

Logistics

- TA Office Hours: Mon & Wed 18:30-19:30pm.
- Same link for office hours and sections.
- All Problem Sets are due before Friday section.
- Please send the Problem Set by email to lc944@cornell.edu

1 Linear programming (continued)¹

- **Fundamental Theorem of LP:** If a linear program in standard form has a feasible solution, then it has a basic feasible solution; if it has an optimal solution, then it has a basic optimal solution.

2 Duality of Linear Programming

2.1 Important concepts:

- **Definition 1: Duality.** Given a linear program in canonical form, we write the *primal problem* and the *dual problem*:

$ \begin{aligned} & \textit{Primal} \\ & V_P(b) = \max c^T x \\ & \text{s.t. } Ax \leq b \\ & x \geq 0 \end{aligned} $	$ \begin{aligned} & \textit{Dual} \\ & V_D(c) = \min y^T b \\ & \text{s.t. } y^T A \geq c^T \\ & y \geq 0 \end{aligned} $
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- **Theorem 1: Duality.** For a primal problem and a dual problem:

1. Both are feasible and have optimal solutions, and $V_P(b) = V_D(c)$.
2. One is unbounded, one is infeasible.
3. Neither is feasible.

¹These notes are based on Jaden Chen's notes from 2020 and Abhi Ananth notes from 2021.

- **Theorem 2: Complementary slackness.** If x and y are feasible solutions for the primal and the dual respectively, then they are optimal if and only if $y^T(b - Ax) = 0$ and $(y^T A - c)x = 0$.

$$x^T (y^T A - c) = 0$$

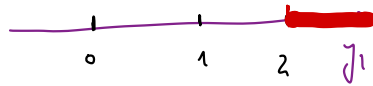
– Example: Consider the following primal problem:

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & (2 \ 1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \text{s.t.} \quad & (1 \ 1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq (1) \\ & x_1, x_2 \geq 0 \end{aligned}$$

The corresponding dual problem is:

$$\begin{aligned} \min \quad & y_1 \\ \text{s.t.} \quad & y_1 \geq 2 \\ & y_1 \geq 1 \\ & y_1 \geq 0 \end{aligned}$$



$$\begin{aligned} \min \quad & y_1 \cdot 1 \\ \text{s.t.} \quad & y_1 \cdot (1 \ 1) \geq \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & y_1 \geq 0 \end{aligned}$$

Clearly, the solution to the dual is $y_1^* = 2$. According to CS, if y_1^* is an optimal solution of the dual, then an optimal solution of the primal must then satisfy:

$$\begin{aligned} y^*(b - Ax^*) &= 0 \\ y_1^*(1 - x_1^* - x_2^*) &= 0 \\ 0 < \checkmark \quad x_1^* + x_2^* &= 1 \quad (1) \end{aligned}$$

Also, CS states that if x_1^*, x_2^* is an optimal solution of the primal, then an optimal solution of the dual must satisfy:

$$\begin{aligned} (y^{*T} A - c)x^* &= 0 \\ (y_1^* - 2)x_1^* + (y_1^* - 1)x_2^* &= 0 \end{aligned}$$

Substituting $y_1^* = 2$, and combining 1 and 2 we obtain the optimal solution to the primal: $(x_1^*, x_2^*) = (1, 0)$.

$$\begin{aligned} & (y_1^* - 2 \quad y_1^* - 1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \\ & \downarrow \text{by CS} \\ & \begin{cases} (y_1^* - 2)x_1 = 0 \\ (y_1^* - 1)x_2 = 0 \end{cases} \end{aligned}$$

then this must hold!

2.2 Questions:

1. (From Section 1). Consider the primal problem:

$$\begin{array}{ll} \max & x_1 + x_2 \\ \text{s.t.} & x_1 + 2x_2 \leq 6 \\ & x_1 - x_2 \leq 3 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{array}$$

- (a) Draw the constraint set and solve graphically.
- (b) State and solve the dual problem.
- (c) Verify that the values coincide and that the complementary slackness conditions hold.

