

Modern Algorithms for Integer Linear Programming

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1 Introduction

This paper surveys new results on the complexity of Integer Linear Programming (ILPs) from Eisenbrand et al. [3] and Jansen et al. [5]. Their work is the first to improve on a classic result from Papadimitriou [6] that is now almost thirty years old.

In section 2, we describe Papadimitriou's algorithm, which introduces a relation between ILPs and the longest path problem. Using this intuition, Papadimitriou develops an algorithm that runs in time $O(\Delta^m \cdot n^{2m+2} \cdot (m\Delta + m\|b\|_\infty)^{(m+1)(2m+1)})$. In section 3, we show how Eisenbrand et al. use the Steinitz lemma from linear algebra to shrink the search space for the longest path, significantly reducing the complexity to $n \cdot O(m\Delta)^{2m} \cdot \|b\|_1^2$. In section 4, we illustrate Jansen et al.'s algorithm which extends Eisenbrand et al.'s results using dynamic programming to achieve a runtime of $O(m\Delta)^{2m} + \text{LP}$, where LP is the time needed to compute the relaxed solution, which is suspected to be almost optimal. In section 5, we sketch Jansen et al.'s proof that assuming there is no $O(n^{2-\delta})$ time algorithm for $(\max, +)$ -convolution, for any $\delta > 0$, their ILP solver runtime is the best possible.

1.1 Integer Linear Programming

Linear programming is a beautiful, general framework that encapsulates many fundamental problems of computer science; shortest path, max flow, and zero-sum games among others. Linear programs can be solved in polynomial time and efficiently in practice. Formally, a linear program is defined by its constraints, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, as well as its objective $c \in \mathbb{R}^n$:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Integer linear programming differs by the added constraints that $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$, and $x \in \mathbb{Z}_{\geq 0}^n$. This subtle difference makes ILPs NP-Complete.

However, if we consider the number of constraints to be fixed, then ILPs can be solved in polynomial time. This makes the following algorithms *pseudopolynomial*.

Additionally, the algorithms will depend on the magnitude of the constraints. We will define Δ to be a bound on the absolute value of the entries of A :

$$\Delta = \max_{i,j} |A_{ij}|$$

Like linear programs, ILPs generalize a number of important problems like planning and scheduling. Even though the algorithms we describe are pseudopolynomial, they are useful for certain problems with a small, fixed number of constraints.

For example, recall the knapsack problem. Given a set of n types of items, i , with values $c_i \in \mathbb{Z}$ and a weights $A_i \in \mathbb{Z}$, determine the quantity of each item $x_i \in \mathbb{Z}_{\geq 0}$ that we should put into our backpack to maximize the total value without exceeding the weight limit $b \in \mathbb{Z}$.

This corresponds to the ILP:

$$\max \sum_{i=1}^n c_i x_i \quad (1)$$

$$\text{s.t.} \quad \sum_{i=1}^n A_i x_i = b \quad (2)$$

$$x \in \mathbb{Z}_{\geq 0}^n \quad (3)$$

Notice that in the knapsack problem, the matrix A is actually of shape $1 \times n$. Since the number of rows, m , is fixed, the knapsack problem can be solved efficiently using the following algorithms as described in section 5.

2 Papadimitriou's Algorithm

In 1981, Papadimitriou introduced an analogy between ILPs and the longest path problem which serves as the inspiration for his algorithm [6]. To illustrate this, realize that the constraint $Ax = b$ is equivalent to saying that b is a weighted sum of the column vectors of A :

$$b = x_1 A_1 + \dots + x_n A_n \quad (4)$$

With this expansion, we can begin to think about a feasible solution $Ax = b$ as a path. This path begins at the origin and makes n steps arriving at b as shown in figure 1. Each step of the path is a column A_i scaled by a positive integer, x_i . To add the objective, $\max c^T x$, we simply need to weight each edge corresponding to the i th step with cost $c_i x_i$. The longest path (according to edge weight, not physical distance) from the origin to b is the optimum of the ILP.

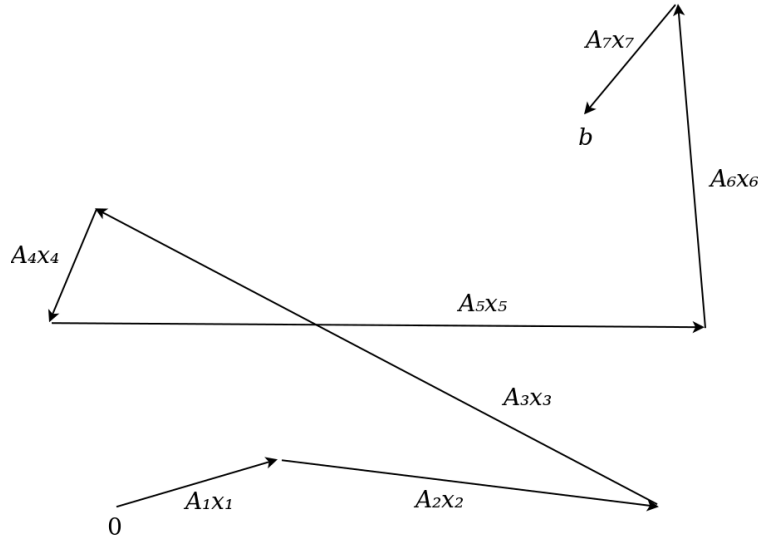


Figure 1: The path in Papadimitriou's algorithm from the origin to b , composed of steps $A_i x_i$.

