Logistics

- TA Office Hours:
  - Mon & Wed 18:30-19:30pm
  - In person (Uris 465) or by Zoom (same link as before).
- All problem sets are due before Friday section.
- Section's material https://github.com/luciacasal/-Cornell_ECON6100.

1 General Leontief Model [1]

1.1 Important concepts:

- **Definition 1:** $n \times m$ input matrix A and $n \times m$ output matrix B.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nm} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ \vdots & b_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \ldots & b_{nm} \end{bmatrix}$$

where: (1) a_{ij} denotes the amount of good i needed to run activity j; b_{ij} means the amount of good i activity j will produce. (2) $a_{ij} \geq 0; b_{ij} \geq 0$.

- **Assumption 1:** There are more activities than goods: $m > n$.
- **Assumption 2:** Every good is produced: for all i, there is a $a_{ij} > 0$.
- **Assumption 3:** No joint-production: for all j, there is a unique i such that $b_{ij} > 0$.
- **Definition 2:** A General Leontief Model (A, B) is productive if there is an $x^* \geq 0$ such that $y^* \equiv (B - A)x^* \gg 0$.
- **Definition 3:** A technology t is a set of n activities such that each good is produced by only one activity.

[1] These notes borrow form Jaden Chen's notes from 2020.

- **Theorem 1:** If a GLM (A, B) is productive and produces $y \gg 0$, then there is a technology that produces y.

- **Theorem 2: Non-substitution Theorem.** If a GLM (A, B) is productive, there is a technology t such that for al $y \geq 0$, $\lambda^t(y) = \lambda(y)$, where:

$$\begin{aligned} \lambda(y) &= min\, a_0 x \\ s.t.\ (B - A)x &\geq y \\ x &\geq 0 \end{aligned} \qquad \begin{aligned} \lambda^t(y) &= min\, a_0 x \\ s.t.\ (B - A)x &\geq y \\ x &\geq 0 \\ x_i &= 0 \text{ if } i \notin t \end{aligned}$$

one technology can achieve minimum labor expenditure for all net outputs.

1.2 Questions:

1. (Linear Model's PS, Question 6).

2. Consider a Generalized Lentief Model with B and A such that:

$$B - A = \begin{bmatrix} 1 & -1/2 & 1/2 \\ -1/2 & 1 & 1/4 \end{bmatrix}$$

$a_0 = (1, 1, 1/2)$.

Is there a technology τ which is efficient for all net outputs $(y_1, y_2) \geq 0$?

Ricardian Trade Model

- 2 countries $\rightarrow$ E, S
- 2 goods $\rightarrow$ v (wine), m (mutton)

- England $\rightarrow$ ave of labor to produce 1 unit of wine
  - ame " " " mutton.
- Spain $\rightarrow$ avs " " " wine
  - ams " " " mutton.

- no intermediate inputs
- Labor endowments $\rightarrow$ Le, Ls
- Labor can't move across countries

a) Describe the production of wine and mutton as a GLM.
Primary Factors? A and B?

A matrix of input requirements $= 0$
Primary input matrix $a^0 = [\text{ave} \quad \text{ame} \quad \text{avs} \quad \text{ams}]$

B output matrix { rows are goods, columns are activities

$$\underset{2\times4}{B} = \begin{array}{cc} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} & \begin{array}{l} \text{wine} \\ \text{mutton} \end{array} \\ \begin{array}{cccc} \text{Ev} & \text{Em} & \text{Sv} & \text{Sm} \end{array} & \end{array}$$

b) Describe the PPS as a convex polyhedron, ie, a solution set to a system of linear inequalities. Show that it is closed and convex.

$$\text{PPS}: \quad P(L) = \{ y \geq 0 ; \underbrace{y}_{\text{output vector}} \leq Bx ; \; a^{\circ} \underbrace{x}_{\text{input vector}} \leq L ; \; x \geq 0 \}$$

$$P(L): \{ v, m \geq 0 ; \quad \underset{2\times1}{\begin{bmatrix} v \\ m \end{bmatrix}} \leq \underset{2\times4}{B} \cdot \begin{bmatrix} x_{ve} \\ x_{me} \\ x_{vs} \\ x_{ms} \end{bmatrix} ; \quad \begin{bmatrix} a_{ve}^{\circ} & a_{me}^{\circ} \end{bmatrix} \cdot \begin{bmatrix} x_{ve} \\ x_{me} \end{bmatrix} \leq L_e$$

