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A Direct Proof of the Continuity of Non-Degenerate Minimum Stable Distributions

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Abstract. *It is a well known result in extreme value theory that if for a sequence $\{\xi_n : n \geq 1\}$ of independent and identically distributed random variables one can find real constants $\alpha_n > 0$ and β_n, $n \geq 1$, such that $\{(\min\{\xi_1, \ldots, \xi_n\} - \beta_n)/\alpha_n \; n \geq 1\}$ has a non-degenerate limiting distribution G, then there are only three possible continuous limiting types for G. The result, due to [Fisher and Tippett 1928] and [Gnedenko 1943], relies on the analysis of certain functional equations whose solutions are necessarily continuous. This fact raises the natural question of wheather the continuity and maybe other properties of minimum stable distributions follow directly from its definition. In this article a direct proof of their necessary continuity is presented.*

Resumo. *Um resultado conhecido da teoria de valores extremos é que se para uma sequência $\{\xi_n : n \geq 1\}$ de variáveis aleatórias independentes e identicamente distribuidas é possível encontrar constantes reais $\alpha_n > 0$ and β_n, $n \geq 1$, tais que $\{(\min\{\xi_1, \ldots, \xi_n\} - \beta_n)/\alpha_n \; n \geq 1\}$ tem uma distribuição limite não-degenerada G, então existem apenas tres tipos contínuos possíveis para a distribição limite G. Este resultado, que se deve a [Fisher and Tippett 1928] and [Gnedenko 1943], decorre da análise de certas equações funcionais cujas soluções são necessáriamente contínuas. Este fato suscita, naturalmente, a seguinte questão: seria possível demonstrar a continuidade e talvez outras propriedades das distribuições mínimo estáveis diretamente da definição? Neste artigo, apresenta-se uma demonstração direta da necessária continuidade dessas distribuições.*

1. Introduction

Extreme value theory deals, among other things, with the derivation of aproximate distributions for the maximum and the minimum values observed in simple random samples. The formal setting is that in which

$$M_n = \min\{\xi_1, \ldots, \xi_n\} \; , \; n \geq 1 \; ,$$

where $\xi_1, \xi_2, \ldots$ are independent and identically distributed random variables defined on the same probability space, $(\Omega, \mathcal{F}, P)$, and one is interested in:

1. whether there are real constants, $\alpha_n > 0$ and β_n, $n \geq 1$, and a distribution function, G, such that

$$\lim_{n\to\infty} P\{\frac{M_n - \beta_n}{\alpha_n} \leq x\} = G(x) ,$$

for all x in the continuity set of G; and, if this is the case,

2. what properties does G have.

In this context we say that G a limiting distribution for $\{M_n\}_{n=1}^{\infty}$ and it is well known that if G is nondegenerate, then it must be one of the following types:

(a) $G_1(x) = 1 - e^{-x^{\lambda}}$, for $x \geq 0$, where $\lambda > 0$;

(b) $G_2(x) = 1 - e^{-(-x)^{-\lambda}}$, for $x \leq 0$, where $\lambda > 0$; or

(c) $G_3(x) = 1 - e^{-e^{x}}$, for $-\infty < x < +\infty$.

A demonstration of the above result, due to [Fisher and Tippett 1928] and [Gnedenko 1943], can be found in [Barlow and Proschan 1975] and it relies on the solution of the following three functional equations:

(d) $\overline{G}^n(\alpha_n x) = \overline{G}(x)$, with $G(x) = 1 - \overline{G}(x) = 0$ for $x \leq 0$;

(e) $\overline{G}^n(\alpha_n x) = \overline{G}(x)$, with $G(x) = 1 - \overline{G}(x) = 1$ for $x \geq 0$;

(f) $\overline{G}^n(x + \beta_n) = \overline{G}(x)$, for $x \in \mathbb{R}$,

derived from the analysis of the normalizing real constants, $\alpha_n > 0$ and β_n, $n \geq 1$, in following definition.

Definition 1. A distribution function G is minimum stable iff there exist real constants $\alpha_n > 0$ and β_n, $n \geq 1$ such that

$$\overline{G}^n(\alpha_n x + \beta_n) = \overline{G}(x) , \quad \text{for all } x \in \mathbb{R} .$$

The defining equality of minimum stability can be rephrased in terms of the concept of distribution type stated below

Definition 2. The distribution functions G and H are of the same type iff there exist real constants $\alpha > 0$ and β, such that

$$\overline{G}(\alpha x + \beta) = \overline{H}(x) , \quad \text{for all } x \in \mathbb{R} .$$

According to definitions 1 and 2, a distribution function G is minimum stable if and only if for every $n \geq 1$, $H_n = 1 - \overline{G}^n$ and G are of the same type.

Limiting distributions and type are connected by the following result, whose proof can be found in [Feller 1966].

