Quantum State Tomography

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Abstract

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1 Introduction

Quantum State Tomography is the process in which a quantum state is reconstructed, in an approximated way, by performing a series of measurement on a large number of identically prepared copies of the original state. The noise of the experimental apparatus leads to some intrinsically problems, such as the uncertainty of the results, and the possible non-physical state as output. The latter is solved by performing a Maximum-Likelihood estimation.

In this experiment we want to infer the density matrix of a quantum state by a series of measurement performed on pairs of qubits.

2 Math tools

2.1 Density matrix

The density matrix $\hat{\rho}$ is an alternative way to describe a quantum state. It allows both the description of pure state and mixed state, under the same formalism. The density matrix is defined as follows:

$$\hat{\rho} = \sum_j p_j \left|\psi_j\right\rangle \left\langle\psi_j\right| = \frac{1}{2}(\mathbb{1} + \vec{\boldsymbol{r}} \cdot \vec{\sigma}) \tag{2.1}$$

For a matrix, in order to represent a physical state, the following properties have to be satisfied:

$$1) \quad \hat{\rho}^\dagger = \hat{\rho} \qquad \text{hermiticity} \tag{2.2}$$

$$2) \quad \mathrm{Tr}(\hat{\rho}) = 1 \qquad \text{normalization} \tag{2.3}$$

$$3) \quad \hat{\rho} \geq 1 \qquad \text{positivity} \tag{2.4}$$

In our case, we have 2 qubits and therefore a 4×4 density matrix. We want to convert it in a 16-dimensional column vector. We use the following conversion:

$$\hat{\rho} = \sum_{j=1}^{16} \hat{\Gamma}_j \, r_j \tag{2.5}$$

Where the $\hat{\Gamma}_j$ matrices are defined as follows:

$$\hat{\Gamma}_j = \frac{1}{2}\,\hat{\sigma}_0 \otimes \hat{\sigma}_j \qquad j = 1, 2, 3 \tag{2.6}$$

$$\hat{\Gamma}_{j+4} = \frac{1}{2}\,\hat{\sigma}_1 \otimes \hat{\sigma}_j \qquad j = 0, 1, 2, 3 \tag{2.7}$$

$$\hat{\Gamma}_{j+8} = \frac{1}{2}\,\hat{\sigma}_2 \otimes \hat{\sigma}_j \qquad j = 0, 1, 2, 3 \tag{2.8}$$

$$\hat{\Gamma}_{j+12} = \frac{1}{2}\,\hat{\sigma}_3 \otimes \hat{\sigma}_j \qquad j = 0, 1, 2, 3 \tag{2.9}$$

$$\hat{\Gamma}_{16} = \frac{1}{2}\,\hat{\sigma}_0 \otimes \hat{\sigma}_0 \tag{2.10}$$

Since these matrices are known, the only element remained to determined $\hat{\rho}$, is the vector $\vec{\boldsymbol{r}}$.

2.2 Determining $\hat{\rho}$

The probability of obtain a certain outcome (number of coincidences detected a part from a normalization factor) is given by:

$$P_\nu = \langle\psi_\nu|\,\hat{\rho}\,|\psi_\nu\rangle = \sum_{j=1}^{16} r_j\,\langle\psi_\nu|\,\hat{\Gamma}_j\,|\psi_\nu\rangle = \sum_{j=1}^{16} r_j\,\hat{B}_{\nu,j} \tag{2.11}$$

Where the $\hat{B}_{\nu,j}$ matrices are 16×16 dimensional introduced in order to simplify the equation.

By inverting Eq.2.11, we find $\vec{\boldsymbol{r}}$:

$$r_j = \sum_{\nu=1}^{16} (\hat{B}^{-1})_{\nu,j}\,P_\nu \tag{2.12}$$

In this equation, the number of coincidences are given by the experiment, while the $\hat{B}^{-1}_{\nu,j}$ matrices can be evaluated from the analytical form of the 16 states $|\psi_\nu\rangle$, reported in Tab.1, obtained as a tensor product of the single states $|H\rangle$, $|V\rangle$, $|R\rangle$, $|D\rangle$ and $|L\rangle$.