$$\underset{2\times4}{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}} \cdot \underset{4\times1}{\begin{bmatrix} x_{ve} \\ x_{me} \\ x_{vs} \\ x_{ms} \end{bmatrix}} ; \quad \begin{bmatrix} a_{vs}^{\circ} & a_{ms}^{\circ} \end{bmatrix} \begin{bmatrix} x_{vs} \\ x_{ms} \end{bmatrix} \leq L_s$$

$$\begin{cases} v - x_{ve} - x_{vs} \leq 0 \\ m - x_{me} - x_{ms} \leq 0 \\ a_{ve}^{\circ} x_{ve} + a_{me}^{\circ} x_{me} \leq L_e \\ a_{vs}^{\circ} x_{vs} + a_{ms}^{\circ} x_{ms} \leq L_s \end{cases} \qquad \begin{bmatrix} x_{ve} \\ x_{me} \\ x_{vs} \\ x_{ms} \end{bmatrix} \geq 0$$

$\begin{bmatrix} a_{ve}^{\circ} & a_{me}^{\circ} \end{bmatrix}$; $\begin{bmatrix} a_{vs}^{\circ} & a_{ms}^{\circ} \end{bmatrix}$ are amounts of primary goods L_e and L_s used in the unit operation of activities.

$$\underset{4\times6}{\begin{bmatrix} 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & a_{ve}^{\circ} & a_{me}^{\circ} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{vs}^{\circ} & a_{ms}^{\circ} \end{bmatrix}} \underset{6\times1}{\begin{bmatrix} v \\ m \\ x_{ve} \\ x_{me} \\ x_{vs} \\ x_{ms} \end{bmatrix}} \leq \underset{4\times1}{\begin{bmatrix} 0 \\ 0 \\ L_e \\ L_s \end{bmatrix}}$$

$$\begin{bmatrix} v & m & x_{ve} & x_{me} & x_{vs} & x_{ms} \end{bmatrix}' \geq 0$$

c) convex support function characterized as

$$k(p) = \max\{p \cdot x \;:\; x \in C\}$$

$$\max \quad \alpha v + \beta m + 0 \cdot x_{ve} + 0 \cdot x_{me} + 0 \cdot x_{vs} + 0\, x_{ms}$$

$$\text{st} \left\{ \begin{array}{l} \underset{4\times 6}{\begin{bmatrix} 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & a_{ve}^0 & a_{me}^0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{vs}^0 & a_{ms}^0 \end{bmatrix}} \overbrace{\underset{6\times 1}{\begin{bmatrix} v \\ m \\ x_{ve} \\ x_{me} \\ x_{vs} \\ x_{ms} \end{bmatrix}}}^{x} \leq \underset{4\times 1}{\begin{bmatrix} 0 \\ 0 \\ L_e \\ L_s \end{bmatrix}} \begin{array}{l} [p_v] \\ [p_m] \\ [w_e] \\ [w_s] \end{array} \\ \\ [v \;\; m \;\; x_{ve} \;\; x_{me} \;\; x_{vs} \;\; x_{ms}]' \geq 0 \end{array} \right.$$

d) Dual.

$$\begin{array}{llll} \text{primal} & \max & c \cdot x & \text{Dual} \quad \min \; yb \\ & \text{st} & Ax \leq b & \text{st} \quad yA \geq c \\ & & x \geq 0 & \qquad\quad y \geq 0 \end{array}$$

$$\min \quad [p_v \quad p_m \quad w_E \quad w_S] \cdot \begin{bmatrix} 0 \\ 0 \\ L_e \\ L_s \end{bmatrix}$$

$$\text{st} \quad \underset{1\times 4}{[p_v \;\; p_m \;\; w_E \;\; w_S]} \underset{4\times 6}{\begin{bmatrix} 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & a_{ve}^0 & a_{me}^0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{vs}^0 & a_{ms}^0 \end{bmatrix}} \geq \begin{bmatrix} \alpha \\ \beta \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{array}{l} [v] \\ [m] \\ [x_{ve}] \\ [x_{me}] \\ [x_{vs}] \\ [x_{ms}] \end{array}$$

$$[p_v \quad p_m \quad w_E \quad w_S] \geq 0$$

e) Using CS, interpret solution to the dual as competitive prices.

