

## Logistics

- TA Office Hours:
  - Mon & Wed 18:30-19:30pm
  - In person (Uris 465) or by Zoom (same link as before).
- All problem sets are due before Friday section.
- Section's material [https://github.com/luciacasal/-Cornell\\_ECON6100](https://github.com/luciacasal/-Cornell_ECON6100).

## 1 Simple Leontief Model <sup>1</sup>

### 1.1 Important concepts:

- **Definition 1: Input requirements (or activity) matrix  $A$ :**

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

where:

- $a_{ij}$  denotes the amount of good  $i$  needed to produce 1 unit of good  $j$ .
- $a_{ij} \geq 0$
- for each  $i$  there exists a  $j$  such that  $a_{ij} > 0$ .
- Column vector  $A^j = \begin{bmatrix} a_{1j} \\ \vdots \\ \vdots \\ a_{nj} \end{bmatrix}$  describes the amount of all goods required to produce 1 unit of good  $j$ .
- Example:

$$A = \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{bmatrix}$$

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<sup>1</sup>These notes are based on Jaden Chen's notes from 2020 and Abhi Ananth notes from 2021.

- **Definition 2: Productive matrix.** Input matrix  $A$  is productive if there exists a  $x^* \geq 0$  such that  $x^* \gg Ax^*$ .

– Example (non- productive matrix):

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- **Theorem 1:** If  $A$  is productive, then for any  $y \geq 0$ , the system  $(I - A)x = y$  has a non-negative solution.
- **Theorem 2:**  $A$  is productive if and only if  $(I - A)$  has full rank and  $(I - A)^{-1}$  is non-negative.

Now, the above analysis imposes no constraint on production, so when  $A$  is productive you can produce any amount you want. Indeed, the missing ingredient we need for the model to be useful is labor, the primary factor of production. The labor requirement vector is given by  $a_0 = (a_{01}, \dots, a_{0n})$ , where  $a_{0j}$  is the amount of labor needed to produce one unit of output  $j$ . Let  $L$  be the supply of labor in the economy. It now makes sense to describe what we can produce:

- **Definition 3:** The set of feasible net output is:

$$Y = \{y \in \mathbb{R}^n : a_0 \cdot (I - A)^{-1}y \leq L, y \geq 0\}$$

We have a constraint on how much we can produce. A natural question then would be how much should we produce? To do this we need to think about costs and revenue, so we need prices. Let  $p = (p_1, \dots, p_n)$  be the prices of goods and let  $w$  denote the wage rate.

- **Definition 4:** The profit from 1 unit of good  $i$  would be:

$$\pi_i = p_i - (wa_{0i} + p_1a_{1i} + \dots + p_na_{ni})$$

Rearranging and putting this in vector form gives the rate of profit  $\pi = p \cdot (I - A) - wa_0$ . With gross output vector  $x$ , the profit is  $p \cdot x$ .

- **Theorem 3:**

– If  $A$  is productive and  $a_0 \gg 0$ ;

OR:

– If  $A$  is productive and irreducible and  $a_0 > 0$ ;

Then there is a  $(w^*, p^*) \gg 0$  such that  $\pi^* = 0$  and  $p^* = w^* a_0 (I - A)^{-1}$ .

$$y = (I - A)x$$

$$x = (I - A)^{-1}y$$

how to interpret  $(I - A)^{-1}$ ?

Take  $\tilde{y} = y + e_1$  with  $e_1 = (1, \dots, 0)$

we have  $x = (I - A)^{-1}y$   $\left\{ \begin{array}{l} \tilde{x} - x = (I - A)^{-1}e_1 \\ \tilde{x} = (I - A)^{-1}(y + e_1) \end{array} \right.$

When you want to produce one more unit of good <sup>im</sup>1,  
(column <sup>im</sup>1 of  $y$ ), then you need an additional  
 $(I - A)^{-1}e_1$  resources (<sup>im</sup>1st column of  $(I - A)^{-1}$ ).

## 1.2 Questions:

1. Suppose the activity matrix for a Simple Leontief Model is  $A = \begin{bmatrix} 1/3 & 1/4 \\ a & 1/2 \end{bmatrix}$  and labor input requirements are  $a_0 = (1/3, 1/2)$ .
- For what values of  $a$  is  $A$  productive?
  - Let  $L = 5$ , describe the set of feasible net outputs in terms of  $a$ .
  - Let  $w = 1$ , describe the set of equilibrium prices in terms of  $a$ .

$$A = \begin{bmatrix} 1/3 & 1/4 \\ a & 1/2 \end{bmatrix} \quad a_0 = (1/3; 1/2)$$

a) For what values of  $a$  is  $A$  productive?

$A$  is productive iff  $(I-A)^{-1}$  exists and is non-negative.

$$(I-A) \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/3 & 1/4 \\ a & 1/2 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/4 \\ -a & 1/2 \end{bmatrix}$$

•  $\det(I-A) = \frac{2}{3} \cdot \frac{1}{2} - \left(\frac{1}{4} \cdot a\right) = \frac{1}{3} - \frac{1}{4}a \neq 0$  iff  $a \neq \frac{4}{3}$ . ① exists.

