Commun Nonlinear Sci Numer Simulat 16 (2011) 1574–1579

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Influence of the finite precision on the simulations of discrete dynamical systems

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ARTICLE INFO

Article history:
Received 11 January 2010
Received in revised form 1 July 2010
Accepted 5 July 2010
Available online 8 August 2010

Keywords:
Simulations of discrete dynamical systems
Numerical precision
Noisy dynamical systems

ABSTRACT

The effect of numerical precision on the mean distance and on the mean coalescence time between trajectories of two random maps was investigated. It was shown that mean coalescence time between trajectories can be used to characterize regions of the phase space of the maps. The mean coalescence time between trajectories scales as a power law as a function of the numerical precision of the calculations in the contracting and transitions regions of the maps. In the contracting regions the exponent of the power law is approximately one for both maps and it is approximately two in the transition regions for both maps. In the chaotic regions, the mean coalescence time between trajectories scales as an exponential law as a function of the numerical precision of the calculations for the maps. For both maps the exponents are of the same order of magnitude in the chaotic regions.

1. Introduction

Computer simulations of dynamical systems models have been intensively and extensively used nowadays in applied research as well as in theoretical ones. Computer simulations and data analysis are present in many fields like economics decision analysis, financial planning management of complex network like electric power grid, various types of industrial design like aircraft design, global climate modeling control and synchronization [1] such as in data transmissions [2,3], etc. To study a real process it is needed to analyze an ensemble of data. Simulating a real process, by means of a dynamical system model, it is also needed an ensemble of data to characterize the behavior of model. In this case, by an ensemble of data, we mean an ensemble of trajectories of the dynamical system model that evolves from an ensemble of numerical typical initial conditions.

The aim of this paper is to contribute in elucidating the relevance of the finite precision in computer calculations by means of characterizing the regions of the phase space of the dynamical system used to model the real process. For this, we use two random maps computer simulated. By a random map we understand a deterministic map either with multiplicative noise parameter or with an additive noise term.

2. Features of computer trajectories

Working analytically a mathematical problem, we imagine operation being performed with infinite precision, while performing simulations computers work with finite precision.

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doi:10.1016/j.cnsns.2010.07.003

Computer trajectories are always eventually periodic because of the finite precision of the computer numbers and also because of the finite amount of computer numbers. Some of these simulated orbits can be real and some can be artificial. The artificial periodic orbits come from aperiodic orbits. These orbits are periodic only by the fact of the finite precision of computers numbers [4]. It is not possible to distinguish between real and artificial periodic orbits in the computer model simulation. The real periodic orbits can be stable or unstable. The stable periodic orbits are attracting whilst the unstable ones are not attracting.

Finite precision may causes the computer orbit be away from the theoretical one. It was shown by Anosov and Bowen [5] that uniformly hyperbolic systems have the shadowing property, that is computer orbit follows the theoretical one. Other systems may not have the shadowing property.

For noisy systems the scenario can be even more complicated to study once the noise term has also finite precision.

The analysis of noisy dynamical systems can be non trivial when noise correlates subsystems [6]. Longa et al. [7] found that for the generalized logistic map (with additive noise) the maximal Lyapunov exponent is not a convenient criterium to distinguish chaotic regions from non chaotic ones. They suggested to use the distance between trajectories as a criterium of identical synchronization. But what could be the distance that characterizes a 'good' synchronization if the distance between trajectories may vary with precision?

3. Coalescence of the random maps

We studied the mean coalescence time and distance between trajectories to know the dynamical regions of phase space of the model. For the random maps we divide the phase space, where the movement of trajectories takes place, in contracting, transition and chaotic regions.

The region is contracting if the map almost always contracts the instantaneous distance between two trajectories for an ensemble of initial conditions. By almost we mean that in almost all iterations the instantaneous distance between two trajectories is contracted. It happens the instantaneous distance between two trajectories might be expanded, in some iterations, because of the noise. But in the contracting areas the rate of contraction is much bigger than the rate of expansions. Therefore, the instantaneous distance between trajectories diminishes quickly as a function of time.

The region is a transition one if the map either contracts or expands the instantaneous distance between two trajectories but, the contracting rate is greater than the expanding rate for the ensemble of initial conditions. Thus, the instantaneous distance diminishes as a function of time, but not as quickly as in the contracting areas.

At last, the region is chaotic if the map contracts and expands the instantaneous distance between trajectories, in such a way that the rate of contractions is comparable to the rate of expansions for the ensemble of typical initial conditions.

These three regions for the maps considered here come from the combined effect of the deterministic part of the map and the variation of the intensity of the noise term.

