Logistics

- TA Office Hours:
  - Mon & Wed 18:30-19:30pm (in person (Uris 465) or by Zoom).
- All problem sets are due before Friday section.
- Section's material https://github.com/luciacasal/-Cornell_ECON6100.

1 Transferable Utility Matching[1]

Setup:

- $\mathcal{L}$ workers, $\mathcal{F}$ firms
- π_f profit of firm f, w_l wage of worker l.
- v_{lf} surplus from matching worker l with firm f.
- $x_{lf} = 1$ if l and f are matched, and 0 otherwise (the TU matching problem is to find this). Each l can only be matched with one f and vice versa. We call x the match vector.
- If l is matched with f then they split the surplus $v_{lf} = w_l + \pi_f$.
- Definition 1: Matching allocation. An allocation is a matching-payoff pair (x, w, π) such that:
  - if $x_{lf} = 1$, then $w_l + \pi_f = v_{lf}$
  - if $x_{lf} = 0$ for all f, then $w_l = 0$
  - if $x_{lf} = 0$ for all l, then $\pi_f = 0$
- Definition 2: A matching is optimal if it maximizes total surplus

$$
\begin{aligned}
v(\mathcal{L} \cup \mathcal{F}) = \max_{l,f} \; & v_{lf} x_{lf} \\
s.t. \quad & \sum_f x_{lf} \leq 1 \quad \forall\, l \in \mathcal{L} \\
& \sum_l x_{lf} \leq 1 \quad \forall\, f \in \mathcal{F} \\
& x_{lf} \in \{0,1\} \; \forall\, l \in \mathcal{L}, f \in \mathcal{F}
\end{aligned}
\quad \overset{\text{Birkhoff VN Theorem}}{\leftrightarrow} \quad
\begin{aligned}
v(\mathcal{L} \cup \mathcal{F}) = \max_{l,f} \; & v_{lf} x_{lf} \\
s.t. \quad & \sum_f x_{lf} \leq 1 \quad \forall\, l \in \mathcal{L} \\
& \sum_l x_{lf} \leq 1 \quad \forall\, f \in \mathcal{F} \\
& x_{lf} \geq 0 \; \forall\, l \in \mathcal{L}, f \in \mathcal{F}
\end{aligned}
$$

[1] These notes are based on Jaden Chen's notes from 2020 and Abhi Ananth notes from 2021.

The solution to the primal tells you how to match workers and firms.

The next natural question is how should we split the surplus among workers and firms? The answer to this involves a notion we refer to as stability.

- Definition 3. An allocation is stable if no worker and firm that are not matched together can increase their welfare by matching with each other and dividing the surplus among themselves. Formally, an allocation is stable if $x_{lf} = 0$, then $w_l + \pi_f \geq v_{lf}$.

Now, how should the surplus be divided to ensure stability? We look at the dual problem:

$$\begin{aligned} & min_{w,\pi} \sum_l w_l + \sum_f \pi_f \\ s.t. \quad & w_l + \pi_f \geq v_{lf} \; \forall \, l, f \\ & w_l, \pi_f \geq 0 \end{aligned}$$

Complementary slackness ensures stability.

Theorem 1.
LP Duality theorem leads to the following theorem:

- If (x, w, π) is a stable allocation, then x is an optimal match.
- If x is an optimal match, then (x, w, π) is a stable allocation where (w, π) solves the dual problem.

If we impose some structure on the surplus that each match can generate, can we say more about the optimal and stable allocation? Think of the surplus of each match as generated by a function $v : L \times F \to \mathbb{R}$. That is, $v_{lf} = v(l, f)$. Further if we suppose that workers in L and firms in F can be ranked, what condition on $v(\cdot, \cdot)$ guarantees that the match is (*positive*) assortative, that is highly ranked workers get matched with highly ranked firms?

Theorem 2. If function v has increasing differences, that is for all $x_0 > x$ and $y_0 > y$ implies $v(x_0, y_0) - v(x, y_0) \geq v(x_0, y) - v(x, y)$, then every stable match is assortative.