By substituting Eq.2.12 into Eq.2.5, we obtain:

$$\hat{\rho} = \sum_{j=1}^{16}\sum_{\nu=1}^{16} \hat{\Gamma}_j\,(\hat{B}^{-1})_{\nu,j}\,P_\nu = \sum_{\nu=1}^{16} \hat{M}_\nu\,P_\nu \tag{2.13}$$

Where we have introduced 16 matrices M_ν in order to simplify the equation.

The probability P_ν is experimentally determined as:

$$P_\nu = \frac{N_\nu}{\sum_{i=1}^{4} N_i} = \frac{N_\nu}{\mathcal{N}} \tag{2.14}$$

The denominator is a summation over just the first 4 terms because the others are null, since it is given by the following equation:

$$\mathcal{N} = \sum_{\nu=1}^{16} \mathrm{Tr}\{\hat{M}_\nu\}\,N_\nu \tag{2.15}$$

Lastly, the density matrix is determined using the following equation:

$$\hat{\rho} = \frac{1}{\mathcal{N}} \sum_{\nu=1}^{16} \hat{M}_\nu\,N_\nu \tag{2.16}$$

2.3 Maximum-Likelihood estimation

We use the Maximum-Likelihood estimation in order to overcome the noise of the experiment (experimental inaccuracies and statistical fluctuations of coincidence counts). In this way, the density matrices $\hat{\rho}$ can represent physical states.

The operation works in these steps:

- Generate a density matrix $\hat{\rho}^{new}$ that represents a physical state, in function of 16 variables t_i.
- Introduce the likelihood function $\chi(t)^2(t_1, ..., t_{16}; N_1, ...N_{16})$ that quantify how much $\hat{\rho}^{new}$ is in according with the experimental data.

- Find a set of t_i that minimize $\chi(t)^2$.

The matrix $\hat{\rho}^{new}$ is given by the following equation:

$$\hat{\rho}^{new} = \frac{\hat{T}^{\dagger}(t)\hat{T}(t)}{\mathrm{Tr}(\hat{T}^{\dagger}(t)\hat{T}(t))} \tag{2.17}$$

Where the matrix $\hat{T(t)}$ is given by:

$$\hat{T}(t) = \begin{pmatrix} t_1 & 0 & 0 & 0 \\ t_5 + it_6 & t_2 & 0 & 0 \\ t_{11} + it_{12} & t_7 + it_8 & t_3 & 0 \\ t_{15} + it_{16} & t_{13} + it_{14} & t_9 + it_{10} & t_4 \end{pmatrix} \tag{2.18}$$

It is possible to invert Eq.2.17 in order to obtain $\hat{T}(t)$ in function of $\hat{\rho}^{new}$.

The likelihood function used is:

$$\chi(t)^2 = \sum_{\nu=1}^{16} \frac{|N_\nu - L_\nu(t)|^2}{N_\nu}. \tag{2.19}$$

Where we have defined the following quantities:

$$N_\nu = \mathcal{N} \langle\psi_\nu| \hat{\rho} |\psi_\nu\rangle \tag{2.20}$$

$$L_\nu = \mathcal{N} \langle\psi_\nu| \hat{\rho^{new}} |\psi_\nu\rangle . \tag{2.21}$$

By finding the set of t_i that minimize Eq.2.19, we transform the previous $\hat{\rho}$ into a new density matrix, called $\hat{\rho^{new}}$, that now represents physical states.

