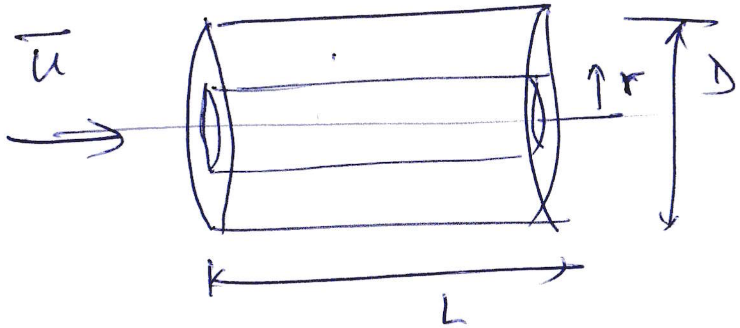
Internal Flows



$\Delta p = ?$

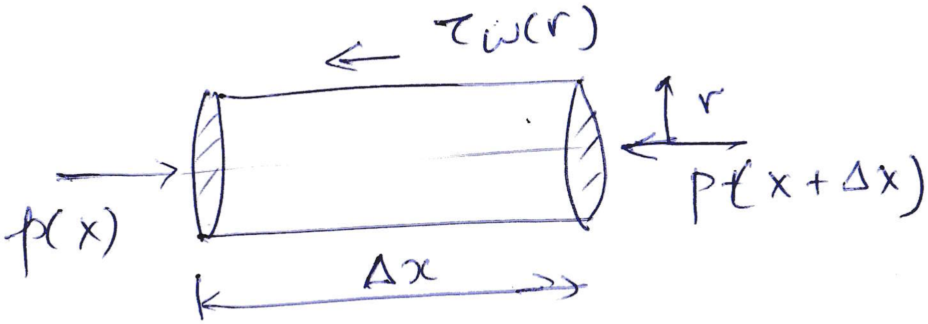
$$\Sigma F_x = 0$$

$$\cancel{p(x) - p(x+\Delta x)} - \tau_w$$

$$p(x)A - p(x+\Delta x)A - \tau_w P \Delta x = 0$$

A = cross-sectional area

P = perimeter



$$p(x)A - \left[p(x) + \frac{dp}{dx}\Delta x\right]A = \tau_w P \Delta x$$

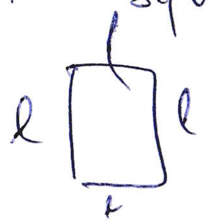
$$-\frac{dp}{dx}\cancel{\Delta x}\, A = \tau_w P \cancel{\Delta x}$$

$$\tau_w = \left(-\frac{dp}{dx}\right)\frac{A}{P}$$

For circular pipe $\dfrac{A}{P} = \left(\dfrac{\pi D^2}{4}\right)\dfrac{1}{\pi D} = \dfrac{D}{4}$

$\dfrac{4A}{P} = D_h$ ← Hydraulic diameter

For square pipe: $D_h = \dfrac{4A}{P} = \dfrac{4l^2}{4l} = l$



$\Rightarrow D_h = l$

For circular pipe:

$$\tau_w = -\frac{dp}{dx}\frac{\pi r^2}{2\pi r}$$

$$= \frac{1}{2}\left(-\frac{dp}{dx}\right)r$$

$$\tau_w = -\mu \frac{du}{dr}$$

$$\mu \frac{du}{dr} = \frac{1}{2}\left(\frac{dp}{dx}\right) r \Rightarrow \frac{du}{dr} = \frac{1}{2\mu}\frac{dp}{dx} r$$