Lemma 1. Let G and H be nondegenerate distribution functions. If $\{F_n\}$ is a sequence of distribution functions such that:

1. there exist real constants $\alpha_n > 0$ and β_n, $n \geq 1$ such that

$$\overline{F_n}(\alpha_n x + \beta_n) \to \overline{G(x)} \quad \text{for all } \ x \in \mathbb{R} ;$$

2. there exist real constants $\gamma_n > 0$ and δ_n, $n \geq 1$ such that

$$\overline{F_n}(\gamma_n x + \delta_n) \to \overline{H}(x) \quad \text{for all } \ x \in \mathbb{R} \ ,$$

then G and H are of the same type.

The following result links the concept of minimum stability to the limiting distribution of the sequence $\{M_n\}_{n=1}^{\infty}$.

Theorem 1. G is a continuous limiting distribution for $\{M_n\}_{n=1}^{\infty}$ if and only if G is minimum stable.

Remarks.

1. The proof of Theorem 1 is elementary. The argument is that of [Barlow and Proschan 1975], page 232, Theorem 2.1, except that continuity of G is not assumed there. The assumption, however, must be made since the defining equality of minimum stability must hold for all $x \in \mathbb{R}$. In order to prove the if part of the theorem we need to show that minimum stable distributions are continuous, which is done in the next section. To the authors knowledge the proof provided is new.

2. For details of the afore mentioned analysis of the normalizing constants see [Barlow and Proschan 1975], pages 232-234, Lemmas 2.2 - 2.6

3. The solution of the above functional equations require the notion of regular variation, introduced by [Karamata 1930]. For details, see [Barlow and Proschan 1975], pages 234-236, Lemma 2.7, definition 2.8, Lemma 2.9 and Theorem 2.10.

2. Main Result

Theorem 2. Every non-degenerate minimum-stable distribution function, G, is continuous.

proof. For any distribution function, G, its continuity on $\{G = 0\}$ is immediate for if $x \in \mathbb{R}$ is such that $G(x) = 0$ then $G(y) = 0$ for any $y < x$, which implies that $G(x - 0) = G(x)$ (Recall that G is always right continuous).

In view of the above remark, to establish the continuity of a non-degenerate minimum-stable distribution function, G, it suffices to prove that G is also left continuous on $\{G > 0\}$ or, equivalently, that $\overline{G}$ is left continuous on $\{\overline{G} < 1\}$.

Recall from Theorem 1 that a distribution function G is minimum stable if and only if, for each $n \in \mathbb{N}$, there are real constants $\alpha_n > 0$ and β_n such that

$$\overline{G}^n(\alpha_n x + \beta_n) = \overline{G}(x) \quad \text{for all} \quad x \in \mathbb{R} \ .$$

Alternatively, G is minimum stable if and only if, for each $n \in \mathbb{N}$, there are real constants $\alpha'_n > 0$ and β'_n such that

$$\overline{G}^n(x) = \overline{G}(\alpha'_n x + \beta'_n) \quad \text{for all} \quad x \in \mathbb{R} \ .$$

Let us now fix $x \in \{\overline{G} < 1\}$ and assume that

$$\overline{G}(x-0) - \overline{G}(x) = \epsilon > 0 \ .$$

If for some $y < x$

$$1 > \overline{G}(y) \geq 1 - \frac{\epsilon}{2} \ ,$$

then

$$\overline{G}^n(y) - \overline{G}^{n+1}(y) = \overline{G}^n(y)[1 - \overline{G}(y)] \leq \frac{\epsilon}{2}$$

and consequently

$$\overline{G}(x) \leq \overline{G}^k(y) = \overline{G}(\alpha'_k y + \beta'_k) \leq \overline{G}(x-0) \ ,$$

for some integer $k \in \mathbb{N}$, since $\lim_{n\to\infty} \overline{G}^n(y) = 0$. But this contradicts the fact that $\overline{G}$ is discontinuous at x since $\overline{G}$ cannot take values between $\overline{G}(x)$ and $\overline{G}(x-0)$.

From the above observation, in order to prove that $\overline{G}$ is left continuous on $\{\overline{G} < 1\}$, it suffices to show that for any $x \in \{\overline{G} < 1\}$ and $\epsilon > 0$, there exists $y < x$ such that

$$1 > \overline{G}(y) \geq 1 - \frac{\epsilon}{2} \ .$$

For that, let

$$z = \sup\{y < x \ : \ \overline{G}(y) \geq 1 - \frac{\epsilon}{2}$$

and observe that

$$\overline{G}(z) = \overline{G}(z+0) \leq 1 - \frac{\epsilon}{2} < 1 \quad (1)$$

and

$$\overline{G}(z) = \overline{G}^2(\alpha_2 z + \beta_2) \geq \overline{G}^2(z) \quad (2) \ ,$$

since $\overline{G}(z) < 1$.

Consequently,

$\alpha_2 z + \beta_2 < z$ (since the fact that G non-degenerate excludes equality

and

$$1 > \overline{G}(\alpha_2 z + \beta_2) \geq 1 - \frac{\epsilon}{2} \ ,$$

for if $\overline{G}(\alpha_2 z + \beta_2) = 1$ would lead to a contradiction between (1) and (2).

QED

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