1) $\frac{dx}{dt} + kx = 0$ ← 1st order linear homogeneous ODE

Solution : $x(t) = \cancel{c}\, e^{-kt}$

2) $\boxed{\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0}$ ← 2nd order linear homogeneous ODE

let $x(t) = e^{\lambda t}$

Assume $x(t) = e^{\lambda t} \Rightarrow \dot{x}(t) = \lambda e^{\lambda t}$

$\Rightarrow \frac{dx}{dt} + kx = 0$

$\lambda \cancel{e^{\lambda t}} + k \cancel{e^{\lambda t}} = 0 \Rightarrow \lambda = -k \quad \therefore x(t) = e^{-kt}$

(3) $\frac{d^3x}{dt} + a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$ Assume $x(t) = e^{\lambda t}$

$\frac{d}{dt}(e^{\lambda t}) = \lambda e^{\lambda t}$

$D \triangleq \frac{d}{dt}$ ← derivative operator

$D(\underbrace{e^{\lambda t}}) = \lambda \underbrace{e^{\lambda t}}$

1) $\frac{dx}{dt} + kx = 0$ ← 1st order linear homogenous ODE

Solution : $x(t) = \cancel{c}\, e^{-kt}$

2) $\boxed{\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0}$ ← 2nd order linear homogenous ODE

Let $x(t) = e^{\lambda t}$

$\Rightarrow \frac{dx}{dt} + kx = 0$ Assume $x(t) = e^{\lambda t} \Rightarrow \dot{x}(t) = \lambda e^{\lambda t}$

$\lambda \cancel{e^{\lambda t}} + k\cancel{e^{\lambda t}} = 0 \Rightarrow \lambda = -k \quad \therefore \quad x(t) = e^{-kt}$

(3) $\frac{d^3x}{dt} + a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$ Assume $x(t) = e^{\lambda t}$

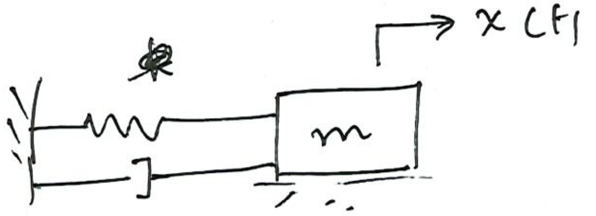
$\underbrace{\frac{d}{dt}}(e^{\lambda t}) = \lambda e^{\lambda t}$

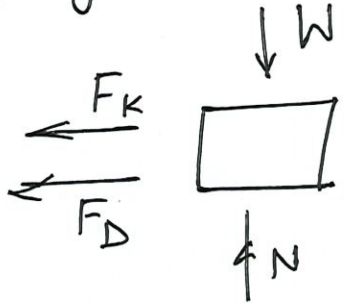
$D \triangleq \frac{d}{dt}$ ← derivative operator

$D(\underbrace{e^{\lambda t}}) = \lambda \underbrace{e^{\lambda t}}$

1

Mass - damper - spring





$$\sum F_y = m \cancel{a_y}^{\,0}$$

$$N - W = 0 \Rightarrow N = W$$

$$\overset{+}{\rightarrow} \sum F_x = m a_x$$

$$m a_x = -F_K - F_D$$

F_K : $\boxed{F_K = -kx}$ $(F_K \sim x)$ ← Hooke's law / linear spring

$= k_1 x + k_2 x^3$ Hardening spring

F_D : Coulomb friction $F \sim N \Rightarrow F = \mu_k N$

Viscous damping $F \sim v \Rightarrow \boxed{F = cv = c\dot{x}}$ (linear damping)

Quadratic damping $F \sim v^2 \Rightarrow F = b\dot{x}^2$ (drag)

Assume linear spring & linear damping.

$$m\ddot{x} = -kx - c\dot{x} \Rightarrow m\ddot{x} + c\dot{x} + kx = 0, \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0$$

$m\ddot{x} + c\dot{x} + kx = 0$; $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$

Let $x(t) = e^{\lambda t} \Rightarrow \dot{x}(t) = \lambda e^{\lambda t} \Rightarrow \ddot{x}(t) = \lambda^2 e^{\lambda t}$

$m\lambda^2 \cancel{e^{\lambda t}} + c\lambda \cancel{e^{\lambda t}} + k \cancel{e^{\lambda t}} = 0$ e

$\Rightarrow m\lambda^2 + c\lambda + k = 0$

$$\Rightarrow \lambda = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} = -\frac{c}{2m} \pm \underbrace{\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}}$$

$$= -\frac{c}{2m} \pm \sqrt{\frac{c^2}{(2m)^2} - \frac{\cancel{4}km}{\cancel{4}m^2}}$$

<u>Case I</u> $\left(\frac{c}{2m}\right)^2 - \frac{k}{m} > 0 \Rightarrow \frac{c^2}{4m^2} > \frac{k}{m}$.