Now, it remains to actually construct the graph that we are going to search over. Clearly the search space is enormous and it is this issue that Eisenbrand et al. realized they could improve in their following paper. Nevertheless, Papadimitriou's "brute force" construction still achieves pseudopolynomial time.

The graph is going to be composed of $n + 1$ layers, the result of n steps. The first layer will contain the origin, and each subsequent layer is formed by adding all possible values of $x_i A_i$ to the layer that precedes it. Specifically, for every node, $(i - 1, v)$, representing a node in layer $i - 1$ corresponding to point v , we

construct nodes $(i, v + x_i A_i)$ for any feasible value of x_i and connect the nodes with a directed edge that has weight $c_i x_i$.

To define how immense this graph is, we will introduce the value B , which will bound the maximum value of any entry of the optimal solution, $\|x^*\|_\infty \leq B$. For any particular point v , if v was constructed by adding the largest possible values of $A_i x_i$ for all n steps, that node could be at most $\|v\|_\infty \leq n\Delta B$. Therefore we can bound the number of nodes in each layer by $(n\Delta B)^m$.

Since the graph is layered we can solve the longest path problem using dynamic programming. Each node has at most B children, so computing longest paths in the next layer takes $O(B(n\Delta B)^m)$ time. Since this update process must be done for each of the n layers, the total running time is $O(\Delta^m (nB)^{m+1})$. Papadimitriou derives that $B = O(n(m\Delta + m\|b\|_\infty)^{2m+1})$, which gives us the claimed running time:

$$O\left(\Delta^m n^{2m+2} (m\Delta + m\|b\|_\infty)^{(m+1)(2m+1)}\right)$$

3 Eisenbrand and Weismantel's Algorithm

Eisenbrand and Weismantel's algorithm [3] improves upon Papadimitriou's algorithm [6] after a gap of almost three decades. Their algorithm also views ILPs as a longest path problem, but they construct the graph in a different way to make it much smaller. This reduction relies on Steinitz's lemma from linear algebra to bound the space of feasible solutions. Once the graph has been constructed, the solution is found with common search algorithms.

First we will describe Steinitz Lemma in section 3.1. Then, we will discuss how we can use this to obtain a feasible solution in section 3.2. With little additional work, we can extend this to an optimal solution as described in section 3.3.

3.1 Steinitz Lemma

Eisenbrand et al.'s work as well as Jansen et al.'s work that follows [5] rely on the Steinitz lemma from linear algebra. The lemma says that there exists a way to rearrange a set of sufficiently small vectors whose sum is zero, so that each partial sum of vectors is bounded.

We can think of this sum as a path from the origin to itself where each vector represents a step. The lemma says that the steps of this path can be reordered so that they are always relatively close to the origin as shown in figure 2.

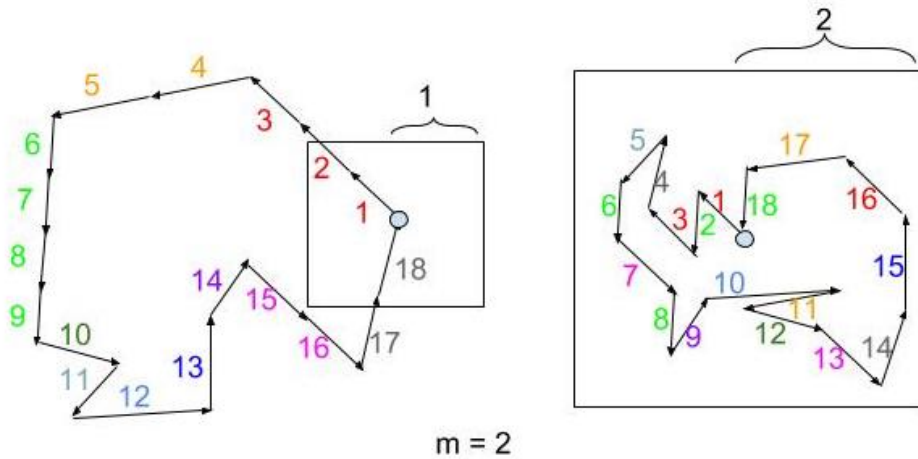


Figure 2: A visualization of the Steinitz lemma for 2 dimensions. The large path on the left is reordered to form more compressed path on the right with an ℓ_∞ norm less than $m = 2$. Steps of the same size are the same color on both sides.

Formally, the Steinitz lemma is given as follows:

Theorem 1 (Steinitz). *Let $w_1, \dots, w_n \in \mathbb{R}^m$ such that:*

$$\sum_{i=1}^n w_i = 0 \quad \text{and} \quad \|w_i\| \leq 1 \quad \text{for all } i.$$

There exists a permutation $\pi \in S_n$ such that all partial sums satisfy

$$\left\| \sum_{j=1}^k w_{\pi(j)} \right\| \leq m \quad \text{for all } k = 1, \dots, n \quad (5)$$

Note that if $\|w_i\|$ is bounded by some other constant, $\|w_i\| \leq \delta$, then through scaling the partial sums are bounded by δm .