2 Questions

1. Suppose we have 2 workers X, Y and 2 firms A, B. The workers and firms can be ranked unambiguously so that X is more productive than Y and A is more productive than B. More precisely, we have $v_{XA} > v_{XB}, v_{YA} > v_{YB}, v_{XA} > v_{YA}$, and $v_{XB} > v_{YB}$.
   (a) Show that if positive assortative matching is stable, then $v_{XA} - v_{YA} \geq v_{XB} - v_{YB}$.
   (b) Give a similar condition for when negative assortative matching is stable.
   (c) If the condition of part (a) is satisfied striclty, can the negative assortative matching be stable?
   (d) Find the values $v_{XA}, v_{XB}, v_{YA}, v_{YB}$ so that both positive assortative matching and negative assortative matching are stable.
2. Suppose there are 3 men (M) and 3 women (W) with the following endowments of labor: $M1 = 80, M2 = 90, M3 = 100, W1 = 90, W2 = 100, W3 = 110$. In this game, a man is matched with a woman and they can produce a final good according to the following production function: $F(M, W) = 100 - (M - W)^2$.
   (a) Find the optimal match.
   (b) Suppose you have not done the previous part. Can you say whether positive assortative matching or negative assortative matching is optimal in this problem.
3. (*Take home*) Find the optimal match and a stable allocation in the following:

	Pa	Pb	Pc
$H1$	5	8	2
$H2$	7	9	6
$H3$	2	3	0

QUESTION 1

a) positive assortative matching

	A	B
x	1	0
y	0	1

Primal

$$\max \; X_{xA} V_{xA} + X_{yB} V_{yB}$$

st

$$X_{xA} + X_{xB} \leq 1$$
$$X_{yA} + X_{yB} \leq 1$$

$$X_{xA} + X_{yA} \leq 1$$
$$X_{xB} + X_{yB} \leq 1$$
$$X_{ij} \geq 0 \quad \forall i j, \quad i = x, y \quad j = A, B$$

Dual

$$\min \; w_x + w_y + \pi_A + \pi_B$$

st

			CS	
$w_x + \pi_A$	$=$	V_{xA}	$[X_{xA} = 1]$	1
$w_x + \pi_B$	$\geq$	V_{xB}	$[X_{xB} = 0]$	2
$w_y + \pi_A$	$\geq$	V_{yA}	$[X_{yA} = 0]$	3
$w_y + \pi_B$	$=$	V_{yB}	$[X_{yB} = 1]$	4

$$w_i ; \pi_j \geq 0 \quad \forall i j$$

Combine 1 2 and 3 4 :

$$V_{xA} - \pi_A + \pi_B \geq V_{xB} \qquad V_{yB} - \pi_B + \pi_A \geq V_{yA}$$

$$\boxed{V_{xA} - V_{xB} \geq \pi_A - \pi_B} \qquad \boxed{\pi_A - \pi_B \geq V_{yA} - V_{yB}}$$

$$V_{xA} - V_{xB} \geq V_{yA} - V_{yB}$$
$$V_{xA} - V_{yA} \geq V_{xB} - V_{yB}$$

max surplus ~ assign the most productive worker with most productive firm.

b) Negative Assortative Matching

	A	B
X	0	1
Y	1	0

analogous to a) we get: $V_{XB} - V_{YB} \geq V_{XA} - V_{YA}$

c) If $V_{XA} - V_{YA} > V_{XB} - V_{YB}$
there is no way that $V_{XB} - V_{YB} \geq V_{XA} - V_{YA}$.

d) we need $V_{XA} - V_{YA} \geq V_{XB} - V_{YB}$
$V_{XB} - V_{YB} \geq V_{XA} - V_{YA}$.

$\rightarrow$ $V_{XA} - V_{YA} = V_{XB} - V_{YB}$

QUESTION 2 conditional on everybody being matched:

	W_1	W_2	W_3
M_1	0	-300	-800
M_2	100	0	-300
M_3	0	100	0

a) 6 possible matchings, the optimal is:

$\{(1,1) \quad (2,2) \quad (3,3)\}$ with total surplus $= 0$

(try stating the LP formally at home)

b) For positive assortative matching we need:

$$\frac{\partial^2 F}{\partial M \partial W} > 0 \quad \text{(suff condition)}.$$

$$\because \frac{\partial F}{\partial M} = -2(M - W) \; ; \quad \frac{\partial F}{\partial M \partial W} = 2 > 0 \checkmark$$