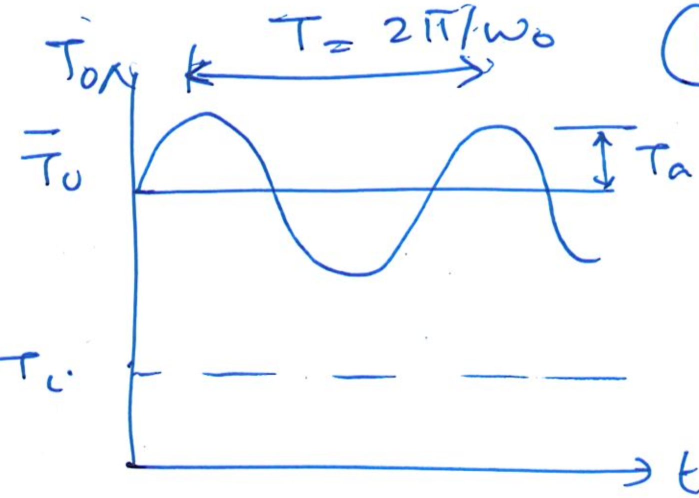
$$\frac{dT}{dt} + kT = kT_o(t) \qquad T(0) = T_i$$

$$T_o(t) = \bar{T}_o + T_a \sin \omega_0 t$$



$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\sin\theta = \mathrm{Im}(e^{j\theta}) \quad ; \quad \cos\theta = \mathrm{Re}(\cancel{\cos\theta}\; e^{j\theta})$$

$$\frac{dT}{dt} + kT = k\bar{T}_o + kT_a \underbrace{\sin\omega_0 t}_{e^{j\omega_0 t}}$$

<u>Integrating factors</u> Multiply by e^{kt}

$$e^{kt}\frac{dT}{dt} + kTe^{kt} = k\bar{T}_o e^{kt} + kT_a e^{j\omega_0 t} e^{kt}$$

$$= k\bar{T}_o e^{kt} + kT_a e^{(k+j\omega_0)t}$$

$$\frac{d}{dt}(Te^{kt}) = k\bar{T}_o e^{kt} + kT_a e^{(k+j\omega_0)t}$$

$$\int_{T_i,\,t=0}^{T,\,t} d(Te^{kt}) = k\bar{T}_o \int_0^t e^{kt}\,dt + kT_a \int_0^t e^{(k+j\omega_0)t}\,dt$$

$$T(t)e^{kt} - T(0)e^{k0} = k\bar{T}_o \frac{e^{kt}}{k}\Big|_0^t + kT_a \frac{e^{(k+j\omega_0)t}}{(k+j\omega_0)}\Big|_0^t$$

$$T(t)e^{kt} - T_i = \bar{T}_o(e^{kt} - 1) + \frac{kT_a}{k+j\omega_0}\left(e^{(k+j\omega_0)t} - 1\right)$$

$$T(t) = T_i e^{-kt} + \bar{T}_0 (1 - e^{-kt}) + \frac{k}{k+j\omega_0} T_a \left(e^{j\omega_0 t} - e^{-kt} \right)$$

$$\frac{k}{k+j\omega_0} e^{j\omega_0 t} = \frac{k(\cos \omega_0 t + j \sin \omega_0 t)}{(k+j\omega_0)} \frac{k-j\omega_0}{k-j\omega_0}$$

$$= \frac{k}{k^2+\omega_0^2} \left[(k \cos\omega_0 t + \omega_0 \sin\omega_0 t) + j (k \sin\omega_0 t - \omega_0 \cos\omega_0 t) \right]$$

$$\mathrm{Im}\left(\frac{k}{k+j\omega_0} e^{j\omega_0 t} \right) = \frac{k}{k^2+\omega_0^2} (k \sin\omega_0 t - \omega_0 \cos\omega_0 t)$$