3.2 Feasibility

In Papadimitriou's algorithm we formulated ILP as a path planning problem by first realizing that the constraint $Ax = b$ is equivalent to saying that b is a weighted sum of the column vectors of A as in equation (4). Because each element, x_i , is an integer we can continue this expansion and just consider b to be the sum of some number of each column of A :

$$b = \underbrace{A_1 + \dots + A_1}_{x_1 \text{ times}} + \underbrace{A_2 + \dots + A_2}_{x_2 \text{ times}} + \dots + \underbrace{A_n + \dots + A_n}_{x_n \text{ times}} \quad (6)$$

Expanded like this, we can think about any feasible solution $Ax = b$ as another path; we begin at the origin and make steps equal to column vectors of A until we reach b . This contrasts with Papadimitriou, whose steps are column vectors of A times integers x_i . While Eisenbrand et al.'s path is going to have many more steps, they will not be divided into layers like Papadimitriou's. This gives us the freedom to do some reorganization using Steiniz's lemma to shrink the search space.

In particular, our search must extend to any vector that is on the path from the origin to b . These vectors are partial sums of the column vectors in equation (6). The Steiniz lemma lets us bound these partial sums. Note that equation (6) already holds two important properties needed to apply the Steiniz lemma:

1. The steps in the path can be made in *any order*. Despite being written in column order there is no reason for this, its just addition.
2. Each step is *small*. Recall that we are assuming that the absolute value of each element of A is bounded by Δ .

The one missing piece is that our path starts from the origin and ends up at b , where as Steiniz's lemma holds for paths that start at the origin and end up back at the origin. We can fix this through some rearrangement. Before doing that, let's simplify the notation. Let t be the number of steps in the path and let $f(j)$ be the index of the column vector used as the j th step. Equation (6) becomes:

$$b = A_{f(1)} + \dots + A_{f(t)} \quad (7)$$

Now, in order to rearrange, we just have to move b onto the same side as the column vectors and then divide it up to retain the small steps.

$$\begin{aligned} 0 &= A_{f(1)} + \dots + A_{f(t)} - b \\ &= (A_{f(1)} - b/t) + \dots + (A_{f(t)} - b/t) \end{aligned}$$

Both $A_{f(j)}$ and b/t are bounded by Δ , so their difference is bounded by 2Δ — still small!

Invoking Steiniz's lemma, for some particular permutation of f , we get that any partial sum with $j \leq t$ is bounded as follows:

$$\|A_{f(1)} + \dots + A_{f(j)} - jb/t\|_\infty \leq 2\Delta m \quad (8)$$

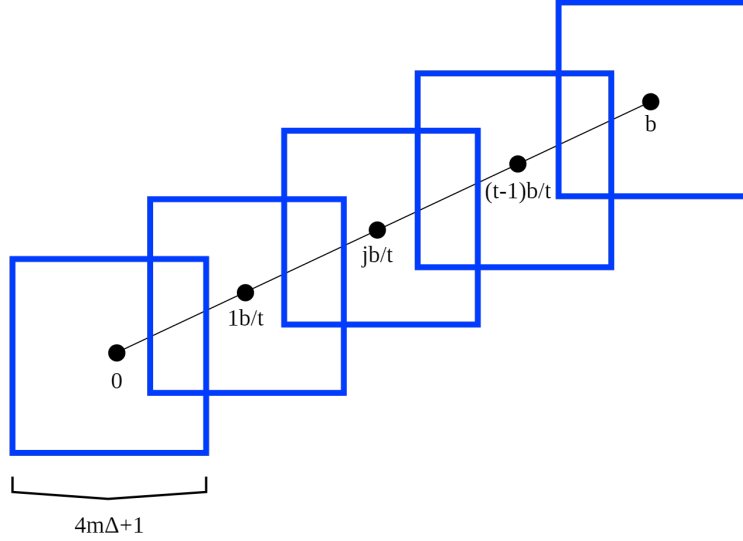


Figure 3: The space of possible paths from the origin to b

This bound limits the entire path to lie within a hypercube whose sides are distance $2\Delta m$ from the point jb/t . This means that each hypercube contains $4\Delta m + 1$ points per side and therefore $(4\Delta m + 1)^m$ points in total. The hypercube lives in m dimensions and there are t possible center points, so the set of all possible vertices the path could pass through is $|V| = t(4\Delta m + 1)^m$. Figure 3 shows this shape in 2 dimensions.

For each of these vertices there are at most n edges from that vertex, one for each column of A . Specifically there will be an edge going from vertex u to vertex v if $v - u$ is a column of A . This means that the graph we have constructed has at most $|E| = n|V| = nt(4\Delta m + 1)^m$ edges.

