Early warning signs of bifurcations - The α-stable case

Lucia S. Layritz ° | Christian Kuehn + | Anja Rammig °

° TUM School of Life Sciences, Munich, Germany
+ TUM Department of Mathematics, Munich, Germany

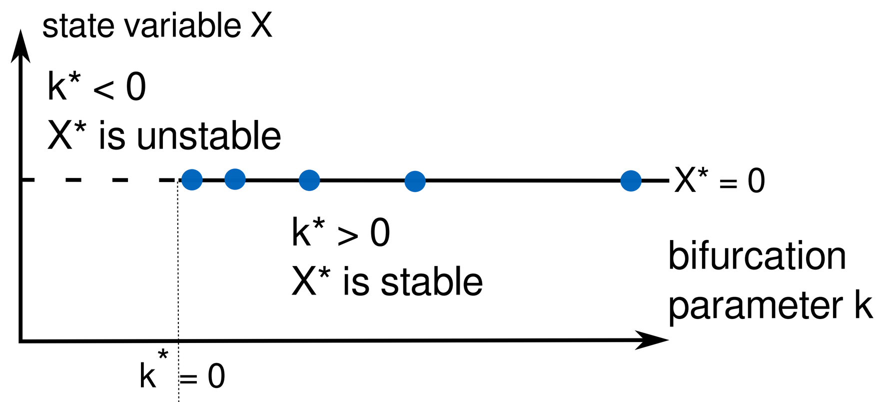
Technical University of Munich

HEINRICH BÖLL STIFTUNG

Rising variance is a commonly used early warning sign of approaching bifurcations in modelled and real-world systems.[1, 2]

The theory of early warning signs is based on the properties of an Ornstein-Uhlenbeck process:

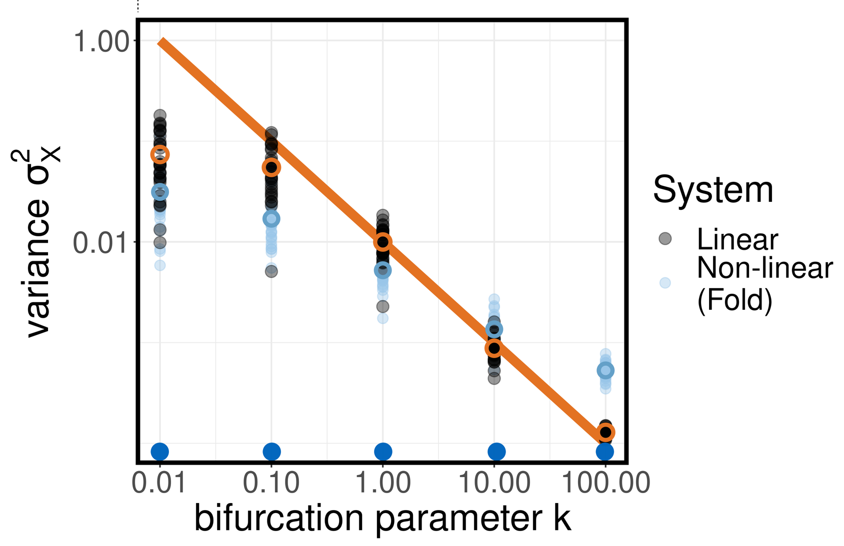
$$\underbrace{dX = -kX\,dt}_{\text{drift term}} + \underbrace{\sigma dW_t}_{\text{diffusion term}}$$



We can obtain an analytical solution of its stationary distribution and variance in dependence of k.

$$\boxed{\sigma_X^2 = \frac{\sigma_W^2}{2k}}$$

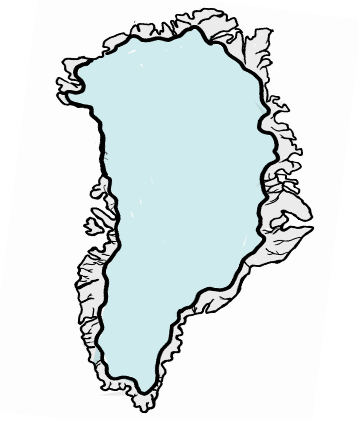
Non-linear applications usually linearize around the fixed point and use the above relationship



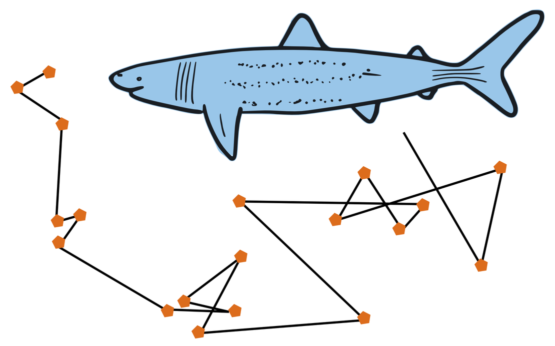
The theory of early warning signs is so far only well described for Gaussian noise.