2.4 Fidelity

Fidelity is a measure of closeness between two quantum states. It is given by the following equation:

$$\mathcal{F}(\hat{\rho}, \hat{\sigma}) = \left(\mathrm{Tr}\sqrt{\sqrt{\hat{\rho}}\hat{\sigma}\sqrt{\hat{\rho}}}\right)^2 \tag{2.22}$$

If $\hat{\rho}$ is pure state, the equation can become:

$$\mathcal{F}(\hat{\rho}) = \langle\Phi| \hat{\rho} |\Phi\rangle \tag{2.23}$$

2.5 Von Neumann entropy

The von Neumann entropy is an important measure of the purity of a quantum state $\hat{\rho}$. It quantifies the amount of information present in a system, and the amount of correlations between quantum systems. It is defined by:

$$\mathcal{S}(\hat{\rho}) = \mathrm{Tr}\{\hat{\rho}\log_2(\hat{\rho})\} = -\sum_{a=1}^{4} p_a \log_2(p_a) \tag{2.24}$$

Where p_a is an eigenvalue of $\hat{\rho}$.

2.6 Concurrence

The concurrence is a way of measuring entanglement. For a mixed state of 2 qubits it is defined as:

$$\mathcal{C}(r) = \mathrm{Max}\{0, \sqrt{r_1} - \sqrt{r_2} - \sqrt{r_3} - \sqrt{r_4}\} \tag{2.25}$$

Where $\sqrt{r_i}$ are the eigenvalues, in decreasing order, of the matrix $\hat{\mathcal{R}}$, defined as:

$$\hat{\mathcal{R}} = \hat{\rho}\,\hat{\Sigma}\,\hat{\rho}^{\mathrm{T}}\,\hat{\Sigma} \tag{2.26}$$

$$\hat{\Sigma} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{2.27}$$

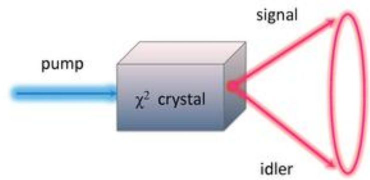

Figure 1: Functioning of Type I SPDC. The input photon p (pump) is separated into 2 output photons: i and s (idler and signal), which have the same polarization but orthogonal to the one of the input beam.

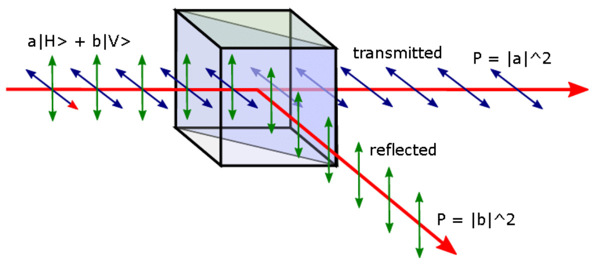

Figure 2: Input and output of a PBS. Usually, two detectors are placed at the output of the light beams.

3 Experimental tools

The following section explain the tools we used in order to perform the experiment.

3.1 Spontaneous Parametric Down Conversion (SPDC) Type I

It is a non-linear crystal used to generate entanglement. The functioning is shown in Fig.1. For the Type I SPDC we have the following two possible situations:

$$|H\rangle_p \rightarrow |VV\rangle_{is} \tag{3.1}$$

$$|V\rangle_p \rightarrow |HH\rangle_{is} \tag{3.2}$$

Where e means the extraordinary polarization, while o the ordinary polarization. It is possible to change the angle of the crystal in order to change the amplitude of the output cone. The entanglement can be created by sending a superposition of H and V through 2 SPDC crystal with perpendicular axis:

$$\frac{1}{\sqrt{2}}\Big(|H\rangle_p + |V\rangle_p\Big) \rightarrow \frac{1}{\sqrt{2}}\Big(|VV\rangle_{is} + |HH\rangle_{is}\Big) \tag{3.3}$$

3.2 Polarizing Beam Splitter (PBS)

It is used in order to separate the two components of a light beam. The horizontal polarization is transmitted, whereas the vertical polarization is reflected, as shown in Fig.2.

