Calculus 1 Workbook Solutions

Squeeze Theorem

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SQUEEZE THEOREM

1. Use the Squeeze Theorem to evaluate the limit.

$$\lim_{x\to 0}\left(x^2\sin\left(\frac{1}{x}\right)-2\right)$$

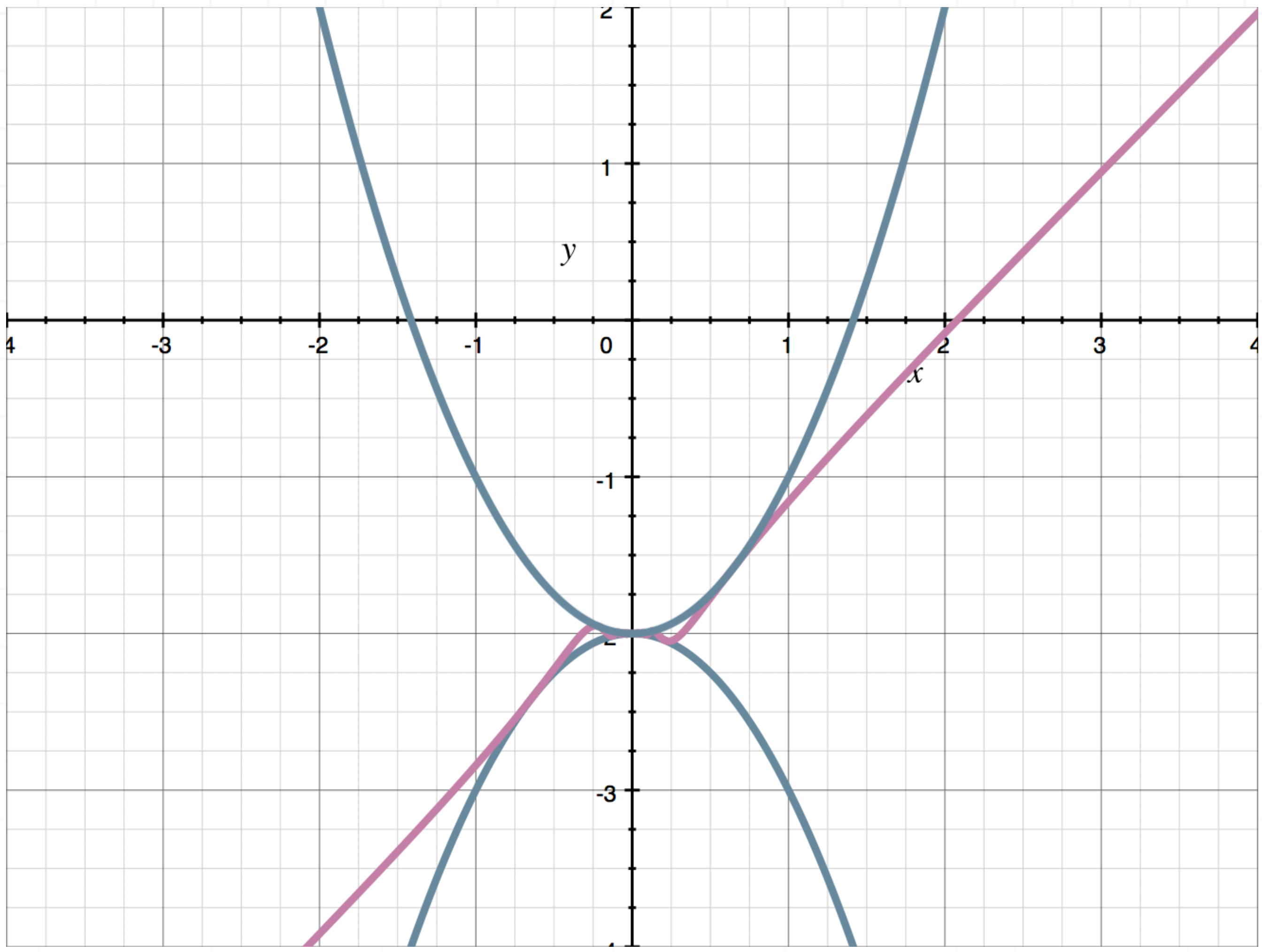
Solution:

Consider the graphs of the three functions shown below.

$$f(x) = -x^2 - 2$$

$$g(x) = x^2\sin\left(\frac{1}{x}\right) - 2$$

$$h(x) = x^2 - 2$$



Notice that $f(x) \leq g(x) \leq h(x)$. Therefore,

$$\lim_{x\to 0} f(x) \leq \lim_{x\to 0} g(x) \leq \lim_{x\to 0} h(x)$$

$$\lim_{x\to 0}(-x^2 - 2) \leq \lim_{x\to 0}\left(x^2 \sin\left(\frac{1}{x}\right) - 2\right) \leq \lim_{x\to 0}(x^2 - 2)$$

We can evaluate the limits on the left and right sides.

$$-0^2 - 2 \leq \lim_{x\to 0}\left(x^2 \sin\left(\frac{1}{x}\right) - 2\right) \leq 0^2 - 2$$

$$-2 \leq \lim_{x\to 0}\left(x^2 \sin\left(\frac{1}{x}\right) - 2\right) \leq -2$$

Therefore, by the Squeeze Theorem, we know that the value of the limit must be -2.

■ 2. Use the Squeeze Theorem to evaluate the limit.

$$\lim_{x \to \infty} \frac{3 \sin x}{4x}$$

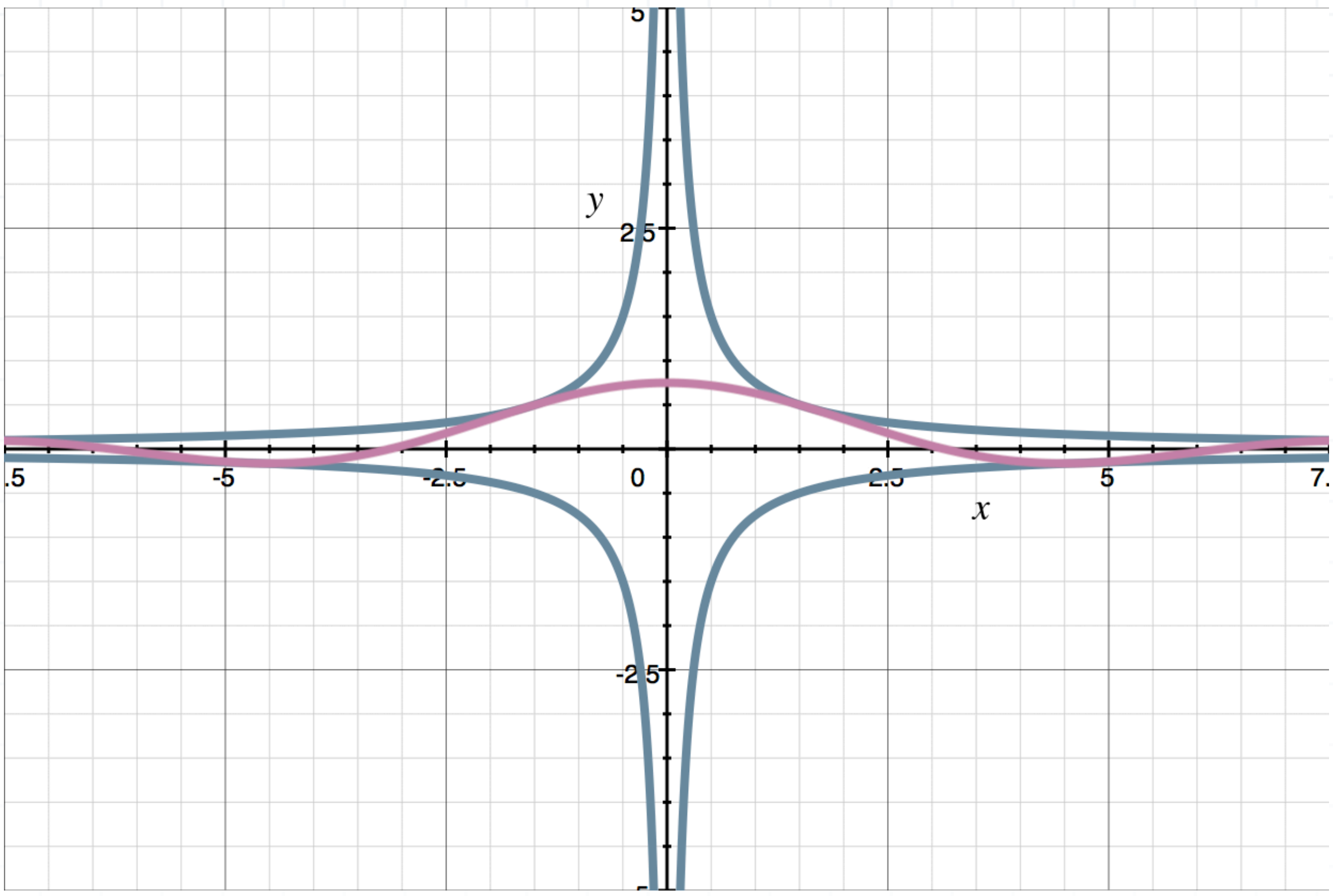
Solution:

Consider the graphs of the three functions shown below.

$$f(x) = -\frac{3}{4x}$$

$$g(x) = \frac{3 \sin x}{4x}$$

$$h(x) = \frac{3}{4x}$$



Notice that $f(x) \leq g(x) \leq h(x)$. Therefore,

$$\lim_{x \to \infty} f(x) \leq \lim_{x \to \infty} g(x) \leq \lim_{x \to \infty} h(x)$$

$$\lim_{x \to \infty} \left(-\frac{3}{4x} \right) \leq \lim_{x \to \infty} \frac{3 \sin x}{4x} \leq \lim_{x \to \infty} \left(\frac{3}{4x} \right)$$

We can evaluate the limits on the left and right sides.

$$0 \leq \lim_{x \to \infty} \frac{3 \sin x}{4x} \leq 0$$

Therefore, by the Squeeze Theorem, we know that the value of the limit must be 0.

■ 3. Use the Squeeze Theorem to evaluate the limit.

$$\lim_{x \to 0} \left(x^2 \cos \left(\frac{1}{x^2} \right) + 1 \right)$$

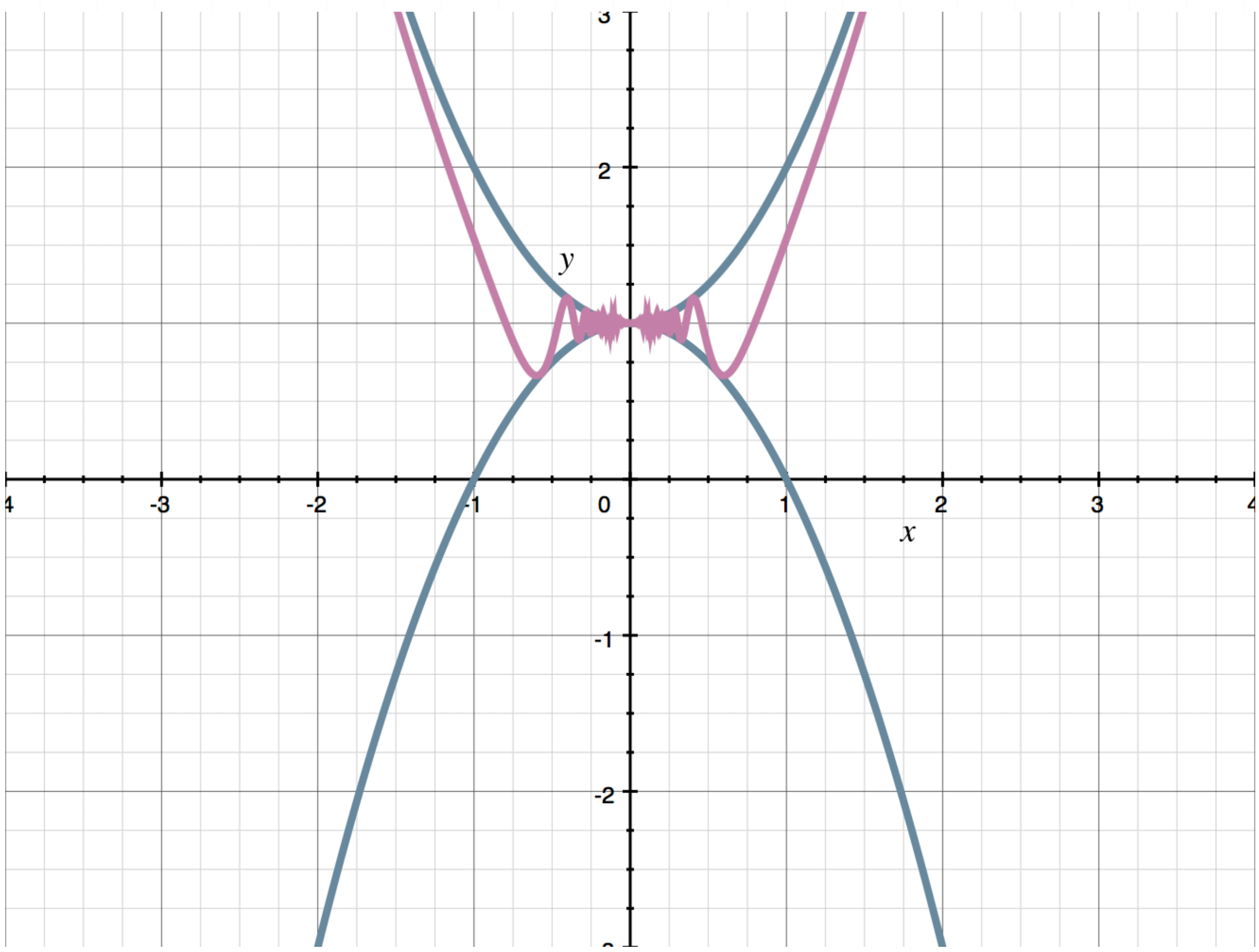
Solution:

Consider the graphs of the three functions shown below.

$$f(x) = -x^2 + 1$$

$$g(x) = x^2 \cos \left(\frac{1}{x^2} \right) + 1$$

$$h(x) = x^2 + 1$$



Notice that $f(x) \leq g(x) \leq h(x)$. Therefore,

$$\lim_{x \to 0} f(x) \le \lim_{x \to 0} g(x) \le \lim_{x \to 0} h(x)$$

$$\lim_{x \to 0} -x^2 + 1 \le \lim_{x \to 0} \left(x^2 \cos\left(\frac{1}{x^2}\right) + 1 \right) \le \lim_{x \to 0} x^2 + 1$$

We can evaluate the limits on the left and right sides.

$$-0^2 + 1 \le \lim_{x \to 0} \left(x^2 \cos\left(\frac{1}{x^2}\right) + 1 \right) \le 0^2 + 1$$

$$1 \le \lim_{x \to 0} \left(x^2 \cos\left(\frac{1}{x^2}\right) + 1 \right) \le 1$$

Therefore, by the Squeeze Theorem, we know that the value of the limit must be 1.