$$\Rightarrow u(r) = \frac{1}{4\mu}\frac{dp}{dx} r^2 + c$$

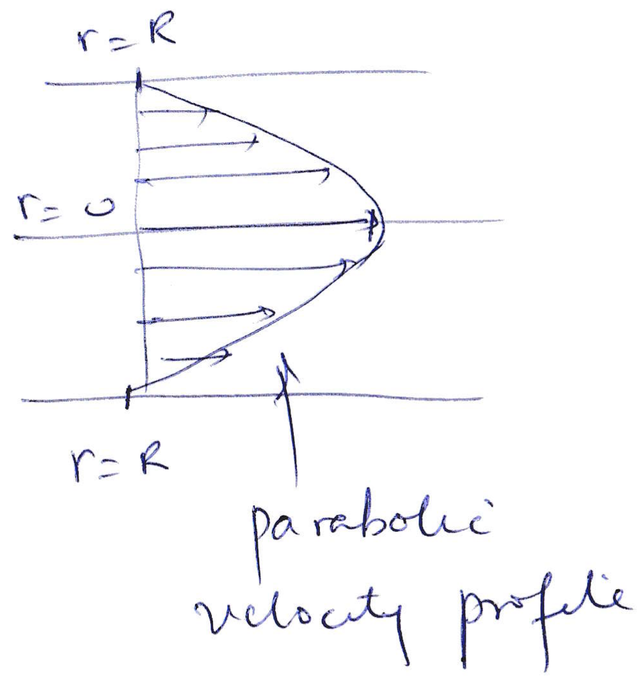
$$u(R) = 0 \Rightarrow 0 = \frac{1}{4\mu}\frac{dp}{dx}R^2 + c \Rightarrow c = -\frac{1}{4\mu}\frac{dp}{dx}R^2$$

$$u(r) = \frac{1}{4\mu}\frac{dp}{dx} r^2 - \frac{1}{4\mu}\frac{dp}{dx}R^2$$

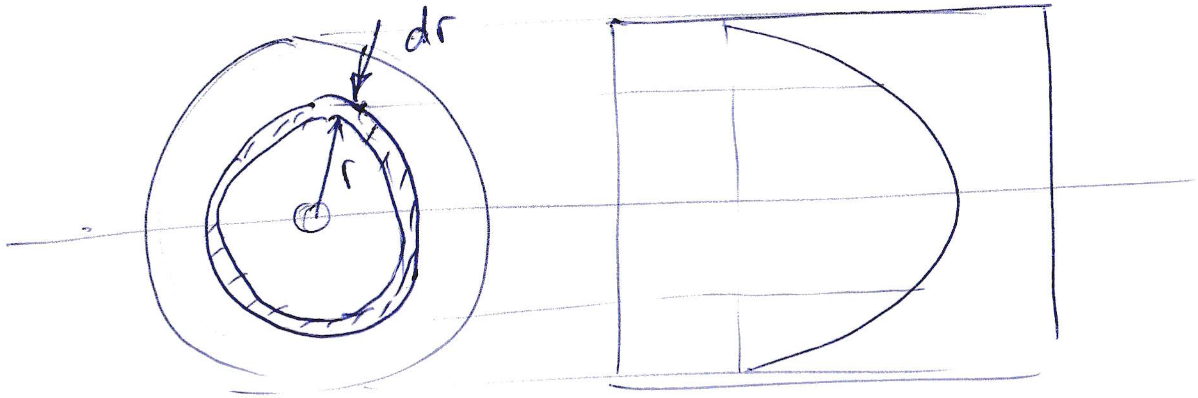
$$= \frac{1}{4\mu}\left(-\frac{dp}{dx}\right)\left[R^2 - r^2\right]$$

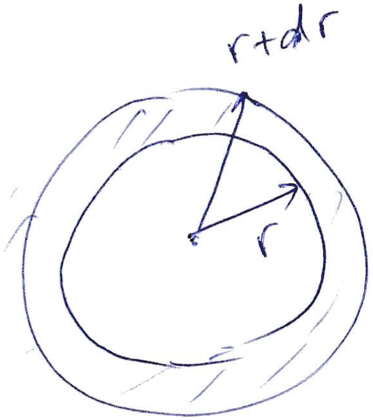
$$= \underbrace{\frac{1}{4\mu}\left(-\frac{dp}{dx}\right)R^2}_{u_{max}}\left[1 - \frac{r^2}{R^2}\right]$$

