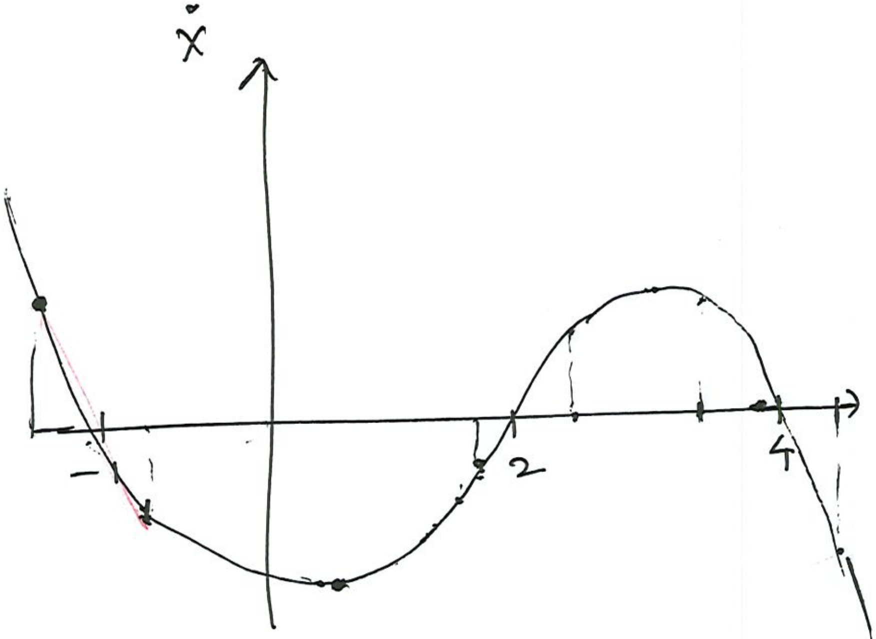
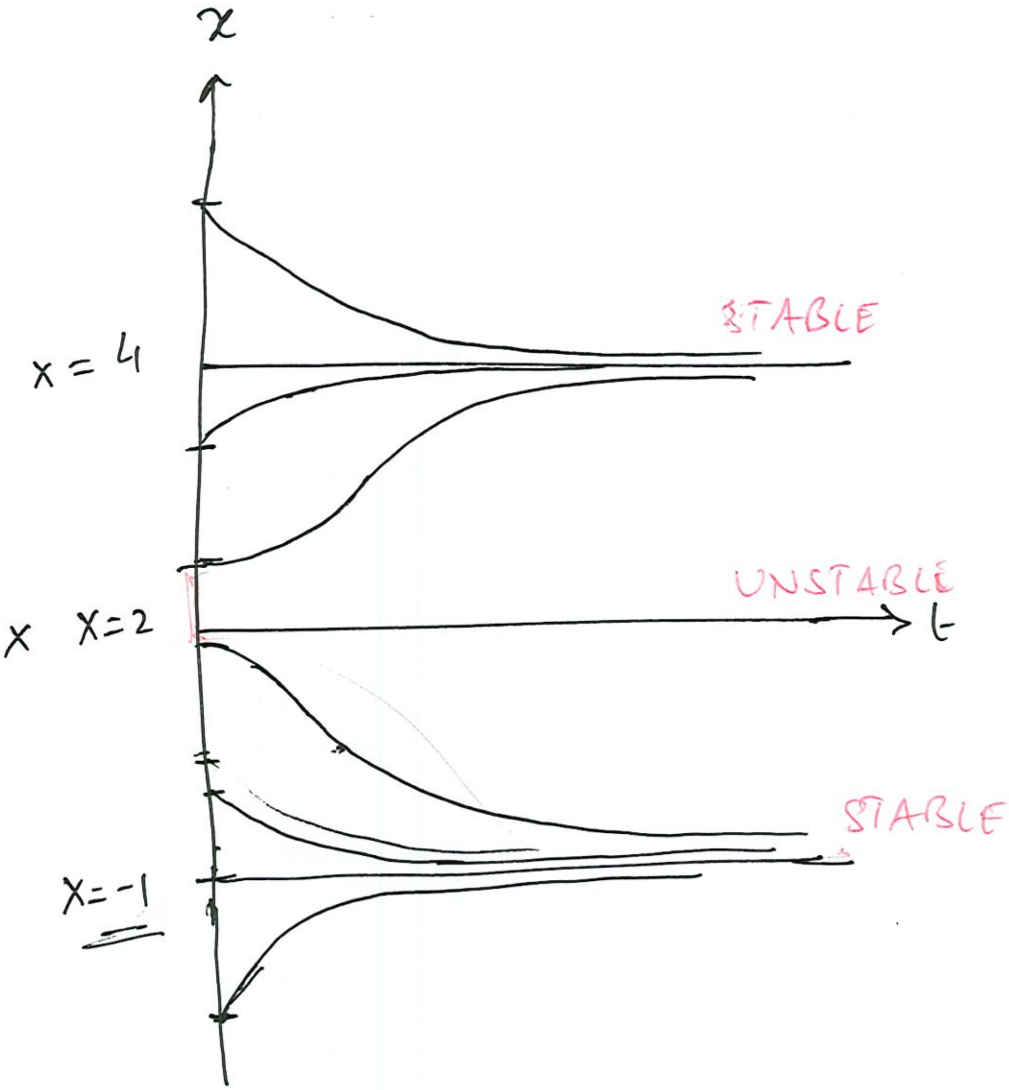
$$\dot{x} = \underbrace{(2-x)(x+1)(x-4)}_{f(x)}$$

