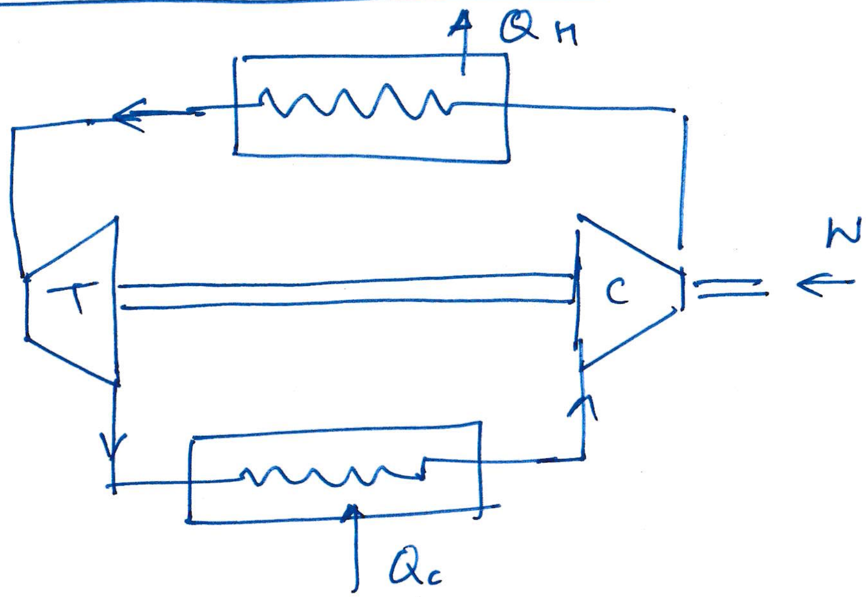
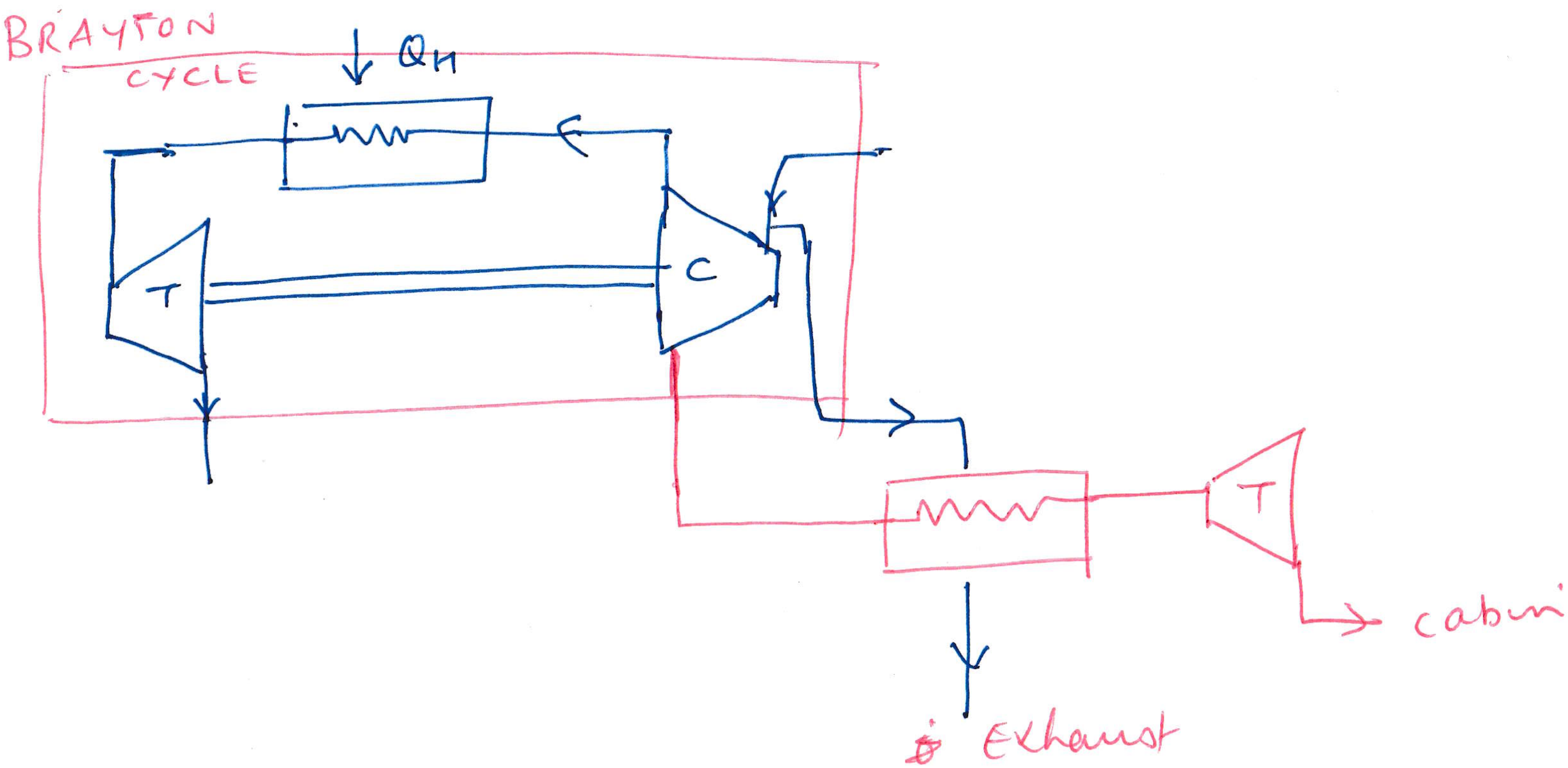
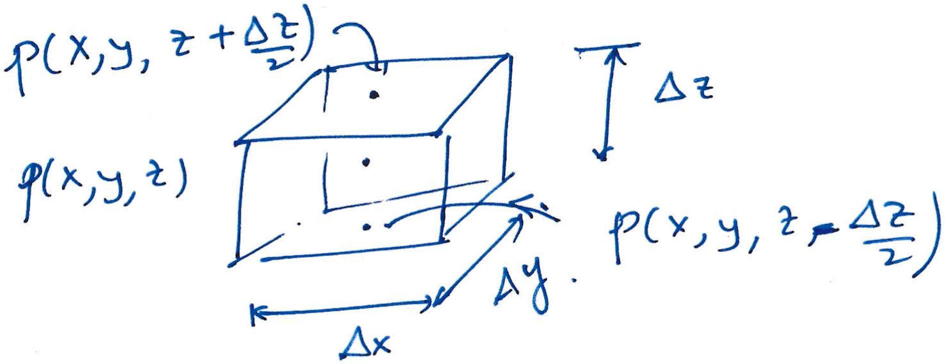
Reversed Brayton Cycle (Refrigeration)