The coalescence between two trajectories of a random map happens in different ways inside these three regions of the phase space of the model. In contracting areas, after a transient time, the trajectories go to a real periodic orbits which are attracting. Within this scenario coalescence happens because of the attraction of the attractor.

Above certain value of the noise term, the trajectories of the map enter the transition region of the phase space. Within the transition areas, trajectories wander either on the real periodic orbits or on the artificial ones. The attraction of the attractor is disturbed by the increasing of the intensity of the noise term. The coalescence between two trajectories may happen in transition areas either because of the attraction of the attractor or because of numerical coincidence due to roundoff of the numerical calculations.

In the chaotic regions, there are no attracting orbits for almost all initial conditions that is, for typical initial conditions. In these regions, the trajectories have always finite distance between them and it happens suddenly they numerically coincide. The coalescence between two trajectories in the chaotic regions happens only because of the numerical roundoff.

It has been shown that the mean coalescence time of two trajectories for the logistic map with additive noise depends upon precision [4]. Based on this reference, we studied through numerical simulations, the behavior of the random logistic map [8] and the behavior of the generalized logistic map [7], by taking into account the mean distance between trajectories and the mean coalescence time between trajectories as a functions of the decimal numerical precision used in the calculations. We found out that the mean coalescence time varies as a power law as function of numerical precision of the calculations in the contracting and transition areas. In the contracting areas the exponent is approximately one, and in the transition areas the exponent is approximately two for both maps. In the chaotic regions the mean coalescence time scales as a exponential law with the exponent of the same order of magnitude for both maps.

Therefore, the mean coalescence time of two trajectories can be used to characterize regions of the phase space for random maps in the sense of depending on the mean coalescence time law, it can be said that the system is either in the contracting or transition or chaotic region.

We also showed that in the contracting and transition areas the mean distance between the trajectories tends to zero as the precision of the calculations increases, and in the chaotic regions the mean distance between trajectories tends to a non-zero value as the precision of the calculations increases.

Calculations were carried out with fortran and MAPLE. The means were performed taking the ensemble of samples in such a way that the value of the variance of the mean distances practically does not change when one more sample is added.

Next, we defined the mean coalescence time and the mean distance between trajectories. We also presented the maps the results and conclusions.

4. Definitions

For each precision (10^{-M}) of M decimal digits we studied the coalescence of two trajectories of the maps using the instantaneous distance between them.

For each precision (10^{-M}), the map was iterated from each two different initial conditions (x_0, y_0) giving rise two trajectories (x_i, y_i). The coalescence between two trajectories was established when the instantaneous distance between these trajectories was of the order of the decimal precision used in the calculations that is, $|x_j - y_j| \leqslant 10^{-M}$. The number of time steps j that is, the number of iterations, needed for the instantaneous distance between trajectories to become of the order of the precision is the coalescence time $t_c = j$ for that initial conditions and for that precision M.

Each two initial conditions define a (α) sample of two trajectories. The mean coalescence time $\bar{t}_c$ of trajectories, among k samples of the ensemble, is the average of the coalescence time of the samples for the precision M:

$$\bar{t}_c = \bar{t}_c(M) = \frac{1}{k}\sum_{\alpha=1}^{k} t_c^{(\alpha)}, \tag{1}$$

where $t_c^{(\alpha)}$ is the coalescence time for the (α) sample.

For each precision M and for each (α) sample the mean distance between two trajectories is the number

$$\bar{d}^{(\alpha)}(M) = \frac{1}{t_c^{(\alpha)}}\sum_{t=0}^{t_c^{(\alpha)}} d_t^{(\alpha)} = \frac{1}{t_c^{(\alpha)}}\sum_{t=0}^{t_c^{(\alpha)}} \left| x_t^{(\alpha)} - y_t^{(\alpha)} \right|. \tag{2}$$

The mean distance between two trajectories for each precision M, considering k samples is defined as

$$\bar{d} = \bar{d}(M) = \frac{1}{k}\sum_{\alpha=1}^{k} \bar{d}^{(\alpha)}. \tag{3}$$

Next, we studied the coalescence of trajectories of the random logistic map and of the generalized logistic map. The random logistic map is the logistic map with multiplicative noise parameter that is, $x_{n+1} = r\, x_n\,(1 - x_n)$, where $r = 4$ with probability p and $r = 1/2$ with probability $1 - p$. This means that when p is small, most of the iterations will have $r = 1/2$ as a coefficient and when p is close to 1, most of the iterations will have $r = 4$ as a coefficient. This map presents three different regimes [8]. For $0 < p < 0.333$ it presents a contracting area. All trajectories go to the stable fixed point irrespective of the initial conditions. For $0.33 < p < 0.47$ it presents a transition area where trajectories go to a fixed point, but this point changes position changing initial conditions. For $0.47 < p < 1$ the map has a chaotic behavior.

