Logistics

- TA Office Hours: Mon & Wed 18:30-19:30pm.
- Same link for office hours and sections.
- All Problem Sets are due before Friday section.
- Please send the Problem Set by email to lc944@cornell.edu

1 Linear programming (continued)[1]

- **Fundamental Theorem of LP:** If a linear program in standard form has a feasible solution, then it has a basic feasible solution; if it has an optimal solution, then it has a basic optimal solution.

2 Duality of Linear Programming

2.1 Important concepts:

- **Definition 1: Duality.** Given a linear program in canonical form, we write the *primal problem* and the *dual problem*:

Primal	*Dual*
$V_P(b) = max\ c^T x$	$V_D(c) = min\ y^T b$
$s.t.\ Ax \leq b$	$s.t.\ y^T A \geq c^T$
$x \geq 0$	$y \geq 0$

- **Theorem 1: Duality.** For a primal problem and a dual problem:
  1. Both are feasible and have optimal solutions, and $V_P(b) = V_D(c)$.
  2. One is unbounded, one is infeasible.
  3. Neither is feasible.

[1] These notes are based on Jaden Chen's notes from 2020 and Abhi Ananth notes from 2021.

- **Theorem 2: Complementary slackness.** If x and y are feasible solutions for the primal and the dual respectively, then they are optimal if and only if $y^T(b - Ax) = 0$ and $(y^T A - c)x = 0$.

$x^T(y^T A - c^T) = 0$ $\quad$ $(b - Ax)y^T = 0$

  - Example: Consider the following primal problem:

$$
\begin{array}{lll}
max & 2x_1 + x_2 & \\
s.t. & x_1 + x_2 & \leq \quad 1 \\
 & x_1 \geq 0 & \\
 & x_2 \geq 0 &
\end{array}
$$

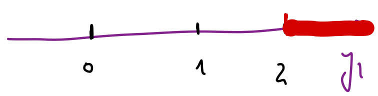
$Ax \leq b$

$$\max \underbrace{\begin{pmatrix} 2 & 1 \end{pmatrix}}_{c} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x}$$

$$\text{st} \quad \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \underbrace{\begin{pmatrix} 1 \end{pmatrix}}_{b}$$

$x_1, x_2 \geq 0$

The corresponding dual problem is:



$$
\begin{array}{llll}
min & y_1 & & \\
s.t. & y_1 & \geq & 2 \\
 & y_1 & \geq & 1 \\
 & y_1 & \geq & 0
\end{array}
$$

min $\quad y_1 \cdot \overbrace{1}^{b}$

$[x_1]$ $\quad$ st $[x_2]$ $\quad y_1 \cdot \overbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}^{A} \geq \overbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}^{c}$

$y_1 \geq 0$

Clearly, the solution to the dual is $y_1^* = 2$. According to CS, if y_1^* is an optimal solution of the dual, then an optimal solution of the primal must then satisfy:

$$
\begin{aligned}
y^*(b - Ax^*) &= 0 \\
y_1^*(1 - x_1^* - x_2^*) &= 0 \\
x_1^* + x_2^* &= 1 \qquad (1)
\end{aligned} \tag{1}
$$

$0 <$

Also, CS states that if x_1^*, x_2^* is an optimal solution of the primal, then an optimal solution of the dual must satisfy:

$$
\begin{aligned}
(y^{*T} A - c)x^* &= 0 \\
(y_1^* - 2)x_1^* + (y_1^* - 1)x_2^* &= 0 \qquad (2)
\end{aligned} \tag{2}
$$

$$\underbrace{\begin{pmatrix} y_1 - 2 & y_1 - 1 \end{pmatrix}}_{y^T A - c} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x^*}$$

by CS

$$\begin{cases} (y_1^* - 2)x_1^* = 0 \\ (y_1^* - 1)x_2^* = 0 \end{cases}$$

then this must hold!

(3)

Substituing $y_1^* = 2$, and combining 1 and 2 we obtain the optimal solution to the primal: $(x_1^*, x_2^*) = (1, 0)$.