$$z = x + jy$$

$$\frac{1}{x+jy} = \frac{1}{(x+jy)} \frac{(x-jy)}{(x-jy)}$$

$$= \underbrace{\frac{x}{x^2+y^2}}_{\mathrm{Re}} - j \underbrace{\frac{y}{x^2+y^2}}_{\mathrm{Im}}$$

$$\frac{k}{k+j\omega_0} = \frac{k}{k+j\omega_0} \frac{k-j\omega_0}{k-j\omega_0}$$

$$= \frac{k}{k^2+\omega_0^2} (k - j\omega_0)$$

$$\mathrm{Im}\left(\frac{k}{k+j\omega_0} \right) = -\frac{k\omega_0}{k^2+\omega_0^2}$$

$$T(t) = T_i e^{-kt} + \bar{T}_0 (1 - e^{-kt}) + \frac{k}{k^2+\omega_0^2} T_a (k \sin\omega_0 t - \omega_0 \cos\omega_0 t) + \frac{k\omega_0 T_a}{k^2+\omega_0^2} e^{-kt}$$

For $t > 5/k$ $\quad e^{-kt} \to 0$

$$T(t) = \bar{T}_0 + \frac{k}{k^2+\omega_0^2} T_a (k \sin\omega_0 t - \omega_0 \cos\omega_0 t)$$

$k \sin\omega_0 t - \omega_0 \cos\omega_0 t$

$$C\sin(\omega_0 t - \phi) = C \sin\omega_0 t \cos\phi - C\cos\omega_0 t \sin\phi$$

Let $C\cos\phi = k \rightarrow C^2\cos^2\phi = k^2$

$C\sin\phi = \omega_0 \rightarrow + \; C^2\sin^2\phi = \omega_0^2$

$$C^2(\cos^2\phi + \sin^2\phi) = k^2 + \omega_0^2$$

$$\Rightarrow \quad C = \sqrt{k^2+\omega_0^2}$$

$$\frac{C\sin\phi}{C\cos\phi} = \frac{\omega_0}{k} \Rightarrow \tan\phi = \frac{\omega_0}{k} \Rightarrow \phi = \tan^{-1}\frac{\omega_0}{k}.$$

$$\begin{aligned} k\sin\omega_0 t - \omega_0\cos\phi &= C\cos\phi \sin\omega_0 t - C\sin\phi\cos\omega_0 t \\ &= C\sin(\omega_0 t - \phi) \\ &= \sqrt{k^2+\omega_0^2}\,\sin\left(\omega_0 t - \tan^{-1}\frac{\omega_0}{k}\right) \\ &= \sqrt{k^2+\omega_0^2}\,\sin(\omega_0 t - \phi) \quad \text{where } \phi = \tan^{-1}\frac{\omega_0}{k} \end{aligned}$$

$$T(t) = T_0 + \frac{k}{k^2+\omega_0^2}\, T_a \sqrt{k^2+\omega_0^2}\,\sin(\omega_0 t - \phi)$$

$$\boxed{T(t) = T_0 + \frac{T_a k}{\sqrt{k^2+\omega_0^2}}\sin(\omega_0 t - \phi)}$$

$t \gg \; t > 5/k$

2

$$\frac{dT}{dt} + kT = k\bar{T}_0 + kT_a e^{j\omega_0 t}$$

Homogenous: $\frac{dT}{dt} + kT = 0 \Rightarrow T_h(t) = e^{-kt}$

Particular $T_p(t) = A + Be^{j\omega_0 t} \Rightarrow \frac{dT_p}{dt} = B(j\omega_0)e^{j\omega_0 t}$

$$(Bj\omega_0)e^{j\omega_0 t} + k(A + Be^{j\omega_0 t}) = k\bar{T}_0 + kT_a e^{j\omega_0 t}$$

Constant: $kA = k\bar{T}_0 \Rightarrow A = \bar{T}_0$

Exponential $Bj\omega_0 + kB = kT_a \Rightarrow B = \frac{k}{k+j\omega_0}T_a$

$$T_p(t) = \bar{T}_0 + \frac{k}{k+j\omega_0}T_a e^{j\omega_0 t}$$

general: $T = T_H(t) + T_p(t) = ce^{-kt} + \bar{T}_0 + \frac{k}{k+j\omega_0}T_a e^{j\omega_0 t}$

