ESCOLA BRASILEIRA DE ECONOMIA E FINANÇAS - EPGE

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Econometrics 1 - Problem Set 4

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Question 1:

1 . We want to show that the distribution of $Y_{ij}|\mu_j$ given $\overline{Y}_j = 1/N\sum_{i=1}^{N} Y_{ij}$ does not depend of μ_j. First, since $\epsilon_{ij}|\mu_j \sim N(0, \sigma_\epsilon^2)$, $Y_{ij}|\mu_j \sim N(\mu_j, \sigma_\epsilon^2)$. Second, since $\{\epsilon_{ij}|\mu_j\}_{i=1}^{N}$ are independent,

$$\begin{aligned}
f(Y_j|\mu_j) &= \prod_{i=1}^{N} f(Y_{ij}|\mu_j) = (2\pi\sigma_\epsilon^2)^{-\frac{n}{2}} exp\left(\frac{-\sum_{i=1}^{N}(Y_{ij}-\mu_j)^2}{2\sigma_\epsilon^2}\right)\\
&= (2\pi\sigma_\epsilon^2)^{-\frac{n}{2}} exp\left(\frac{-\sum_{i=1}^{N}(Y_{ij}-\overline{Y}_j+\overline{Y}_j-\mu_j)^2}{2\sigma_\epsilon^2}\right)\\
&= (2\pi\sigma_\epsilon^2)^{-\frac{n}{2}} exp\left(\frac{-\sum_{i=1}^{N}(Y_{ij}-\overline{Y}_j)^2+(\overline{Y}_j-\mu_j)^2+2(Y_{ij}-\overline{Y}_j)(\overline{Y}_j-\mu_j)}{2\sigma_\epsilon^2}\right)\\
&= (2\pi\sigma_\epsilon^2)^{-\frac{n}{2}} exp\left(\frac{-\sum_{i=1}^{N}(Y_{ij}-\overline{Y}_j)^2}{2\sigma_\epsilon^2}\right) exp\left(\frac{-N(\overline{Y}_j-\mu_j)^2}{2\sigma_\epsilon^2}\right)
\end{aligned}$$

Last, by independence of the $\epsilon_{ij}|\mu_j$, $\overline{Y}_j|\mu_j \sim N(\mu_j, \sigma_\epsilon^2/N)$, with pdf

$$g(\overline{Y}_j|\mu_j) = N^{\frac{1}{2}}(2\pi\sigma_\epsilon^2)^{-\frac{1}{2}} exp\left(\frac{-N(\overline{Y}_j-\mu_j)^2}{2\sigma_\epsilon^2}\right)$$

Thus, if we let

$$h(Y_j) = N^{-\frac{1}{2}}(2\pi\sigma_\epsilon^2)^{-\frac{n-1}{2}} exp\left(\frac{-\sum_{i=1}^{N}(Y_{ij}-\overline{Y}_j)^2}{2\sigma_\epsilon^2}\right)$$

a function that does not depend on μ_j, we have $f(Y_j|\mu_j) = h(Y_j)g(\overline{Y}_j|\mu_j)$. Therefore, by the Factorization Theorem, $\overline{Y}_j|\mu_j$ is a sufficient statistic for μ_j

2 . By independence of the $\epsilon_{ij}|\mu_j, \overline{Y}_j|\mu_j \sim N(\mu_j, \sigma_\epsilon^2/N)$. Moreover, we have

