Fourier Series

Let $f(t)$ be a periodic function with period T

$$f(t) = a_0 + a_1 \cos \frac{2\pi t}{T} + a_2 \cos 2 \cdot \frac{2\pi t}{T} + \ldots b_1 \sin \frac{2\pi t}{T} + b_2 \sin 2 \cdot \frac{2\pi t}{T} + \ldots$$

$$= a_0 + \sum_{k=1}^{\infty} \left(a_k \cos k \frac{2\pi t}{T} + b_k \sin k \frac{2\pi t}{T} \right)$$

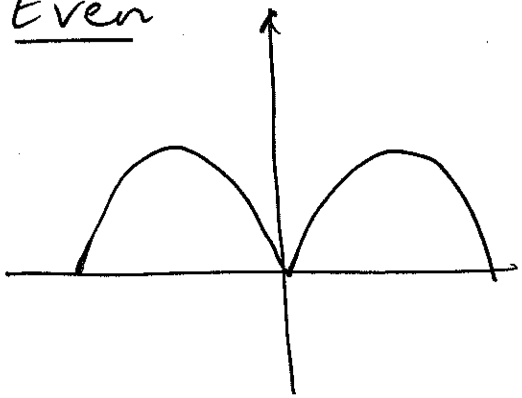
$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, dt$$

$$a_k = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos k \frac{2\pi t}{T}\, dt$$

$$b_k = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin k \frac{2\pi t}{T}\, dt$$

0

Even



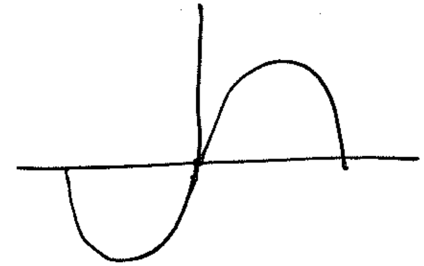
$$f(t) = f(-t)$$

$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt$$

$$a_k = \frac{4}{T}\int_0^{T/2} f(t)\cos k\frac{2\pi t}{T}\,dt$$

$$b_k = 0$$

Odd



$$f(t) = -f(-t)$$

$$a_0 = 0$$

$$a_k = 0$$

$$b_k = \frac{4}{T}\int_0^{T/2} f(t)\sin k\frac{2\pi t}{T}\,dt$$

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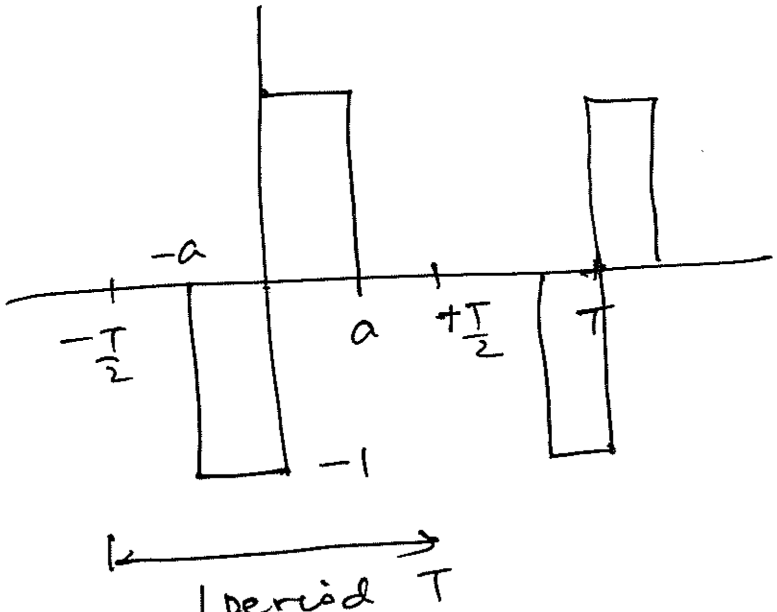

$$f(t) = 0 \quad \left(-\frac{T}{2} < t < -a\right)$$
$$= -1 \quad (-a < t < 0)$$
$$= 1 \quad (0 < t < a)$$
$$= 0 \quad \left(a < t < \frac{T}{2}\right)$$