$$= u_{max}\left[1 - \frac{r^2}{R^2}\right]$$



Volumetric flow rate





$$dA = \pi(r+dr)^2 - \pi r^2$$

$$= \pi\left[r^2 + 2r\,dr + dr^2\right] - \pi r^2$$

$$= 2\pi r\,dr + \pi\,dr^2$$

If $dr << r \Rightarrow dr^2 << r\,dr$

$$dA \simeq 2\pi r\,dr$$

$$\dot{V} = \bar{u}\cancel{\pi}R^{\cancel{2}} = \cancel{\pi}\,u_m \frac{R^{\cancel{2}}}{2}$$

$$\Rightarrow \boxed{\bar{u} = \frac{u_m}{2} \quad \text{or} \quad u_m = 2\bar{u}}$$

$$d\dot{V} = u(r)\,\overbrace{2\pi r\,dr}^{dA}$$

$$\dot{V} = \int_0^R u(r)\,2\pi r\,dr$$

$$= 2\pi u_m \int_0^R r\left(1 - \frac{r^2}{R^2}\right)dr$$

$$= 2\pi u_m \left[\int_0^R \left(r - \frac{r^3}{R^2}\right)dr\right]$$

$$= 2\pi u_m \left[\left(\frac{r^2}{2} - \frac{r^4}{4R^2}\right)\right]_0^R$$

$$= 2\pi u_m \left(\frac{R^2}{2} - \frac{R^4}{4R^2}\right)$$

$$= 2\pi u_m \frac{R^2}{4}$$

$$= \pi u_m R^2/2$$

$$u_{max} = \frac{1}{4\mu}\left(-\frac{dp}{dx}\right)R^2$$

$$\left(-\frac{dp}{dx}\right) = \frac{4\mu\, u_{max}}{R^2}$$

$$= 4\mu\,(2\bar{u})\left(\frac{4}{D^2}\right)$$

$$\frac{(P_1 - P_2)}{L} = \cancel{4}\, 32\,\mu\,\frac{\bar{u}}{D^2}$$

$$\frac{P_1 - P_2}{\rho} = 32\,\frac{\mu}{\rho}\,\frac{\bar{u}}{D}\,\frac{L}{D}$$

$$= 32\,\nu\,\frac{\bar{u}^2}{(\bar{u}D)}\,\frac{L}{D}$$

$$= 64\left(\frac{\nu}{\bar{u}D}\right)\frac{\bar{u}^2}{2}\,\frac{L}{D}$$

$$= \frac{64}{Re}\,\frac{\bar{u}^2}{2}\,\frac{L}{D}$$

Darcy-Weisbach friction factor $\boxed{f_D = 64/Re}$

$$\frac{P_1 - P_2}{\rho} = f_D\,\frac{u^2}{2}\,\frac{L}{D} \Rightarrow h_L = \frac{P_1 - P_2}{\rho g} = f_D\,\frac{u^2}{2g}\,\frac{L}{D} \leftarrow \text{head loss}$$

In the last class $\tau_w = \frac{1}{2} C_f \rho \bar{u}^2 \rightarrow (1)$ (5)

Sub $\tau_w = -\mu \left.\frac{du}{dr}\right|_{r=R}$ & $u = u_m\left(1 - \frac{r^2}{R^2}\right)$

$$\tau_w = -\mu u_m \left.\left(-\frac{2r}{R^2}\right)\right|_{r=R}$$

$$= \mu u_m \frac{2}{R}$$

$$= \mu \frac{4\bar{u}}{R} \quad \left(\bar{u} = \frac{u_m}{2}\right)$$

$$= \mu \frac{8\bar{u}}{D} \rightarrow (2)$$

Equating (1) & (2)

$$\frac{1}{2} C_f \rho \bar{u}^2 = \mu \frac{8\bar{u}}{D}$$

$$C_f = \left(\frac{\mu}{\rho}\right) \frac{16}{\bar{u}D}$$

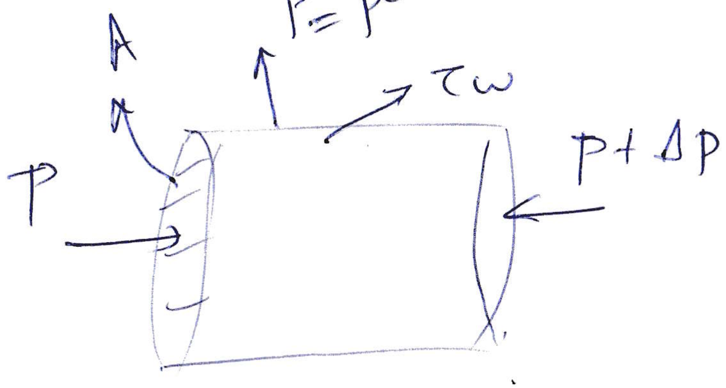
$$= 16 \left(\frac{\nu}{\bar{u}D}\right)$$