•  $(I-A)^{-1} = \frac{1}{\left(\frac{1}{3} - \frac{a}{4}\right)} \cdot \begin{bmatrix} 1/2 & 1/4 \\ a & 2/3 \end{bmatrix} \geq 0$  iff  $\left\{ \begin{array}{l} a \geq 0 \\ \frac{1}{3} - \frac{a}{4} > 0 \rightarrow a < \frac{4}{3} \end{array} \right.$  ② non-neg.

$\Rightarrow$  We need  $a \in \left[0; \frac{4}{3}\right)$  for  $A$  to be productive.

b) Let  $L=5$ ; set of feasible net outputs in terms of  $a$ .

$$Y = \{ y \geq 0; y \in \mathbb{R}^n : a^0 (I-A)^{-1} y \leq L \}$$

$$\rightarrow (I-A)^{-1} = \frac{1}{\left(\frac{1}{3} - \frac{a}{4}\right)} \begin{bmatrix} 1/2 & 1/4 \\ a & 2/3 \end{bmatrix} = \frac{12}{4-3a} \begin{bmatrix} 1/2 & 1/4 \\ a & 2/3 \end{bmatrix}$$

$$(I-A)^{-1} = \begin{bmatrix} \frac{6}{4-3a} & \frac{3}{4-3a} \\ \frac{12a}{4-3a} & \frac{8}{4-3a} \end{bmatrix}$$

$$\rightarrow \overbrace{\begin{pmatrix} 1/3 & 1/2 \end{pmatrix}}^{a_0} \cdot \underbrace{(I-A)^{-1}}_{2 \times 2} = \begin{bmatrix} \frac{1}{3} \cdot \frac{6}{(4-3a)} + \frac{1}{2} \left( \frac{12a}{4-3a} \right) & \frac{1}{3} \left( \frac{3}{4-3a} \right) + \frac{1}{2} \left( \frac{8}{4-3a} \right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2+6a}{4-3a} & \frac{5}{4-3a} \end{bmatrix}$$

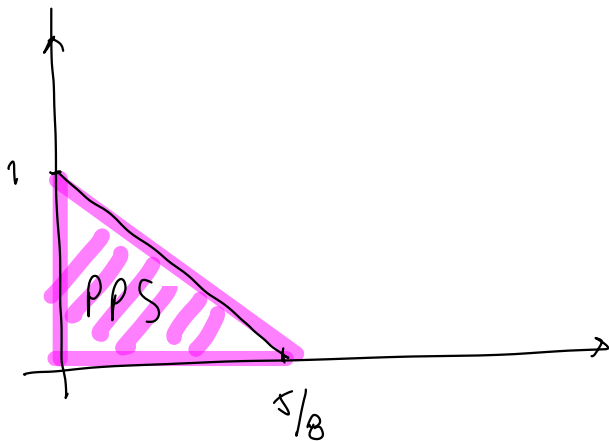
$$\rightarrow a_0 (I-A)^{-1} \cdot y = \begin{bmatrix} \frac{2+6a}{4-3a} & \frac{5}{4-3a} \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{2+6a}{4-3a} y_1 + \frac{5}{4-3a} y_2$$

$$\rightarrow Y = \left\{ y \geq 0; y \in \mathbb{R}^2 : \frac{2+6a}{4-3a} \cdot y_1 + \frac{5}{4-3a} \cdot y_2 \leq 5 \right\}$$

e.g.  $a=1$

$$Y = \{ y \geq 0; y \in \mathbb{R}^2: 8y_1 + 5y_2 \leq 5 \}$$

$$y_2 = 1 - \frac{8}{5}y_1$$



c) Let  $w=1$  set of eq prices in terms of  $a$ .

Use Thm 3. if  $a_0 \gg 0$  ✓ and  $A$  is productive ✓  
 $a \in [0; 4/3)$

then  $\pi^* = 0$  and  $p^* = \underbrace{w^* a_0}_{\gg 0} (I-A)^{-1}$

$$p^* = \begin{bmatrix} \frac{2+6a}{4-3a} & \frac{5}{4-3a} \end{bmatrix} \gg 0$$

2. (From LM problem set) Are the following matrices productive? Prove your answers.

$$A_1 = \begin{bmatrix} 0.6 & 0.2 & 0.1 \\ 0.3 & 0.2 & 0.4 \\ 0.2 & 0.4 & 0.3 \end{bmatrix} \quad \text{for } x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad A_1 \text{ is productive}$$

$$A_2 = \begin{bmatrix} 0.6 & 0.5 \\ 0.1 & 0.5 \end{bmatrix} \quad \text{for } x_2 = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} \quad A_2 \text{ is productive}$$

(it is enough to find one vector  $x \geq 0$  for which the matrix is productive!)