$f(t)$ is odd $\Rightarrow$ $a_0 = 0$

$a_k = 0$

$$b_k = \frac{4}{T}\int_0^{T/2} f(t) \sin k\frac{\pi t}{T}\,dt = \frac{4}{T}\int_0^{\cancel{T/2}\, a} (1) \sin k\frac{2\pi t}{T}\,dt + \frac{4}{T}\int 0 \cdot \sin k\frac{2\pi t}{T}\,dt$$

$$= \frac{\cancel{4}^{2}}{\cancel{T}} \frac{-\cos k\frac{2\pi t}{T}\Big|_0^{T/2}}{\frac{2\pi k}{\cancel{T}}}$$

$$= \frac{2}{\pi k}\left(1 - \cos k\pi\right)$$

$$f(t) = \sum_{k=1}^{\infty} \frac{2}{\pi k}(1 - \cos k\pi) \sin k\frac{2\pi t}{T}$$

$$= \frac{2}{\pi}\cdot 2 \sin\frac{2\pi t}{T} + \frac{2}{3\pi}\cdot 2 \sin 3\,\frac{2\pi t}{T} + \frac{2}{5\pi}\, 2 \sin 5\cdot\frac{2\pi t}{T} + \dots$$

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$$b_k = \frac{4}{T}\int_0^{T/2} f(t)\sin k\frac{2\pi t}{T}\,dt = \frac{4}{T}\left[\int_0^a (1)\sin k\frac{2\pi t}{T}\,dt + \underbrace{\int_a^{T/2} 0\cdot\sin k\frac{2\pi t}{T}\,dt}_{0}\right]$$

$$= \frac{\cancel{4}^{\,2}}{\cancel{T}}\;\frac{-\cos k\frac{2\pi t}{T}}{\frac{k\,2\pi}{\cancel{T}}}\Bigg|_0^a$$

$$= \frac{2}{k\pi}\left[1-\cos k\frac{2\pi a}{T}\right]$$

$$f(t) = \sum_{k=1}^{\infty}\frac{2}{k\pi}\left(1-\cos k\frac{2\pi a}{T}\right)\sin k\frac{2\pi t}{T}$$

$$= \frac{2}{\pi}\left(1-\cos\frac{2\pi a}{T}\right)\sin\frac{2\pi t}{T} + \frac{2}{2\pi}\left(1-\cos 2\,\frac{2\pi a}{T}\right)\sin 2\,\frac{2\pi t}{T}$$

$$+ \frac{2}{3\pi}\left(1-\cos 3\cdot\frac{2\pi a}{T}\right)\sin 3\cdot\frac{2\pi t}{T} + \cdots$$

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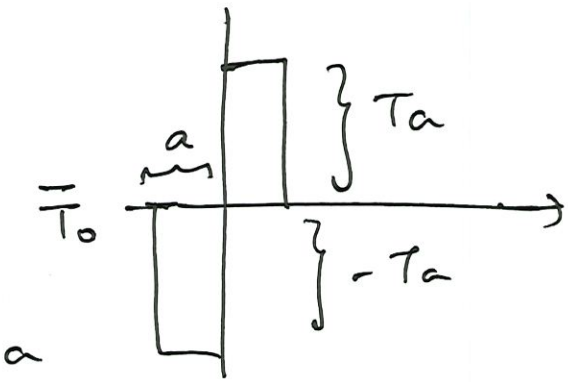
$$\frac{dT}{dt} + kT = kT_0(t)$$

where

$$\begin{aligned} T_0(t) &= \cancel{\bar{T}_0 + T_a} \quad -0 \\ &= \bar{T}_0 \quad -\tfrac{T}{2} < t < -a \\ &= \bar{T}_0 \cancel{+} T_a \quad -a < t < 0 \\ &= \bar{T}_0 + T_a \quad 0 < t < a \\ &= \bar{T}_0 \quad a < t < \tfrac{T}{2} \end{aligned}$$



~~We~~ We know the solution to

$$\frac{dT}{dt} + kT = k\bar{T}_0 + kT_a \sin\omega_0 t$$

$$T(t) = \bar{T}_0 + \frac{k}{\sqrt{k^2+\omega_0^2}} T_a \sin(\omega_0 t - \phi)$$

where $\phi = \tan^{-1}\frac{\omega_0}{k}$

Solution:

(Steady state $t > 5/k$)

Fourier series of $T_0(t)$

$$T_0(t) = \bar{T}_0 + T_a\frac{2}{\pi}\left(1-\cos\frac{2\pi a}{T}\right)\sin\frac{2\pi t}{T} + \frac{1}{\pi}T_a\left(1-\cos 2\cdot\frac{2\pi a}{T}\right)\sin 2\cdot\frac{2\pi t}{T}$$

$$+ \frac{2}{3\pi}T_a\left(1-\cos 3\cdot\frac{2\pi a}{T}\right)\sin 3\cdot\frac{2\pi t}{T} + \dots$$

Fourier Series

Let $f(t)$ be a periodic function with period T

$$f(t) = a_0 + a_1 \cos \frac{2\pi t}{T} + a_2 \cos 2 \cdot \frac{2\pi t}{T} + \dots b_1 \sin \frac{2\pi t}{T} + b_2 \sin 2 \cdot \frac{2\pi t}{T} + \dots$$

$$= a_0 + \sum_{k=1}^{\infty} \left(a_k \cos k \frac{2\pi t}{T} + b_k \sin k \frac{2\pi t}{T} \right)$$

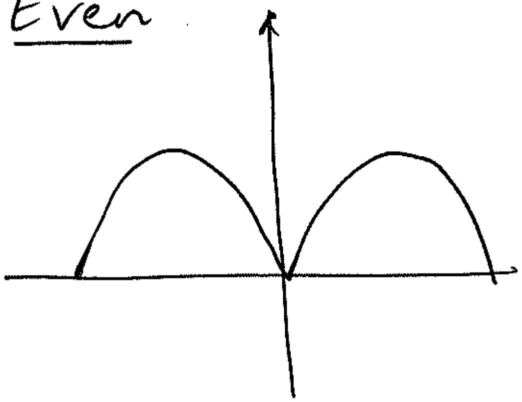
$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, dt$$

$$a_k = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos k \frac{2\pi t}{T}\, dt$$

$$b_k = \frac{2}{T} \int_{-\frac{T}{2}}^{T/2} f(t) \sin k \frac{2\pi t}{T}\, dt$$

5

Even



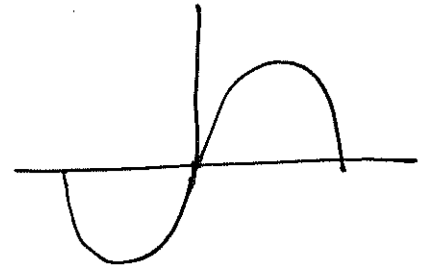
$f(t) = f(-t)$

$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt$$

$$a_k = \frac{4}{T}\int_0^{T/2} f(t)\cos k\frac{2\pi t}{T}\,dt$$

$$b_k = 0$$

Odd



$f(t) = -f(-t)$

$$a_0 = 0$$

$$a_k = 0$$

$$b_k = \frac{4}{T}\int_0^{T/2} f(t)\sin k\frac{2\pi t}{T}\,dt$$

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Fourier Series

Let $f(t)$ be a periodic function with period T

$$f(t) = a_0 + a_1 \cos\frac{2\pi t}{T} + a_2 \cos 2\cdot\frac{2\pi t}{T} + \dots \; b_1 \sin\frac{2\pi t}{T} + b_2 \sin 2\cdot\frac{2\pi t}{T} + \dots$$

$$= a_0 + \sum_{k=1}^{\infty}\left(a_k \cos k\frac{2\pi t}{T} + b_k \sin k\frac{2\pi t}{T}\right)$$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt$$

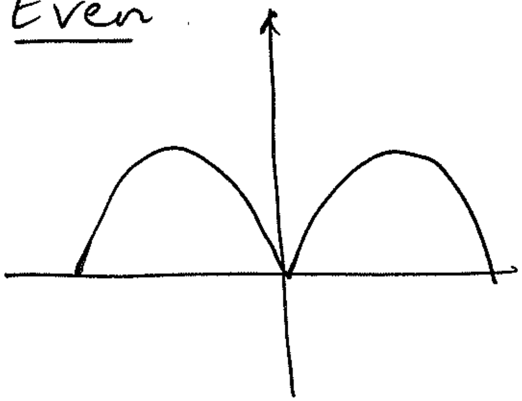
$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos k\frac{2\pi t}{T}\,dt$$

$$b_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin k\frac{2\pi t}{T}\,dt$$