$$\begin{aligned}
f(\mu_j|\overline{Y}_j)f(\overline{Y}_j) &= f(\overline{Y}_j, \mu_j) = g(\overline{Y}_j|\mu_j)f(\mu_j) \\
&= N^{\frac{1}{2}}(2\pi\sigma_\epsilon^2)^{-\frac{1}{2}} exp\left(\frac{-N(\overline{Y}_j - \mu_j)^2}{2\sigma_\epsilon^2}\right) \cdot (2\pi\sigma_\mu^2)^{-\frac{1}{2}} exp\left(\frac{-(\mu_j - \mu_0)^2}{2\sigma_\mu^2}\right) \\
&= (2^2\pi^2 N^{-1}\sigma_\epsilon\sigma_\mu^2)^{-\frac{1}{2}} exp\left(\frac{-1}{2}\left(\frac{\sigma_\mu^2(\overline{Y}_j - \mu_j)^2 + N_{-1}\sigma_\epsilon(\mu_j - \mu_0)^2}{\sigma_\mu^2 N^{-1}\sigma_\epsilon^2}\right)\right) \\
&= (2^2\pi^2 N^{-1}\sigma_\epsilon\sigma_\mu^2)^{-\frac{1}{2}} exp\left(\frac{-1}{2}\left(\frac{(\mu_j - \frac{\sigma_\mu^2\overline{Y}_j + N^{-1}\sigma_\epsilon^2\mu_0}{\sigma_\mu^2 + N^{-1}\sigma_\epsilon^2})^2}{\frac{\sigma_\mu^2 N^{-1}\sigma_\epsilon}{\sigma_\mu^2 + N^{-1}\sigma_\epsilon^2}} + \frac{(\overline{Y}_j - \mu_0)^2}{\sigma_\mu^2 + N^{-1}\sigma_\epsilon^2}\right)\right) \\
&= (2\pi(\sigma_\mu^2 + N^{-1}\sigma_\epsilon^2))^{-\frac{1}{2}} exp\left(\frac{-1}{2}\left(\frac{(\overline{Y}_j - \mu_0)^2}{\sigma_\mu^2 + N^{-1}\sigma_\epsilon^2}\right)\right) \\
&\times \left(2\pi\left(\frac{\sigma_\mu^2 N^{-1}\sigma_\epsilon}{\sigma_\mu^2 + N^{-1}\sigma_\epsilon^2}\right)\right)^{-\frac{1}{2}} exp\left(\frac{-1}{2}\left(\frac{(\mu_j - \frac{\sigma_\mu^2\overline{Y}_j + N^{-1}\sigma_\epsilon^2\mu_0}{\sigma_\mu^2 + N^{-1}\sigma_\epsilon^2})^2}{\frac{\sigma_\mu^2 N^{-1}\sigma_\epsilon}{\sigma_\mu^2 + N^{-1}\sigma_\epsilon^2}}\right)\right)
\end{aligned}$$

See that the first term of the multiplication is the pdf of a $N(\mu_0, (\sigma_\mu^2 + N^{-1}\sigma_\epsilon^2))$ distribution, while the second term is the pdf of a $N\left(\frac{\sigma_\mu^2\overline{Y}_j + N^{-1}\sigma_\epsilon^2\mu_0}{\sigma_\mu^2 + N^{-1}\sigma_\epsilon^2}, \frac{\sigma_\mu^2 N^{-1}\sigma_\epsilon}{\sigma_\mu^2 + N^{-1}\sigma_\epsilon^2}\right)$. Thus we get that

$$\overline{Y}_j \sim N(\mu_0, (\sigma_\mu^2 + N^{-1}\sigma_\epsilon^2))$$

and

$$\mu_j|\overline{Y}_j \sim N\left(\frac{\sigma_\mu^2\overline{Y}_j + N^{-1}\sigma_\epsilon^2\mu_0}{\sigma_\mu^2 + N^{-1}\sigma_\epsilon^2}, \frac{\sigma_\mu^2 N^{-1}\sigma_\epsilon}{\sigma_\mu^2 + N^{-1}\sigma_\epsilon^2}\right)$$

3 . As seen in item 2,

$$\mu_j|\overline{Y}_j \sim N\left(\frac{\sigma_\mu^2\overline{Y}_j + N^{-1}\sigma_\epsilon^2\mu_0}{\sigma_\mu^2 + N^{-1}\sigma_\epsilon^2}, \frac{\sigma_\mu^2 N^{-1}\sigma_\epsilon}{\sigma_\mu^2 + N^{-1}\sigma_\epsilon^2}\right)$$

See that $\mathbb{E}[\mu_j|\overline{Y}_j] = \dfrac{\sigma_\mu^2\overline{Y}_j + N^{-1}\sigma_\epsilon^2\mu_0}{\sigma_\mu^2 + N^{-1}\sigma_\epsilon^2} = \dfrac{\sigma_\mu^2}{\sigma_\mu^2 + N^{-1}\sigma_\epsilon^2}\overline{Y}_j + \dfrac{N^{-1}\sigma_\epsilon^2}{\sigma_\mu^2 + N^{-1}\sigma_\epsilon^2}\mu_0$, i.e, the expectation if a weighted mean of $\overline{Y}_j$ and μ_0, with more weight given to $\overline{Y}_j$, the sample mean, when the unconditional variance of μ_j is bigger.

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Question 2:

1 . Let d_i be an $n \times 1$ vector of dummy variables that indicates the individual, i.e., d_i is a vector with 1 in the ith element and 0 in the others. Consider the model $Y_{it} = \alpha d_i + \beta X_{it} + \epsilon_{it}$. Applying the FWL theorem, we first regress Y_{it} on d_i to get $Y_{it} = \tilde{\alpha} d_i + \tilde{\epsilon}_{it}$. Since $\tilde{\alpha}_i = d_i \overline{Y_i}$, we have that $\tilde{\epsilon}_{it} = Y_{it} - \overline{Y}_i$. Second, we regress X_{it} on d_i, and get $X_{it} = \tilde{\delta} d_i + \tilde{u}_{it}$, and have $\tilde{u}_{it} = X_{it} - \overline{X}_i$. Finally, we regress $\tilde{\epsilon}_{it} = Y_{it} - \overline{Y}_i$ on $\tilde{u}_{it} = X_{it} - \overline{X}_i$, i.e. we regress the model $Y_{it} - \overline{Y}_i = (X_{it} - \overline{X}_i)\beta + \eta_{ij} \Rightarrow Y_{it} - \overline{Y}_i = (X_{it} - \overline{X}_i)\beta + \epsilon_{it} - \overline{\epsilon}_i$. Since we assumed strict exogeneity, $E(\epsilon_{it} - \overline{\epsilon}_i | X_{it} - \overline{X}_i) = 0$, which implies that the LSDV estimator of β is consistent.

2 . We simulate a model with $X_{it} \sim N(2,4)$, $\epsilon_{it} \sim N(0,1)$, $\alpha_i \sim N(0,1)$, and $Y_{it} = \alpha_i + 3 \cdot X_{it} + \epsilon_{it}$. As we see in the code, the estimators are identical.

3 . By item 1, we have that $\hat{\beta}_{LSDV}$ comes from the regression of $Y_{it} - \overline{Y}_i$ on $(X_{it} - \overline{X}_i)$, so $\hat{\beta}_{LSDV} = \dfrac{\hat{\text{cov}}(X_{it} - \overline{X}_i, Y_{it} - \overline{Y}_i)}{\hat{\text{var}}(X_{it} - \overline{X}_i)}$.

Now, consider the model

$$Y_{it} = a_0 + a_1 \overline{X}_i + \beta X_{it} + u_{it}$$

where $u_{it} = v_i + \epsilon_{it}$. Consider first the regression of $X_{it} = \lambda_0 + \lambda_1 \overline{X}_i + \gamma_{it}$. We have that $\hat{\lambda}_1 = \dfrac{\sum_i \sum_t (X_{it} - \overline{\overline{X}})(\overline{X}_i - \overline{\overline{X}})}{\sum_i (\overline{X}_i - \overline{\overline{X}})^2} = 1$, and $\hat{\lambda}_0 = \overline{\overline{X}} - \overline{\overline{X}}\hat{\lambda}_1 = 0$. Thus, $\tilde{\gamma}_{it} = X_{it} - \hat{\lambda}_0 - \hat{\lambda}_1 \overline{X}_i = X_{it} - \overline{X}_i$

Consider also the linear projection $Y_{it} = \tau_0 + \tau_1 \overline{X}_i + \zeta_{it}$ and let $\tilde{\zeta}_{it} = Y_{it} - \hat{\tau}_0 + \hat{\tau}_1 \overline{X}_i$. By the FWL theorem, we have that

$$\begin{aligned}
\hat{\beta} &= \frac{\hat{\text{cov}}(\tilde{\gamma}_{it}, \tilde{\zeta}_{it})}{\hat{\text{var}}(\tilde{\gamma}_{it})} \\
&= \frac{\hat{\text{cov}}(X_{it} - \overline{X}_i, Y_{it} - \hat{\tau}_0 - \hat{\tau}_1 \overline{X}_i)}{\hat{\text{var}}(X_{it} - \overline{X}_i)} \\
&= \frac{\hat{\text{cov}}(X_{it} - \overline{X}_i, (Y_{it} - \overline{\overline{Y}}) - \hat{\tau}_1(\overline{X}_i - \overline{\overline{X}}))}{\hat{\text{var}}(X_{it} - \overline{X}_i)} \\
&= \frac{\hat{\text{cov}}(X_{it} - \overline{X}_i, Y_{it} - \overline{\overline{Y}})}{\hat{\text{var}}(X_{it} - \overline{X}_i)} \\
&= \frac{\hat{\text{cov}}(X_{it} - \overline{X}_i, Y_{it})}{\hat{\text{var}}(X_{it} - \overline{X}_i)} \\
&= \frac{\hat{\text{cov}}(X_{it} - \overline{X}_i, Y_{it} - \overline{Y}_i)}{\hat{\text{var}}(X_{it} - \overline{X}_i)} \\
&= \hat{\beta}_{LSDV}
\end{aligned}$$

4 . To test for correlation between the person effect and X_{it}, we can test $\tilde{\delta} = 0$ in the regression of X_{it} in d_i, like in item 1, using a t-statistic.

For $Var(v_i)$, we can assume a random effects model to estimate α_i. With these estimates, we can estimate $Var(v_i)$ as we would usually do in the regression $\hat{\alpha}_i = a_0 + a_1\overline{X}_i + v_i$. With $Var(\alpha_i)$ in hands, we can estimate $Var(\alpha_i) = a_1^2 Var(\overline{X}_i) + Var(v_i)$.