2. Consider the primal problem:

$$\begin{array}{ll} \max & x_1 + 2x_2 \\ \text{s.t.} & x_1 + x_2 \leq 4 \\ & x_1 + 3x_2 \leq b \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{array}$$

- (a) Draw the constraint set.
- (b) Solve the problem and plot $V_P(b)$.
- (c) State and solve the dual problem. How does the solution of the dual problem depend on b ?
- (d) Let $b = 6$, verify the complementary slackness conditions.

3. Solve the following linear program:

$$\begin{array}{ll} \min & 4y_1 + 12y_2 + y_3 \\ \text{s.t.} & y_1 + 4y_2 - y_3 \geq 1 \\ & 2y_1 + 2y_2 + y_3 \geq 1 \\ & y_1 \geq 0 \\ & y_2 \geq 0 \\ & y_3 \geq 0 \end{array}$$

2.2 Questions:

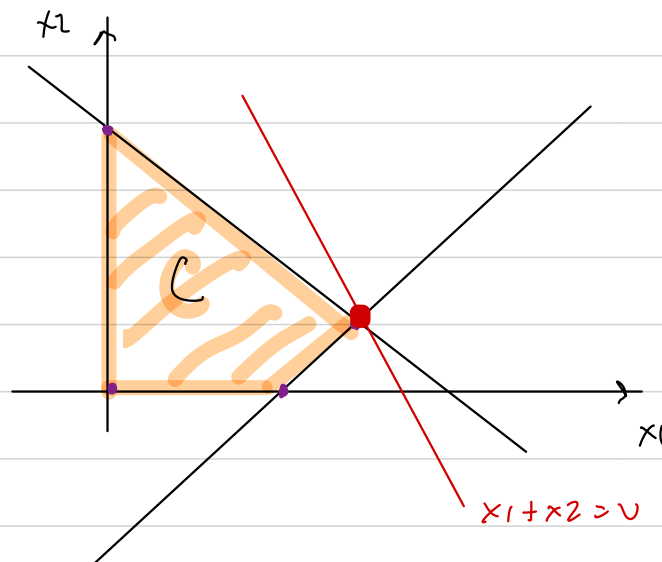
1. (From Section 1). Consider the primal problem:

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 6 \\ & x_1 - x_2 \leq 3 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

- Draw the constraint set and solve graphically.
- State and solve the dual problem.
- Verify that the values coincide and that the complementary slackness conditions hold.

(a)

Primal



$$(x_1^*, x_2^*) = (4, 1)$$

(b)

Dual?

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 6 \\ & x_1 - x_2 \leq 3 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

In matrix form: PRIMAL

$$\max \quad \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

$$x_1, x_2 \geq 0$$

DUAL:

$$\begin{aligned} \min \quad & (y_1 \ y_2) \begin{pmatrix} 6 \\ 3 \end{pmatrix} \\ \text{s.t.} \quad & (y_1 \ y_2) \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ & y_1, y_2 \geq 0 \end{aligned}$$

DUAL

$$\begin{aligned} \rightarrow \quad & \min \quad 6y_1 + 3y_2 \\ & \text{s.t.} \quad y_1 + y_2 \geq 1 \quad [x_1] \\ & \quad \quad 2y_1 - y_2 \geq 1 \quad [x_2] \\ & \quad \quad y_1, y_2 \geq 0 \end{aligned}$$

because CS:
 $x^*(yA - c) = 0$

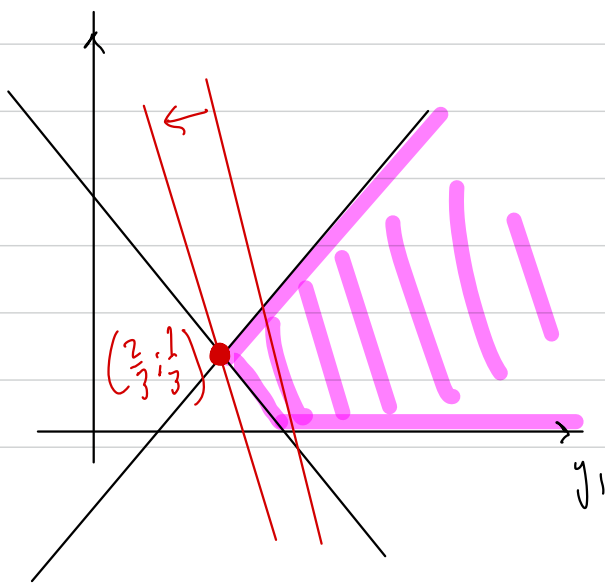
$$\text{CS} \quad \begin{cases} (y_1 + y_2 - 1) x_1^* = 0 \\ (2y_1 - y_2 - 1) x_2^* = 0 \end{cases}$$