2.2 Questions:

1. (From Section 1). Consider the primal problem:

$$\begin{array}{lll} max & x_1 + x_2 & \\ s.t. & x_1 + 2x_2 & \leq 6 \\ & x_1 - x_2 & \leq 3 \\ & x_1 \geq 0 & \\ & x_2 \geq 0 & \end{array}$$

   (a) Draw the constraint set and solve graphically.
   (b) State and solve the dual problem.
   (c) Verify that the values coincide and that the complementary slackness conditions hold.

2. Consider the primal problem:

$$\begin{array}{lll} max & x_1 + 2x_2 & \\ s.t. & x_1 + x_2 & \leq 4 \\ & x_1 + 3x_2 & \leq b \\ & x_1 \geq 0 & \\ & x_2 \geq 0 & \end{array}$$

   (a) Draw the constraint set.
   (b) Solve the problem and plot $V_P(b)$.
   (c) State and solve the dual problem. How does the solution of the dual problem depend on b?
   (d) Let $b = 6$, verify the complementary slackness conditions.

3. Solve the following linear program:

$$\begin{array}{lll} min & 4y_1 + 12y_2 + y_3 & \\ s.t. & y_1 + 4y_2 - y_3 & \geq 1 \\ & 2y_1 + 2y_2 + y_3 & \geq 1 \\ & y_1 \geq 0 & \\ & y_2 \geq 0 & \\ & y_3 \geq 0 & \end{array}$$

2.2 Questions:

1. (From Section 1). Consider the primal problem:

$$\begin{array}{lccc} max & x_1 + x_2 & & \\ s.t. & x_1 + 2x_2 & \leq & 6 \\ & x_1 - x_2 & \leq & 3 \\ & x_1 \geq 0 & & \\ & x_2 \geq 0 & & \end{array}$$

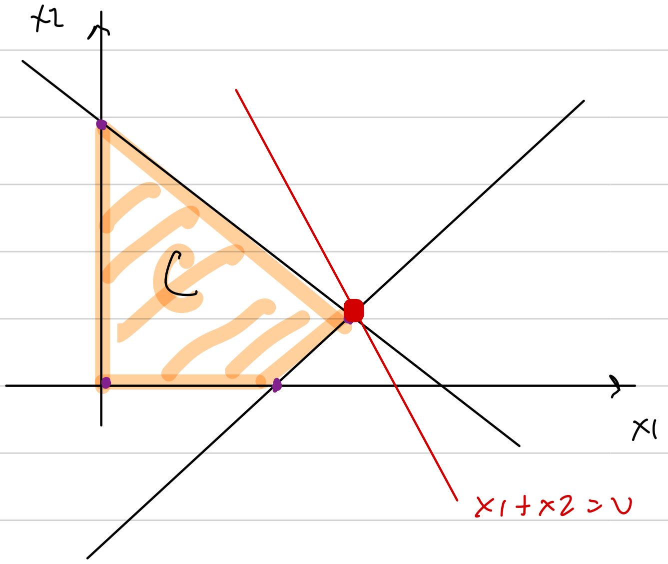
   (a) Draw the constraint set and solve graphically.

   (b) State and solve the dual problem.

   (c) Verify that the values coincide and that the complementary slackness conditions hold.

(a)

Primal



$(x_1^*; x_2^*) = (4, 1)$

(b)

Dual?

