ESCOLA BRASILEIRA DE ECONOMIA E FINANÇAS - EPGE

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Econometrics 1 - Problem Set 1

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Question 1: Assume that $y_i = x_i'\beta + u_i$. Using $E(u_i|z_i) = 0$, prove that

$$E(z_i(y_i - x_i'\beta)) = 0$$

Solution: By the Law of Iterated Expectations, $E(z_i(y_i - x_i'\beta)) = E(z_i u_i) = E(E(z_i u_i|z_i)) = E(z_i E(u_i|z_i)) = E(0) = 0$.

Question 2: Suppose you have some i.i.d. data (x_i, y_i). Due to your economic expertise and experience, you strongly suspect that the relationship between x_i and y_i is $y_i = \beta log(x_i) + u_i$. You assume that $E(u_i|x_i) = 0$.

1. Following the steps outlined in class, set up the moment equations for a Method of Moments estimation to get at the parameter β.
2. Solve for the β_{MM} estimator.

Solution:

1. Let $g_i(\beta) = y_i - \beta log(x_i)$. We have that $E(g_i(\beta)) = E(y_i - \beta log(x_i)) = E(u_i) = E(E(u_i|x_i)) = E(0) = 0$, and since $\beta \in \mathbb{R}$, the model is just-identified. Thus the moment equation is $\frac{1}{n}\sum_{i=1}^{n} g_i(\beta_{\text{MM}}) = 0 \Leftrightarrow \frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_{\text{MM}} log(x_i)) = 0$.
2. Solving for β_{MM}, we get $\beta_{\text{MM}} = \dfrac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} log(x_i)}$.

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Question 3: Remember that instead of x as the factor in a moment equation $E(xu) = 0$, we can pick some other transformation f, as long as the transformation does not depend on y, i.e, $E(f(x)u) = 0$. The non-linear least squares estimator β_{NLLS} is defined as an estimator that minimizes the residual sum of squares

$$\min_{\beta} \; RSS(\beta) = \frac{1}{n}\sum_{i=1}^{n}(y_i - g(x_i, \beta))^2$$

Assume that g is differentiable in β. Show that the first order condition of this minimization problem is equivalent to a Method of Moments estimation where as a 'convenient transformation' f of x_i you pick $f = \frac{\partial g}{\partial \beta}$.

Solution: The First Order Conditions are

$$\frac{-2}{n}\sum_{i=1}^{n}\frac{\partial g(x_i,\beta_{\text{NLLS}})}{\partial\beta}(y_i - g(x_i,\beta_{\text{NLLS}})) = 0 \Leftrightarrow$$
$$\frac{1}{n}\sum_{i=1}^{n}\frac{\partial g(x_i,\beta_{\text{NLLS}})}{\partial\beta}(y_i - g(x_i,\beta_{\text{NLLS}})) = 0$$

With $f = \frac{\partial g(x_i,\beta)}{\partial\beta}$ and assuming $E(u_i|x_i) = 0$, we have that $E(\frac{\partial g(x_i,\beta)}{\partial\beta}u_i) = 0$, by the Law of Iterated Expectations, so $E(\frac{\partial g(x_i,\beta)}{\partial\beta}(y_i - g(x_i,\beta)) = 0$. Note that the model is just-identified. Now, using the sample moments, we have $\dfrac{1}{n}\sum_{i=1}^{n}\dfrac{\partial g(x_i,\beta_{\text{MM}})}{\partial\beta}(y_i - g(x_i,\beta_{\text{MM}})) = 0$. See that the equations are the same, thus $\beta_{\text{NLLS}} = \beta_{\text{MM}}$.

Question 4: Assume $y_i = x_i'\beta + u_i$ and $E(u_i|x_i) = 0$ for a i.i.d. sample $\{(x_1,y_1),\ldots,(x_n,y_n)\}$. Prove that the MM estimator is consistent.

Solution: Let $f:\mathbb{R}^q \to \mathbb{R}^q$ and β_{MM} the MM estimator that solves the equation $\frac{1}{n}\sum_{i=1}^{n} f(x_i)(y_i - x_i'\beta_{MM}) = 0 \Leftrightarrow \frac{1}{n}\sum_{i=1}^{n} f(x_i)x_i'\beta_{MM} = \frac{1}{n}\sum_{i=1}^{n} f(x_i)y_i \Leftrightarrow \beta_{MM} = \left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)x_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)y_i\right)$. Assuming that $E(f(x_i)x_i)$ is invertible, since $\{(x_i,y_i)\}_{i=1,\ldots,n}$ are i.i.d, by the Law of Large Numbers, $\beta_{MM} \xrightarrow{P} (E(f(x_i)x_i'))^{-1}E(f(x_i)y_i)$. Now, $y_i = x_i'\beta + u_i \Rightarrow f(x_i)y_i = f(x_i)x_i'\beta + f(x_i)u_i$. By the Law of Iterated Expectations, $E(f(x_i)y_i|x_i) = E(f(x_i)x_i'|x_i)\beta + E(f(x_i)u_i|x_i) = E(f(x_i)x_i'|x_i)\beta \Rightarrow E(f(x_i)y_i|x_i) = E(f(x_i)x_i'|x_i)\beta \Rightarrow E(f(x_i)y_i) = E(f(x_i)x_i')\beta$, so $\beta = (E(f(x_i)x_i'))^{-1}E(f(x_i)y_i)$, and we have established that plim $\beta_{MM} = \beta$.

Question 5: Can an estimator be unbiased **and** inconsistent? Find an example or argue verbally.

Solution: An estimator can be unbiased and inconsistent. Let $\{x_i\}_i$ be i.i.d Bernoulli with parameter $p = \frac{1}{2}$, i.e, $x_i = 0$ with probability $\frac{1}{2}$ and $x_i = 1$ with probability $\frac{1}{2}$. Thus, $\mu = E(x_i) = \frac{1}{2}$. Consider now the estimator $\hat{\mu} = x_1$ for all n. We have that $E(\hat{\mu}) = \mu$, but $P(|\hat{\mu} - \mu| > \epsilon) =$

$P(|x_1 - \mu| > \epsilon) = 1$ for $\epsilon < \frac{1}{2}$ and all n. Therefore, $\hat{\mu}$ is not consistent for μ, even though is unbiased.