we know from part a) $x_1^* = 4$
 $x_2^* = 1$

DUAL:

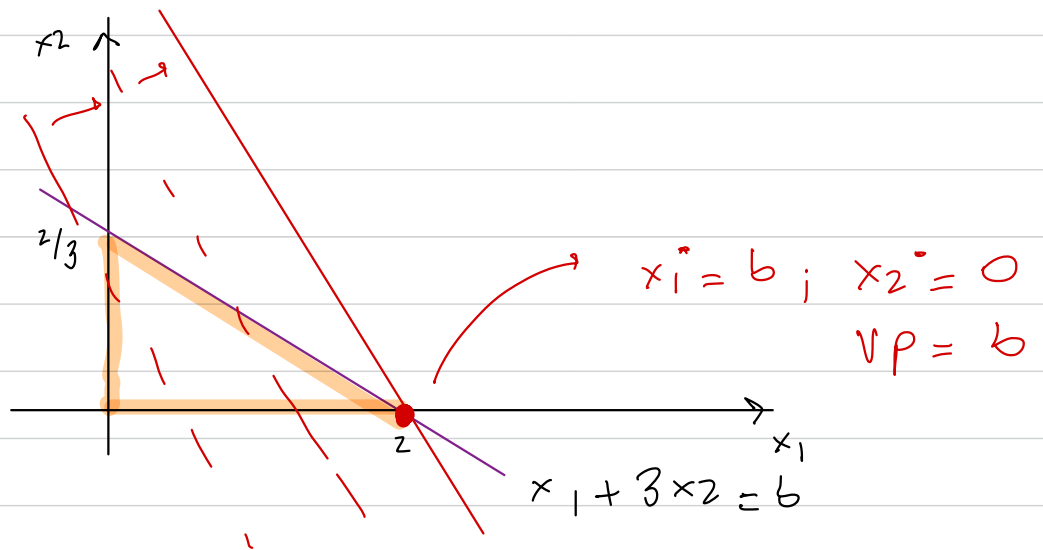
$$\text{By CS} \rightarrow \begin{aligned} y_1 + y_2 &= 1 \\ 2y_1 - y_2 &= 1 \end{aligned} \rightarrow (y_1^*, y_2^*) = \left(\frac{2}{3}, \frac{1}{3} \right)$$

Graphically:



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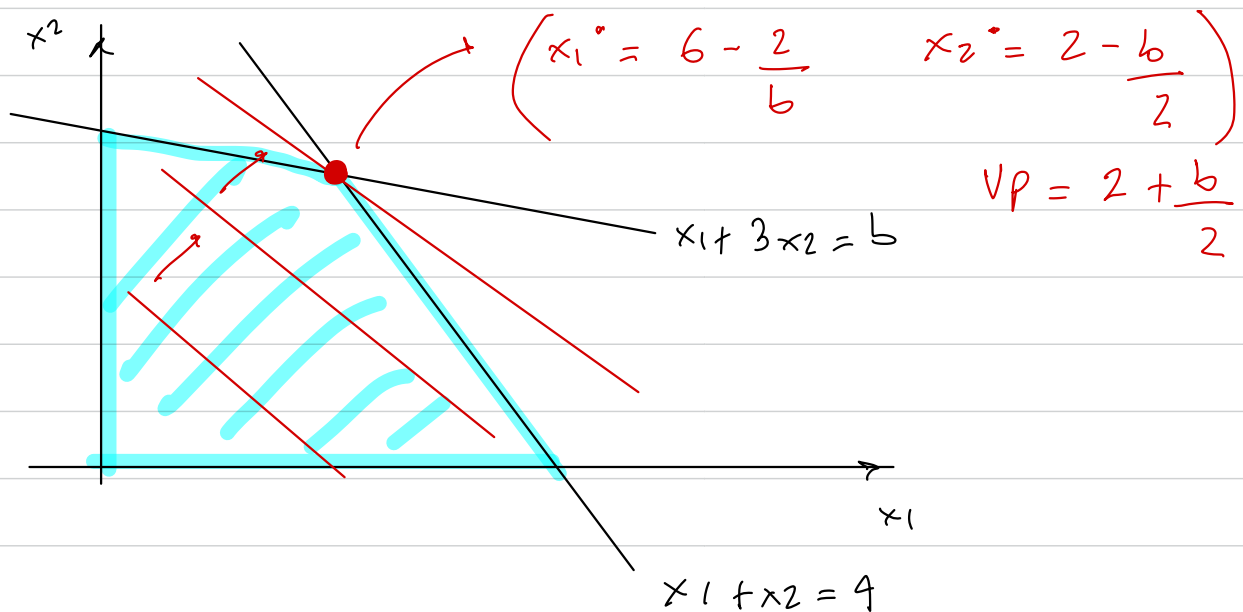
b) case i) $b \leq 4$



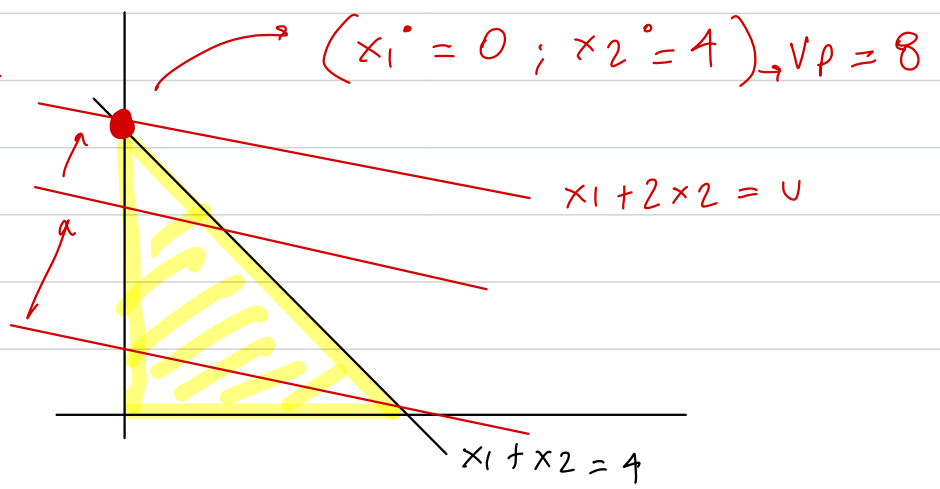
case ii) $4 < b \leq 12$

$$x_1^* + 3x_2^* = b$$

$$x_1^* + x_2^* = 4$$



case iii) $b > 12$



c)

PRIMAL

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 4 \\ & x_1 + 3x_2 \leq b \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

st.

$$\max \quad \overbrace{(1 \ 2)}^C \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \overbrace{\begin{pmatrix} 4 \\ b \end{pmatrix}}^b$$

$$x_1, x_2 \geq 0$$

DUAL:

$$\begin{cases} \min \quad (y_1 \ y_2) \begin{pmatrix} 4 \\ b \end{pmatrix} \\ (y_1 \ y_2) \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ y_1, y_2 \geq 0 \end{cases}$$