For feasibility, we just need to prove that a path exists. This can be done using breadth first search in $O(|E|) = O(nt(3\Delta m + 1)^m)$ time. To bound t , we can assume that the path does not contain zero columns, or sequences of columns that sum to zero as these are unnecessary to the final results. Therefore every step of the path must increase the ℓ_1 norm of our partial sum by at least 1, so $t \leq \|b\|_1$. This gives us the bound:

$$n \cdot O(m\Delta)^m \cdot \|b\|_1$$

3.3 Optimization

Optimization is not much different from feasibility. Rather than simply finding a path, we now need to find the path that optimizes $c^T x$. If we assign weights c_i to each edge corresponding edge, then the maximum weight path corresponds to the solution maximizing $c^T x$. Bellman-Ford can solve the problem with complexity $O(|V||E|) = O(nt^2(3\Delta m + 1)^{2m}) = n \cdot O((m\Delta)^{2m}) \cdot \|b\|_1^2$

Some care needs to be made with positive cycles. Just like negative cycles in the shortest path problem, these cycles can be looped around forever with every solution being better than the last. If these occur, they indicate that the solution is unbounded. Fortunately, Bellman Ford can detect these cycles and we can output “unbounded” if we find any.

4 Jansen and Rohwedder’s Algorithm

In the Eisenbrand et al.’s algorithm, we learned that our solution could be considered as a path from the origin to b . Moreover, this path is constrained to lie within hypercubes with sidelengths $4\Delta m + 1$ centered around the the points jb/t for $j \leq t$ as we showed in figure 3. This series of center points lies exactly on the line from the origin to b , so one way of looking at the bound is that the path never strays far from this line.

Using this intuition Jansen and Rohwedder developed another, more efficient, solution to ILP using dynamic programming [5].

Specifically, they realized that if every point on the path is pretty close to the line from 0 to b , and the step sizes are small, then there must be some point that lies pretty close to the point $b/2$. If we guess the vector $b^{(1)}$ that is near to $b/2$, then we can solve the smaller problems $Ax = b^{(1)}$ and $Ax = b - b^{(1)}$ and then merge the solutions to get the full result. This sounds like dynamic programming!

In particular, the vectors we are “guessing” as halfway points can be described as the set of all $b^{(i)}$ such that $b^{(i)}$ is within distance $4m\Delta$ of a midpoint:

$$\left\| b^{(i)} - \frac{b}{2^i} \right\|_{\infty} \leq 4m\Delta \quad (9)$$

If we know the solution to $Ax = b^{(i+1)}$ for all possible $b^{(i+1)}$ s satisfying (9), we can combine the results to get solutions to all $b^{(i)}$ s, which are twice as far away the $b^{(i+1)}$ s. Repeating this procedure until $i = 0$, we get the solution to $b = b^{(0)}$ which solves our problem.

First of all we will define the base case in section 4.1: how do we begin to get solutions to the problem for some large i ? We also prove the dynamic programming step in section 4.2, which describes how we are able to compute solutions $i - 1$ from i . Then, we will discuss checking if the solution is unbounded in section 4.3 and conclude with an analysis of the runtime in section 4.4.

4.1 Base Case

In Papadimitriou’s algorithm, we used a value B , to bound the ℓ_{∞} -norm of the optimal solution x^* . In this case we will be using a value of K to bound the ℓ_1 norm of the optimal solution. Specifically, $\|x^*\|_1 = 2^K$. As we divide b in half, the solution x will also be divided roughly in half. We can, in fact constraint the solution to do so:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b^{(i)} \\ & \|x\|_1 = 2^{K-i} \\ & x \in \mathbb{Z}_{\geq 0}^m \end{aligned}$$

With this constraint, when $i = K$ we have $\|x\|_1 = 1$; each such x is zero except for a single row with a 1. This means that these solutions are selecting single columns of A . We can easily iterate over each columns of A for each $b^{(K)}$ to find any matches. $b^{(K)}$ s with matches will get value $c^T x$ and ones without will get value $-\infty$.

Now that we have solutions for a large $i = K$, we can use dynamic programming to iteratively find the solutions for smaller and smaller i until we reach the desired $i = 0$. Jansen et al. derive that $K = O(m \log(m\Delta) + \log(\|b\|_{\infty}))$. As an intuition for this bound, we recall again figure 3. This figure shows that x^* is constrained to live in one of at most t hypercubes, each with volume $(4m\Delta + 1)^m$. With $t \leq \|b\|_{\infty} + 1$, this means that the ℓ_1 norm is at most $\|x^*\| \leq O(m\Delta)^m (\|b\|_{\infty} + 1)$. Taking the log of this bound gives us K .

4.2 Induction

In order to apply dynamic programming, we need to prove that if we have solutions to $Ax = b^{(i+1)}$ for all $b^{(i+1)}$ satisfying equation (9), then we can construct solutions for all $b^{(i)}$ satisfying (9). To do that, we are actually going to go backwards. We are going to start with the path to some $b^{(i)}$ and prove that each half of that path is in the set of $b^{(i+1)}$ s. Using the same notation we used in (7), we can write $b^{(i)}$ as columns of A :

$$b^{(i)} = A_{f(1)} + \dots + A_{f(t)}$$

We claim that each half of the path is in the set of all $b^{(i+1)}$ s. In other words, we are claiming that:

$$\begin{aligned} \left\| A_{f(1)} + \dots + A_{f(t/2)} - \frac{b}{2^{i+1}} \right\|_{\infty} &\leq 4m\Delta && \text{The left half is a } b^{(i+1)} \\ \left\| A_{f(t/2+1)} + \dots + A_{f(t)} - \frac{b}{2^{i+1}} \right\|_{\infty} &\leq 4m\Delta && \text{The right half is a } b^{(i+1)} \end{aligned}$$

