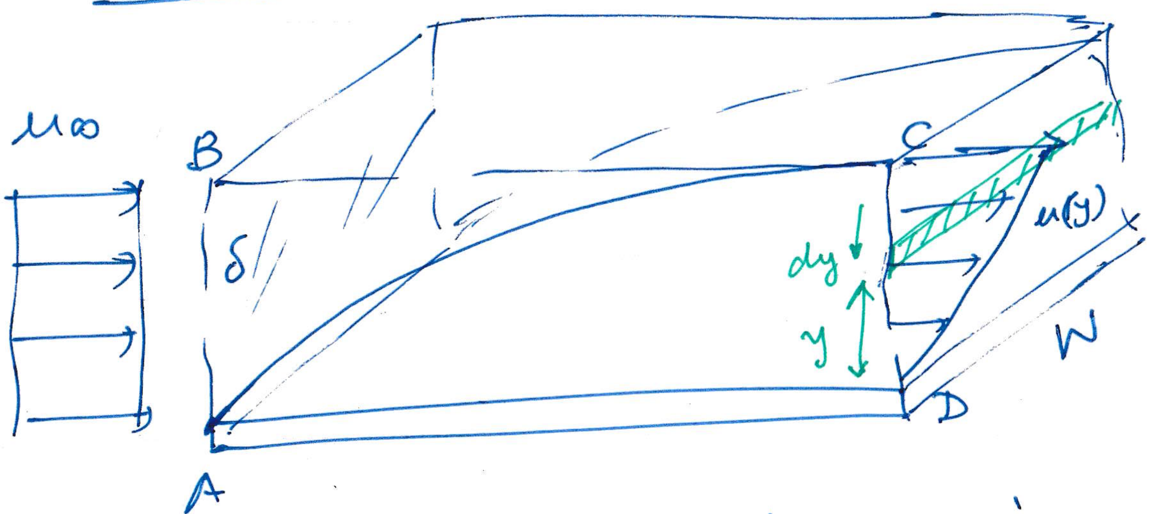
Integral Momentum Method

$\dot{m} = \rho A V$



Mass Balance:

$\dot{m}_{AB} - \dot{m}_{CD} - \dot{m}_{BC} = 0 \Rightarrow \dot{m}_{BC} = \dot{m}_{AB} - \dot{m}_{CD}$

$$\dot{m}_{AB} = \rho(\delta W)u_\infty$$

$$\dot{m}_{CD} = \int d\dot{m}_{CD} = \int_0^\delta \rho(dy)W u(y)$$

$$\dot{m}_{BC} = \dot{m}_{AB} - \dot{m}_{CD} = \rho W u_\infty \delta - \int_0^\delta \rho W u(y)\,dy$$

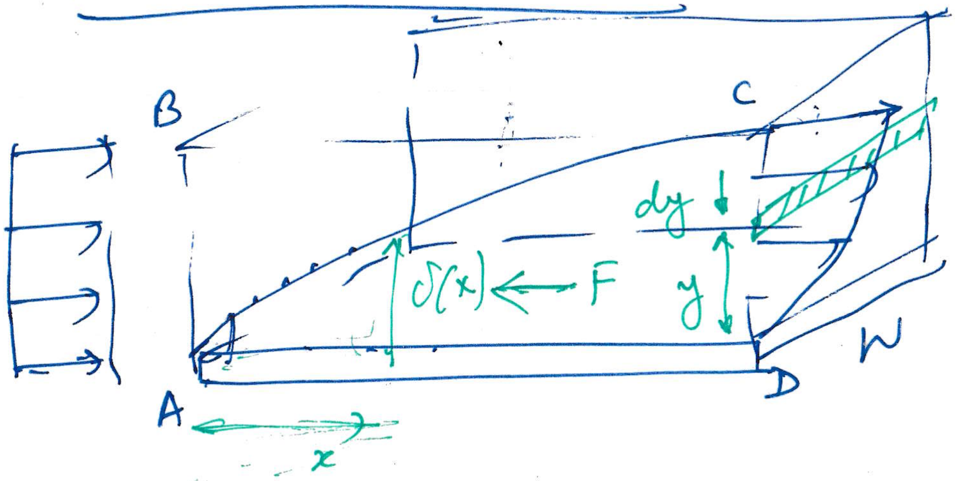
$$= \rho W u_\infty \int_0^\delta dy - \int_0^\delta \rho W u(y)\,dy$$