IC: $T(0) = T_i = c + \bar{T}_0 + \frac{k}{k+j\omega_0}T_a \Rightarrow c = T_i - \bar{T}_0 - \frac{k}{k+j\omega_0}T_a$

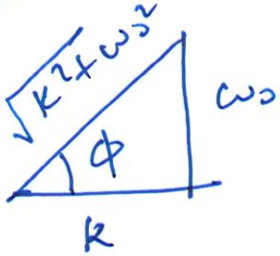
$$T(t) = \left(T_i - \bar{T}_0 - \frac{k}{k+j\omega_0}T_a\right)e^{-kt} + \bar{T}_0 + \frac{k}{k+j\omega_0}T_a e^{j\omega_0 t}$$

$$= T_i e^{-kt} + \bar{T}_0(1 - e^{-kt}) + \frac{k}{k+j\omega_0}T_a e^{j\omega_0 t} - \frac{k}{k+j\omega_0}T_a e^{-kt}$$

3

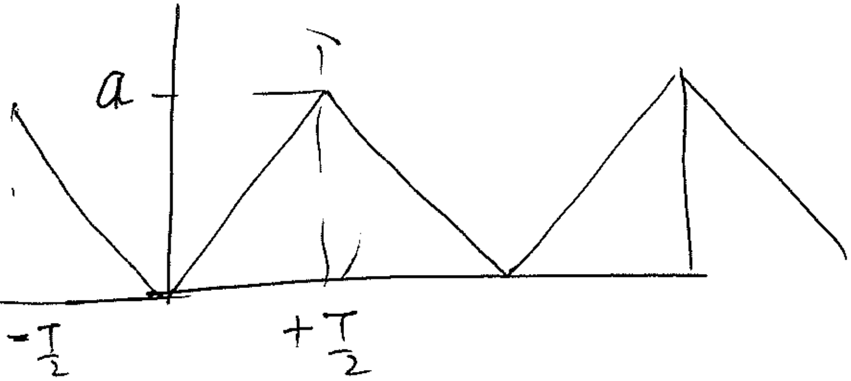
$$\frac{k}{k+j\omega_0} e^{j\omega_0 t} = \frac{k}{\sqrt{k^2+\omega_0^2}\; e^{j\phi}}\, e^{j\omega_0 t} = \frac{k}{\sqrt{k^2+\omega_0^2}}\, e^{j(\omega_0 t-\phi)}$$

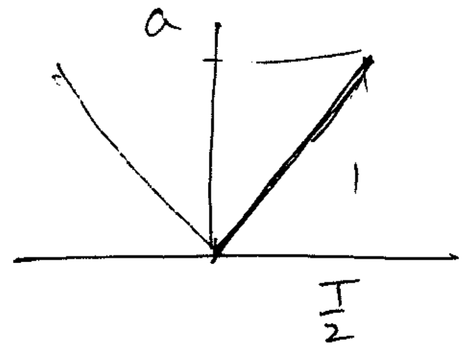
$$= \frac{k}{\sqrt{k^2+\omega_0^2}}\, \cos(\omega_0 t-\phi) + j\sin(\omega_0 t-\phi)$$



$$k+j\omega_0 = \sqrt{k^2+\omega_0^2}\, e^{j\phi} = \sqrt{k^2+\omega_0^2}\, e^{j\tan^{-1}\frac{\omega_0}{k}}$$

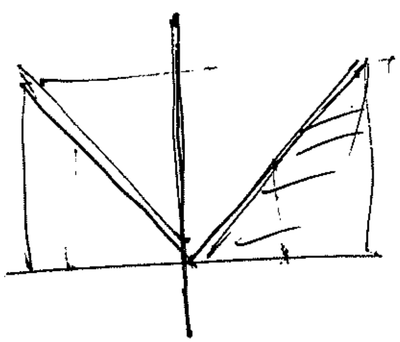
$$Im\left(\frac{k}{k+j\omega_0} e^{j\omega_0 t}\right) = \frac{k}{\sqrt{k^2+\omega_0^2}}\, \sin(\omega_0 t-\phi)$$