Primal

$$\begin{array}{lccc} max & x_1 + x_2 & & \\ s.t. & x_1 + 2x_2 & \leq & 6 \\ & x_1 - x_2 & \leq & 3 \\ & x_1 \geq 0 & & \\ & x_2 \geq 0 & & \end{array}$$

In matrix form: PRIMAL

$$\max \ \overbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}^{c} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\overbrace{\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \overbrace{\begin{pmatrix} 6 \\ 3 \end{pmatrix}}^{b}$$

$x_1, x_2 \geq 0$

DUAL:

$$\min \quad (y_1 \;\; y_2)\begin{pmatrix}6\\3\end{pmatrix}$$

$$\text{st} \quad (y_1 \;\; y_2)\begin{pmatrix}1 & 2\\1 & -1\end{pmatrix} \geq \begin{pmatrix}1\\1\end{pmatrix}$$

$$y_1, y_2 \geq 0$$

DUAL $\rightarrow$

$$\begin{cases} \min \quad 6y_1 + 3y_2 \\ \text{st} \quad y_1 + y_2 \geq 1 \quad [x_1] \\ \qquad 2y_1 - y_2 \geq 1 \quad [x_2] \\ \qquad y_1, y_2 \geq 0 \end{cases}$$

Because CS: $x^*(yA - c) = 0$

$$\text{CS} \begin{cases} (y_1 + y_2 - 1)\, x_1^* = 0 \\ (2y_1 - y_2 - 1)\, x_2^* = 0 \end{cases}$$

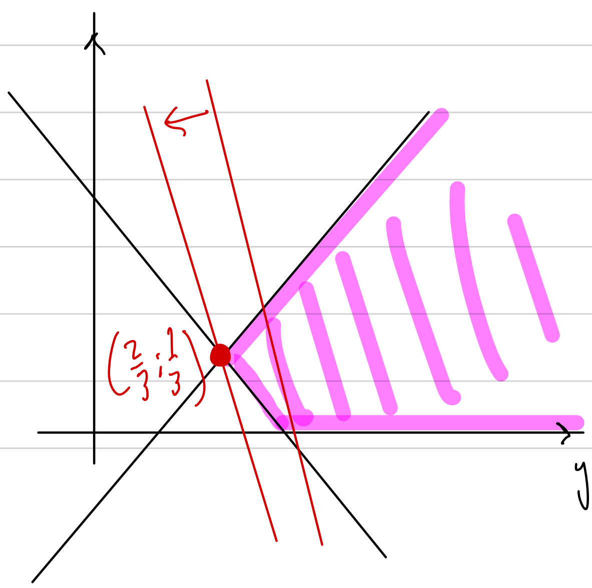
we know from part a) $x_1^* = 4$

$x_2^* = 1$

By CS $\rightarrow$ $y_1 + y_2 = 1$ $\longrightarrow$ DUAL: $(y_1^*; y_2^*) = \left(\frac{2}{3}; \frac{1}{3}\right)$

$2y_1 - y_2 = 1$

Graphically:



2. Consider the primal problem:

$$\begin{array}{lrcl} max & x_1 + 2x_2 & & \\ s.t. & x_1 + x_2 & \leq & 4 \\ & x_1 + 3x_2 & \leq & b \\ & x_1 \geq 0 & & \\ & x_2 \geq 0 & & \end{array}$$

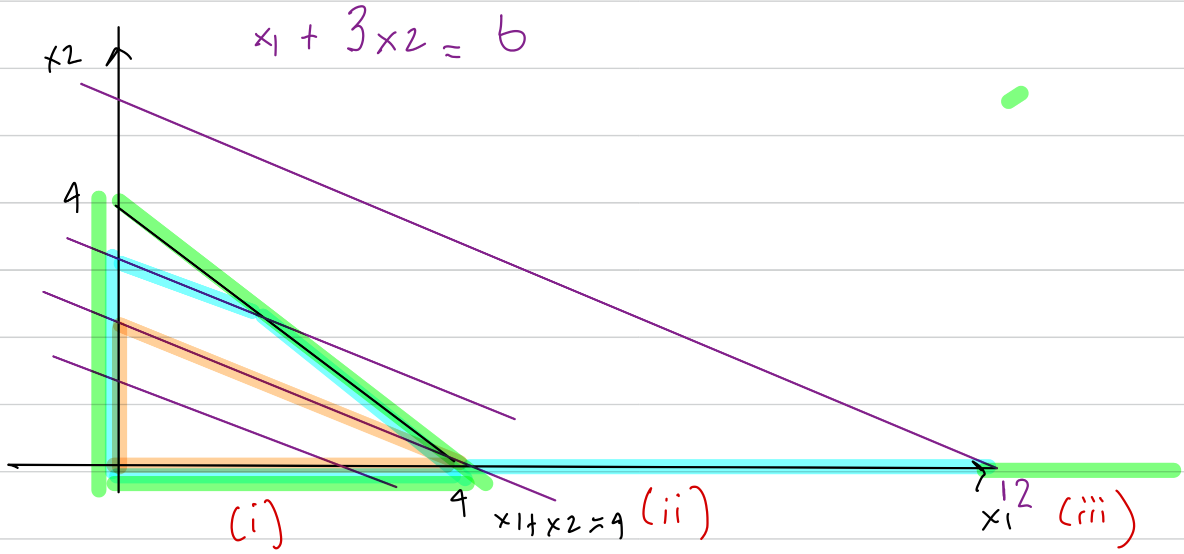
(a) Draw the constraint set.