$\int_0^\delta dy = \delta$

$$= \rho W \int_0^\delta (u_\infty - u(y))\,dy$$

$$\boxed{\dot{m}_{BC} = \rho W u_\infty \int_0^\delta \left(1 - \frac{u(y)}{u_\infty}\right) dy}$$

Momentum Balance X!



$$\dot M_{AB} - \dot M_{CD} - \dot M_{BC} - F = 0 \quad (2)$$

$$\dot M_{AB} = \dot m_{AB} u_\infty = (\rho u_\infty W \delta) u_\infty = \rho u_\infty^2 W \delta$$

$$\dot M_{CD} = \int d\dot M = \int d\dot m \, u(y) = \int_0^\delta \rho W u(y)\, dy \, u(y) = \int_0^\delta \rho W u^2(y)\, dy$$

$$\dot M_{BC} = (u_\infty)\, \dot m_{BC} = \rho W \int_0^\delta u_\infty (u_\infty - u(y))\, dy = \rho W u_\infty^2 \delta - \rho W \int_0^\delta u_\infty u(y)\, dy$$

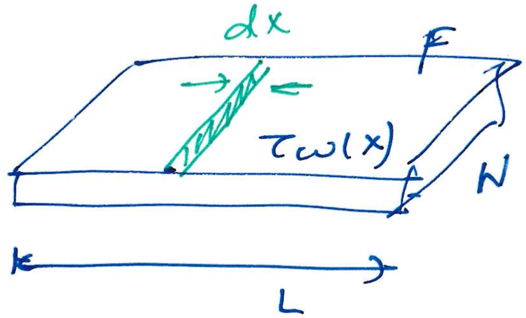
$$F = \dot M_{AB} - \dot M_{CD} - \dot M_{BC}$$
$$= (\cancel{\rho u_\infty^2 W \delta}) - \int_0^\delta \rho W u^2(y)\, dy - \left(\cancel{\rho u_\infty^2 W \delta} - \rho W \int_0^\delta u_\infty u(y)\, dy\right)$$

$$= \rho W \int_0^\delta \left(u_\infty u(y) - u^2(y)\right) dy$$

$$= \rho W u_\infty^2 \int_0^\delta \left(\frac{u(y)}{u_\infty} - \frac{u^2(y)}{u_\infty^2}\right) dy$$

$$\boxed{F = \rho W u_\infty^2 \underbrace{\int_0^{\delta(x)} \frac{u(y)}{u_\infty}\left(1 - \frac{u(y)}{u_\infty}\right) dy}_{\triangleq\, \theta(x)}}$$

$\triangleq \theta(x) \leftarrow$ Momentum layer thickness



$$F = \rho W u_\infty^2 \theta(x)$$

$$F = \int \tau_w \, dA = \int_0^L \tau_w(x) \, dx \, W$$

$$\Rightarrow \quad \tau_w(x) = \frac{1}{W}\frac{dF}{dx}$$

$$\tau_w(x) = \frac{1}{W}\frac{d}{dx}\left(\rho W u_\infty^2 \theta(x)\right)$$

$$\boxed{\tau_w(x) = \rho u_\infty^2 \frac{d\theta(x)}{dx}} \leftarrow ①$$

where $\boxed{\theta(x) = \int_0^\delta \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right) dy}$

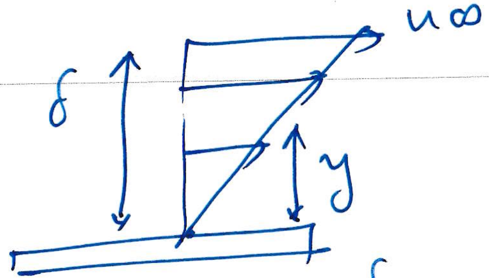
$=$

$$\boxed{\tau_w = \mu \left.\frac{du}{dy}\right|_{y=0}} \leftarrow ②$$

1) Assume a velocity profile
2) Compute τ_w from Eq. ①
3) Compute τ_w from Eq ②
(4) Equating the two τ_w's ,

Example: Assume linear velocity profile.