■ 4. Use the Squeeze Theorem to evaluate the limit.

$$\lim_{x \to \infty} \frac{e^{-x}}{x}$$

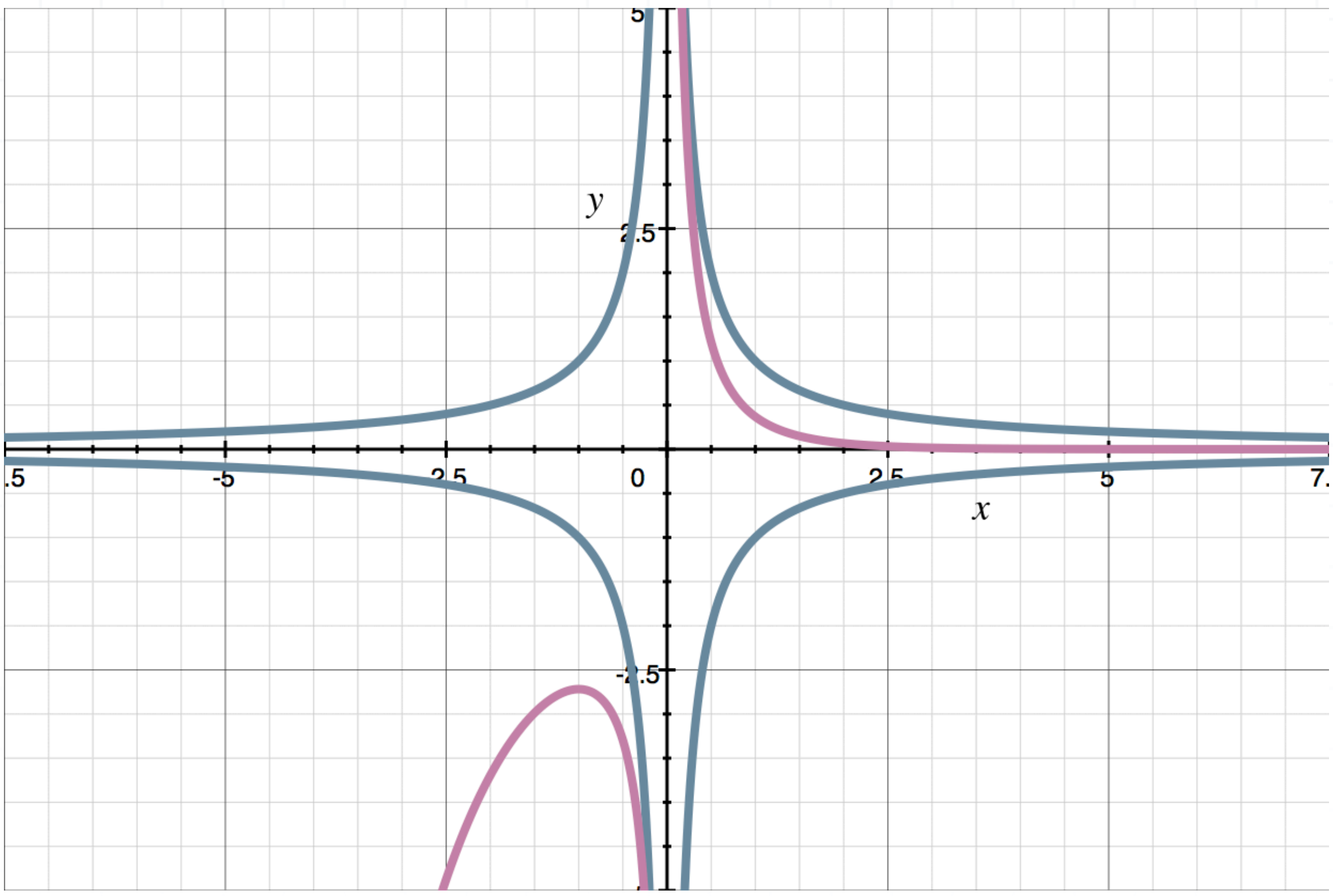
Solution:

Consider the graphs of the three functions shown below.