(b) Solve the problem and plot $V_P(b)$.

(c) State and solve the dual problem. How does the solution of the dual problem depend on b?

(d) Let $b = 6$, verify the complementary slackness conditions.

(a)

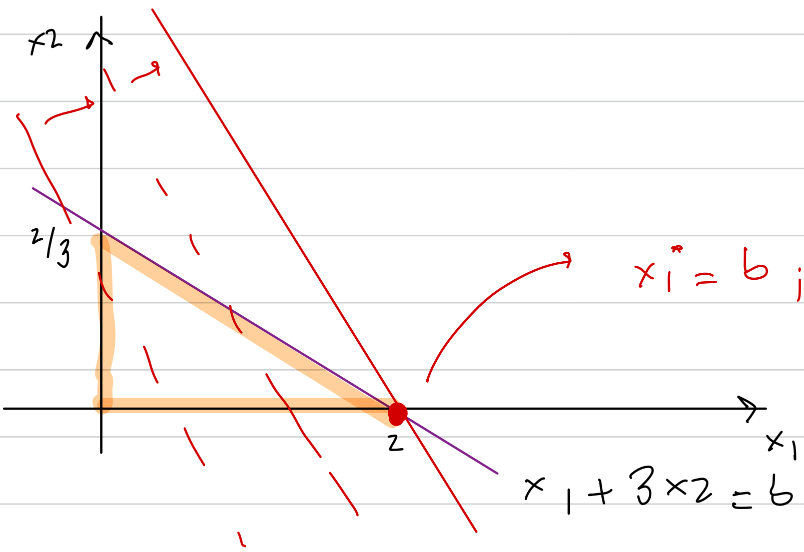


3 regions: (i) $b \leq 4$

(ii) $4 < b \leq 12$

(iii) $b > 12$

b) case i) $b \leq 4$



$x_1^* = b \; ; \; x_2^* = 0$

$VP = b$

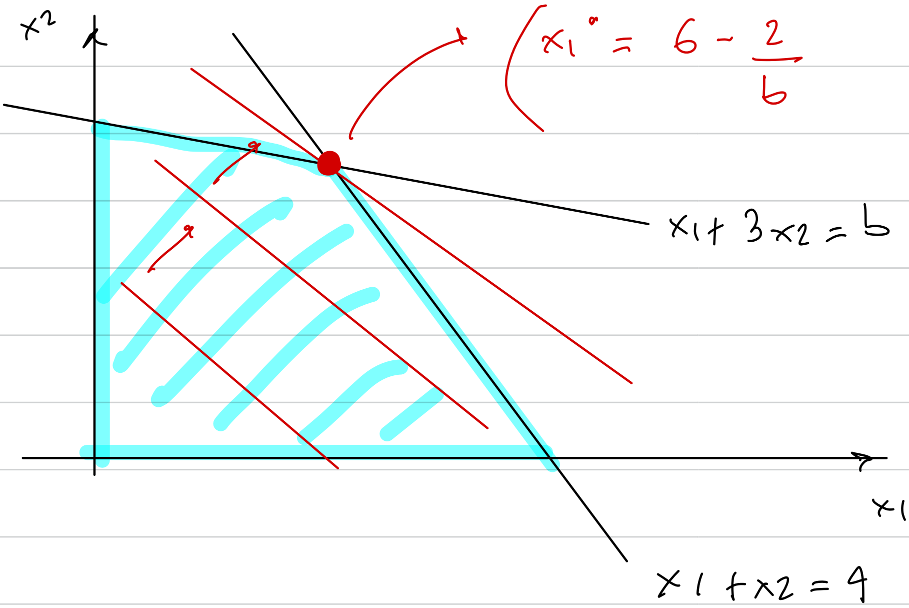
case ii) $4 < b \leq 12$

$x_1^* + 3x_2^* = b$

$x_1^* + x_2^* = 4$

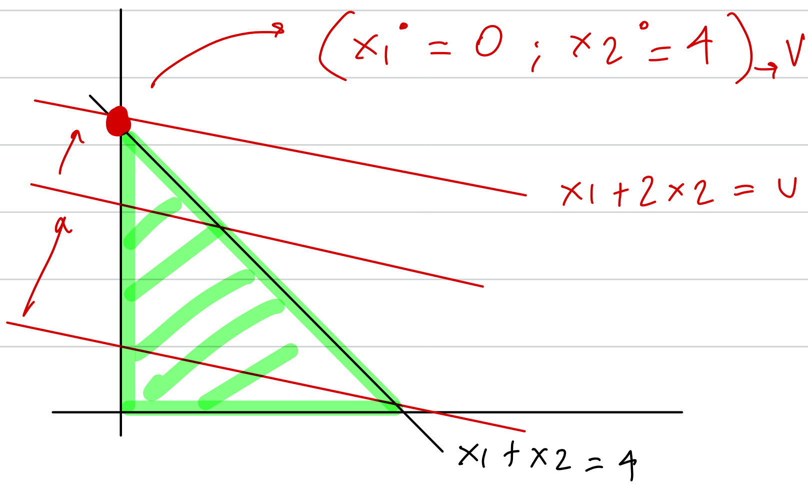
$\left(x_1^* = 6 - \frac{2}{b} \qquad x_2^* = 2 - \frac{b}{2} \right)$

$VP = 2 + \frac{b}{2}$



case iii) $b > 12$

$(x_1^* = 0 \; ; \; x_2^* = 4) \rightarrow VP = 8$



c)

PRIMAL

$$\begin{array}{lrcl} max & x_1 + 2x_2 & & \\ s.t. & x_1 + x_2 & \leq & 4 \\ & x_1 + 3x_2 & \leq & b \\ & x_1 \geq 0 & & \\ & x_2 \geq 0 & & \end{array}$$

et.

$$\max \overbrace{\begin{pmatrix} 1 & 2 \end{pmatrix}}^{c} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\overbrace{\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \overbrace{\begin{pmatrix} 4 \\ b \end{pmatrix}}^{b}$$

