Exercise 1. *Suppose individual's valuations for houses are as follows: Find the optimal match, and prices for individuals and houses.*

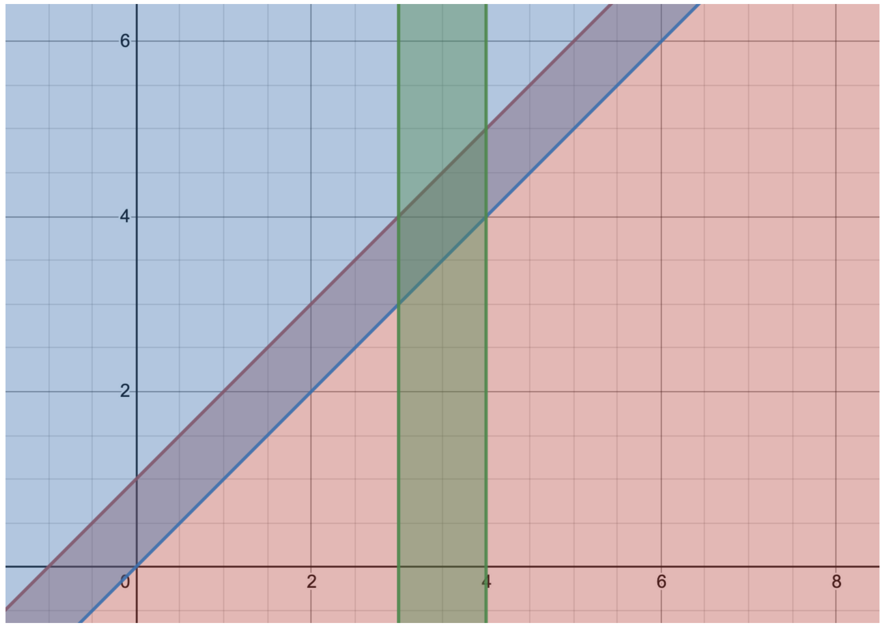
		Houses		
		A	*B*	*C*
	X	5	7	1
Workers	*Y*	2	3	1
	Z	5	4	4

		Houses		
		A	*B*	*C*
	X	5	7	1
Workers	*Y*	2	3	1
	Z	5	4	4

Proof. The two optimal matches are marked in red above. Each of them gives a surplus of 13. We illustrate how to find the surplus for individual and houses using the first optimal match. We find the dual solutions which satisfy the dual constraints and the complementary slackness condition:

$$
\begin{aligned}
w_X + \pi_A &\geq 5 \\
w_X + \pi_B &= 7 \\
w_X + \pi_C &\geq 1 \\
w_Y + \pi_A &\geq 2 \\
w_Y + \pi_B &\geq 3 \\
w_Y + \pi_C &= 1 \\
w_Z + \pi_A &= 5 \\
w_Z + \pi_B &\geq 4 \\
w_Z + \pi_C &\geq 4 \\
w_i, \pi_i &\geq 0 \ \forall i
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
w_X + 5 - w_Z &\geq 5 \\
w_X + 1 - w_Y &\geq 1 \\
w_Y + 5 - w_Z &\geq 2 \\
w_Y + 7 - w_X &\geq 3 \\
w_Z + 7 - w_X &\geq 4 \\
w_Z + 1 - w_Y &\geq 4 \\
w_i, \pi_i &\geq 0 \ \forall i
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
w_X &\geq w_Z \\
w_X &\geq w_Y \\
w_Z - w_Y &\leq 3 \\
w_X - w_Y &\leq 4 \\
w_X - w_Z &\leq 3 \\
w_Z - w_Y &\geq 3 \\
w_i, \pi_i &\geq 0 \ \forall i
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
w_Z - w_Y &= 3 \\
0 \leq w_Y &\leq 1 \\
3 \leq w_Z &\leq 4 \\
0 \leq w_X - w_Z &\leq 1 \\
\pi_A &= 5 - w_Z \\
\pi_B &= 7 - w_X \\
\pi_C &= 1 - w_Y
\end{aligned}
$$

Graphically, the range of stable wages and profits is shown in the plot below. Note that the y-axis represents w_X and the x-axis w_Z, so the overlapping region give all the stable (w_Z, w_X) pairs from which we can deduce all the other wages and profits.