$$f(x) = -\frac{1}{x}$$

$$g(x) = \frac{e^{-x}}{x}$$

$$h(x) = \frac{1}{x}$$



Notice that $f(x) \le g(x) \le h(x)$. Therefore,

$$\lim_{x \to \infty} f(x) \le \lim_{x \to \infty} g(x) \le \lim_{x \to \infty} h(x)$$

$$\lim_{x \to \infty} \left(-\frac{1}{x}\right) \le \lim_{x \to \infty} \frac{e^{-x}}{x} \le \lim_{x \to \infty} \left(\frac{1}{x}\right)$$

We can evaluate the limits on the left and right sides.

$$0 \le \lim_{x \to \infty} \frac{e^{-x}}{x} \le 0$$

Therefore, by the Squeeze Theorem, we know that the value of the limit must be 0.

■ 5. Use the Squeeze Theorem to evaluate the limit.

$$\lim_{x\to\infty} \frac{x^2 + x\sin\sqrt{x}}{4x^2 + 7}$$

Solution:

We know that the value of the sine function oscillates back and forth between -1 and 1, so we'll start with

$$-1 \le \sin\sqrt{x} \le 1$$

Multiply each part of the inequality by x.

$$-x \le x\sin\sqrt{x} \le x$$

Add x^2 to each part of the inequality.

$$x^2 - x \le x^2 + x\sin\sqrt{x} \le x + x^2$$

Divide through the inequality by $4x^2 + 7$ to get the function at the center of the inequality to match the one we were given.

$$\frac{x^2 - x}{4x^2 + 7} \le \frac{x^2 + x\sin\sqrt{x}}{4x^2 + 7} \le \frac{x + x^2}{4x^2 + 7}$$

Apply the limit throughout the inequality.

$$\lim_{x \to \infty} \frac{x^2 - x}{4x^2 + 7} \le \lim_{x \to \infty} \frac{x^2 + x \sin \sqrt{x}}{4x^2 + 7} \le \lim_{x \to \infty} \frac{x + x^2}{4x^2 + 7}$$

$$\frac{1}{4} \le \lim_{x \to \infty} \frac{x^2 + x \sin \sqrt{x}}{4x^2 + 7} \le \frac{1}{4}$$

Because we were able to squeeze the limit between the same $1/4$ value, the value of the limit is

$$\lim_{x \to \infty} \frac{x^2 + x \sin \sqrt{x}}{4x^2 + 7} = \frac{1}{4}$$

■ 6. Use the Squeeze Theorem to evaluate the limit.

$$\lim_{x \to \infty} \frac{5x - \sin x}{\cos x + 2x}$$

Solution:

We know that the value of the sine function oscillates back and forth between -1 and 1, so we'll start with

$$-1 \le \sin x \le 1$$

Multiply through the inequality by -1, then add $5x$ to each part.

$$5x + 1 \le 5x - \sin x \le 5x - 1$$

We can follow a similar process for the cosine function.

$$-1 \leq \cos x \leq 1$$

$$2x - 1 \leq \cos x + 2x \leq 2x + 1$$

Now we'll divide the first inequality (for sine) by the second inequality (for cosine).

$$\frac{5x+1}{2x-1} \leq \frac{5x - \sin x}{\cos x + 2x} \leq \frac{5x-1}{2x+1}$$

Apply the limit throughout the inequality.

$$\lim_{x\to\infty} \frac{5x+1}{2x-1} \leq \lim_{x\to\infty} \frac{5x - \sin x}{\cos x + 2x} \leq \lim_{x\to\infty} \frac{5x-1}{2x+1}$$

$$\frac{5}{2} \leq \lim_{x\to\infty} \frac{5x - \sin x}{\cos x + 2x} \leq \frac{5}{2}$$

Because we were able to squeeze the limit between the same $5/2$ value, the value of the limit is

$$\lim_{x\to\infty} \frac{5x - \sin x}{\cos x + 2x} = \frac{5}{2}$$

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