Calculus 1 Workbook Solutions

Related rates

krista king
MATH

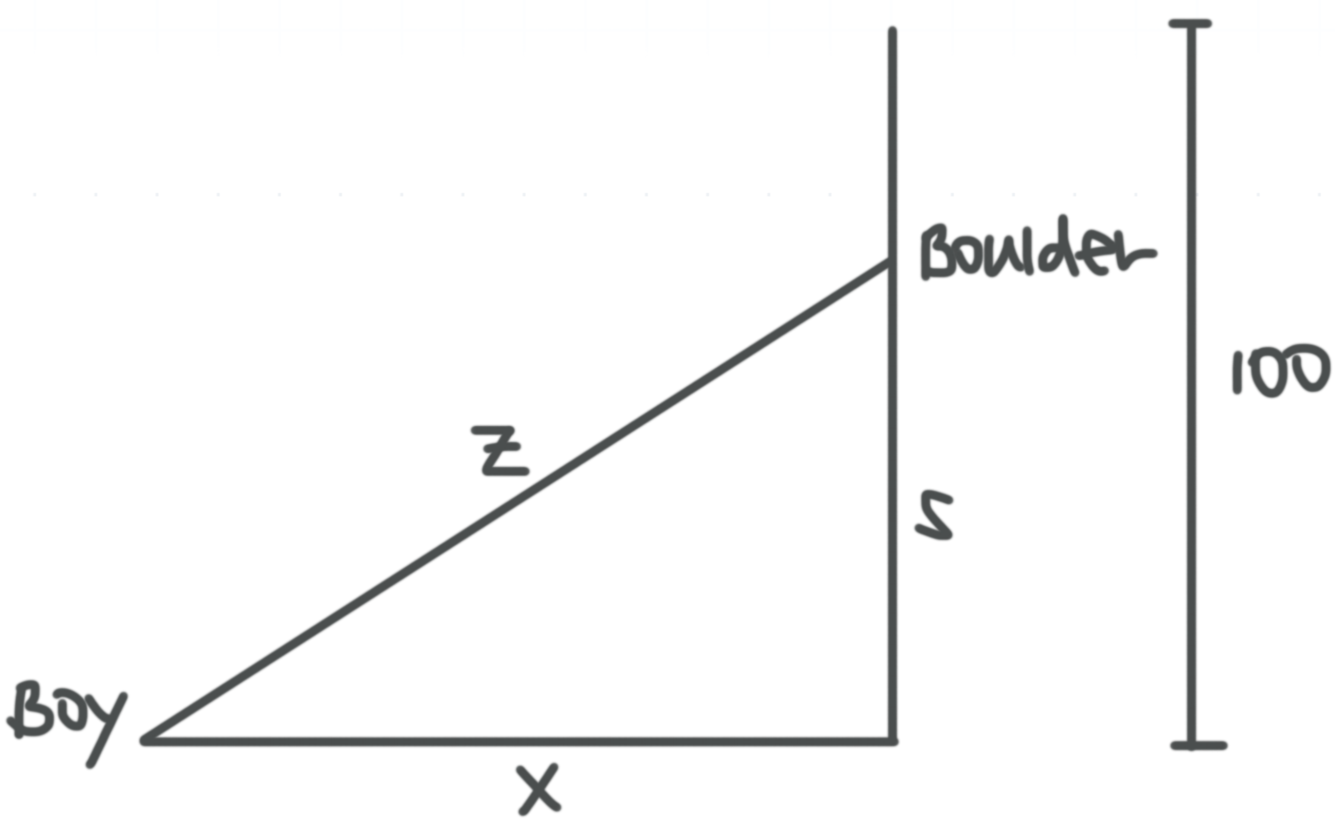
RELATED RATES

■ 1. A boy is standing 15 feet from the base of a 100 feet cliff. As a boulder rolls off the cliff, the boy begins running away at 8 ft/s. At what rate is the distance between the boy and the boulder changing after 2 seconds?

The height of the falling boulder is modeled by the position function $s = -16t^2 + v_0t + s_0$, where s_0 is the initial height and v_0 is the initial velocity of the boulder.

Solution:

Draw a diagram.