3.3 Half Waveplate (HWP) and Quarter Waveplate (QWP)

They can be used in order to change the polarization of the light. It is possible to rotate the HWP and the QWP. For the HWP, the rotation matrix is:

$$M_\phi^{HWP} = \begin{pmatrix} cos(2\phi) & sin(2\phi) \\ sin(2\phi) & -cos(2\phi) \end{pmatrix} \tag{3.4}$$

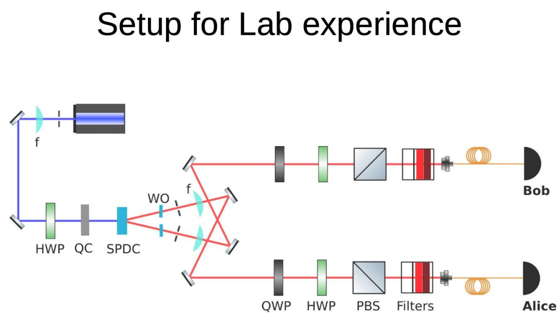

Figure 3: Laboratory setup.

While for the QWP:

$$M_{\phi}^{QWP} = \begin{pmatrix} cos^2(\phi) + isin^2(\phi) & (1-i)sin(\phi)cos(\phi) \\ (1-i)sin(\phi)cos(\phi) & sin^2(\phi) + icos^2(\phi) \end{pmatrix} \tag{3.5}$$

We have the following conversion:

- $\phi_{HWP} = 0°$, $\phi_{QWP} = 0°$: $|H\rangle \rightarrow |H\rangle$
- $\phi_{HWP} = 45°$, $\phi_{QWP} = 0°$: $|H\rangle \rightarrow |V\rangle$
- $\phi_{HWP} = 45°$, $\phi_{QWP} = 45°$: $|H\rangle \rightarrow |R\rangle$
- $\phi_{HWP} = 22.5°$, $\phi_{QWP} = 45°$: $|H\rangle \rightarrow |D\rangle$
- $\phi_{HWP} = 22.5°$, $\phi_{QWP} = -45°$: $|H\rangle \rightarrow |D\rangle$
- $\phi_{HWP} = 45°$, $\phi_{QWP} = -45°$: $|H\rangle \rightarrow |L\rangle$

By putting a HWP+QWP before a PBS, we can change the measurement basis.

4 Setup of the experiment

The setup of the experiment is shown in Fig.3. Photons are emitted from a laser with frequency 405 nm. The light beam pass through a lens in order to focus the light, and then it is reflected by 2 mirrors. Then there is a HWP that is used to produce a light beam with Horizontal (H) polarization. After this, the light pass through a 2 Type I SPDCs in order to create entanglement. Later, the beams pass throughout a series of collider, because photons have to be indistinguishable each others. At this point the entangled photons follow two equal and parallel paths made up by a HWP and a QWP (that can be rotated in order to create the different states), a PBS, some filters and a photon detector.

The two HWP and QWP are calibrated as shown in Tab.1. Three different initial states are used: $|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle)$, $|\Phi_-\rangle = \frac{1}{\sqrt{2}}(|HH\rangle - |VV\rangle)$ and $|\Phi_{dec}\rangle$.

5 Determine $\hat{\rho}$ and check for positivity

Using Eq.2.16, $\hat{\rho}$ is determined for the 3 different initial state.

$$\hat{\rho}_+ = \begin{pmatrix} 0.4875 & 0.0082 + 0.0049i & 0.0468 - 0.0104i & 0.3962 - 0.1237i \\ 0.0082 - 0.0049i & 0.0031 & 0.0076 + 0.0394i & -0.0206 - 0.0338i \\ 0.0468 + 0.0104i & 0.0076 - 0.0394i & 0.0044 & -0.0387 + 0.0155i \\ 0.3962 + 0.1237i & -0.0206 + 0.0338i & -0.0387 - 0.0155i & 0.5049 \end{pmatrix} \tag{5.1}$$

