times 1}{b_{m\times 1}}$

set $\underset{4-2}{n-m}$ variables equal to 0

$\rightarrow \underset{m\times m}{\hat{A}} \cdot \underset{m\times 1}{\hat{x}} = b_{m\times 1}$

basis matrix

- **Theorem 1.** A solution x is basic if and only if it is a vertex.

- **Theorem 2.** *Vertex Theorem.* For a linear program in standard form with feasible solutions:

  1. A vertex exists.
  2. If $v_P(b) < \infty$ and $x \in C$, then there is a vertex x' such that $c \cdot x' \geq c \cdot x$.

2.2 Questions

1. Consider the following linear program

$$
\begin{array}{lrcl}
max & x_1 + x_2 & & \\
s.t. & x_1 + 2x_2 & \leq & 6 \\
 & x_1 - x_2 & \leq & 3 \\
 & x_1 \geq 0 & & \\
 & x_2 \geq 0 & &
\end{array}
$$

   (a) Express the linear program in canonical form and draw the constraint set and solve the problem graphically.

   (b) Express the linear program in standard form.

   (c) Use Simplex Algorithm to solve this problem.

Q2

$$x_1 + x_2$$

$$\text{st} \quad x_1 + 2x_2 \leq 6$$
$$x_1 - x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

a) Canonical $\underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{c} \cdot \underbrace{\begin{pmatrix} x_1 & x_2 \end{pmatrix}}_{x}$

$$\text{st} \quad \underbrace{\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \underbrace{\begin{pmatrix} 6 \\ 3 \end{pmatrix}}_{b}$$

$$(x_1 \; x_2) \geq 0$$

Graph

$x_1 + x_2 = U$

$x_1 + 2x_2 \leq 6$

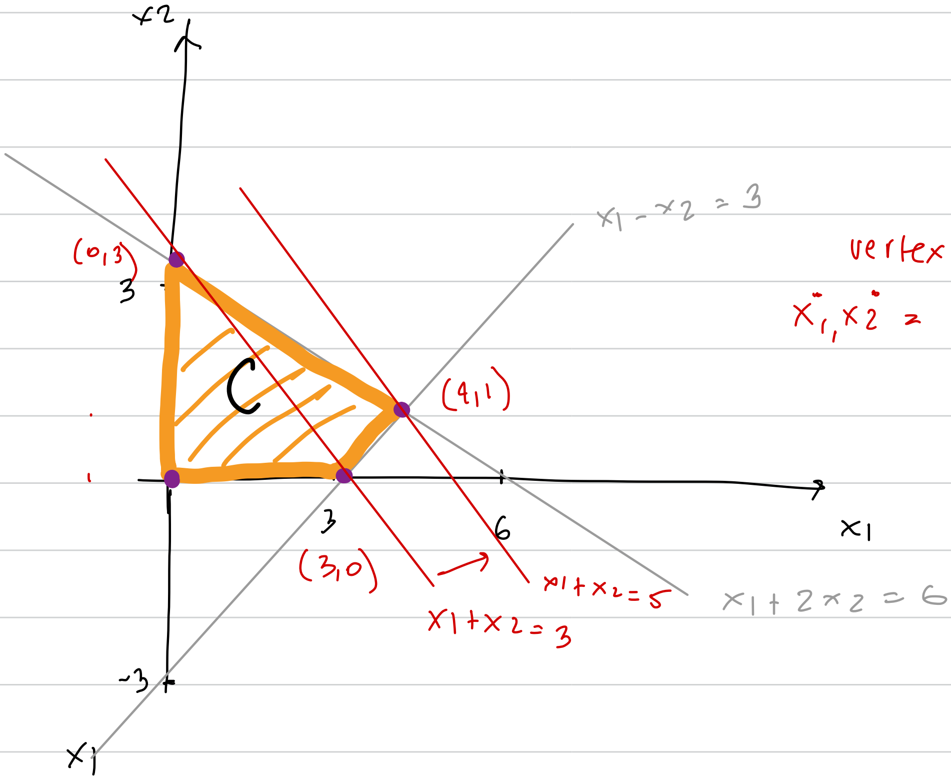
$x_1 - x_2 \leq 3$

$x_1\ x_2 \geq 0$

$(3,0)$ $(0,3)$

$x_1 + x_2 =$

$x_2 =$	$U - x_1$	
0	3	3
3	3	0



vertex theorem

$x_1^*, x_2^* = (4,1)$

b) Standard form:

$$x_1 + x_2 + S_1 \cdot 0 + S_2 \cdot 0$$

st. $x_1 + 2x_2 + S_1 + S_2 \cdot 0 = 6$

$x_1 - x_2 + S_1 \cdot 0 + S_2 = 3$

$x_1\ x_2 \geq 0$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 & x_2 & S_1 & S_2 \end{pmatrix}$$

st
$$\begin{array}{cccc} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} \\ x_1 \quad x_2 \quad S_1 \quad S_2 \end{array} \cdot \begin{pmatrix} x_1 \\ x_2 \\ S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