$$x_1, x_2 \geq 0$$

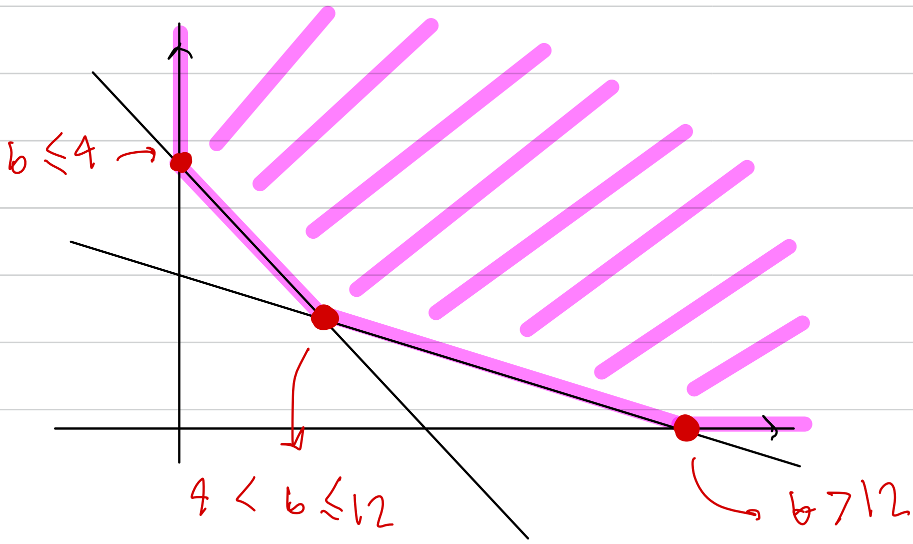
DUAL:

$$\begin{cases} \min \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} 4 \\ b \end{pmatrix} \\ \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ y_1, y_2 \geq 0 \end{cases}$$

DUAL:

$$\begin{cases} \min & 4y_1 + by_2 \\ st & y_1 + y_2 \geq 1 \quad [x_1] \\ & y_1 + 3y_2 \geq 2 \quad [x_2] \\ & y_1, y_2 \geq 0 \end{cases}$$

case (i)	case (ii)	case (iii)
$x_1^* = b$	$x_1^* > 0$	$x_1^* = 0$
$x_2^* = 0$	$x_2^* > 0$	$x_2^* = 4$
$y_1 + y_2 - 1 = 0$	$y_1 + y_2 - 1 = 0$	$y_1 + 3y_2 - 2 = 0$
	$y_1 + 3y_2 - 2 = 0$	



3. Solve the following linear program:

Primal

$$\begin{array}{lrcl} min & 4y_1 + 12y_2 + y_3 & & \\ s.t. & y_1 + 4y_2 - y_3 & \geq & 1 \quad [x1] \\ & 2y_1 + 2y_2 + y_3 & \geq & 1 \quad [x2] \\ & y_1 \geq 0 & & \\ & y_2 \geq 0 & & \\ & y_3 \geq 0 & & \end{array}$$

primal: $\min \overbrace{\begin{pmatrix} 4 & 12 & 1 \end{pmatrix}}^{c} \begin{pmatrix} y1 \\ y2 \\ y3 \end{pmatrix}$

st $\overbrace{\begin{pmatrix} 1 & 4 & -1 \\ 2 & 2 & 1 \end{pmatrix}}^{A} \begin{pmatrix} y1 \\ y2 \\ y3 \end{pmatrix} \geq \overbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}^{b}$

$y_1, y_2 \geq 0$

Dual: max $\begin{pmatrix} x1 & x2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

st $\begin{pmatrix} x1 \\ x2 \end{pmatrix} \begin{pmatrix} 1 & 4 & -1 \\ 2 & 2 & 1 \end{pmatrix} \leq \begin{pmatrix} 4 \\ 12 \\ 1 \end{pmatrix}$