The generalized logistic map is the logistic map with additive noise term [7]:

$$\begin{aligned} x\prime &= X_M(\tilde{x}, \tilde{y}) + \eta\widetilde{W}(\tilde{x}, \tilde{y}, W), \\ y\prime &= Y_M(\tilde{x}, \tilde{y}) + \eta\widetilde{W}(\tilde{x}, \tilde{y}, W), \\ \tilde{x} &= 4x(1-x), \quad \tilde{y} = 4y(1-y). \end{aligned} \tag{4}$$

$\eta\widetilde{W}$ is the noise term, η is an independent uniform random variable within the interval $[-1, 1]$. The functions X_M e Y_M are:

$$X_M = \frac{T+B}{2}, \quad Y_M = X_M + \tilde{y} - \tilde{x}, \quad \widetilde{W} = \frac{T-B}{2}, \tag{5}$$

where

$$T = \min\left\{1, 1 - (\tilde{y} - \tilde{x}), \frac{1}{2}\left[1 + \frac{e}{4} - (\tilde{y} - \tilde{x})\right], \tilde{x} + W\right\} \tag{6}$$

$$B = \max\left\{0, -(\tilde{y} - \tilde{x}), \frac{1}{2}\left[1 - \frac{e}{4} - (\tilde{y} - \tilde{x})\right], \tilde{x} - W\right\}. \tag{7}$$

The e parameter varies in the $(0,4]$ interval. The generalized logistic map presents for $W = 1$, a contracting region when $0 < e < 2.717$, and a chaotic region when $2.717 < e \leqslant 4$. The transition region for the generalized logistic map is characterized only by the value $e = 2.717$.

5. Numerical simulations and results

Calculations have been done from precision 4 to 32 in contracting and transitions areas. For the chaotic areas, calculations have been done from precision 4 to 16. Higher precisions are impracticable because coalescence time becomes to long in the chaotic regions and programs would run for years to obtain the necessary number of samples.

We performed simulations of the evolution of the trajectories of the maps within the contracting, transition and chaotic regions, varying the precision M for each value of the parameters considered for the maps. From the evolutions of the trajectories, we got the mean distance as a function of the precision M within the three regions for both maps. We can see this in Fig. 1 for the random logistic map. Curves of this figure show that for each value of the p parameter of the map, the mean distance between trajectories diminishes as the precision M increases. Also, for each precision, the value of the mean distance between trajectories increases as the value of the p parameter increases.

In the contracting areas, curves are for values of the probability from $p = 0.01$ to $p = 0.33$, and in the transition areas from $p = 0.33$ to $p = 0.47$. In these areas, the mean distance tends to zero when M tends to infinity. The convergence of the mean distance is slower in the transitions areas as compared as the convergence in the contracting areas, as it was expected. In the contracting areas, for each precision M, the coalescence of two trajectories happens, for each sample, because the trajectories become more and more closer to each other as the time step evolves. Also, the greater the precision, the closer to each other trajectories are when they coalesce. We may say this is true coalescence of the trajectories. This behavior of the trajectories characterizes the contracting areas. In the transition areas, the coalescence happens either in the same way as in the contracting areas or because of the numerical roundoff.

In the chaotic region, the curve of Fig. 1 is for the probability value $p = 0.5$. We can see that, for the chaotic region, the mean distance between trajectories tends to a non-zero value with increasing precision of the calculations. This means that the trajectories are always far from each other irrespective of the precision used in calculations, except by the very moment – of the time step– when they coalesce due to numerical roundoff. This is the behavior of trajectories that characterizes chaotic regions. The same results we obtained for the probability value $p = 0.6$.

The generalized logistic map has similar behavior within the respective three regions of the phase space. In the contracting and transition areas, the mean distance obeys a power law ($\bar{d} \sim M^{-\rho},\ \rho > 0$) as a function of the precision M. This can be seen in Fig. 2 for contracting areas, for the random logistic map. For both maps $\rho \approx 1$ in the contracting areas. In the transition areas, from $p = 0.33$ to $p = 0.47$, there is a tendency of increasing of the mean distance as a function of M, when the