α-stable variables are a wide class of heavy-tailed probability distributions that are found in many applications, e.g. in ecology and climatology:

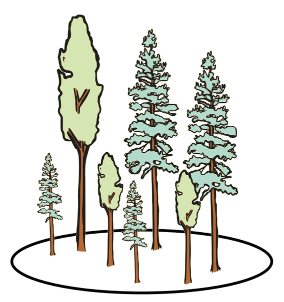
Greenland ice core temperature record:[3] α = 1.75



Foraging behaviour of marine predators[4]: α = 0.93



Aboveground carbon of modelled vegetation[7]: α = 1.8



However, the variance of non-Gaussian α-stable processes is non-converging and therefore not a reliable early warning sign[5].

We propose the scaling factor γ as an alternative early warning sign applicable to systems driven by α-stable noise ...

We construct an Ornstein-Uhlenbeck *type* process driven by α-stable noise L ~ S(α, β = 0, γ, δ = 0)

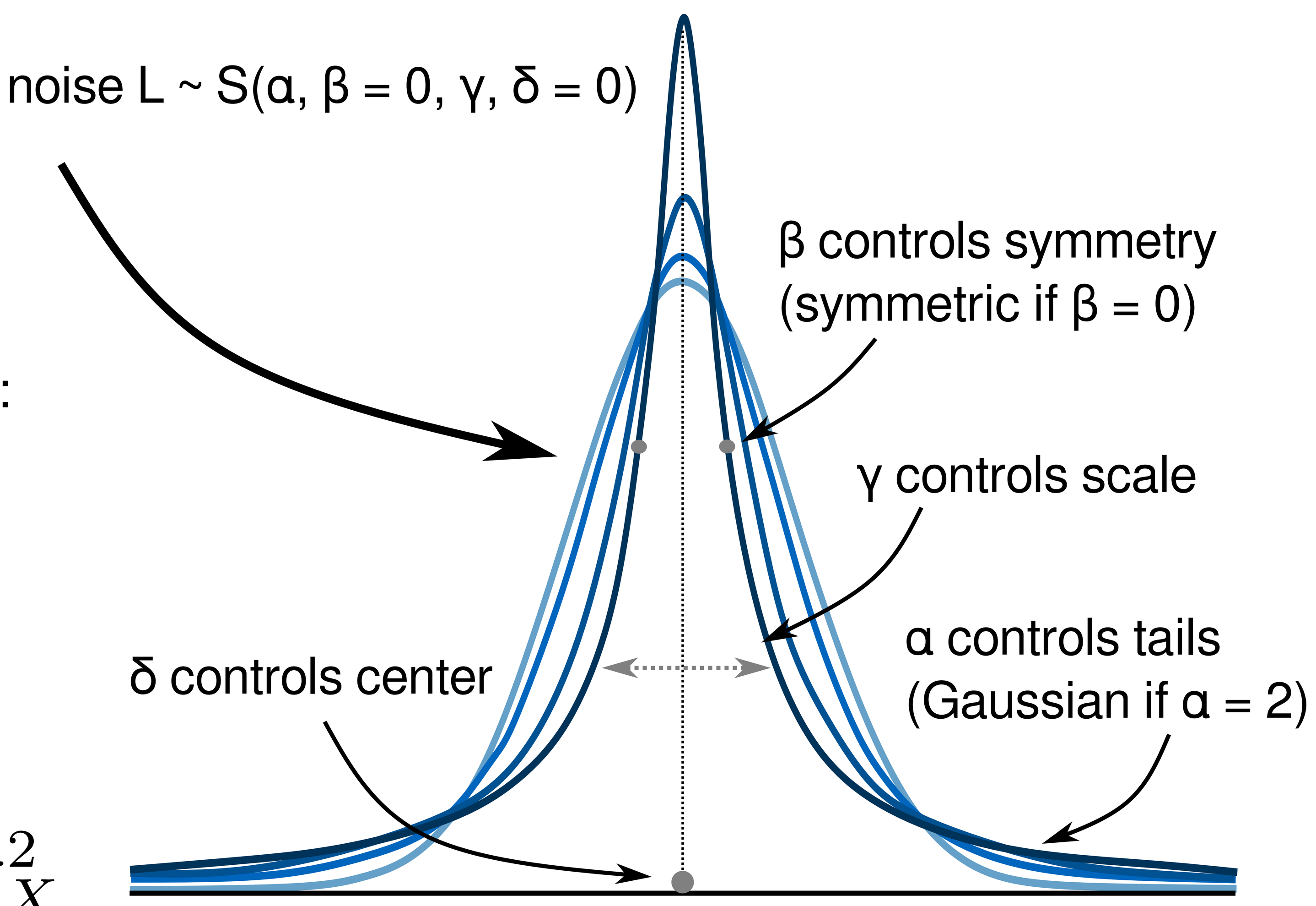
$$dX = -kX\,dt + \mu dL_t^\alpha$$

One can show that X is α-stable as well with the following properties[6]:

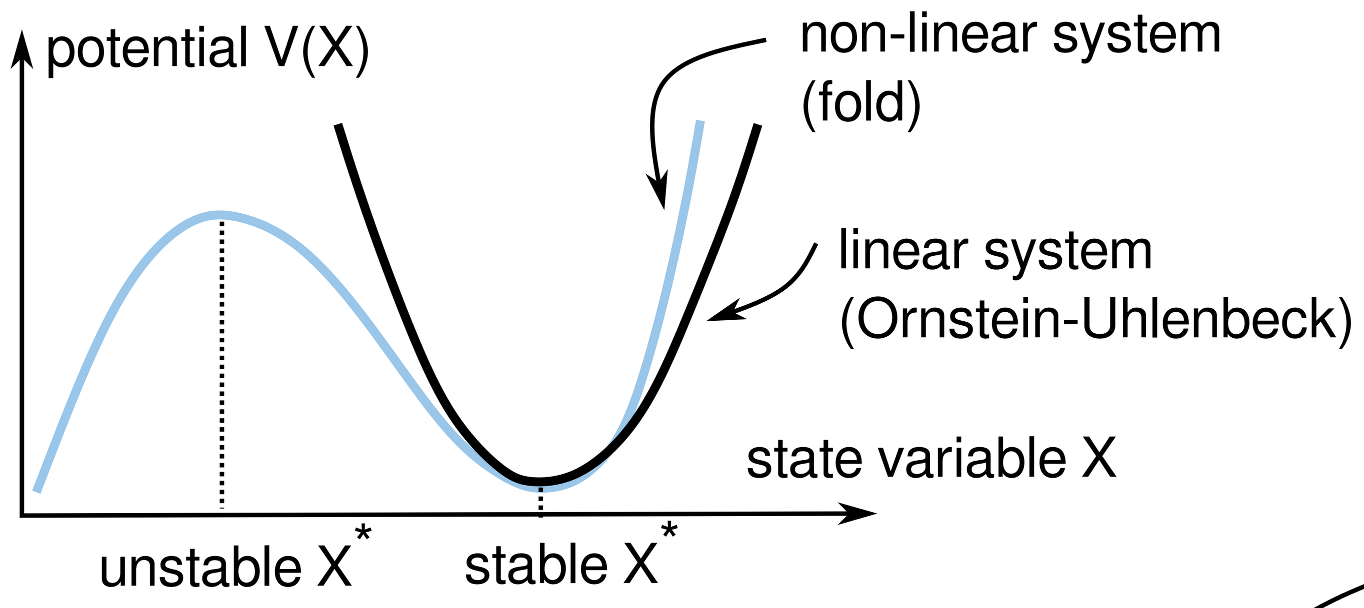
$$\alpha_X = \alpha_L$$

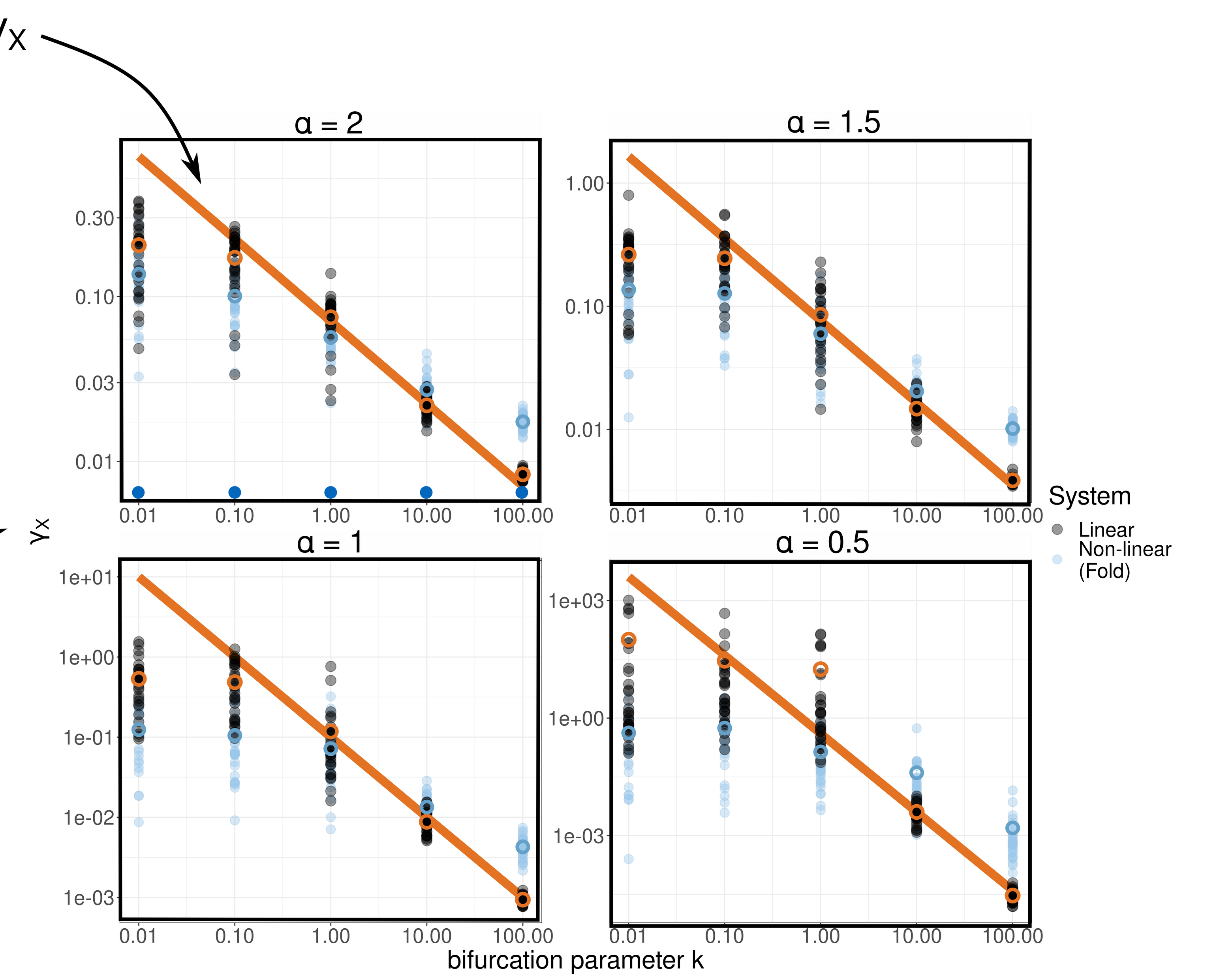
$$\boxed{\gamma_X = \gamma_L \sqrt[\alpha]{\frac{1}{k\alpha}}}$$

In the Gaussian case γ is closely related to the variance $2\gamma_X^2 = \sigma_X^2$



We confirm our results by numerical simulations for the linear and non-linear case[7].





Each point represents one estmate of γ_X across 5 individual trajectories with 70 timesteps each, fitted with *scipy.stats.levy_stable.fit()*.
γ_L is 1 and μ is 0.1.

siam. 2023 Conference on Applications of Dynamical Systems

1. Kuehn, C.. (2011). A Mathematical Framework for Critical Transitions: Normal Forms, Variance and Applications. Journal of Nonlinear Science.
2. Scheffer, M. (2009). Early-Warning Signals for Critical Transitions. Nature.
3. Dilevsen, P. D. (1999). Observation of alpha-stable noise and a bistable climate potential in an ice-core record. Geophysical Research Letters.
4. Sims et al. (2008). Scaling Laws of Marine Predator Search Behaviour. Nature.
5. Nolan, J.. (2014). Stable Distribution: Models for Heavy-Tailed data.
6. Sato, K. (1999) Lévy processes and infinitely divisible distributions
7. Layritz et al. (2023) *In preparation*

Contact me:
@lucialayr
lucia.layritz@tum.de