7

②

Even



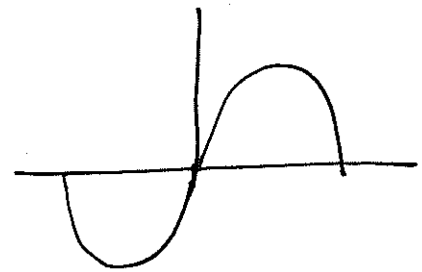
$f(t) = f(-t)$

$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt$$

$$a_k = \frac{4}{T}\int_0^{T/2} f(t)\cos k\frac{2\pi t}{T}\,dt$$

$$b_k = 0$$

Odd



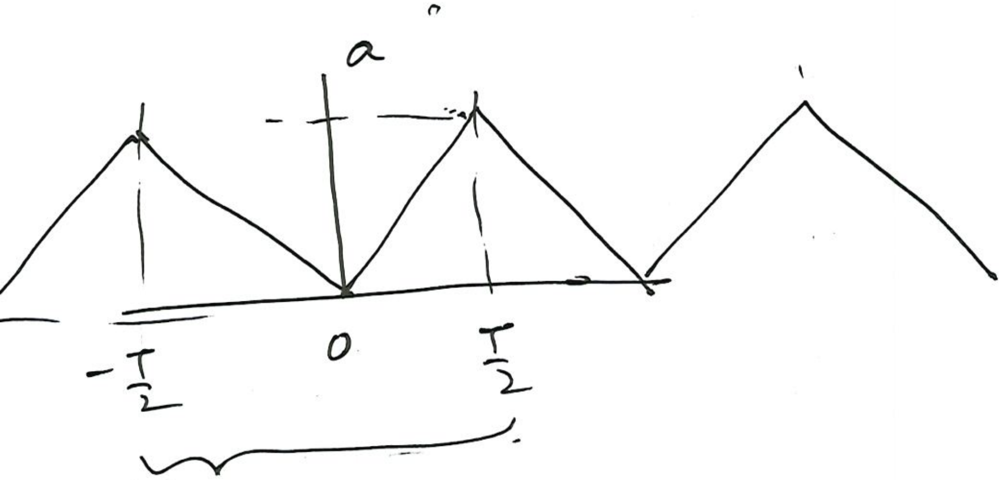
$f(t) = -f(-t)$

$$a_0 = 0$$

$$a_k = 0$$

$$b_k = \frac{4}{T}\int_0^{T/2} f(t)\sin k\frac{2\pi t}{T}\,dt$$

8



$$f(t) = \frac{2at}{T} \qquad 0 < t < \frac{T}{2}$$

$$= -\frac{2at}{T} \qquad -\frac{T}{2} < t \leq 0$$

$f(t)$ is even

$$\Rightarrow \quad b_k = 0$$

$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt = \frac{2}{T}\int_0^{T/2} \frac{2a}{T} t\,dt = \frac{4a}{T^2}\int_0^{T/2} t\,dt = \frac{4a}{T^2}\,\frac{t^2}{2}\Big|_0^{\frac{T}{2}}$$

$$= \frac{a}{2}$$

$$a_k = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos k\frac{2\pi t}{T}\,dt = \frac{4}{T}\int_0^{T/2} \frac{2at}{T}\cos k\frac{2\pi t}{T}\,dt$$

$$= \frac{8a}{T^2}\int_0^{T/2} \underbrace{t}_{u}\,\underbrace{\cos k\frac{2\pi t}{T}\,dt}_{dv} = \frac{8a}{T^2}\left[\frac{\sin k\frac{2\pi t}{T}}{\frac{2\pi k}{T}}\Bigg|_0^{T/2} - \frac{\int_0^{T/2}\sin k\frac{2\pi t}{T}\,dt}{\frac{2\pi k}{T}}\right]$$

$$= \frac{8a}{T^2}\left[\left(\frac{-T}{2\pi k}\right)\frac{-\cos k\frac{2\pi t}{T}\Big|_0^{T/2}}{\frac{2\pi k}{T}}\right]$$

$$= \frac{\overset{2}{\cancel{8}}a}{\cancel{T^2}}\,\frac{\cancel{T^2}}{\cancel{2^2}\pi^2 k^2}\left[\cos k\pi - 1\right]$$