<u>Case II</u> $\left(\frac{c}{2m}\right)^2 - \frac{k}{m} < 0 \Rightarrow \frac{c^2}{4m^2} < \frac{k}{m}$.

<u>Case III</u> $\left(\frac{c}{2m}\right)^2 = \frac{k}{m}$

Case I : $\left(\frac{c}{2m}\right)^2 - \frac{k}{m} > 0 \Rightarrow$ 2 real roots.

$$\lambda_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \; ; \qquad \lambda_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

We have 2 solutions. $x(t) = e^{\lambda_1 t}$; $x(t) = e^{\lambda_2 t}$

general Solution $x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \Rightarrow \dot{x}(t) = \lambda_1 c_1 e^{\lambda_1 t} + \lambda_2 c_2 e^{\lambda_2 t}$

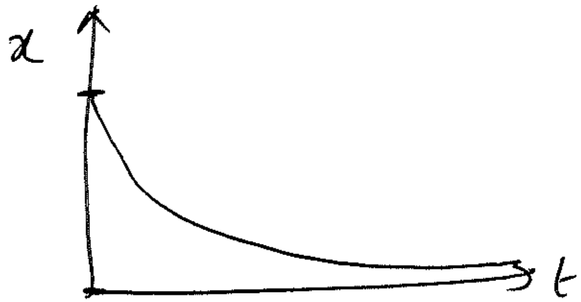
IC. $x(0) = x_0 = c_1 + c_2$

$\dot{x}(0) = \dot{x}_0 = \lambda_1 c_1 + \lambda_2 c_2$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix}$$

$$c_1 = \frac{\begin{vmatrix} x_0 & 1 \\ \dot{x}_0 & \lambda_2 \end{vmatrix}}{\lambda_2 - \lambda_1} = \frac{x_0 \lambda_2 - \dot{x}_0}{\lambda_2 - \lambda_1} \qquad c_2 = \frac{\begin{vmatrix} 1 & x_0 \\ \lambda_1 & \dot{x}_0 \end{vmatrix}}{\lambda_2 - \lambda_1} = \frac{\dot{x}_0 - \lambda_1 x_0}{\lambda_2 - \lambda_1}$$

$$x(t) = \left(\frac{x_0 \lambda_2 - \dot{x}_0}{\lambda_2 - \lambda_1}\right) e^{\left(-\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right)t} + \left(\frac{\dot{x}_0 - \lambda_1 x_0}{\lambda_2 - \lambda_1}\right) e^{\left(-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right)t}$$



$\boxed{\text{OVERDAMPED}}$

4

$\underline{\text{Case II}}$: $\left(\frac{c}{2m}\right)^2 - \frac{k}{m} < 0 \leftarrow \boxed{\text{UNDERDAMPED}}$

$$\lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = -\frac{c}{2m} \pm \sqrt{(-1)\left(\frac{k}{m} - \left(\frac{c}{2m}\right)^2\right)}$$

$$= -\frac{c}{2m} \pm j\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$$

Define $\omega_d \triangleq \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$

$$\lambda_1 = -\frac{c}{2m} + j\omega_d \quad ; \quad \lambda_2 = -\frac{c}{2m} - j\omega_d .$$

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{+\lambda_2 t} = c_1 e^{-\frac{c}{2m}t} \cdot e^{j\omega_d t} + c_2 e^{-\frac{ct}{2m}} \cdot e^{-j\omega_d t}$$

$$= e^{-\frac{c}{2m}t}\left[\cancel{c_1 \cos}\; c_1 e^{j\omega_d t} + c_2 e^{-j\omega_d t}\right]$$

$$= e^{-\frac{c}{2m}t}\left[c_1 \cos\omega_d t + jc_1 \sin\omega_d t + c_2 \cos\omega_d t - jc_2 \sin\omega_d t\right]$$

$$= e^{-\frac{c}{2m}t}\left[(c_1 + c_2)\cos\omega_d t + (jc_1 - jc_2)\sin\omega_d t\right]$$

5

Let $A \triangleq c_1 + c_2$; $B \triangleq jc_1 - jc_2$

$$x(t) = e^{-\frac{c}{2m}t}\left[A\cos\omega_d t + B\sin\omega_d t\right]$$

$$\ldots B\omega_d \cos\omega_d t) - \frac{c}{2m} e^{-\frac{c}{2m}t}\left[A\cos\omega_d t + B\sin\omega_d t\right]$$