In this case, $s_0 = 100$ and $v_0 = 0$, so we'll plug these into the position function.

$$s = -16t^2 + v_0t + s_0$$

$$s = -16t^2 + 100$$

Applying the Pythagorean Theorem to the diagram we drew, we get

$$x^2 + s^2 = z^2$$

$$x^2 + (-16t^2 + 100)^2 = z^2$$

Differentiate with respect to time.

$$2x\frac{dx}{dt} + 2(-16t^2 + 100)(-32t) = 2z\frac{dz}{dt}$$

$$x\frac{dx}{dt} + (-16t^2 + 100)(-32t) = z\frac{dz}{dt}$$

$$x\frac{dx}{dt} + 512t^3 + -3200t = z\frac{dz}{dt}$$

Since the boy initially starts at $x = 15$, after 2 seconds the value of x is

$$x = 15 + 8t$$

$$x = 15 + 8(2)$$

$$x = 31$$

And the height of the boulder after 2 seconds is

$$s(2) = -16(2)^2 + 100$$

$$s(2) = 36$$

We can use the Pythagorean Theorem to find the distance between the boy and the boulder, z, at the time when $x = 31$ and $s = 36$.

$$x^2 + s^2 = z^2$$

$$31^2 + 36^2 = z^2$$

$$961 + 1{,}296 = z^2$$

$$2{,}257 = z^2$$

$$z \approx 47.5$$

Substitute what we know into the derivative, then solve for dz/dt.

$$31(8) + 512(2)^3 + -3{,}200(2) = 47.5\frac{dz}{dt}$$

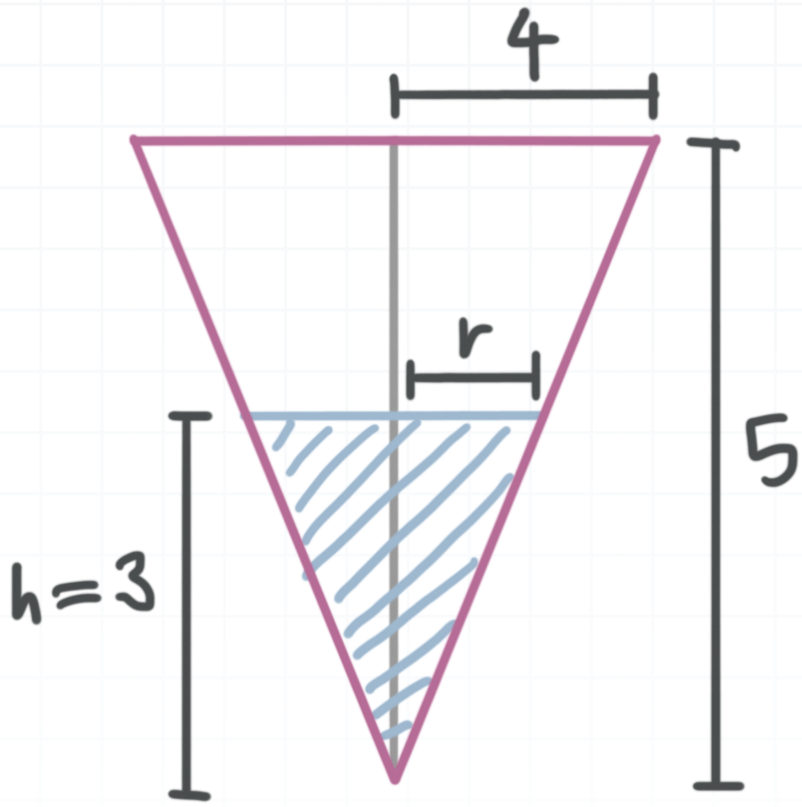
$$-2{,}056 = 47.5\frac{dz}{dt}$$

$$\frac{dz}{dt} = -\frac{2{,}056}{47.5} \approx -43.3$$

■ 2. Water is flowing out of a cone-shaped tank at a rate of 6 cubic inches per second. If the cone has a height of 5 inches and a base radius of 4 inches, how fast is the water level falling when the water is 3 inches deep?

Solution:

Draw a diagram.