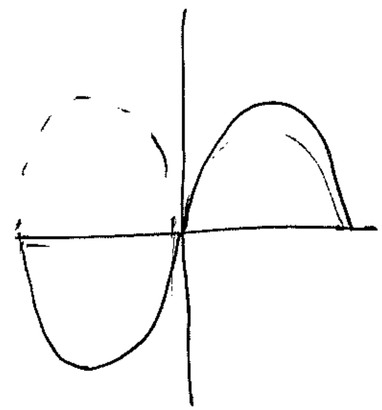
$$f(t) = \frac{2at}{T} \quad 0 < t < \frac{T}{2}$$
$$= -\frac{2at}{T}$$

Even



$$f(t) = +f(-t)$$

Odd



$$f(t) = -f(-t)$$

odd

$$\int_{-a}^{a} f(t)\,dt = 0$$

$$\int_{-T}^{T} f(t)\,dt = \int_{-T}^{0} f(t)\,dt + \int_{0}^{T} f(t)\,dt$$

$$= \int_{-T}^{0} f(-t)\,dt + \int_{0}^{T} f(t)\,dt$$

Let $u = -t \Rightarrow du = -dt$ (in 1st integral)

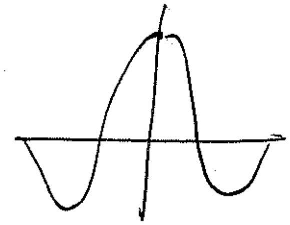
Even

$$\int_{T}^{0} f(u)\,(-du) + \int_{0}^{T} f(t)\,dt = \int_{0}^{T} f(u)\,du + \int_{0}^{T} f(t)\,dt = 2\int_{0}^{T} f(t)\,dt$$

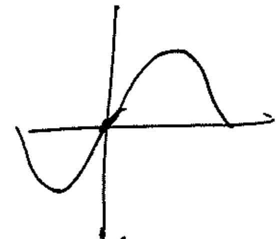
Even × Even $\longrightarrow$ Even

Odd × odd $\longrightarrow$ Even

Even × odd $\longrightarrow$ odd



$\rightarrow$ even



If $f(t)$ is even

$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt$$

$$a_k = \frac{4}{T}\int_0^{T/2} f(t)\cos\left(k\frac{2\pi t}{T}\right) dt$$

$$b_k = 0$$

<u>If $f(t)$ is odd</u>

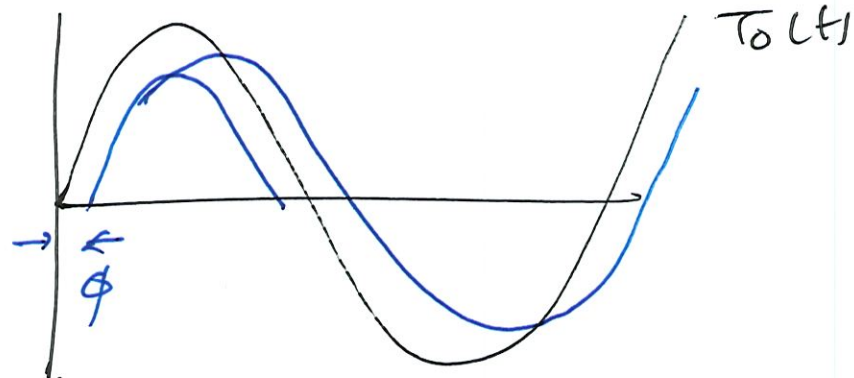
$$a_0 = 0$$

$$a_k = 0$$

$$b_k = \frac{4}{T}\int_0^{T/2} f(t)\sin k\frac{2\pi t}{T}\,dt$$

$$T_0(t) = \bar{T}_0 + T_a \sin\omega_0 t \longrightarrow T(t) = \bar{T}_0 + T_a \frac{k}{\sqrt{k^2+\omega_0^2}} \sin(\omega_0 t - \phi)$$