1) $\frac{dx}{dt} + kx = 0$ ← 1st order linear homogenous ODE

Solution : $x(t) = \cancel{c}\, e^{-kt}$

2) $\boxed{\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0}$ ← 2nd order linear homogenous ODE

Let $x(t) = e^{\lambda t}$

$\Rightarrow \frac{dx}{dt} + kx = 0$ Assume $x(t) = e^{\lambda t} \Rightarrow \dot{x}(t) = \lambda e^{\lambda t}$

$\lambda \cancel{e^{\lambda t}} + k \cancel{e^{\lambda t}} = 0 \Rightarrow \lambda = -k \quad \therefore x(t) = e^{-kt}$

(3) $\frac{d^3x}{dt} + a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$ Assume $x(t) = e^{\lambda t}$

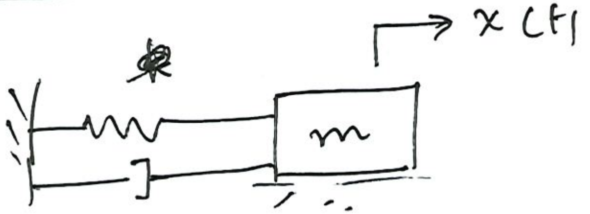
$\underbrace{\frac{d}{dt}}(e^{\lambda t}) = \lambda e^{\lambda t}$

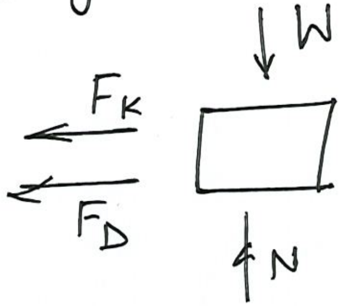
$D \triangleq \frac{d}{dt}$ ← derivative operator

$D(\underbrace{e^{\lambda t}}) = \lambda \underbrace{e^{\lambda t}}$

6

Mass - damper - spring





$$\sum F_y = m\cancel{a_y}^{\,0}$$

$$N - W = 0 \Rightarrow N = W$$

$$\overset{+}{\rightarrow} \sum F_x = m a_x$$

$$m a_x = -F_K - F_D$$

F_K : $\boxed{F_K = kx}$ $(F_K \sim x)$ ← Hooke's law / linear spring

$= k_1 x + k_2 x^3$ Hardening spring

F_D : Coulomb friction $F \sim N \Rightarrow F = \mu_K N$

Viscous damping $F \sim v \Rightarrow \boxed{\begin{aligned} F &= cv \\ &= c\dot{x} \end{aligned}}$ (linear damping)

Quadratic damping $F \sim v^2 \Rightarrow F = b\dot{x}^2$ (drag)

Assume linear spring & linear damping.

$$m\ddot{x} = -kx - c\dot{x} \Rightarrow m\ddot{x} + c\dot{x} + kx = 0, \quad \begin{aligned} x(0) &= x_0 \\ \dot{x}(0) &= \dot{x}_0 \end{aligned}$$

$m\ddot{x} + c\dot{x} + kx = 0$; $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$

Let $x(t) = e^{\lambda t} \Rightarrow \dot{x}(t) = \lambda e^{\lambda t} \Rightarrow \ddot{x}(t) = \lambda^2 e^{\lambda t}$

$m\lambda^2 \cancel{e^{\lambda t}} + c\lambda e^{\lambda t} + k e^{\lambda t} = 0$

$\Rightarrow m\lambda^2 + c\lambda + k = 0$

$$\Rightarrow \lambda = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} = -\frac{c}{2m} \pm \underbrace{\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}}$$

$$= -\frac{c}{2m} \pm \sqrt{\frac{c^2}{(2m)^2} - \frac{\cancel{4}km}{\cancel{4}m^2}}$$

<u>Case I</u> $\left(\frac{c}{2m}\right)^2 - \frac{k}{m} > 0 \Rightarrow \frac{c^2}{4m^2} > \frac{k}{m}$.

<u>Case II</u> $\left(\frac{c}{2m}\right)^2 - \frac{k}{m} < 0 \Rightarrow \frac{c^2}{4m^2} < \frac{k}{m}$.