(1)



$$u(y) = u_\infty \frac{y}{\delta} \Rightarrow \frac{u(y)}{u_\infty} = \frac{y}{\delta}$$

(2)
$$\theta(x) = \int_0^\delta \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right) dy$$

$$= \int_0^\delta \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right) dy = \int_0^\delta \left(\frac{y}{\delta} - \frac{y^2}{\delta^2}\right) dy$$

$$= \frac{y^2}{2\delta} - \frac{y^3}{3\delta^2}\Big|_0^\delta$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right)\delta = \frac{1}{6}\delta$$

$$F = \rho u_\infty^2 \theta W = \rho u_\infty^2 \frac{1}{6} \delta W$$

$$\tau_w = \rho u_\infty^2 \frac{d\theta}{dx} = \rho u_\infty^2 \frac{d}{dx}\left(\frac{1}{6}\delta\right) = \rho u_\infty^2 \frac{1}{6} \frac{d\delta}{dx}$$

$=$

(3) Compute τ_w from $\tau_w = \mu \frac{du}{dy}\Big|_{y=0} = \mu \frac{u_\infty}{\delta}\Big|_{y=0} = \mu \frac{u_\infty}{\delta(x)}$

$$\rho u_\infty^2 \frac{1}{6}\frac{d\delta}{dx} = \mu \frac{u_\infty}{\delta}$$

$$\delta\, d\delta = \left(\frac{\mu}{\rho}\right)\frac{1}{u_\infty} 6\, dx$$

$$\int \delta\, d\delta = \int 6 \frac{\nu}{u_\infty} dx$$

$$\frac{\delta^2}{2} = \frac{6\nu x}{u_\infty} \Rightarrow \frac{\delta^2}{x}\frac{1}{x} = \frac{12\nu}{u_\infty}\cdot\frac{1}{x}$$

$$\Rightarrow \left(\frac{\delta}{x}\right)^2 = 12\left(\frac{\nu}{u_\infty x}\right)$$

$$\left(\frac{\delta}{x}\right)^2 = \frac{12}{Re_x} \qquad Re_x = \frac{u_\infty x}{\nu}$$

local Reynold's number

$$\Rightarrow \boxed{\delta = \sqrt{12}\, Re^{-1/2}\, x}$$

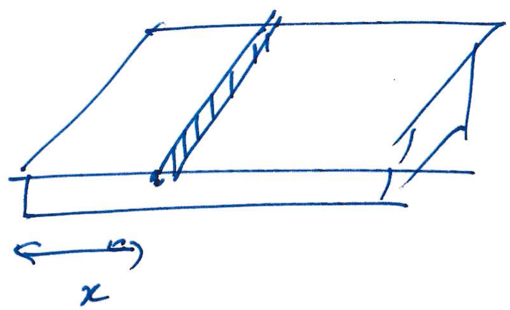
$$\tau_w = \frac{1}{2} C_f \rho u_\infty^2 = \mu \frac{u_\infty}{\delta}$$

$$\Rightarrow \frac{C_f}{2} = \left(\frac{\mu}{\rho}\right)\frac{1}{u_\infty \delta} \Rightarrow \frac{C_f}{2} = \frac{\nu}{u_\infty x}\frac{x}{\delta} \Rightarrow \frac{C_f}{2} = \frac{1}{Re}\frac{x}{\delta}$$

$$\Rightarrow \frac{C_f}{2} Re = \frac{x}{\delta}$$

$$\Rightarrow \quad \frac{C_f}{2} Re = \frac{1}{\sqrt{12}} Re^{-1/2} \quad \Rightarrow \quad \frac{C_f}{2} Re = \frac{1}{\sqrt{12}} Re^{1/2}$$