DUAL

max $x_1 + x_2$

st

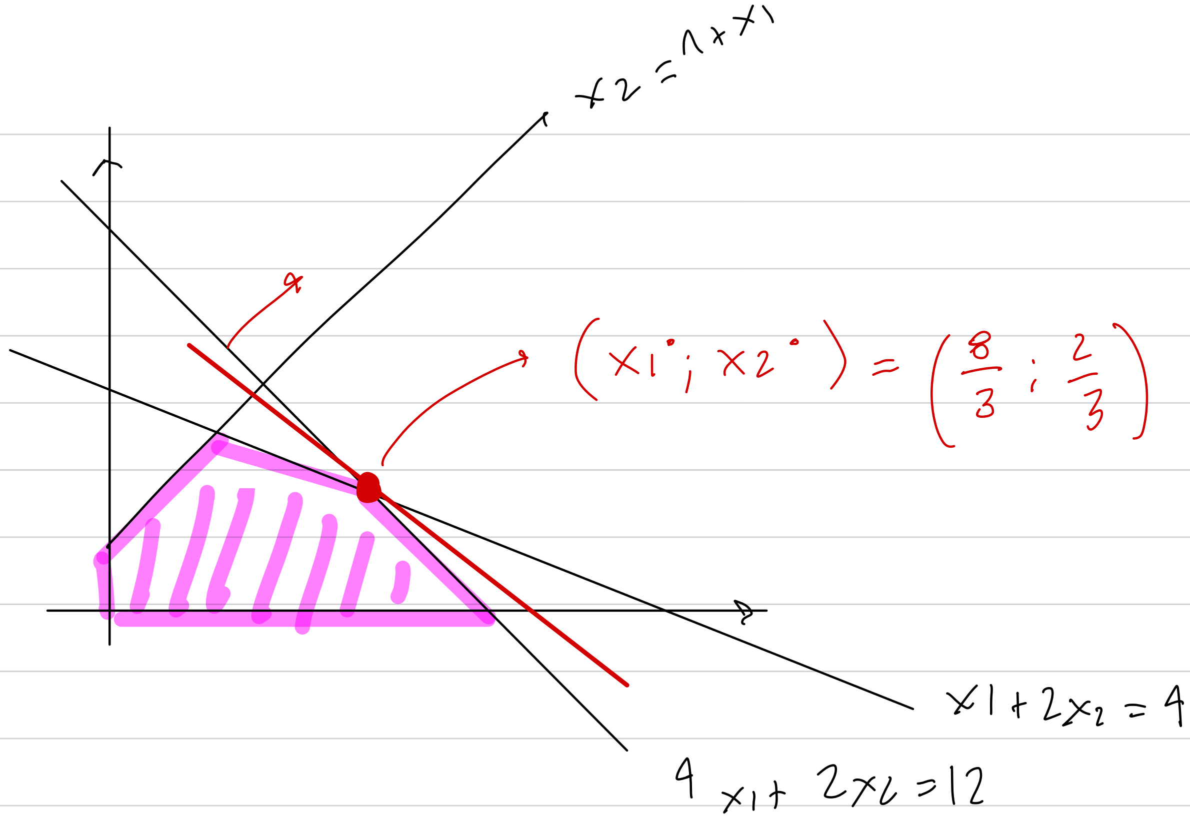
$x_1 + 2x_2 \leq 4$ [y1]

$4x_1 + 2x_2 \leq 12$ [y2]

$-x_1 + x_2 \leq 1$ [y3]

$x_1, x_2 \geq 0$

by CS: $\underbrace{(-x_1 + x_2 - 1)}_{\neq 0} y_3 = 0$



Going back to the primal:

program.

$$
\begin{array}{llll}
min & 4y_1 + 12y_2 + y_3 & & \\
s.t. & y_1 + 4y_2 - y_3 & \geq 1 & [x_1] \\
 & 2y_1 + 2y_2 + y_3 & \geq 1 & [x_2] \\
 & y_1 \geq 0 & & \\
 & y_2 \geq 0 & & \\
 & y_3 \geq 0 & &
\end{array}
$$

$\because\ x_1^* > 0 \quad x_2^* > 0 \xrightarrow{by}$ CS :

$$
\begin{cases} y_1 + 4y_2 - y_3 - 1 = 0 \\ 2y_1 + 2y_2 + y_3 - 1 = 0 \end{cases}
$$

$\because\ -x_1 + x_2 < 1 \rightarrow \boxed{y_3 = 0}$

$$
\rightarrow \begin{cases} y_1 + 4y_2 = 1 \\ 2y_1 + 2y_2 = 1 \end{cases} \rightarrow \boxed{\begin{array}{l} y_1^* = 1/3 \\ y_2^* = 1/6 \end{array}}
$$