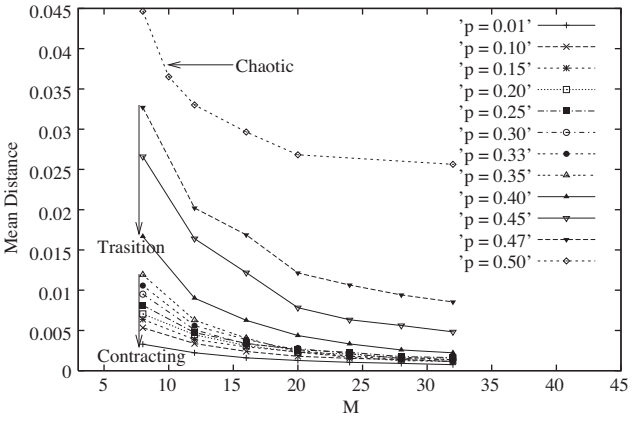

Fig. 1. Mean distance versus precision M for the random logistic map.

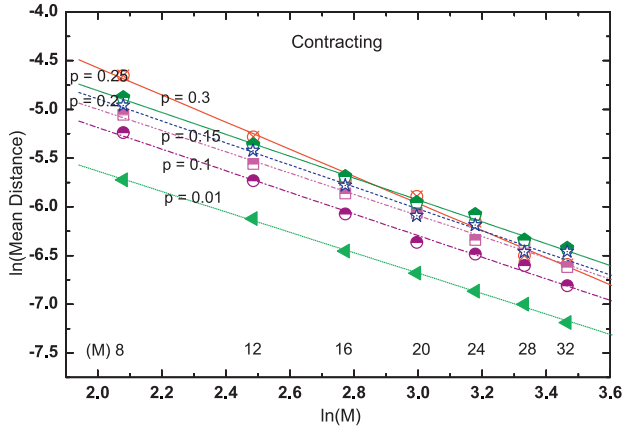

Fig. 2. Logarithm of the mean distance versus logarithm of the precision M for the random logistic map for contracting areas: $p = 0.01, 0.1, 0.15, 0.2, 0.25, 0.3$.

probability values increase. This transition behavior is due to the increasing of the number of samples (as p increases), whose trajectories go through artificial periodic orbits that are not attracting. This is also true for the generalized logistic map in the transition area.

We now turn to the coalescence time analysis. We observe that for both maps in the contracting and transition areas, the mean coalescence time between two trajectories behaves as a power law $\bar{t}_c \sim M^{\gamma}$ as function of the precision M. We can see this in Fig. 3 for the random logistic map. We observed for both maps that $\gamma \approx 1$, in the contracting areas and $\gamma \approx 2$ in the transitions areas. The behavior of the mean coalescence time in the chaotic regions is showed in Fig. 4 for the random logistic map. For both maps, the mean coalescence time between two trajectories obeys an exponential law $\bar{t}_c \sim e^{\eta M}$ as a function of the precision M in the chaotic regions. It can be concluded from Fig. 4, for the random logistic map, that $\eta \approx 0.17$ for $p = 0.5$ and $\eta \approx 0.86$ for $p = 0.6$. We also observed for the generalized logistic map (when $W = 1$) that $\eta \approx 0.45$ for $e = 3.0$, $\eta \approx 0.86$ for $e = 3.5$ and $\eta \approx 1.14$ for $e = 4.0$. The exponents are of the same order of magnitude for both maps in the chaotic regions.

We also got for the contracting areas, a relation between the mean distance and the mean coalescence time between the trajectories. We saw that in the contracting areas for both maps it holds $\bar{t}_c \sim M^{\gamma}$, with $\gamma \approx 1$ and also $\bar{d} \sim M^{-\rho}$, with $\rho \approx 1$. We drew a graphic of logarithm of the mean distance versus logarithm of the mean coalescence time. We concluded from it that $\bar{d} \sim \bar{t}_c^{-\zeta}$ where $\zeta \approx 1$ for both maps within contracting areas.

6. Discussions and conclusions

We concluded from these experiments that the movement of the trajectories through real periodic orbits and through a mixed of real periodic orbits and artificial ones induces a polynomial behavior on the mean coalescence time between

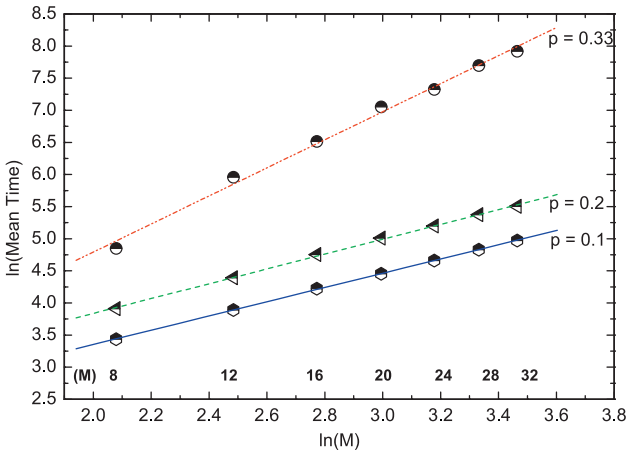

Fig. 3. Logarithm of the mean coalescence time versus logarithm of the precision M for the random logistic map for contracting ($p = 0.1, 0.2,$) and transition ($p = 0.33$) areas.