ψ_ν	Composed by	$\mathrm{HWP_a}$	$\mathrm{QWP_a}$	$\mathrm{HWP_b}$	$\mathrm{QWP_b}$	$\mathrm{N}_\nu \lvert\Phi_+\rangle$	$\mathrm{N}_\nu \lvert\Phi_-\rangle$	$\mathrm{N}_\nu \lvert\Phi_{dec}\rangle$
$\lvert\psi_1\rangle$	$\lvert HH\rangle$	36° (0°)	−7° (0°)	39° (0°)	128° (0°)	2959	3214	3544
$\lvert\psi_2\rangle$	$\lvert HV\rangle$	36° (0°)	−7° (0°)	84° (45°)	128° (0°)	19	59	27
$\lvert\psi_3\rangle$	$\lvert VV\rangle$	81° (45°)	−7° (0°)	84° (45°)	128° (0°)	3065	3090	3509
$\lvert\psi_4\rangle$	$\lvert VH\rangle$	81° (45°)	−7° (0°)	39° (0°)	128° (0°)	27	170	69
$\lvert\psi_5\rangle$	$\lvert RH\rangle$	81° (45°)	38° (45°)	39° (0°)	128° (0°)	1430	170	69
$\lvert\psi_6\rangle$	$\lvert RV\rangle$	81° (45°)	38° (45°)	84° (45°)	128° (0°)	1337	170	69
$\lvert\psi_7\rangle$	$\lvert DV\rangle$	58.5° (22.5°)	38° (45°)	84° (45°)	128° (0°)	1417	170	69
$\lvert\psi_8\rangle$	$\lvert DH\rangle$	58.5° (22.5°)	38° (45°)	39° (0°)	128° (0°)	1777	170	69
$\lvert\psi_9\rangle$	$\lvert DR\rangle$	58.5° (22.5°)	38° (45°)	39° (0°)	83° (−45°)	1164	170	69
$\lvert\psi_{10}\rangle$	$\lvert DD\rangle$	58.5° (22.5°)	38° (45°)	61.5° (22.5°)	83° (−45°)	2730	356	2206
$\lvert\psi_{11}\rangle$	$\lvert RD\rangle$	81° (45°)	38° (45°)	61.5° (22.5°)	83° (−45°)	1035	356	2206
$\lvert\psi_{12}\rangle$	$\lvert HD\rangle$	36° (0°)	−7° (0°)	61.5° (22.5°)	83° (−45°)	1539	356	2206
$\lvert\psi_{13}\rangle$	$\lvert VD\rangle$	81° (45°)	−7° (0°)	61.5° (22.5°)	83° (−45°)	1311	356	2206
$\lvert\psi_{14}\rangle$	$\lvert VL\rangle$	81° (45°)	−7° (0°)	84° (45°)	83° (−45°)	1452	356	2206
$\lvert\psi_{15}\rangle$	$\lvert HL\rangle$	36° (0°)	−7° (0°)	84° (45°)	83° (−45°)	1459	356	2206
$\lvert\psi_{16}\rangle$	$\lvert RL\rangle$	81° (0°)	83° (0°)	84° (45°)	83° (−45°)	2501	376	2185

Table 1: Summary of the experiment. The first six columns defined the states, the last three the number of coincidences detected for each different initial states.

$$\hat{\rho}_- = \begin{pmatrix} 0.4920 & -0.0149+0.0272i & 0.0130-0.0401i & -0.2851-0.1150i \\ -0.0149-0.0272i & 0.0090 & -0.0311+0.0720i & -0.0241-0.0364i \\ 0.0130+0.0401i & -0.0311-0.0720i & 0.0260 & -0.0488+0.0272i \\ -0.2851+0.1150i & -0.0241+0.0364i & -0.0488-0.0272i & 0.4730 \end{pmatrix} \tag{5.2}$$

$$\hat{\rho}_{dec} = \begin{pmatrix} 0.4957 & -0.0345+0.0457i & -0.0041-0.0527i & 0.2944-0.1050i \\ -0.0345-0.0457i & 0.0038 & -0.0227+0.1089i & -0.0491-0.0603i \\ -0.0041+0.0527i & -0.0227-0.1089i & 0.0097 & -0.0669+0.0471i \\ 0.2944+0.1050i & -0.0491+0.0603i & -0.0669-0.0471i & 0.4908 \end{pmatrix} \tag{5.3}$$

These density matrices satisfy conditions 1) and 2), but not condition 3). Their eigenvalues are reported in the second, third and fourth columns of Tab.2. Since the last eigenvalues are negative, the matrices do not satisfy the positivity condition, and therefore they represent non-physical states.