Let $\bar{T}_0 = 0$ $\quad T_0(t) = T_a \sin\omega_0 t$ $\quad T(t) = T_a \frac{k}{\sqrt{k^2+\omega_0^2}} \sin(\omega_0 t - \phi)$



$$|T| = \frac{T_a k}{\sqrt{k^2+\omega_0^2}}$$

$\omega_0 \to 0 \qquad |T| = T_a$

$\omega_0 \to \infty \qquad |T| \approx \frac{T_a k}{\sqrt{\omega_0^2}} \approx T_a \frac{k}{\omega_0} \to 0$

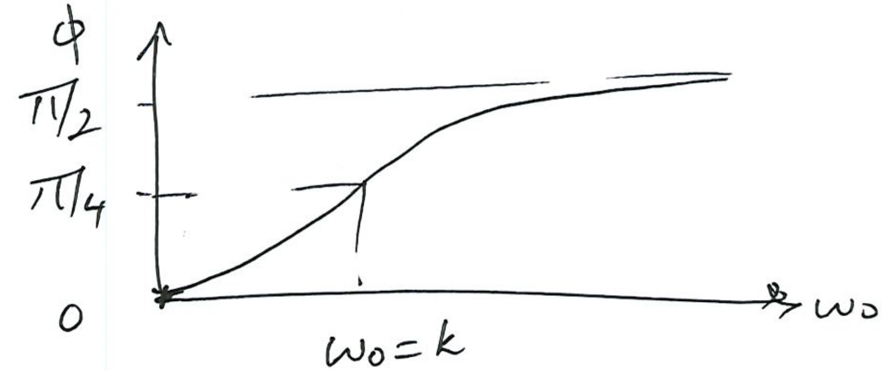
$\omega_0 = k \qquad |T| = T_a \frac{k}{\sqrt{k^2+k^2}} = \frac{T_a}{\sqrt{2}}$

$$\phi = \tan^{-1}\frac{\omega_0}{k}$$

$\omega_0 \to 0 \qquad \phi \to 0$

$\omega_0 \to \infty \qquad \phi \to \pi/2$

$\omega_0 = k \qquad \phi = \pi/4$



Let $f(t)$ be a periodic function with period T

$$f(t) = a_0 + a_1 \cos \frac{2\pi t}{T} + a_2 \cos 2 \cdot \frac{2\pi t}{T} + a_3 \cos 3 \cdot \frac{2\pi t}{T} + \cdots$$

$$+ b_1 \sin \frac{2\pi t}{T} + b_2 \sin 2 \cdot \frac{2\pi t}{T} + b_3 \sin 3 \cdot \frac{2\pi t}{T} + \cdots$$

$$= a_0 + \sum_{k=1}^{\infty} a_k \cos\left(k \frac{2\pi t}{T}\right) + \sum_{k=1}^{\infty} b_k \sin\left(k \frac{2\pi t}{T}\right)$$