The volume of a cone is $V=(1/3)\pi r^2 h$. We want to express volume as a function of h only.

$$\frac{r}{h}=\frac{4}{5}$$

$$r=\frac{4}{5}h$$

Then the volume of the cone of water is

$$V=\frac{1}{3}\pi r^2 h$$

$$V=\frac{1}{3}\pi\left(\frac{4}{5}h\right)^2 h$$

$$V=\frac{1}{3}\pi\left(\frac{16}{25}h^2\right)h$$

$$V = \frac{16}{75}\pi h^3$$

Differentiate the volume equation with respect to t.

$$\frac{dV}{dt} = \frac{16}{75}\pi(3h^2)\frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{16}{25}\pi h^2\frac{dh}{dt}$$

The problem states that $dV/dt = -6$. Substitute this and $h = 3$ into the derivative equation and solve for dh/dt.

$$-6 = \frac{16}{25}\pi(3)^2\frac{dh}{dt}$$

$$-6 = \frac{144}{25}\pi\frac{dh}{dt}$$

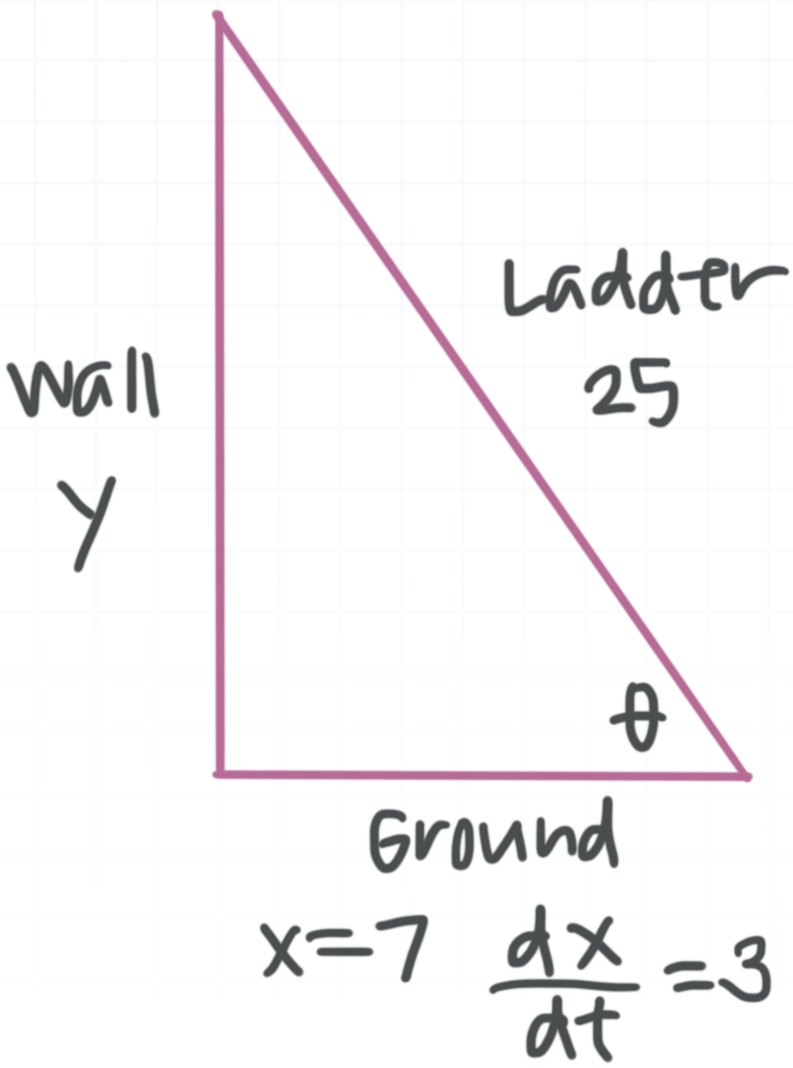
$$\frac{dh}{dt} = -6\frac{25}{144\pi}$$

$$\frac{dh}{dt} = -\frac{25}{24\pi} \approx -0.33 \text{ in/s}$$

■ 3. A ladder 25 feet long leans against a vertical wall of a building. If the bottom of the ladder is pulled away horizontally from the building at 3 feet per second, how fast is the angle formed by the ladder and the horizontal ground decreasing when the bottom of the ladder is 7 feet from the base of the wall?

Solution:

Draw a diagram.