<u>Case III</u> $\left(\frac{c}{2m}\right)^2 = \frac{k}{m}$

8

Case i: $\left(\frac{c}{2m}\right)^2 - \frac{k}{m} > 0 \Rightarrow$ 2 real roots.

$$\lambda_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \; ; \qquad \lambda_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

We have 2 solutions. $x(t) = e^{\lambda_1 t}$; $x(t) = e^{\lambda_2 t}$

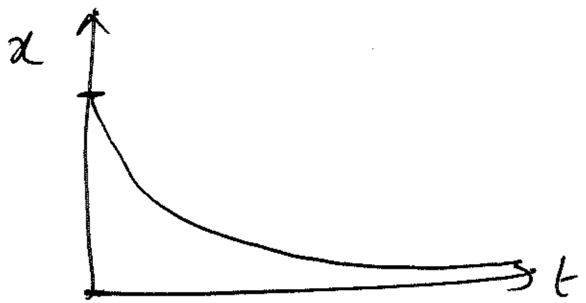
general Solution $x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \Rightarrow \dot{x}(t) = \lambda_1 c_1 e^{\lambda_1 t} + \lambda_2 c_2 e^{\lambda_2 t}$

IC. $x(0) = x_0 = c_1 + c_2$

$\dot{x}(0) = \dot{x}_0 = \lambda_1 c_1 + \lambda_2 c_2 \Rightarrow \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix}$

$$c_1 = \frac{\begin{vmatrix} x_0 & 1 \\ \dot{x}_0 & \lambda_2 \end{vmatrix}}{\lambda_2 - \lambda_1} = \frac{x_0 \lambda_2 - \dot{x}_0}{\lambda_2 - \lambda_1} \qquad c_2 = \frac{\begin{vmatrix} 1 & x_0 \\ \lambda_1 & \dot{x}_0 \end{vmatrix}}{\lambda_2 - \lambda_1} = \frac{\dot{x}_0 - \lambda_1 x_0}{\lambda_2 - \lambda_1}$$

$$x(t) = \left(\frac{x_0 \lambda_2 - \dot{x}_0}{\lambda_2 - \lambda_1}\right) e^{\left(-\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right)t} + \left(\frac{\dot{x}_0 - \lambda_1 x_0}{\lambda_2 - \lambda_1}\right) e^{\left(-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right)t}$$



$\boxed{\text{OVERDAMPED}}$

9

Case II: $\left(\frac{c}{2m}\right)^2 - \frac{k}{m} < 0 \leftarrow$ $\boxed{\text{UNDERDAMPED}}$

$$\lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = -\frac{c}{2m} \pm \sqrt{(-1)\left(\frac{k}{m} - \left(\frac{c}{2m}\right)^2\right)}$$

$$= -\frac{c}{2m} \pm j\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$$

Define $\omega_d \triangleq \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$

$$\lambda_1 = -\frac{c}{2m} + j\omega_d \quad ; \quad \lambda_2 = -\frac{c}{2m} - j\omega_d.$$

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{+\lambda_2 t} = c_1 e^{-\frac{c}{2m}t} \cdot e^{j\omega_d t} + c_2 e^{-\frac{ct}{2m}} \cdot e^{-j\omega_d t}$$

$$= e^{-\frac{c}{2m}t}\left[\cancel{c_1 \cos}\; c_1 e^{j\omega_d t} + c_2 e^{-j\omega_d t}\right]$$

$$= e^{-\frac{c}{2m}t}\left[c_1 \cos\omega_d t + jc_1 \sin\omega_d t + c_2 \cos\omega_d t - jc_2 \sin\omega_d t\right]$$

$$= e^{-\frac{c}{2m}t}\left[(c_1+c_2)\cos\omega_d t + (jc_1 - jc_2)\sin\omega_d t\right]$$

10

Let $A \triangleq c_1 + c_2$; $B \triangleq jc_1 - jc_2$

$$x(t) = e^{-\frac{c}{2m}t}\left[A\cos\omega_d t + B\sin\omega_d t\right]$$

$$\ldots \cos\omega_d t) - \frac{c}{2m} e^{-\frac{c}{2m}t}\left[A\cos\omega_d t + B\sin\omega_d t\right]$$

$$x(0) = x_0 = A$$

$$\dot{x}(0) = B\omega_d - \frac{c}{2m}A = \dot{x}_0 \Rightarrow B\omega_d = \dot{x}_0 - \frac{c}{2m}x_0$$

$$\Rightarrow B = \left(\frac{\dot{x}_0}{\omega_d} - \frac{c\,x_0}{2m\,\omega_d}\right)$$

$$x(t) = e^{-\frac{c}{2m}t}\Big[\underbrace{x_0}_{\Downarrow\; P\cos\phi}\cos\omega_d t + \underbrace{\left(\frac{\dot{x}_0}{\omega_d} - \frac{c}{2m}\frac{x_0}{\omega_d}\right)}_{P\sin\phi}\sin\omega_d t\Big]$$