We'll focus on proving the left half, because the right half is similar. Without loss of generality, let's assume that the ordering of the columns, f , is done by Steinitz lemma. Under this assumption, equation (8) gives us this bound:

$$\left\| A_{f(1)} + \dots + A_{f(t/2)} - \frac{b^{(i)}}{2} \right\|_{\infty} \leq 2\Delta m$$

Now, with a little bit of algebra, we can do the proof:

$$\begin{aligned} \left\| A_{f(1)} + \dots + A_{f(t/2)} - \frac{b}{2^{i+1}} \right\|_{\infty} &= \left\| A_{f(1)} + \dots + A_{f(t/2)} - \frac{b^{(i)}}{2} + \frac{b^{(i)}}{2} - \frac{b}{2^{i+1}} \right\|_{\infty} \\ &\leq \left\| A_{f(1)} + \dots + A_{f(t/2)} - \frac{b^{(i)}}{2} \right\|_{\infty} + \left\| \frac{b^{(i)}}{2} - \frac{b}{2^{i+1}} \right\|_{\infty} \\ &\leq 2\Delta m + \frac{1}{2} \left\| b^{(i)} - \frac{b}{2^i} \right\|_{\infty} \\ &\leq 4\Delta m \end{aligned}$$

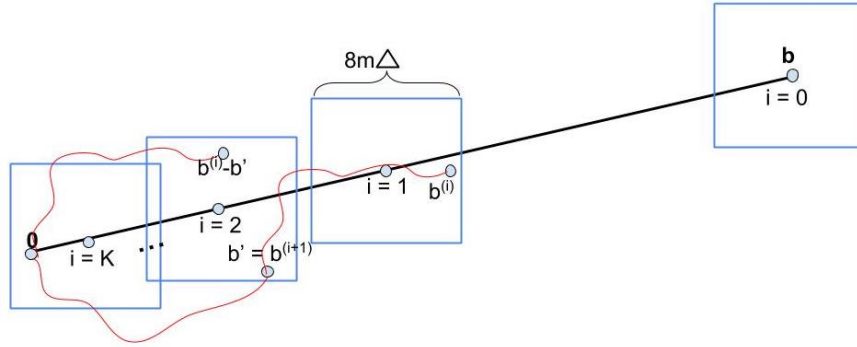


Figure 4: The path $b^{(i)}$ is the sum of paths $b' = b^{(i+1)}$ and $b^{(i)} - b'$.

4.3 Unbounded

So far, we have established how to find solutions to small subproblems, and how to combine those subproblems using dynamic programming to get a complete solution. However, what we have described so far will not be able to check if the solution is unbounded. Unlike in Eisenbrand et al.'s case, where checking if the solution was unbounded was simply built into Bellman-Ford, we will need to add some machinery here to do the check explicitly.

In particular, we are going to be solving a different ILP that is necessarily bounded. The solution to this

ILP will indicate whether the original ILP is unbounded. The new problem is:

$$\min \|x\|_1 \quad (10)$$

$$\text{s.t. } Ax = 0 \quad (11)$$

$$c^T x > 0 \quad (12)$$

$$x \in \mathbb{Z}_{\geq 0}^n \quad (13)$$

If the ILP is feasible then the original ILP is unbounded, otherwise it is bounded. Essentially, what this ILP is doing is checking for positive cycles. The constraint $Ax = 0$ forces the steps to form a cycle that departs from the origin and returns to the origin. The constraint $c^T x > 0$ forces the cycle to be positive.

Now we need to make sure that we can solve this problem is itself bounded. Suppose that $t > (2\Delta m + 1)^m$. Then realize that from our bound in equation (8) from the Steinitz lemma, the steps are bounded by

$$\|A_{f(1)} + \dots + A_{f(j)}\|_\infty \leq 2\Delta m$$

This means that the partial sum of steps can take on at most $(2\Delta m + 1)^m$ unique values. If the number of steps is larger than $(2\Delta m + 1)^m$ there must be some value that has been repeated. This pins the total cycle together to make two cycles as shown in figure 5. Since $c^T x > 0$, then at least one of the cycles must be a positive cycle, which means there is a solution with a smaller $\|x\|_1$ that satisfies the ILP. This contradicts the minimality of $\|x\|_1$, so $t \leq (2\Delta m + 1)^m$. This means that we can solve this ILP with a bounded number of steps $t \leq (2\Delta m + 1)^m$. This is less than the number needed for the complete problem, so it will not affect the running time.

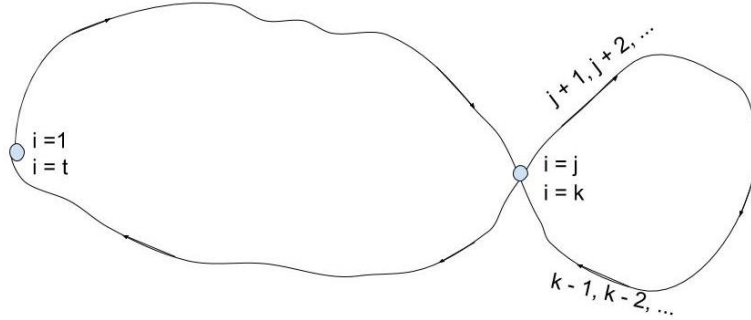


Figure 5: Positive cycle x^* decomposed into two cycles, where one of them is positive.