$$\Rightarrow \quad \boxed{C_{f_x} = \frac{2}{\sqrt{12}} Re_x^{-1/2}}$$



$$\tau_\omega = \frac{1}{2} C_f \rho u_\infty^2 \longrightarrow F = A \frac{1}{2} \rho u_\infty^2 C_f$$

$$F = \underset{\substack{\downarrow \\ WL}}{A} \cancel{\tfrac{1}{2}} \rho \cancel{u_\infty^2} \bar{C}_f = \int_0^L \tau_\omega \, dA$$

$$= \int_0^L \cancel{\tfrac{1}{2}} C_f(x) \cancel{\rho u_\infty^2} (W dx)$$

$$\bar{C}_f = \frac{1}{L} \int_0^L C_f(x) \, dx$$

$$\bar{C}_f = \frac{1}{L} \int_0^L \frac{2}{\sqrt{12}} \left(\frac{u_\infty x}{\nu} \right)^{-1/2} dx \qquad \int_0^L x^{-1/2} dx = \frac{x^{1/2}}{1/2}$$

$$= \frac{1}{L} \frac{2}{\sqrt{12}} \frac{u_\infty^{-1/2}}{\nu^{-1/2}} \int_0^L x^{-1/2} dx \qquad = 2x^{1/2} \Big|_0^L$$

$$= 2L^{1/2}$$

$$= \frac{1}{L} \frac{2}{\sqrt{12}} \frac{u_\infty^{-1/2}}{\nu^{-1/2}} 2 L^{1/2} \Rightarrow \bar{C}_f = \frac{4}{\sqrt{12}} \left(\frac{u_\infty L}{\nu} \right)^{-1/2} \Rightarrow \boxed{\bar{C}_f = 1.1547 Re^{-1/2}}$$

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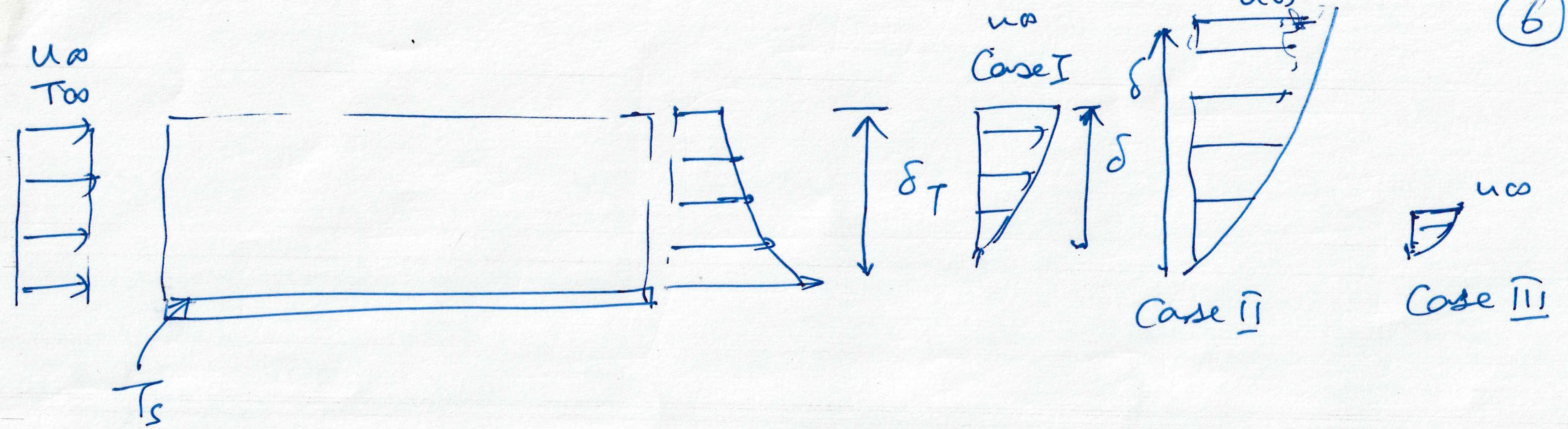

u_∞
T_∞
T_s
δ_T
u_∞
Case I
δ
u_∞
δ
Case II
u_∞
Case III