Find y using the Pythagorean theorem.

$$y^2 + 7^2 = 25^2$$

$$y = 24$$

Use the cosine function, which gives the equation

$$\cos\theta = \frac{x}{25}$$

Differentiate with respect to t.

$$-\sin\theta\frac{d\theta}{dt}=\frac{1}{25}\frac{dx}{dt}$$

Use the sine function, which gives the equation

$$\sin\theta=\frac{y}{25}=\frac{24}{25}$$

Substitute what we know.

$$-\frac{24}{25}\cdot\frac{d\theta}{dt}=\frac{1}{25}\cdot 3$$

$$\frac{d\theta}{dt}=\frac{3}{25}\cdot\left(-\frac{25}{24}\right)=-\frac{3}{24}=-\frac{1}{8}\text{ ft/s}$$

4. The radius of a spherical balloon is increasing at a rate of 4.5 ft/hr. At what rate are the sphere's surface area and volume increasing when the surface area is 36π ft^2?

Solution:

The formula for the surface area of a sphere is

$$S=4\pi r^2$$

First we need to find the radius of the ballon when the surface area is 36π ft^2.

$$36\pi=4\pi r^2$$

$$r^2 = 9$$

$$r = 3$$

Use implicit differentiation to take the derivative of both sides of the standard surface area formula.

$$(1)\frac{dS}{dt} = 4\pi(2r)\frac{dr}{dt}$$

$$\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$$

From the question, we know that $dr/dt = 4.5$ and that $r = 3$, so we'll plug those in.

$$\frac{dS}{dt} = 8\pi(3)(4.5)$$

$$\frac{dS}{dt} = 108\pi$$

The formula for the volume of a sphere is

$$V = \frac{4}{3}\pi r^3$$

Use implicit differentiation to take the derivative of both sides.

$$(1)\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

From the question, we know that $dr/dt = 4.5$ and that $r = 3$, so we'll plug those in.

$$\frac{dV}{dt} = 4\pi(3^2)(4.5)$$

$$\frac{dV}{dt} = 162\pi$$

Therefore, at the moment when the surface area is 36π ft^2, the surface area is increasing at 108π ft/hr, while the volume is increasing at 162π ft/hr.

■ 5. A price p and demand q for a product are related by $q^2 - 2qp + 30p^2 = 10{,}125$. If the price is increasing at a rate of 2.5 dollars per month when the price is 15 dollars, find the rate of change of the demand.

Solution:

First we need to find the value of demand. Substitute $p = 15$ into the equation, then solve for q.

$$q^2 - 2qp + 30p^2 = 10125$$

$$q^2 - 2q(15) + 30(15)^2 = 10125$$

$$q^2 - 30q + 6750 = 10125$$

$q = -45$ and $q = 75$

The value we select has to be greater than 0, so $q = 75$.

Use implicit differentiation to take the derivative of both sides of the quantity equation.

$$q^2 - 2qp + 30p^2 = 10125$$

$$2q\frac{dq}{dt} - 2p\frac{dq}{dt} - 2q\frac{dp}{dt} + 60p\frac{dp}{dt} = 0$$

$$(2q - 2p)\frac{dq}{dt} + (60p - 2q)\frac{dp}{dt} = 0$$

From the question, we know that $p = 15$, $q = 75$ and $dp/dt = 2.5$, so we'll plug those in.