$$f(x) = (2-x)(x+1)(x-4)$$





0

$\dot{x} = (2-x)(x+1)(x-4)$

$f(x) = (2-x)(x+1)(x-4)$

$f'(x) = -(x+1)(x-4) + (2-x)(x-4) + (2-x)(x+1)$

$f(x) \approx f(x_{eq}) + f'(x_{eq})(x - x_{eq})$

At $x = -1$

$f'(-1) = -15$

$f(x) \approx -15(x-(-1))$

$$\frac{dx}{dt} = -15\underbrace{(x-(-1))}_{\tilde{x}}$$ $\leftarrow$ deviation of x from $x_{eq} = -1$

$\tilde{x} \triangleq x-(-1) \Rightarrow \frac{d\tilde{x}}{dt} = \frac{dx}{dt}$

$\frac{d\tilde{x}}{dt} = -15\tilde{x} \Rightarrow \tilde{x}(t) = (e^{-15t})\,\tilde{x}(0)$

As $t \to \infty$ $\tilde{x}(t) \to 0 \Rightarrow x-(-1) \to 0$

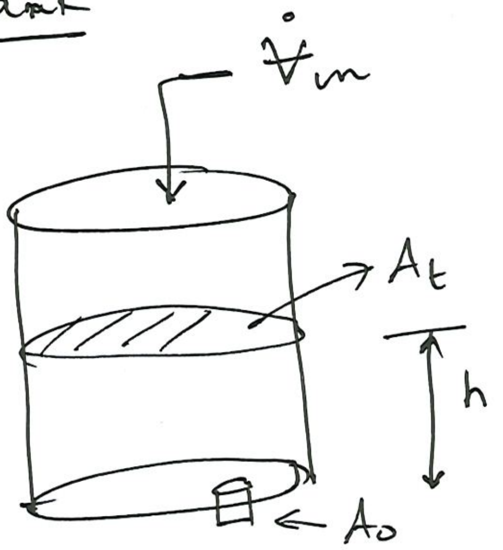
$\Rightarrow x \to -1$

At $x=2$ $\quad f'(2) = -(2+1)(2-4) = 6$

$f(x) \approx 6(x-2)$

Let $\tilde{x} = x-2 \Rightarrow \dot{x} \approx 6(x-2) \Rightarrow \dot{\tilde{x}} = 6\tilde{x} \Rightarrow \tilde{x}(t) = x(0)e^{6t}$

Tank



$$\dot{m}_{in} - \dot{m}_{out} = \frac{d m_{tank}}{dt}$$

1) ~~$\dot{m} =$~~ $m = \rho V$ (ρ = density)

$$\dot{m} = \rho \dot{V}$$

2) $\dot{V}_o = A_o V$

3) ~~$\dot{V} =$~~ $V = \sqrt{2gH}$ ← Consequence of Bernoulli's equation

(4) $V_t = A_t h$

$$\rho \dot{V}_{in} - \rho \dot{V}_o = \frac{d}{dt} \rho V_t$$

$$\dot{V}_{in} - A_o\sqrt{2gh} = \frac{d}{dt} A_t h$$

$$\frac{dh}{dt} = \underbrace{\frac{\dot{V}_{in}}{A_t}}_{\alpha} - \underbrace{\frac{A_o}{A_t}\sqrt{2g}}_{\beta}\sqrt{h}$$

$$\alpha \triangleq \frac{\dot{V}_{in}}{A_t}; \quad \beta \triangleq \frac{A_o}{A_t}\sqrt{2g}$$

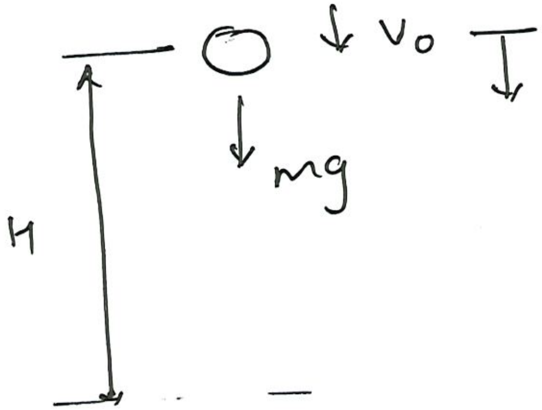
$$\frac{dh}{dt} = \alpha - \beta\sqrt{h}$$

2

Set $\frac{dh}{dt} = 0$

$$\alpha - \beta\sqrt{h} = 0 \Rightarrow h_{eq} = \frac{\alpha^2}{\beta^2} = \frac{\dot{V}_{in}^2}{A_t^2} \frac{A_t^2}{A_o^2\, 2g} = \frac{\dot{V}_{in}^2}{A_o^2\, 2g}.$$