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$$a_K = \frac{2a}{\pi^2 k^2}\left[\cos k\pi - 1\right]$$

$$f(t) = \frac{a}{2} + \sum_{k=1}^{\infty} \frac{2a}{\pi^2 k^2}(\cos k\pi - 1)\cos k\frac{2\pi t}{T}$$

$$= \frac{a}{2} +$$

$$a_1 = \frac{2a}{\pi^2}(\cos\pi - 1) = \frac{2a}{\pi^2}(-2) = \frac{-4a}{\pi^2}$$

$$a_2 = \frac{2a}{\pi^2(2)^2}(\cos 2\pi - 1) = 0$$

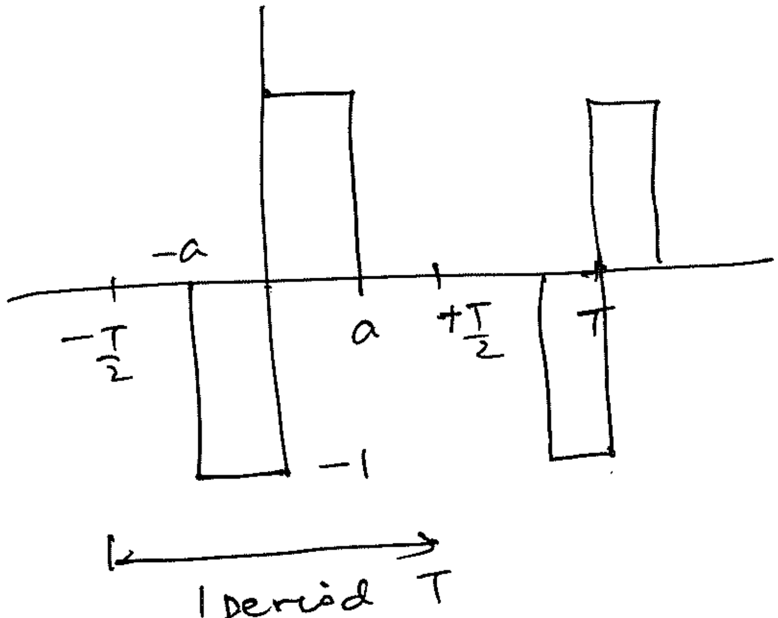
$$a_3 = \frac{2a}{\pi^2(3)^2}(\cos 3\pi - 1) = \frac{-4a}{9\pi^2}$$

$$a_4 = 0$$

$$a_5 = \frac{-4a}{25\pi^2}$$

$$f(t) = \frac{a}{2} - \frac{4a}{\pi^2}\cos\frac{2\pi t}{T} - \frac{4a}{9\pi^2}\cos 3\cdot\frac{2\pi t}{T} - \frac{4a}{25\pi^2}\cos 5\cdot\frac{2\pi t}{T} \cdots -$$

10



$$f(t) = 0 \quad \left(-\frac{T}{2} < t < -a\right)$$
$$= -1 \quad (-a < t < 0)$$
$$= 1 \quad (0 < t < a)$$
$$= 0 \quad \left(a < t < \frac{T}{2}\right)$$