The process to find the stable wages and profits for the second optimal match is analogous, so it is omitted. ☐

Exercise 2. *For each $10 > \theta > 5$, valuations increase in workers from bottom to top, for each firm, and valuations increase in firms from right to left for each worker. For what values of θ is the optimal match assortative?*

		Firms		
		A	*B*	*C*
	X	*10*	θ	*4*
Workers	*Y*	*7*	*5*	*3*
	Z	*4*	*3*	*2*

Proof. For $\theta \in (6, 8)$, the optimal match is assortative. Recall the theorem in the section in the case of a complete order among workers and firms: Every stable matching is assortative if the value function V has (strict) increasing differences, i.e., for all $l' > l$ and $f' > f$,

$$V(l', f') - V(l, f') > V(l', f) - V(l, f)$$

To make V to have (strict) increasing differences, we need:

$$\begin{aligned} 10 - \theta > 7 - 5 > 4 - 3 \\ \theta - 4 > 5 - 3 > 3 - 2 \\ 10 - 7 > \theta - 5 > 4 - 3 \end{aligned}$$

which reduces to $8 > \theta > 6$. ☐

Exercise 3. *Suppose that there are three populations: Low-skilled workers $l \in \mathcal{L}$, high-skilled workers $h \in \mathcal{H}$, and robots $r \in \mathcal{R}$. It takes one of each to to generate surplus: v_{lhr} is the surplus from the triple lhr. Define a competitive equilibrium, conjecture a relationship between competitive and optimal matches, and prove your conjecture.*

Proof. Let x_{lhr} equals 1 if l, h, and r are matched together, and 0 otherwise. Let the payoff vector (w_l, π_h, k_r) be the payoff to the low-skilled worker, high-skilled worker, and robot in a match, respectively. Mimicing the definition for a stable allocation in class, we define the following notion of a 'competitive equilibrium/stable match':

Definition. *An allocation (x, w, π, k) is* ***stable*** *if no currently unmatched threesome can increase their total surplus by matching together. That is,*

$$\text{If } x_{lhr} = 0, \text{then } w_l + \pi_h + k_r \geq v_{lhr}$$

$$\text{If } x_{lhr} = 1, \text{then } w_l + \pi_h + k_r = v_{lhr}$$

Under this definition, we conjecture that the following theorem still holds:

Theorem 0.1.

(a) If (x, w, π, k) is a ***stable*** *allocation, then x is an* ***optimal*** *matching.*

(b) If x is an ***optimal*** *matching, then $\exists(w, \pi, k)$ such that (x, w, π, k) is a* ***stable*** *allocation.*

We proved the original theorem for the worker-firm case using the duality theorem. Therefore, if we can still write out the primal and dual problem of this problem, the proof should follow.

The primal problem: A matching is **optimal** if it maximizes total surplus, i.e., it solves the following optimization problem (with Birkhoff-von Neuman theorem holds):

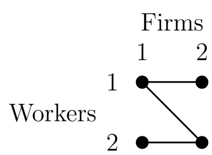
$$\begin{aligned} v_P(\mathcal{L} \cup \mathcal{H} \cup \mathcal{R}) = \max_{l,h,r} & \sum_{l,h,r} v_{lhr} x_{lhr} \\ \text{s.t.} \qquad & \sum_{h,r} x_{lhr} \leq 1 \text{ for all } l \in \mathcal{L}, \\ & \sum_{l,r} x_{lhr} \leq 1 \text{ for all } h \in \mathcal{H}, \\ & \sum_{l,h} x_{lhr} \leq 1 \text{ for all } r \in \mathcal{R}, \\ & x_{lhr} \geq 0 \text{ for all } l \in \mathcal{L}, h \in \mathcal{H}, r \in \mathcal{R}. \end{aligned} \tag{1}$$