4.4 Runtime

So far, as we have constructed it, each layer of our dynamic programming table is limited by equation (9) to be contained in a hypercube with side lengths $8\Delta m + 1$. Because this is an m dimensional shape, its volume is $8\Delta m + 1 = O(\Delta m)^m$. Naively, updating each entry of the table requires iterating over all of the elements in the previous layer, requiring $(O(\Delta m)^m)^2 = O(\Delta m)^{2m}$ time.

In total, this table has $K + 1$ layers, which makes the total running time:

$$O(m\Delta)^{2m} \cdot (K + 1) = O(m\Delta)^{2m} (\log \Delta + \log \|b\|_\infty)$$

Jansen et al. propose several additional ways to reduce this running time further, removing the logarithmic factors. We will discuss them briefly here, along with their implications.

4.4.1 (max, +)-convolution

The (max, +)-convolution problem takes in two input sequences $r_1 \dots r_n$ and $s_1 \dots s_n$. Given these, the goal is to compute sequence $t_1 \dots t_n$ such that

$$t_k = \max_{i+j=k} r_i + s_j$$

Surprisingly, this simple little problem encapsulates the hardness of many other problems [2]. An algorithm exists with complexity $O(n^2/\log n)$ but it conjectured that no algorithm exists with complexity $O(n^{2-\epsilon})$ for any $\epsilon > 0$.

Bringing this back to Jansen et al.'s ILP algorithm, one of the areas for improvement is the dynamic programming step which iterates over every element of the previous layer for each element of the current layer in order to find the path with the maximum weight. With a little massaging, this problem is just like iterating over the sequences in the (max, +)-convolution problem. Applying the algorithm, the complexity does not square with the size of the layer s , but instead has a slightly lower growth, $s^2/\log s$. This makes the complexity:

$$\frac{O(m\Delta)^{2m}}{2m \log O(m\Delta)} (K+1) = O(m\Delta)^{2m} (1 + \log(\|b\|_\infty)/\log(\Delta))$$

4.4.2 Proximity

This is a technique used by both Eisenbrand et al. and Jansen et al. Essentially, it uses the solution to the linear program to help bound the solution to the ILP even further. Eisenbrand et al. proved that the distance from the relaxed solution z^* to the actual solution x^* is

$$\|z^* - x^*\|_1 \leq m(2m\Delta + 1)^m$$

By using this bound in combination with the (max, +)-convolution algorithm, the logarithmic factors vanish completely, giving an algorithm that runs in time:

$$O(m\Delta)^{2m} + \text{LP},$$

where LP is the time needed to compute the relaxed solution.

5 Unbounded Knapsack

As we discussed in the introduction, the unbounded knapsack problem with item values c_i and weights A_i can be written as the ILP:

$$\max \sum_{i=1}^n c_i x_i \tag{14}$$

$$\text{s.t.} \sum_{i=1}^n A_i x_i = b \tag{15}$$

$$x \in \mathbb{Z}_{\geq 0}^n \tag{16}$$

In this problem, the A matrix is of shape $1 \times n$, meaning that $m = 1$. Having such a small m makes are otherwise exponential algorithms look much less scary! The best running time we have for ILP is $O(m\Delta)^{2m} + \text{LP}$ from Jansen et al. using proximity as mentioned in section 4.4.2. This makes the running time for unbounded knapsack $O(\Delta^2)$.

A classic dynamic programming algorithm for unbounded knapsack achieve $O(nC)$, where C is the sum of all item weights [1]. Another result achieves $O(n\Delta^2)$ [3]. So for $\Delta \ll C$ — which is reasonable if there are many items — the running time achieved by Jansen et al.'s ILP algorithm is the state of the art.

6 Unbounded Knapsack and Conditional Runtime Lower Bound

Jansen and Rohwedder give a conditional lower bound¹ on the runtime of ILP solvers using a previous lower bound on the runtime of Unbounded Knapsack solvers. We attempt to relay the key ideas of their proof.

6.1 Problem Setting

Unbounded Knapsack Problem: Given a knapsack of weight capacity $C \in \mathbb{Z}_{\geq 0}$ and an unlimited² quantity of a set of items $\{X_1, \dots, X_n\}$, where X_i has weight $w_i \in \mathbb{Z}$ and profit $p_i \in \mathbb{Z}$, we seek a choice of items with maximum total profit such that the total weight of the items does not exceed C . Let x where the entry x_i equals the number of item X_i taken. Then we would like to optimize the following ILP:

$$\begin{aligned} \max \quad & p^T x \\ \text{s.t.} \quad & w^T x \leq C \\ & x \in \mathbb{Z}_{\geq 0}^n. \end{aligned}$$

For the sake of consistency with the rest of the paper, we convert the inequality constraint to an equality constraint by adding an item X_0 of profit 0 and weight 1. This allows us to pad any feasible set of items with sufficiently many of the item X_0 to fill the knapsack to capacity, while maintaining the same total profit. Thus we can optimize the following ILP, which we will refer to as UKS1.