6 Maximum-Likelihood estimation

Using the Maximum-Likelihood estimation, by minimizing Eq.2.19 we get the following density matrices:

$$\hat{\rho}_+^{new} = \begin{pmatrix} 0.5140 & 0.0016+0.0202i & -0.0009-0.0003i & 0.3778-0.1381i \\ 0.0016-0.0202i & 0.0031 & -0.0001+0.0001i & -0.0004-0.0143i \\ -0.0009+0.0003i & -0.0001-0.0001i & 0.0001 & -0.0041+0.0010i \\ 0.3778+0.1381i & -0.0004+0.0143i & -0.0041-0.0010i & 0.4828 \end{pmatrix} \tag{6.1}$$

$$\hat{\rho}_-^{new} = \begin{pmatrix} 0.4912 & -0.0062+0.0150i & 0.0060-0.0273i & -0.3114-0.1295i \\ -0.0062-0.0150i & 0.0101 & -0.0038+0.0146i & -0.0270-0.0172i \\ 0.0060+0.0273i & -0.0038-0.0146i & 0.0272 & -0.0406+0.0240i \\ -0.3114+0.1295i & -0.0270+0.0172i & -0.0406-0.0240i & 0.4715 \end{pmatrix} \tag{6.2}$$

$$\hat{\rho}_{dec}^{new} = \begin{pmatrix} 0.5048 & -0.0238+0.0294i & -0.0379-0.0387i & 0.2505-0.0877i \\ -0.0238-0.0294i & 0.0060 & 0.0022+0.0072i & -0.0382-0.0348i \\ -0.0379+0.0387i & 0.0022-0.0072i & 0.0122 & -0.0527+0.0247i \\ 0.2505+0.0877i & -0.0382+0.0348i & -0.0527-0.0247i & 0.4770 \end{pmatrix} \tag{6.3}$$

The eigenvalues of these density matrices are shown in the last three columns of Tab.2. As we can see, the eigenvalues are now all positive, so the density matrices represent physical states.

Eigenvalue	$\hat{\rho}_+$	$\hat{\rho}_-$	$\hat{\rho}_{dec}$	$\hat{\rho}_+^{new}$	$\hat{\rho}_-^{new}$	$\hat{\rho}_{dec}^{new}$
λ_1	0.912	0.791	0.827	0.901	0.820	0.769
λ_2	0.130	0.224	0.209	0.090	0.169	0.229
λ_3	0.002	0.046	0.071	0.002	0.017	0.001
λ_4	-0.046	-0.062	-0.107	0.00001	0.0003	0.000003

Table 2: Eigenvalues of the density matrices before and after the Maximum-Likelihood estimation.

	$\vert\Phi_+\rangle$	$\vert\Phi_-\rangle$	$\vert\Phi_{dec}\rangle$
Fidelity ($\mathcal{F}$)	0.88 $\pm$ 0.05	0.79 $\pm$ 0.01	x
von Neumann entropy ($\mathcal{S}$)	0.48 $\pm$ 0.10	0.74 $\pm$ 0.03	0.79 $\pm$ 0.07
Concurrence ($\mathcal{C}$)	0.80 $\pm$ 0.10	0.68 $\pm$ 0.02	0.52 $\pm$ 0.09

Table 3: Fidelity, von Neumann entropy and Concurrence for each initial quantum states.

7 Fidelity, von-Neumann entropy and Concurrence

Using Eq.2.23, Eq.2.24 and Eq.2.25 for each initial state, we get the results reported in Tab.3

8 Statistical errors by simulation

In order to evaluate the errors of the 3 parameters defined above, we performed 200 simulations of the experiment. The only difference between each simulations is the number of coincidence detected. By considering the experimental number of coincidences as the mean value of a Poissonian distribution, we get 200 new values for each detection. As error, we consider the standard deviation of the results obtained.

References

[JKMW01] Daniel F. V. James, Paul G. Kwiat, William J. Munro, and Andrew G. White. Measurement of qubits. *Phys. Rev. A*, 64:052312, Oct 2001.

[JKMW01]