The dual problem:

$$
\begin{aligned}
v_D(\mathcal{L} \cup \mathcal{H} \cup \mathcal{R}) = \min_{\pi, w, k} & \sum_l w_l + \sum_h \pi_h + \sum_r k_r \\
\text{s.t.} \quad & w_l + \pi_h + k_r \geq v_{lhr} \text{ for all } l \in \mathcal{L}, h \in \mathcal{H}, r \in \mathcal{R}, \\
& w_l, \pi_h, k_r \geq 0 \text{ for all } l \in \mathcal{L}, h \in \mathcal{H}, r \in \mathcal{R}.
\end{aligned} \tag{2}
$$

Using the duality theorem, if one of the problem is feasible, then both are feasible, both have optimal solutions, and their optimal value coincide. Then, it is clear that Theorem 0.1 is an immediate consequence. □

Exercise 4. *Suppose we are given workers $\mathcal{L}$ and firms $\mathcal{F}$, and a set of edges $\mathcal{E}$ connecting workers with firms. This is an example of a bi-partite graph. The edges describe the set of feasible matches, so that not all matches are possible. Here is an example:*



Worker 1 can match with both firms, but worker 2 can match only with firm 2 . Denote by $\delta(l)$ and $\delta(v)$ the set of firms (workers) worker l (firm f) can match with.

(a) Write down inequalities describing the set of feasible matches.

(b) On the assumption that the Birkhoff-von Neumann Theorem remains true (it does), how does the theory differ? Write down the primal and dual problems for the graph in the figure.

Proof.

(a)

$$
\begin{aligned}
x_{11} + x_{12} &\leq 1 \\
x_{22} &\leq 1 \\
x_{11} &\leq 1 \\
x_{12} + x_{22} &\leq 1 \\
x_{ij} &\geq 0 \\
(x_{21} &= 0)
\end{aligned}
$$

(b) The primal is

$$
\max \begin{pmatrix} v_{11} & v_{12} & v_{22} \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{22} \end{pmatrix}
$$

s.t.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{22} \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$x_{ij} \geq 0$$

$$x_{21} = 0$$

The dual is

$$\min \begin{pmatrix} w_1 & w_2 & \pi_1 & \pi_2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

s.t.

$$\begin{pmatrix} w_1 & w_2 & \pi_1 & \pi_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \geq \begin{pmatrix} v_{11} & v_{12} & v_{22} \end{pmatrix}$$

$$w_i, \pi_j \geq 0$$

Rewritten,

$$\min w_1 + w_2 + \pi_1 + \pi_2$$

s.t.

$$w_1 + \pi_1 \geq v_{11}$$

$$w_1 + \pi_2 \geq v_{12}$$

$$w_2 + \pi_2 \geq v_{22}$$

$$w_i, \pi_j \geq 0$$

We observe that:

(i) The total surplus of the population is weakly smaller. It can been seen from two perspectives. First, the primal problem is more constrained (with the additional $x_{21} = 0$ constraint), so the maximization problem has a weakly lower optimal value. Second, the dual problem is less constrained (missing the $w_2 + \pi_1 \geq v_{21}$ constraint), so the minimization problem could have a weakly lower value. But by duality, this equals the value of the primal, i.e., the social surplus.

(ii) This constrained matching problem cannot be a Pareto improvement of the unconstrained problem. This is because if that is the case, the total surplus of the constrained problem must exceeds that of the unconstrained problem, which contradicts (i).

(iii) Finally, note that even if the optimal matching in both cases are the same ($l_1 \leftrightarrow f_1$ and $l_2 \leftrightarrow f_2$), the stable wage and profit may still differ. Specifically, in the constrained case we should have

$$
\begin{aligned}
w_1 &= v_{11} \\
\pi_2 &= v_{22} \\
v_{11} + v_{22} &\geq v_{12}
\end{aligned}
$$

i.e., worker 1 and firm 2 are able to extract all the surplus because their counterparts do not have an outside option. The last ineuquality gives the condition under which the matching $l_1 \leftrightarrow f_1$ and $l_2 \leftrightarrow f_2$ is stable.

$\square$