$f(t)$ is add $\Rightarrow$ $a_0 = 0$

$a_k = 0$

$$b_k = \frac{4}{T}\int_0^{T/2} f(t) \sin k\frac{2\pi t}{T}\, dt = \frac{4}{T}\int_0^{a} (1) \sin k \frac{2\pi t}{T}\, dt + \frac{4}{T}\int_a^{\cancel{T/2}} 0 \cdot \sin k\frac{2\pi t}{T}\, dt$$

$$= \frac{\cancel{4}^{2}}{\cancel{T}} \frac{-\cos k\frac{2\pi t}{T}\Big|_0^{T/2}}{\frac{2\pi k}{\cancel{T}}}$$

$$= \frac{2}{\pi k}\left(1 - \cos k\pi\right)$$

$$f(t) = \sum_{k=1}^{\infty} \frac{2}{\pi k}(1-\cos k\pi) \sin k\frac{2\pi t}{T}$$

$$= \frac{2}{\pi} \cdot 2 \sin \frac{2\pi t}{T} + \frac{2}{3\pi} \cdot 2 \sin 3\frac{2\pi t}{T} + \frac{2}{5\pi}\, 2 \sin 5 \cdot \frac{2\pi t}{T} + \ldots$$

$$b_k = \frac{4}{T}\int_0^{T/2} f(t)\sin k\frac{2\pi t}{T}\,dt = \frac{4}{T}\left[\int_0^a (1)\sin k\frac{2\pi t}{T}\,dt + \cancel{\int_a^{T/2} 0\cdot\sin k\frac{2\pi t}{T}\,dt}\right]$$ 5

$$= \frac{\cancel{4}^{2}}{\cancel{T}}\left.\frac{-\cos k\frac{2\pi t}{T}}{\frac{k\cancel{2}\pi}{\cancel{T}}}\right|_0^a$$

$$= \frac{2}{k\pi}\left[1-\cos k\frac{2\pi a}{T}\right]$$

$$f(t) = \sum_{k=1}^{\infty}\frac{2}{k\pi}\left(1-\cos k\frac{2\pi a}{T}\right)\sin k\frac{2\pi t}{T}$$

$$= \frac{2}{\pi}\left(1-\cos\frac{2\pi a}{T}\right)\sin\frac{2\pi t}{T} + \frac{2}{2\pi}\left(1-\cos 2\frac{2\pi a}{T}\right)\sin 2\frac{2\pi t}{T}$$

$$+ \frac{2}{3\pi}\left(1-\cos 3\cdot\frac{2\pi a}{T}\right)\sin 3\cdot\frac{2\pi t}{T} + \cdots$$

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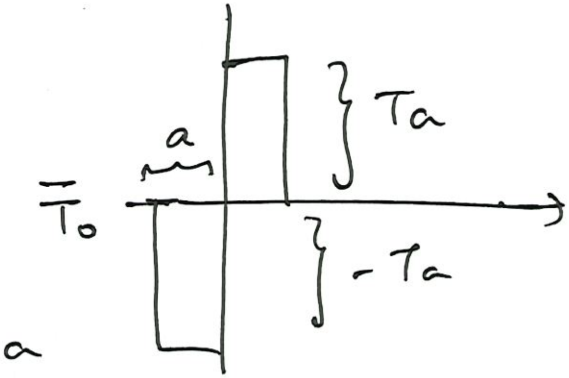
$$\frac{dT}{dt} + kT = kT_o(t)$$

where $T_o(t) = \cancel{\bar{T}_o + T_a \quad -0}$



$= \bar{T}_o \qquad -\frac{T}{2} < t < -a$

$= \bar{T}_o - T_a \qquad -a < t < 0$

$= \bar{T}_o + T_a \qquad 0 < t < a$

$= \bar{T}_o \qquad a < t < \frac{T}{2}$

~~Wke~~ We know. the solution to

$$\frac{dT}{dt} + kT = k\bar{T}_o + kT_a \sin \omega_o t$$

$$T(t) = \bar{T}_o + \frac{k}{\sqrt{k^2 + \omega_o^2}} T_a \sin(\omega_o t - \phi)$$

where $\phi = \tan^{-1} \frac{\omega_o}{k}$

Solution:

(steady state $t > 5/k$)