$$C_f = \frac{16}{Re} \leftarrow \text{Fanning friction factor}$$

Friction velocity u_*

τ_w is ~~ex~~ expressed as $\tau_w = \rho u_*^2$

$$\Rightarrow u_* = \sqrt{\frac{\tau_w}{\rho}}$$



$$pA - (p+\Delta p)A - \tau_w P L = 0$$

$$\Delta p\, A = \tau_w P L$$

$$\Delta p\, \cancel{\pi} R^{\cancel{2}} = \tau_w\, 2 \cancel{\pi} \cancel{R} L$$

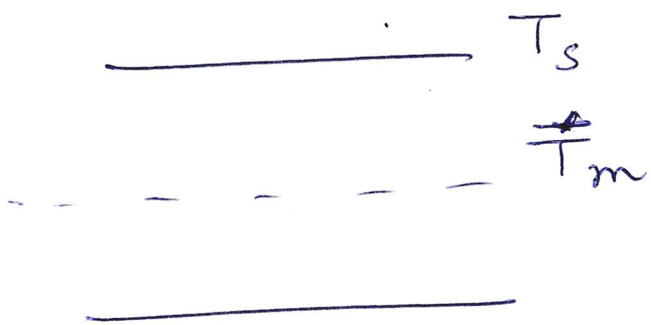
$$\Delta p\, R = \tau_w\, 2\, R L$$

$$\Delta p = \tau_w\, 2 \frac{L}{R}$$

$$\rho f_D \frac{\bar{u}^2}{2} \frac{L}{D} = \tau_w\, 4 \frac{L}{D} \quad (D = 2R)$$

$$\cancel{\rho} f_D \frac{\bar{u}^2}{2} \cancel{\frac{L}{D}} = (\cancel{\rho} u_*^2)\, 4 \cancel{\frac{L}{D}}$$

$$\boxed{f_D = \left(\frac{u_*}{\bar{u}}\right)^2 8}$$



$$\frac{T - \bar{T}_m}{T_s - \bar{T}} = \left(1 - \frac{r^2}{R^2}\right)$$

$\bar{T}$ $r = 0$ $\quad$ $T - \bar{T} = T_s - \bar{T}$

$T =$

$$\frac{T - T_s}{\bar{T}_m - T_s} = \left(1 - \frac{r^2}{R^2}\right)$$

$$q = h(T_m - T_s) = -k \left.\frac{dT}{dr}\right|_{r=R}$$

$$T - T_s = (T_m - T_s)\left(1 - \frac{r^2}{R^2}\right) \Rightarrow \frac{dT}{dr} = (T_m - T_s)\left(-\frac{2r}{R^2}\right)$$

$$\Rightarrow \left.\frac{dT}{dr}\right|_{r=R} = -(T_m - T_s)\frac{2}{R}$$

$$q = h(T_m - T_s) = -k\left(-(T_m - T_s)\right)\frac{2}{R}$$

$$h = k\frac{2}{R} \Rightarrow h = k\frac{4}{D} \Rightarrow \boxed{\begin{array}{c} \frac{hD}{k} = 4 \\ Nu = 4 \end{array}}$$

$$\frac{C_f}{2} Re\, \delta = Nu\, \delta_T = Nu_M\, \delta_c = L$$

For laminar flow in pipe $\delta = \delta_T = \delta_c = R$

$$\Rightarrow \frac{C_f}{2} Re = Nu = Nu_M$$

$$C_f = \frac{16}{Re} \Rightarrow \frac{C_f}{2} = \frac{8}{Re} \Rightarrow \frac{C_f}{2} Re \overset{8}{=} \text{constant}$$