$$(2(75) - 2(15))\frac{dq}{dt} + (60(15) - 2(75))(2.5) = 0$$

$$(150 - 30)\frac{dq}{dt} + (900 - 150)(2.5) = 0$$

$$120\frac{dq}{dt} + 1875 = 0$$

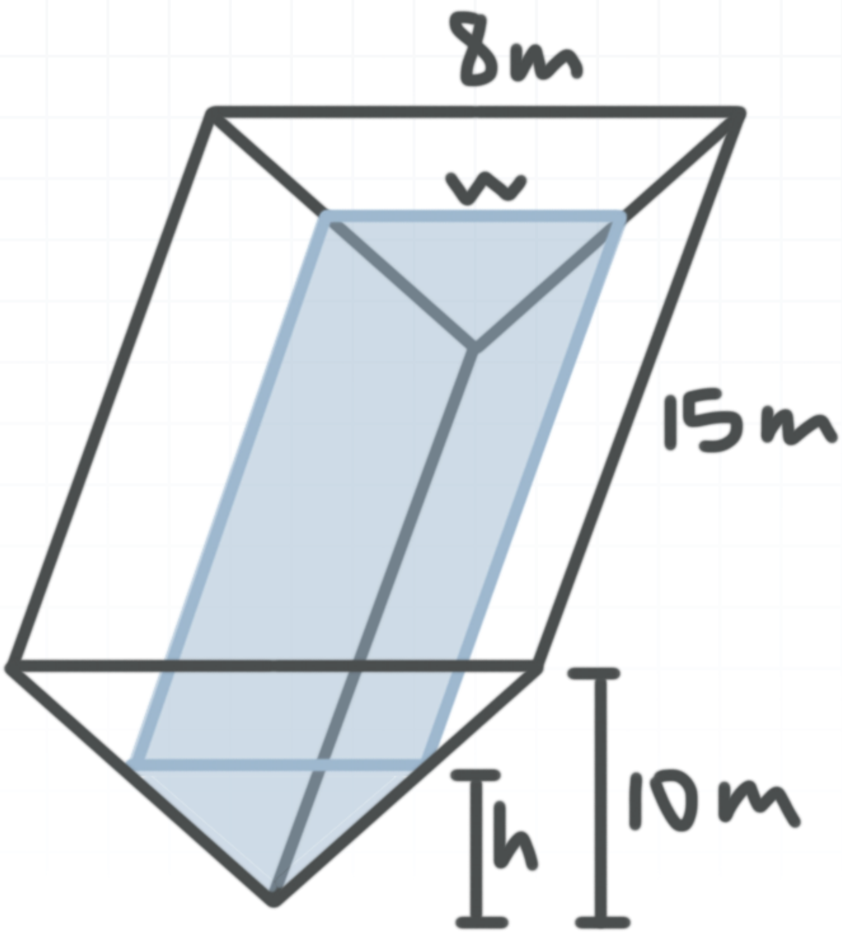
$$\frac{dq}{dt} = -\frac{1875}{120}$$

$$\frac{dq}{dt} \approx -15.6$$

■ 6. A trough of water 15 meters long, 8 meters wide, and 10 meters high has ends shaped like isosceles triangles. If water is being pumped in at a constant rate of 6 m^3/s, how fast are the height and width of the water changing when the water has a height of 250 cm?

Solution:

Sketch a diagram.



The volume of a triangular prism is $V = (1/2)whl$. To find the width of the water, we'll use similar triangles.

$$\frac{w}{h} = \frac{8}{10}$$

$$w = \frac{4}{5}h$$

The volume of the triangular prism of water is

$$V = \frac{1}{2}whl$$

We know that $l = 15$, so

$$V = \frac{1}{2}wh(15)$$

$$V = \frac{15}{2}wh$$

$$V = \frac{15}{2}\left(\frac{4}{5}h\right)h$$

$$V = 6h^2$$

Differentiate the volume equation with respect to t.

$$\frac{dV}{dt} = 6(2h)\frac{dh}{dt}$$

$$\frac{dV}{dt} = 12h\frac{dh}{dt}$$

The problem states that $dV/dt = 6$ and $h = 2.5$, so we'll substitute these into the derivative equation and solve for dh/dt, the rate at which the height of the water is changing when the water has a height of 250 cm.

$$6 = 12(2.5)\frac{dh}{dt}$$

$$6 = 30\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{6}{30}$$

$$\frac{dh}{dt} = \frac{1}{5} \text{ m/s}$$

We can find the relationship between w and h,

$$w = \frac{4}{5}h$$

and then differentiate with respect to t.

$$\frac{dw}{dt} = \frac{4}{5}\frac{dh}{dt}$$

Substitute $dh/dt = 1/5$ into the derivative equation and solve for dw/dt, the rate at which the width of the water is changing when the water has a height of 250 cm.

$$\frac{dw}{dt} = \frac{4}{5} \cdot \frac{1}{5}$$

$$\frac{dw}{dt} = \frac{4}{25} \text{ m/s}$$

W W W . K R I S T A K I N G M A T H . C O M