Fourier series of $T_o(t)$

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$$T_o(t) = \bar{T}_o + T_a \frac{2}{\pi}\left(1 - \cos \frac{2\pi a}{T}\right) \sin \frac{2\pi t}{T} + \frac{1}{\pi} T_a \left(1 - \cos 2 \cdot \frac{2\pi a}{T}\right) \sin 2 \cdot \frac{2\pi t}{T}$$

$$+ \frac{2}{3\pi} T_a \left(1 - \cos 3 \cdot \frac{2\pi a}{T}\right) \sin 3 \cdot \frac{2\pi t}{T} + \ldots$$

$$\frac{dT}{dt} + kT = k\,T_o(t)$$

$$= k$$

(7)

$$T_o(t) = \bar{T}_o$$

$$+ \; T_a \frac{2}{\pi}\left(1-\cos\frac{2\pi a}{T}\right)\sin\frac{2\pi t}{T}$$

$$+ \; \frac{1}{\pi}(T_a)\left(1-\cos 2\cdot\frac{2\pi a}{T}\right)\sin 2\cdot\frac{2\pi t}{T}$$

$$+ \; T_a \cdot \frac{2}{3\pi} T_a\left(1-\cos 3\cdot\frac{2\pi a}{T}\right)\sin 3\cdot\frac{2\pi t}{T}$$

$$T(t) = \bar{T}_o$$

$$+ \; \frac{k}{\sqrt{k^2+\left(\frac{2\pi}{T}\right)^2}}\frac{2T_a}{\pi}\left(1-\cos\frac{2\pi a}{T}\right)\sin\left(\frac{2\pi}{T}t-\phi_1\right) \qquad \phi_1 = \tan^{-1}\frac{2\pi}{Tk}$$

$$+ \; \frac{k}{\sqrt{k^2+\left(\frac{4\pi}{T}\right)^2}}\frac{T_a}{\pi}\left(1-\cos\frac{4\pi a}{T}\right)\sin\left(2\cdot\frac{2\pi t}{T}-\phi_2\right) \qquad \phi_2 = \tan^{-1}\frac{2\cdot 2\pi}{Tk}$$

$$+ \; \frac{k}{\sqrt{k^2+\left(\frac{6\pi}{T}\right)^2}}\frac{2T_a}{3\pi}\left(1-\cos 3\cdot\frac{2\pi a}{T}\right)\sin\left(3\cdot\frac{2\pi t}{T}-\phi_3\right)$$

$$\phi_3 = \tan^{-1}\frac{6\pi}{Tk}$$

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$$\frac{dT}{dt} + kT = k\bar{T}_0 + kT_a \sin\omega_0 t \rightarrow T(t) = \bar{T}_0 + \frac{k}{\sqrt{k^2+\omega_0^2}}\, T_a \sin(\omega_0 t - \phi)$$

$$= k\left(\bar{T}_0 + T_a \sin(\omega_0 t)\right)$$

$$\phi = \tan^{-1}\frac{\omega_0}{k}$$

$$T_0(t) = \bar{T}_0 + T_a\frac{2}{\pi}\left(1-\cos\frac{2\pi a}{T}\right)\sin\frac{2\pi t}{T} + \frac{1}{\pi}T_a\left(1-\cos 2\cdot\frac{2\pi a}{T}\right)\sin 2\cdot\frac{2\pi t}{T} + \frac{2}{3\pi}T_a\left(1-\cos 3\cdot\frac{2\pi a}{T}\right)\sin 3\cdot\frac{2\pi t}{T} + \cdots$$

$$\longrightarrow$$

$$T(t) = \bar{T}_0 + \frac{k}{\sqrt{k^2+\left(\frac{2\pi}{T}\right)^2}}\frac{2T_a}{\pi}\left(1-\cos\frac{2\pi a}{T}\right)\sin\left(\frac{2\pi}{T}t-\phi_1\right) \quad \phi_1 = \tan^{-1}\frac{2\pi}{Tk}$$

$$+ \frac{k}{\sqrt{k^2+\left(\frac{4\pi}{T}\right)^2}}\frac{T_a}{\pi}\left(1-\cos\frac{4\pi a}{T}\right)\sin\left(\frac{4\pi}{T}t-\phi_2\right) \quad \phi_2 = \tan^{-1}\frac{4\pi}{Tk}$$

$$+ \frac{k}{\sqrt{k^2+\left(\frac{6\pi}{T}\right)^2}}\frac{2T_a}{3\pi}\left(1-\cos 6\frac{\pi a}{T}\right)\sin\left(\frac{6\pi t}{T}-\phi_3\right) \quad \phi_3 = \tan^{-1}\frac{6\pi}{Tk}$$

$$+ \cdots$$