$x_1, x_2 \geq 0$

c) SIMPLEX:

General Rules
- In a system of n variables and m constraints, we will have n-m basis variables
- Basic variables $\neq 0$; non-basic var. $= 0$
- write the objective function in terms of the non-basic variables
- In the constraint set, put the basic var in the LHS and the non-basic in the RHS
- change one non-basic variable at a time (keep the other one $=0$)

Step 1: S_1, S_2 basic variables

set $x_1 = x_2 = 0$ (set non-basic variables to 0)

$$\begin{cases} \max & x_1 + x_2 \\ \text{st} & S_1 = 6 - x_1 - 2x_2 \\ & S_2 = 3 - x_1 + x_2 \\ & \underbrace{x_1\ x_2}_{\text{Basic RHS}}\ \underbrace{S_1\ S_2 \geq 0}_{\text{non-Basic LHS}} \end{cases}$$

$$\left[\begin{array}{l} S_1 = 6 \\ S_2 = 3 \end{array}\right.$$

notice that obj function is increasing in x_1, x_2, so take any of the 2 and replace its 0 value for its max value

$$\left[\begin{array}{l} 0 = 6 - x_1 - 0 \longrightarrow x_1 = 6 \longrightarrow s_2 = -3 \ \times \\ 0 = 3 - x_1 \longrightarrow x_1 = 3 \longrightarrow s_1 = 3 \ \checkmark \\ \qquad\qquad\qquad\qquad\qquad\qquad s_2 = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad x_2 = 0 \end{array}\right.$$

set $x_1 = 3$; $x_2 = 0$; $s_1 = 3$; $s_2 = 0$

step 2: rewrite the LP in terms of the new basic and non basic var.

non-basic

$$\left\{\begin{array}{ll} \max & x_1 + x_2 \\ \text{st} & s_1 = 6 - x_1 - 2x_2 \\ & s_2 = 3 - x_1 + x_2 \\ & x_1\ x_2\ s_1\ s_2 \geq 0 \end{array}\right.$$

$s_1 = 6 - x_1 - 2x_2 \longrightarrow s_1 = 6 - (3 + x_2 - s_2) - 2x_2$

$s_2 = 3 - x_1 + x_2 \longrightarrow x_1 = 3 + x_2 - s_2$

$$\Longrightarrow \left\{\begin{array}{l} s_1 = 3 - 3x_2 + s_2 \\ x_1 = 3 + x_2 - s_2 \end{array}\right.$$

$$\Longrightarrow \max \quad 3 + x_2 - s_2 + x_2$$

$$3 + 2x_2 - s_2$$

$$\text{st} \left\{\begin{array}{l} s_1 = 3 - 3x_2 + s_2 \\ x_1 = 3 + x_2 - s_2 \end{array}\right. \qquad s_1, x_1 \geq 0 \quad x_2, s_2$$

Repeat process. set NB to 0 $\longrightarrow$ $x_2 = 0$; $s_2 = 0$

$\longrightarrow$ $s_1 = 3$; $x_1 = 3$

notice that objective function is increasing in x_2; so we can ↑ x_2 and it will be optimal.

$$0 = 3 - 3x_2 + 0 \rightarrow x_2 = 1$$
$$0 = 3 + x_2 \rightarrow x_2 = -3$$

$$(x_1 = 4 \ ; \ x_2 = 1 \ ; \ 0 \ ; \ 0)$$

step 3

$$\max \quad x_1 + x_2$$
$$\text{st} \quad s_1 = 3 - 3x_2 + s_2 \rightarrow 3x_2 = 3 + s_2 - s_1$$
$$x_1 = 3 + x_2 - s_2 \qquad x_2 = 1 + \frac{1}{3}s_2 - \frac{1}{3}s_1$$

$$x_1 = 3 + 1 + \frac{1}{3}s_2 - \frac{1}{3}s_1 - s_2$$
$$x_1 = 4 - \frac{1}{3}s_1 - \frac{2}{3}s_2$$

$$\max \quad 4 - \frac{1}{3}s_1 - \frac{2}{3}s_2 + 1 + \frac{1}{3}s_2 - \frac{1}{3}s_1$$
$$= 5 - \frac{2}{3}s_1 - \frac{1}{3}s_2$$
$$\text{st} \quad x_1 = 4 - \frac{1}{3}s_1 - \frac{2}{3}s_2$$
$$x_2 = 1 - \frac{1}{3}s_1 + \frac{1}{3}s_2 \qquad s_1, s_2 \geq 0$$
$$x_1, x_2$$

notice that we can not increase the value of the objective function by increasing s_1 or s_2, and neither we can decrease s_1 or s_2 because of the non-negativity constraints.

$\longrightarrow (4, 1)$ is the optimal solution
$x_1 \quad x_2$