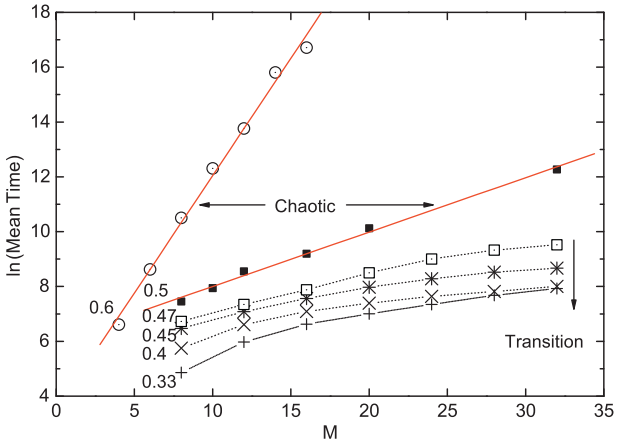

Fig. 4. Logarithm of the mean coalescence time versus precision M for the random logistic map the transition ($p = 0.33, 0.4, 0.45, 0.47$) and chaotic areas ($p = 0.5, 0.6$).

trajectories as a function of the precision M. This happens in the contracting and in the transition areas. On the other hand, the movement of the trajectories through artificial periodic orbits that is, in chaotic regions, yields an exponential behavior on the mean coalescence time as a function of the precision M.

In the contracting areas, the mean coalescence time between the trajectories increases with the precision of the calculations scaling as a power law ($\bar{t}_c \sim M^{\gamma}$) and the exponent is practically the same ($\gamma \approx 1$) for both maps. In the transition areas, the mean coalescence time between the trajectories increases with the precision of the calculations, also scales as a power law and the exponent is practically the same ($\gamma \approx 2$) for both maps. The fact that, in contracting and transition areas, the exponent of the mean coalescence time becomes practically the same for both maps, should not be a coincidence but, a more general result, once the two random maps tested are not related to each other. These results should be applicable to other random maps. In the chaotic areas, the mean coalescence time between the trajectories increases with the precision, having an exponential dependence $\bar{t}_c \sim e^{\eta M}$ on the precision of the calculations for both maps, as it was foreseen by Longa et al. [4]. The exponents have the same order of magnitude for both maps. We were able to see that, the coalescence time between the trajectories characterizes well the regions of the phase space of the maps from an ensemble of typical initial conditions.

For both maps, the random logistic map and the generalized logistic map, the mean distance between the trajectories scales as a power law ($\bar{d} \sim M^{-\rho},\ \rho > 0$) and decreases tending to zero with the increasing of the numerical precision used in the calculations in the contracting and transition areas. The exponent of the mean distance between the trajectories as a function of the precision (power law) is practically the same for both maps ($\rho \approx 1$) in the contracting areas. For chaotic areas, the mean distance between the trajectories decreases tending to a non-zero value, as the precision of the calculations increases. The magnitude of the mean distance gives an idea of the size of the phase space that the maps occupy. In modeling climate variables, as rainfall precipitation, for instance, the asymptotic value of the mean distance may indicate the most probably intensity of the rain. In the chaotic regions this may say: ‘flood is coming’.

Acknowledgments

This work was supported by CAPES in Brasil and ICTP in Italy.

References

[1] Ciésla M, Dias SP, Longa L, Oliveira FA. Phys Rev E 2001;63:065202(R).
[2] Hu G, Zhang Y, Cerdeira HA, Chen S. Phys Rev Lett 2000;85:3377.
[3] Perez G, Cerdeira H. Phys Rev Lett 1995;74:1970.
[4] Longa L, Curado EMF, Oliveira FA. Phys Rev E 1996;54:R2201.
[5] Anosov DV. Proc Steklov Inst Math 1967;90:1;
Bowen R. J Diff Eq 1975;18:333.
[6] Pikovsky AS. Phys Lett A 1992;165:33.
[7] Longa L, Dias SP, Curado EMF. Phys Rev E 1997;56:259.
[8] Loreto L, Paladin G, Pasquini M, Vulppiani A. Physica A 1996;232:189.