DUAL:

$$\begin{cases} \min \quad 4y_1 + by_2 \\ \text{s.t.} \quad y_1 + y_2 \geq 1 \\ y_1 + 3y_2 \geq 2 \\ y_1, y_2 \geq 0 \end{cases}$$

$$x_1 = b \quad x_1 > 0 \quad x_1 = 0$$

$$x_2 = 0 \quad x_2 > 0 \quad x_2 = 4$$

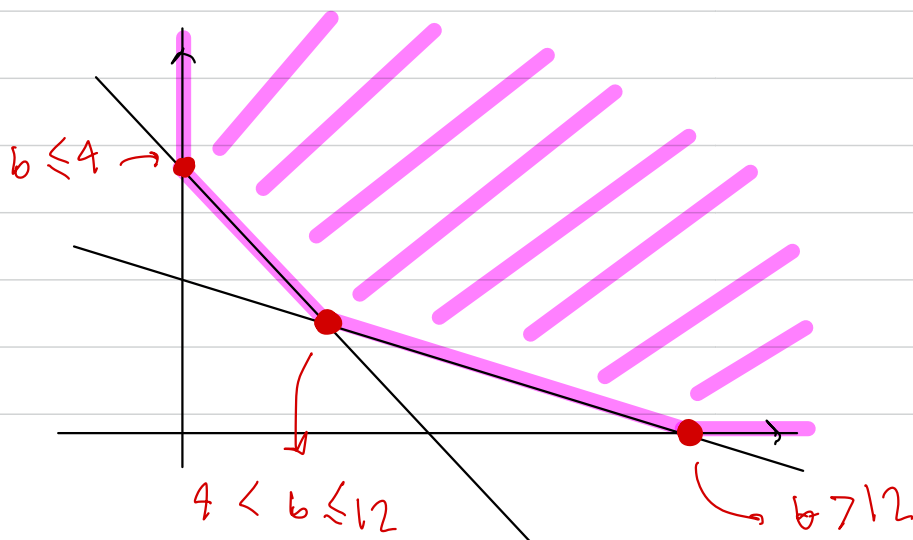
case (i) case (ii) case (iii)

$$[x_1] \quad y_1 + y_2 - 1 = 0 \quad \downarrow$$

$$y_1 + y_2 - 1 = 0$$

$$y_1 + 3y_2 - 2 = 0$$

$$y_1 + 3y_2 - 2 = 0$$



3. Solve the following linear program:

$$\begin{array}{ll}
 \text{Primal} & \min \quad 4y_1 + 12y_2 + y_3 \\
 & \text{s.t.} \quad y_1 + 4y_2 - y_3 \geq 1 \quad [x_1] \\
 & \quad \quad 2y_1 + 2y_2 + y_3 \geq 1 \quad [x_2] \\
 & \quad \quad y_1 \geq 0 \\
 & \quad \quad y_2 \geq 0 \\
 & \quad \quad y_3 \geq 0
 \end{array}$$