Variation of pressure, temperature, density with altitude

Hydrostatic equation



$$\sum F_z = 0$$

$$-p(x,y,z+\tfrac{\Delta z}{2})\,\Delta x\,\Delta y + p(x,y,z-\tfrac{\Delta z}{2})\,\Delta x\,\Delta y - \rho\,\Delta x\,\Delta y\,\Delta z\,g = 0$$

$$p(x,y,z+\tfrac{\Delta z}{2}) \approx p(x,y,z) + \frac{\partial p}{\partial z}\frac{\Delta z}{2}$$

$$p(x,y,z-\tfrac{\Delta z}{2}) \approx p(x,y,z) - \frac{\partial p}{\partial z}\frac{\Delta z}{2}$$

$$-\left(p(x,y,z) + \frac{\partial p}{\partial z}\frac{\Delta z}{2}\right)\Delta x\,\Delta y + \left(p(x,y,z) - \frac{\partial p}{\partial z}\frac{\Delta z}{2}\right)\Delta x\,\Delta y - \rho\,\Delta x\,\Delta y\,\Delta z\,g = 0$$

$$-\frac{\partial p}{\partial z}\,\Delta x\,\Delta y\,\Delta z - \rho g\,\Delta x\,\Delta y\,\Delta z = 0$$

$$\Rightarrow \quad \boxed{\frac{\partial p}{\partial z} = -\rho g}$$

For air (ideal gas) $pv = RT \Rightarrow p = \frac{1}{v} RT$

$= \rho RT \Rightarrow \rho = \frac{p}{RT}$

$$\frac{dp}{dz} = -\frac{p}{RT} g \Rightarrow \frac{dp}{p} = -\frac{g}{RT} dz$$

Assume $T(z) = T_0 - \Gamma z$ where Γ = lapse rate

$$\int \frac{dp}{p} = \int \frac{-g}{R(T_0 - \Gamma z)} dz$$

$$\ln p \Big|_{p_0}^{p(z)} = \frac{-g}{R} \frac{\ln(T_0 - \Gamma z)}{-\Gamma}\Bigg|_{z=0}^{z=z} \rightarrow \frac{g}{R\Gamma}\left(\ln(T_0 - \Gamma z) - \ln T_0\right)$$

$$= \frac{g}{R\Gamma} \ln\left(\frac{T_0 - \Gamma z}{T_0}\right)$$

$$\ln \frac{p(z)}{p_0} = \frac{g}{R\Gamma} \ln\left(\frac{T_0 - \Gamma z}{T_0}\right)$$

$$= \ln\left(1 - \frac{\Gamma z}{T_0}\right)^{g/R\Gamma}$$

$$\Rightarrow \boxed{\frac{p(z)}{p_0} = \left(1 - \frac{\Gamma z}{T_0}\right)^{g/R\Gamma}}$$

$$pv^r = c \Rightarrow p = \left(\frac{1}{v}\right)^r c \Rightarrow p = \rho^r c$$