UKS1:

$$\begin{aligned} \max \quad & p^T x \\ \text{s.t.} \quad & w^T x = C \\ & x \in \mathbb{Z}_{\geq 0}^n. \end{aligned}$$

6.2 Conditional Lower Bound on Runtime

At the time of writing of this article, there is no known algorithm for computing a $(\max, +)$ -convolution in $O(n^{2-\delta})$, for any $\delta > 0$. After several decades of work, we still do not know if it is possible to perform $\text{poly}(n)$ better than the naive $O(n^2)$ algorithm. We consider the implications of there being no polynomially subquadratic algorithm for $(\max, +)$ -convolution [5].

Theorem 2. *If there is no polynomially subquadratic algorithm for $(\max, +)$ -convolution, then there exists no $O(n + C)^{2-\delta}$ time algorithm for UKS, for any $\delta > 0$.*

Jansen and Rohwedder use Theorem 2 to prove the following conditional lower bound on runtimes for solving ILP. We provide a sketch of their argument.

Theorem 3. *If there is no polynomially subquadratic algorithm for $(\max, +)$ -convolution, then for fixed m , there is no algorithm that can solve ILPs on m constraints in $f(m) \cdot (n^{2-\delta} + (\Delta + \|b\|_\infty)^{2m-\delta})$, for some $\delta > 0$ and some computable function³ f .*

Proof Sketch. We would like to prove a lower bound on ILP solver runtime when m is fixed. We can do this by showing that if we have a solver that runs faster than our proposed lower bound, then we would be able to solve Unbounded Knapsack faster than its proven lower bound in Theorem 2.

As before, we consider ILPs of the standard form

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \in \mathbb{Z}_{\geq 0}^n, \end{aligned}$$

¹i.e. a lower bound that is only true given a condition is true - in our case, the existence of a polynomially subquadratic algorithm for $(\max, +)$ -convolution.

²We call this knapsack problem *unbounded* because the quantity of each item available is unbounded.

³i.e. a function for which there exists an algorithm to compute its output for any input in its domain.

where $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$, and let Δ be the maximum over the absolute values of the entries of A . We make the following assumptions:

1. Given an input set X of items X_0, \dots, X_n , the only relevant items have weight at most C . Thus we can preprocess the input in $O(n + C)$ time to isolate the relevant items. We can thus assume that we have $w_i \leq C$ for all i and $n \leq C$.
2. There exists an algorithm SOLVER that solves standard form ILPs in time $f(m) \cdot (n^{2-\delta} + \Delta + \|b\|_\infty)^{2m-\delta}$.

6.2.1 Case $m = 1$

Note that our previously formulated ILP UKS1 is equivalent to the standard form ILP with $m = 1$. From assumption 1, $\Delta \leq C$ and $\|b\|_\infty = C$, so SOLVER takes

$$\begin{aligned} f(m) \cdot \left(n^{2-\delta} + (\Delta + \|b\|_\infty)^{2m-\delta} \right) &\leq f(1) \cdot \left(n^{2-\delta} + (C + C)^{2-\delta} \right) \\ &= O \left(n^{2-\delta} + (2C)^{2-\delta} \right). \end{aligned}$$

6.2.2 Case $m > 1$

The key idea is to formulate UKS as an m -constraint standard form ILP UKSm with small Δ and $\|b\|_\infty$. We can then bound the runtime of SOLVER on this ILP by $O((n + C)^{2-\delta})$ to generate a contradiction of Theorem 2.

We force Δ to be small by first choosing a target value of Δ and then constructing A to fit this target value. UKSm needs to express the fact that $w^T x = C$ in m separate constraints. We are thus vaguely motivated to break the w_i and C into m parts and somehow relate these parts to Δ . We might then stumble upon a critical realization: a number written in base Δ has digits of absolute value $< \Delta$. Therefore, we might try to write the w_i and C as m -digit numbers⁴ in base Δ . Since for all i , $C \geq w_i$, we find Δ in terms of C rather than the w_i . In particular, for C to have m digits in base Δ , we choose $\Delta = \lfloor C^{1/m} \rfloor + 1$.

We define some notation before moving forward. Let a be a positive integer $\leq \Delta^m$. Then we can write the base Δ representation of a as

$$a = a^{(0)}\Delta^0 + a^{(1)}\Delta^1 + \dots + a^{(m)}\Delta^m,$$

where each $a^{(k)} \in \{0, \dots, \Delta - 1\}$. Define $a^{(k)}$ to be the coefficient of the Δ^k term in the base Δ representation of a . Note that we allow any of the a_k to be 0, and we enforce that there are exactly m of them.