CS for the dual: $[yA - c]x = 0$

① $(p_v - \alpha)v = 0$ ⎫ doesn't have much meaning
② $(p_m - \beta)m = 0$ ⎭
③ $(-p_v + w_e a_{ve}^\circ)x_{ve} = 0$
④ $(-p_m + w_e a_{me}^\circ)x_{me} = 0$
⑤ $(-p_v + w_s a_{vs}^\circ)x_{vs} = 0$
⑥ $(-p_m + w_s a_{ms}^\circ)x_{ms} = 0$

③, ④, ⑤ and ⑥ state that if production of any good (v or m) produced in country i (e or s) is positive, then the price of that good will be equal to the marginal cost of producing it (ie; if $x_{vi} > 0 \rightarrow p_v = w_i \cdot a_{vi}^\circ$)

f) Under what conditions can all activities be used?

If all activities are used $\rightarrow$ all goods are produced.
$(v \quad m \quad x_{ve} \quad x_{me} \quad x_{vs} \quad x_{ms}) >> 0$.

$\rightarrow$ By CS of the dual we need:

① $p_v = \alpha$
② $p_m = \beta$
③ $p_v = w_e a_{ve}^\circ$ ⎫
④ $p_m = w_e a_{me}^\circ$ ⎭ $\boxed{\dfrac{\alpha}{a_{ve}^\circ} = \dfrac{\beta}{a_{me}^\circ}}$ (A)
⑤ $p_v = w_s a_{vs}^\circ$ ⎫
⑥ $p_m = w_s a_{ms}^\circ$ ⎭ $\boxed{\dfrac{\alpha}{a_{vs}^\circ} = \dfrac{\beta}{a_{ms}^\circ}}$ (B)

Combine (A) and (B):

$$\boxed{\frac{a_{ve}^{\circ}}{a_{me}^{\circ}} = \frac{a_{vs}^{\circ}}{a_{ms}^{\circ}}}$$

relative productivity of wine wrt cotton in England has to be equal to relative productivity of wine wrt cotton in Spain for all products to be produced by all countries.

$$RPS = RPE$$

if $\nexists$ a single country that has a relative CA then it's possible that they both produce all goods because they are indifferent.

g) Suppose there is an optimal solution with $x_{me} > 0$ and $x_{vs} > 0$. What can you infer about the ratios $\frac{a_{me}}{a_{ve}}$ and $\frac{a_{ms}}{a_{vs}}$?

By CS of the dual:

- If $x_{me} > 0 \longrightarrow w_e = \frac{\beta}{a_{me}^{\circ}}$

- If $x_{vs} > 0 \longrightarrow w_s = \frac{\alpha}{a_{vs}^{\circ}}$

2 cases $\longrightarrow$ I all activities are used; $x_{me}; x_{ve}; x_{ms}; x_{vs} >> 0$ then we know from f) that that implies $\frac{a_{ve}}{a_{me}} = \frac{a_{vs}}{a_{ms}}$

→ II If $x_{me} > 0$ and $x_{vs} > 0$, but $x_{ve} = 0$ and $x_{ms} = 0$

$\begin{cases} E \text{ only produces } m \\ S \text{ only produces } v \end{cases}$

By CS of dual:

(3) $(-p_v + w_e a_{ve}^\circ) x_{ve} = 0 \rightarrow$ if $x_{ve} = 0 \rightarrow w_e \geq \frac{p_v}{a_{ve}^\circ} = \frac{\alpha}{a_{ve}^\circ}$

(4) $(-p_m + w_e a_{me}^\circ) x_{me} = 0 \rightarrow$ if $x_{me} > 0 \rightarrow w_e = \beta / a_{me}^\circ$

(5) $(-p_v + w_s a_{vs}^\circ) x_{vs} = 0 \rightarrow$ if $x_{vs} > 0 \rightarrow w_s = \alpha / a_{vs}^\circ$

(6) $(-p_m + w_s a_{ms}^\circ) x_{ms} = 0 \rightarrow$ if $x_{ms} = 0 \rightarrow w_s \geq \beta / a_{ms}^\circ$

combine $\rightarrow \quad \frac{\beta}{a_{me}^\circ} \geq \frac{\alpha}{a_{ve}^\circ} \rightarrow \frac{\alpha}{\beta} \leq \frac{a_{ve}^\circ}{a_{me}^\circ}$

$\rightarrow \quad \frac{\alpha}{a_{vs}^\circ} \geq \frac{\beta}{a_{ms}^\circ} \rightarrow \frac{\alpha}{\beta} \geq \frac{a_{vs}^\circ}{a_{ms}^\circ}$

$$\Rightarrow \boxed{\frac{a_{ve}^\circ}{a_{ms}^\circ} \geq \frac{a_{vs}^\circ}{a_{ms}^\circ}}$$

The relative cost of producing wine wrt mutton is cheaper in Spain → Spain should produce wine.
Same argument for mutton in England.