$$p_0 = \rho_0^r c$$ [$p_0, \rho_0 \to$ values at $z = 0$)

$$\Rightarrow \frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^r \Rightarrow \rho = \rho_0 \frac{p^{1/r}}{p_0^{1/r}}$$

$$\frac{dp}{dz} = -\rho g \Rightarrow \frac{dp}{dz} = -\rho_0 \frac{p^{1/r}}{p_0^{1/r}} g$$

$$\Rightarrow \int_{p_0}^{p} p^{-1/r}\, dp = \int_0^z -\frac{\rho_0 g}{p_0^{1/r}}\, dz \Rightarrow \frac{p^{\left(-\frac{1}{r}+1\right)} - p_0^{\left(-\frac{1}{r}+1\right)}}{-\frac{1}{r}+1} = -\frac{\rho_0}{p_0^{1/r}} z g$$

$$\Rightarrow p^{\frac{r-1}{r}} - p_0^{\frac{r-1}{r}} = -\frac{\rho_0}{p_0^{1/r}} \frac{r-1}{r} z g$$

$$\Rightarrow p^{\frac{r-1}{r}} = p_0^{\frac{r-1}{r}} - \frac{\rho_0}{p_0^{1/r}} \frac{r-1}{r} z g$$

$$= p_0^{\frac{r-1}{r}} - \rho_0 p_0^{-1/r} \left(\frac{r-1}{r}\right) z \left(\frac{p_0}{p_0}\right) g$$

$$= p_0^{\frac{r-1}{r}} - \rho_0 p_0^{\frac{r-1}{r}} \left(\frac{r-1}{r}\right) z \frac{1}{p_0} g.$$

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$$= p_0^{\frac{\gamma-1}{\gamma}}\left[1 - \frac{\rho_0}{p_0}\left(\frac{\gamma-1}{\gamma}\right) z g\right]$$

$$p_0 = \rho_0 R T_0$$

$$\frac{\rho_0}{p_0} = \frac{1}{R T_0}$$

$$\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}} = \left[1 - \frac{g}{R T_0}\left(\frac{\gamma-1}{\gamma}\right) z\right]$$

$$\frac{p}{p_0} = \left[1 - \frac{g}{R T_0}\left(\frac{\gamma-1}{\gamma}\right) z\right]^{\frac{\gamma}{\gamma-1}}$$

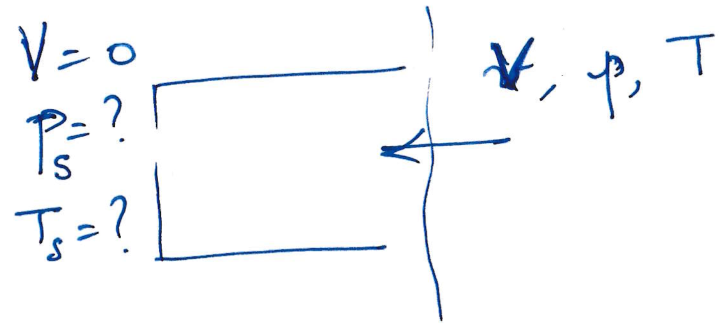
$$\frac{\Gamma}{T_0} = \frac{g}{R T_0}\left(\frac{\gamma-1}{\gamma}\right) \Rightarrow \boxed{\Gamma = \frac{g}{R}\left(\frac{\gamma-1}{\gamma}\right)}$$

$g = 9.81 \frac{m}{s}$; $R = 287$ J/kg °C $\gamma = 1.4$.

$$\Gamma = \frac{9.81}{287}\left(\frac{1.4-1}{1.4}\right) = 9.7\times10^{-3}\ °C/m$$

$$= 9.7\ °C/km \leftarrow$$ Adiabatic lapse rate.

Lapse rate $\Gamma = 6 \sim 7\ °C/km$.



$$h + \frac{V^2}{2} = h_s$$

$$\frac{V^2}{2} = h_s - h$$

$$= c_p(T_s - T)$$

$$\boxed{T_s = T + \frac{V^2}{2c_p}}$$

$$\frac{p_s}{p} = \left(\frac{T_s}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

$$c_p = R\frac{\gamma}{\gamma-1}.$$

$$T_s = T + \frac{V^2(\gamma-1)}{2\gamma R}$$

$$= T\left[1 + V^2\left(\frac{\gamma-1}{2}\right)\frac{1}{\gamma R T}\right]$$

Speed of sound $c^2 = \gamma R T \Rightarrow c = \sqrt{\gamma R T}$

$$= T\left[1 + \frac{\gamma-1}{2}\frac{V^2}{c^2}\right]$$

Define Mach number $Ma \triangleq \frac{V}{c}$

$$\boxed{T_s = T\left[1 + \frac{\gamma-1}{2}Ma^2\right]}$$