Then consider the following base Δ decompositions of C and w_i :

$$\begin{aligned} C &= C^{(0)} + \Delta C^{(1)} + \dots + \Delta^{m-1} C^{(m-1)} \\ w_i &= w_i^{(0)} + \Delta w_i^{(1)} + \dots + \Delta^{m-1} w_i^{(m-1)}, \end{aligned}$$

which give rise to

$$w_i x_i = w_i^{(0)} x_i + \Delta w_i^{(1)} x_i + \dots + \Delta^{m-1} w_i^{(m-1)} x_i$$

for all $i \in [n]$. Summing the $w_i x_i$ equations columnwise, we get

$$C = \sum_{i=1}^n w_i x_i = \sum_{i=1}^n w_i^{(0)} x_i + \Delta \sum_{i=1}^n w_i^{(1)} x_i + \dots + \Delta^{m-1} \sum_{i=1}^n w_i^{(m-1)} x_i.$$

Then matching terms of like superscript and canceling factors of Δ^k between the two equations, we might be tempted to write the following ILP, which we will call `almostUKSm`.

⁴(allowing leading 0s)

almostUKSm:

$$\begin{aligned} \max \quad & p^T x \\ \text{s.t.} \quad & \sum_{i=1}^n w_i^{(k)} x_i = C^{(k)}, \quad k \in \{0, \dots, m-1\} \\ & x \in \mathbb{Z}_{\geq 0}^n. \end{aligned}$$

Then the sum of the m equality constraints gives rise to the single constraint in UKS1, so a feasible solution x to almostUKSm is a feasible solution to UKS1. However, a feasible solution to UKS1 is not necessarily a feasible solution to almostUKSm. For $w_1 = 1$, setting $x_1 = C$, $x_i = 0$ for $i \neq 1$ produces a feasible solution to UKS1 but not to almostUKSm if $\Delta \neq 1$. We need to modify almostUKSm to allow each of its m equality constraints to be violated. Thus, we construct the slack variables y_1, \dots, y_{m-1} . However, we would like to preserve the property that summing the m equality constraints recovers the single equality constraint in UKS1, so we are motivated to set up a telescoping sum. Define UKSm as follows:

UKSm:

$$\begin{aligned} \max \quad & p^T x \\ \text{s.t.} \quad & \Delta^0 \sum_{i=1}^n w_i^{(0)} x_i - \Delta \cdot y_1 = C^{(0)} \\ & \Delta^k \sum_{i=1}^n w_i^{(k)} x_i + \Delta^k \cdot y_k - \Delta^{k+1} \cdot y_{k+1} = \Delta^k C^{(k)} \quad \forall k = 1, \dots, m-2 \\ & \Delta^{m-1} \sum_{i=1}^n w_i^{(m-1)} x_i + \Delta^{m-1} \cdot y_{m-1} = \Delta^{m-1} C^{(m-1)} \\ & x \in \mathbb{Z}_{\geq 0}^n \\ & y \in \mathbb{Z}_{\geq 0}^m. \end{aligned}$$

Summing the equality constraints, we get

$$\begin{aligned} \sum_i x_i \sum_{k=0}^{m-1} \Delta^k w^{(k)} &= \sum_{k=0}^{m-1} \Delta^k C^{(k)} \\ &= \sum_{i=1}^n w_i x_i = C, \end{aligned}$$

as desired. Some additional mathematical rigor is necessary to verify that indeed we need only $m-1$ slack variables for the m equations, but for the sake of brevity and clarity we omit the fine details.

6.2.3 Runtime:

Canceling out powers of Δ from the constraints, we get

UKSm:

$$\begin{aligned}
& \max && p^T x \\
& \text{s.t.} && \sum_{i=1}^n w_i^{(0)} x_i - \Delta \cdot y_1 = C^{(0)} \\
& && \sum_{i=1}^n w_i^{(k)} x_i + y_k - \Delta \cdot y_{k+1} = C^{(k)} \quad \forall k = 1, \dots, m-2 \\
& && \sum_{i=1}^n w_i^{(m-1)} x_i + \Delta \cdot y_{m-1} = C^{(m-1)} \\
& && x \in \mathbb{Z}_{\geq 0}^n \\
& && y \in \mathbb{Z}_{\geq 0}^m.
\end{aligned}$$

If we write the equality constraints in the form $Ax = b$, then we have that the absolute values of the entries of A and b are all bounded by $\Delta = O(C^{1/m})$. Thus this ILP can be solved in time

$$\begin{aligned}
f(m) \cdot \left(n^{2-\delta} + O(C^{1/m})^{2m-\delta} \right) &\leq f(m) \cdot O(1)^{2m-\delta} \cdot (n + C)^{2-\delta/m} \\
&= O\left((n + C)^{2-\delta/m} \right),
\end{aligned}$$

a contradiction of Theorem 2. We conclude that if there is no polynomially subquadratic algorithm for $(\max, +)$ -convolution, then for fixed m there is no algorithm that solves ILPs in time

$$f(m) \cdot \left(n^{2-\delta} + (\Delta + \|b\|_{\infty})^{2m-\delta} \right).$$

□

7 Conclusion

We have described the progress of integer linear programming algorithms that has flourished in the past couple of years. Eisenbrand et al.'s insight to use Steinitz's lemma made great improvements over the long-held bound established by Papadimitriou. Jansen et al. built on Eisenbrand et al.'s work to achieve a running time as good as $O(m\Delta)^{2m} + \text{LP}$. Fomin et al. [4] have showed that if we assume the exponential time hypothesis, then Jansen et al.'s results are nearly optimal. In application, these results give state of the art bounds on problems with a limited number of constraints, such as the knapsack problem.

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