Homework 4

Consider the linear system of equations:

(i) Determine the existence of solutions. (1 point)

(ii) Determine the uniqueness of solutions (if applicable). (1 point)

(iii) Find the set of all solutions. (1 points)

(iv) Find a vector *x* that minimizes the norm of the error

How to Determine if a solution exist ?

Given a system of linear equations represented by the matrix equation $\mathbf{Ax} = \mathbf{b}$, where$\mathbf{A}$is the coefficient matrix and $\mathbf{b}$

is the constant vector, we can determine the existence of solutions by comparing the rank of $\mathbf{A}$ and the rank of the augmented matrix[

- *If* $\mathrm{rank}(\mathbf{A}) = \mathrm{rank}([\mathbf{A}|\mathbf{b}]) = \mathrm{n}$ (where n is the number of unknowns), then the system has a unique solution.
- *If* $\mathrm{rank}(\mathbf{A}) = \mathrm{rank}([\mathbf{A}|\mathbf{b}]) <$ n, then the system has infinitely many solutions.
- *If* $\mathrm{rank}(\mathbf{A}) < \mathrm{rank}([\mathbf{A}|\mathbf{b}])$, then the system has no solution.

Problem 1

$$\begin{cases} 4x_2 = 4 \\ 3x_1 + x_2 = -2 \\ 3x_1 - 2x_2 = 3 \end{cases}$$

Hint: For the matrix A in this question, you can compute its pseudo-inverse by hand using the formula $A^+ = (A^T A)^{-1} A^T$

1. Determining solution: Ans: There is no solution
2. There is no uniqueness since there is no solution
3. There is no solution so we dont need to determine the set
4. Finding a vector that minimizes the norm of the vector

Solution to 1

```
% Define matrix A and vector b
A = [0 4 0; 3 1 0; 3 -2 0];
b = [4; -2; 3];

% Determine rank of matrix A
rank_A = rank(A);

% Create the augmented matrix [A | b]
augmented_matrix = [A, b];

% Determine rank of the augmented matrix
rank_augmented = rank(augmented_matrix);
```

```
% Display the ranks
fprintf('Rank of matrix A: %d\n', rank_A);
```

```
Rank of matrix A: 2
```

```
fprintf('Rank of the augmented matrix [A | b]: %d\n', rank_augmented);
```

```
Rank of the augmented matrix [A | b]: 3
```

```

% Check the principle to determine the existence of solutions
if rank_A == rank_augmented
    if rank_A == size(A, 2)
        fprintf('The system has a unique solution.\n');
    else
        fprintf('The system has infinitely many solutions.\n');
    end
else
    fprintf('The system has no solution.\n');
end
```

```
The system has no solution.
```

Solution to 4

```
% Initialization
clear all;
clc;
close all;

% Define matrix A and vector b
A = [0 4; 3 1; 3 -2];
b = [4; -2; 3];

% Calculate A transpose
A_transpose = A';

% Display determinant of the inverse of A_transpose*A
fprintf('Determinant of the inverse: %.2f\n', det((A_transpose*A)^-1));
```

```
Determinant of the inverse: 0.00
```

```

% Compute the solution using the formula x = (A^T A)^-1 A^T b
X = ((A_transpose*A)^-1) * A_transpose * b;

% Display the solution
fprintf('Solution using the formula: \n');
```

```
Solution using the formula:
```

```
disp(X);
```

```
    0.2358
    0.4146
```

```
% Compute the solution using MATLAB's pseudo-inverse function
X_star = pinv(A) * b;

% Display the solution using pseudo-inverse
fprintf('Solution using MATLAB pseudo-inverse: \n');
```

```
Solution using MATLAB pseudo-inverse:
```

```
disp(X_star);
```

```
    0.2358
    0.4146
```

```
% Calculate the error
error_star = A * X_star - b;

% Compute and display the norm of the error
error_norm = norm(error_star);
fprintf('Norm of the error: %.2f\n', error_norm);
```

```
Norm of the error: 5.00
```

Problem 2

$$\begin{cases} 4x_2 + 4x_3 = 4 \\ 3x_1 + x_2 + 2x_3 = -2 \end{cases}$$

Hint: For the matrix A in this question, you can compute its pseudo-inverse by hand using the formula $A^{+} = A^{T}(AA^{T})^{-1}$

1. The system contains a solution and its an infinite amount
2. There is no uniqueness of a solution since theres an infinite amount due to the free variable
3. To find all the sets of solutions we just need to define the free variable
4. Since there is a solution we do not need to compute the error

```
% Define matrix A and vector b
A = [0 4 4;3 1 2];
b = [4 ;-2]
```

```
b = 2×1
     4
    -2
```

```
% Determine rank of matrix A
rank_A = rank(A);
```

```
% Create the augmented matrix [A | b]
augmented_matrix = [A, b];

% Determine rank of the augmented matrix
rank_augmented = rank(augmented_matrix);

% Display the ranks
fprintf('Rank of matrix A: %d\n', rank_A);
```

```
Rank of matrix A: 2
```

```
fprintf('Rank of the augmented matrix [A | b]: %d\n', rank_augmented);
```

```
Rank of the augmented matrix [A | b]: 2
```

```
% Check the principle to determine the existence of solutions
if rank_A == rank_augmented
    if rank_A == size(A, 2)
        fprintf('The system has a unique solution.\n');
    else
        fprintf('The system has infinitely many solutions.\n');
    end
else
    fprintf('The system has no solution.\n');
end
```

```
The system has infinitely many solutions.
```

```
% Form the augmented matrix
augmented_matrix = [A, b];

% Perform RREF on the augmented matrix
R = rref(augmented_matrix);

% Display RREF result
disp('Row-reduced echelon form of the augmented matrix: from here we can declare a
free variable and set up the solution');
```

```
Row-reduced echelon form of the augmented matrix:
```

```
disp(R);
```

```
    1.0000         0    0.3333   -1.0000
         0    1.0000    1.0000    1.0000
```