Motion of a particle



$$\cancel{a=} \quad v = \frac{dx}{dt} \; ; \quad a = \frac{dv}{dt}$$

$$\Sigma F_y = ma$$

$$ma = mg$$

$$\Rightarrow a = g$$

$$\Rightarrow \frac{dv}{dt} = g$$

$$\Rightarrow \int_{v_0}^{v} dv = \int_0^t g\, dt$$

$$\Rightarrow v - v_0 = gt$$

$$\Rightarrow \boxed{v(t) = v_0 + gt}$$

$$\frac{dx}{dt} = v = v_0 + gt$$

$$\Rightarrow \int_0^x dx = \int_0^t (v_0 + gt)\, dt$$

$$x - 0 = v_0 t + \frac{gt^2}{2}\Big|_0^t \Rightarrow \boxed{x = v_0 t + \frac{gt^2}{2}}$$

3

Motion of particle projected upwards in resistive medium

$$m\frac{dv}{dt} = -mg - F_R$$

$$= -mg - bv \qquad \text{(linear drag)}$$

$$m\frac{dv}{dt} = -(mg + bv)$$

$$\cancel{dt}\ \frac{dv}{dt} = -\left(g + \frac{b}{m}v\right)$$

$$\int_{v_0}^{v} \frac{dv}{\left(g + \frac{b}{m}v\right)} = \int_0^t -dt$$

$$\left.\frac{\ln\left(g + \frac{b}{m}v\right)}{b/m}\right|_{v_0}^{v} = -t$$

$$\ln\frac{g + b/m\, v}{g + b/m\, v_0} = -\frac{b}{m}t$$

$$\frac{g + b/m\, v}{g + b/m\, v_0} = e^{-b/m\, t} \Rightarrow g + b/m\, v = \left(g + b/m\, v_0\right)e^{-b/m\, t}$$

$$\Rightarrow \boxed{v = \frac{m}{b}\left(-g + \left(g + \frac{b}{m}v_0\right)e^{-\frac{b}{m}t}\right)}$$

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$$\ln \frac{v - v_t}{v_0 - v_t} = -\frac{b}{m} t$$

$$\Rightarrow \quad v - v_t = (v_0 - v_t) e^{-b/m\, t}$$

$$\Rightarrow \quad \boxed{v(t) = v_t + (v_0 - v_t) e^{-b/m\, t}}$$ where $v_t = \frac{mg}{b}$

~~whe~~ $t = 0 \Rightarrow v(t) = v_t + (v_0 - v_t) e^0$

$= v_0$ ✓

$t \to \infty \quad v(t) = v_t + (v_0 - v_t) 0$

$= v_t$ ✓

$$\left[\frac{b}{m}\right] = \frac{1}{s}$$

$$\left[\frac{m}{b}\right] = s$$ ← Time constant.

5

④

$$\frac{dh}{dt} = \underbrace{\alpha - \beta\sqrt{h}}_{f(h)\,g(t)} \qquad f(h) = \alpha - \beta\sqrt{h}$$

$$g(t) = 1$$

$$\int_{h_0}^{h} \frac{dh}{\alpha - \beta\sqrt{h}} = \int_0^t dt$$

Let $\frac{\alpha^2 u^2}{\beta^2} = h \Rightarrow dh = du\,\frac{\alpha^2}{\beta^2}\,2u$.

When $h = h_0$

$$u = \frac{\beta}{\alpha}\sqrt{h_0}$$

When $h = h$

$$u = \frac{\beta}{\alpha}\sqrt{h}$$

$$\frac{1}{\alpha - \beta\sqrt{h}} = \frac{1}{\alpha - \beta\frac{\alpha u}{\beta}} = \frac{1}{\alpha(1-u)}$$

$$\int_{\beta/\alpha\sqrt{h_0}}^{\beta/\alpha\sqrt{h}} \frac{u\,du\,\frac{2\alpha^2}{\beta^2}}{\alpha(1-u)} = t$$

$$\frac{2\alpha}{\beta^2}\int_{\beta/\alpha\sqrt{h_0}}^{\frac{\beta}{\alpha}\sqrt{h}} \frac{u}{1-u}\,du = t$$

6

$$\int \frac{u}{1-u}\,du$$

$$= -\int \frac{u}{u-1}\,du$$

$$= -\int\left(\frac{u-1+1}{u-1}\right)du = -\int\left(1+\frac{1}{u-1}\right)du$$

$$= -\left(u+\ln(u-1)\right)$$

$$-\frac{2\alpha}{\beta^2}\left(u+\ln(u-1)\right)\Big|_{\beta/\alpha\sqrt{h_0}}^{\beta/\alpha\sqrt{h}} = t$$

$$\left(\frac{\beta}{\alpha}\sqrt{h}-\frac{\beta}{\alpha}\sqrt{h_0}\right)+\ln\left(\frac{\beta/\alpha\sqrt{h}-1}{\beta/\alpha\sqrt{h_0}-1}\right) = -\frac{\beta^2}{2\alpha}t$$

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