where

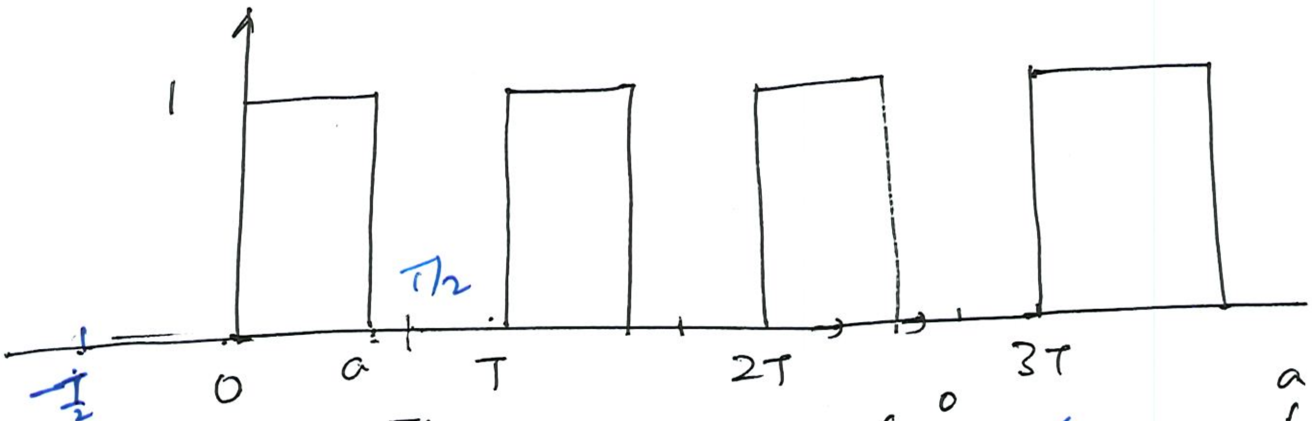
$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, dt$$

$$a_k = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos\left(k \frac{2\pi t}{T}\right) dt$$

$$b_k = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin k \frac{2\pi t}{T}\, dt$$

Periodic input

$$f(t) = \begin{cases} 0 & -T/2 < t < 0 \\ 1 & 0 < t < a \\ 0 & a < t < \frac{T}{2} \end{cases}$$



$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt = \frac{1}{T}\left[\int_{-T/2}^{0} \underbrace{f(t)}_{0}\,dt + \int_0^a f(t)\,dt + \int_a^{T/2} \underbrace{f(t)}_{0}\,dt\right]$$

$$= \frac{1}{T}\left[\int_0^a 1\,dt\right] = \frac{a}{T}$$

$$a_k = \frac{2}{T}\int_{-\frac{T}{2}}^{T/2} f(t)\cos k\frac{2\pi t}{T}\,dt = \frac{2}{T}\left[\int_0^a (1)\cos k\frac{2\pi t}{T}\,dt\right] = \frac{2}{T}\left.\frac{\sin k\frac{2\pi t}{T}}{\frac{2\pi k}{T}}\right|_0^a$$

$$= \left(\frac{1}{\pi k}\right)\sin k\frac{2\pi a}{T}$$

$$b_k = \frac{2}{T}\int_{-\frac{T}{2}}^{T/2} f(t)\sin k\frac{2\pi t}{T}\,dt = \frac{2}{T}\int_0^a 1\sin k\frac{2\pi t}{T}\,dt$$

$$= \frac{2}{T}\left.\frac{-\cos k\frac{2\pi t}{T}}{\frac{2\pi k}{T}}\right|_0^a = \frac{1}{\pi k}\left(1 - \cos k\frac{2\pi a}{T}\right)$$

$$f(t) = \frac{a}{T} + \underbrace{\frac{1}{\pi}\sin\frac{2\pi a}{T}}_{a_1}\cos\frac{2\pi t}{T} + \underbrace{\frac{1}{2\pi}\sin 2\cdot\frac{2\pi a}{T}}_{a_2}\cos 2\cdot\frac{2\pi t}{T}$$

$$+ \underbrace{\frac{1}{3\pi}\sin 3\cdot\frac{2\pi a}{T}}_{a_3}\cos 3\cdot\frac{2\pi t}{T} + \dots$$

$$+ \frac{1}{\pi}\left(1-\cos\frac{2\pi a}{T}\right)\sin\frac{2\pi t}{T} + \frac{1}{2\pi}\left(1-\cos 2\,\frac{2\pi a}{T}\right)\sin 2\cdot\frac{2\pi t}{T}$$

$$+ \frac{1}{3\pi}\left(1-\cos 3\cdot\frac{2\pi a}{T}\right)\sin 3\cdot\frac{2\pi t}{T} + \dots$$