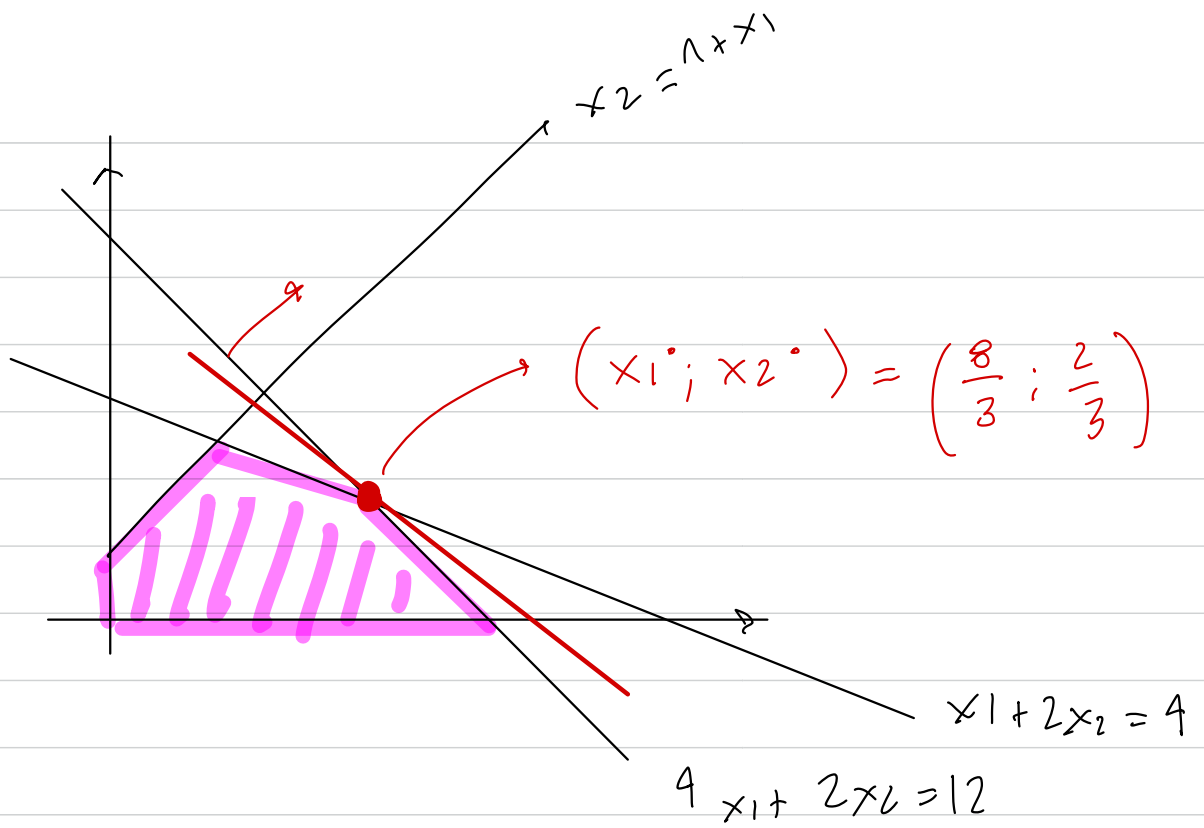
$$\left\{ \begin{array}{l}
 \text{primal: } \min \quad \overbrace{(4 \quad 12 \quad 1)}^c \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\
 \text{s.t.} \quad \begin{pmatrix} 1 & 4 & -1 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \geq \overbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}^b \\
 \quad \quad \quad y_1, y_2 \geq 0
 \end{array} \right.$$

$$\begin{array}{ll}
 \text{Dual:} & \max \quad (x_1 \quad x_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
 & \text{s.t.} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} 1 & 4 & -1 \\ 2 & 2 & 1 \end{pmatrix} \leq \begin{pmatrix} 4 \\ 12 \\ 1 \end{pmatrix}
 \end{array}$$

DUAL

$$\left\{ \begin{array}{ll}
 \max & x_1 + x_2 \\
 \text{s.t.} & x_1 + 2x_2 \leq 4 \quad [y_1] \\
 & 4x_1 + 2x_2 \leq 12 \quad [y_2] \\
 & -x_1 + x_2 \leq 1 \quad [y_3] \\
 & x_1, x_2 \geq 0
 \end{array} \right.$$

by Cs: $(-x_1 + x_2 - 1) y_3 = 0$
 $\neq 0$



Going back to the primal:

$$\begin{array}{ll}
 \min & 4y_1 + 12y_2 + y_3 \\
 \text{s.t.} & y_1 + 4y_2 - y_3 \geq 1 \quad [x_1] \\
 & 2y_1 + 2y_2 + y_3 \geq 1 \quad [x_2] \\
 & y_1 \geq 0 \\
 & y_2 \geq 0 \\
 & y_3 \geq 0
 \end{array}$$

$$\therefore x_1^* > 0 \quad x_2^* > 0 \rightarrow \text{CS:}$$

$$\begin{cases}
 y_1 + 4y_2 - y_3 - 1 = 0 \\
 2y_1 + 2y_2 + y_3 - 1 = 0
 \end{cases}$$

$$\therefore -x_1 + x_2 < 1 \rightarrow y_3 = 0$$

$$\rightarrow \begin{cases}
 y_1 + 4y_2 = 1 \\
 2y_1 + 2y_2 = 1
 \end{cases} \rightarrow \begin{cases}
 y_1^* = 1/3 \\
 y_2^* = 1/6
 \end{cases}$$