$$P\cos\phi = x_0$$

$$P\sin\phi = \frac{\left(\dot{x}_0 - \frac{c}{2m}x_0\right)}{\omega_d}$$

$$P = \sqrt{x_0^2 + \frac{\left(\dot{x}_0 - \frac{c}{2m}x_0\right)^2}{\omega_d^2}} \qquad \phi = \tan^{-1}\left(\frac{\dot{x}_0 - \frac{c}{2m}x_0}{\omega_d}\right)\frac{1}{x_0}$$

$$\boxed{x(t) = e^{-\frac{c}{2m}t}\,P\cos(\omega_d t - \phi)}$$

11

1) $\frac{dx}{dt} + kx = 0$ ← 1st order linear homogeneous ODE

Solution : $x(t) = e^{-kt}$

2) $\boxed{\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0}$ ← 2nd order linear homogeneous ODE

Let $x(t) = e^{\lambda t}$

$\Rightarrow \frac{dx}{dt} + kx = 0$ Assume $x(t) = e^{\lambda t} \Rightarrow \dot{x}(t) = \lambda e^{\lambda t}$

$\lambda \cancel{e^{\lambda t}} + k\cancel{e^{\lambda t}} = 0 \Rightarrow \lambda = -k \quad \therefore x(t) = e^{-kt}$

(3) $\frac{d^3x}{dt} + a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$ Assume $x(t) = e^{\lambda t}$

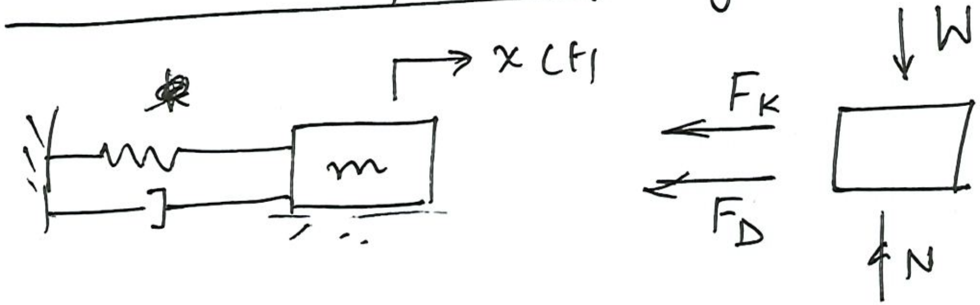
$\frac{d}{dt}(e^{\lambda t}) = \lambda e^{\lambda t}$

$D \triangleq \frac{d}{dt}$ ← derivative operator

$D(\underbrace{e^{\lambda t}}) = \lambda \underbrace{e^{\lambda t}}$

12

Mass - damper - spring



$\overset{+}{\rightarrow} \sum F_x = m a_x$

$\sum F_y = m \cancel{a_y}^{0}$

$N - W = 0 \Rightarrow N = W$

$m a_x = -F_K - F_D$

F_K : $\boxed{F_K = -kx}$ $(F_K \sim x) \Leftarrow$ Hooke's law / Linear spring

$= k_1 x + k_2 x^3$ Hardening spring

F_D : Coulomb friction $F \sim N \Rightarrow F = \mu_k N$

Viscous damping $F \sim v \Rightarrow \boxed{F = cv = c\dot{x}}$ (linear damping)

Quadratic damping $F \sim v^2 \Rightarrow F = b\dot{x}^2$ (drag)

Assume linear spring & linear damping.

$m\ddot{x} = -kx - c\dot{x} \Rightarrow m\ddot{x} + c\dot{x} + kx = 0$, $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$

$$m\ddot{x} + c\dot{x} + kx = 0 \quad ; \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0$$

Let $x(t) = e^{\lambda t} \Rightarrow \dot{x}(t) = \lambda e^{\lambda t} \Rightarrow \ddot{x}(t) = \lambda^2 e^{\lambda t}$

$$m\lambda^2 \cancel{e^{\lambda t}} + c\lambda e^{\lambda t} + k e^{\lambda t} = 0 \qquad \epsilon$$

$$\Rightarrow m\lambda^2 + c\lambda + k = 0$$

$$\Rightarrow \lambda = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} = -\frac{c}{2m} \pm \underbrace{\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}}$$

$$= -\frac{c}{2m} \pm \sqrt{\frac{c^2}{(2m)^2} - \frac{\cancel{4}km}{\cancel{4}m^2}}$$

Case I $\quad \left(\frac{c}{2m}\right)^2 - \frac{k}{m} > 0 \Rightarrow \frac{c^2}{4m^2} > \frac{k}{m}.$

Case II $\quad \left(\frac{c}{2m}\right)^2 - \frac{k}{m} < 0 \Rightarrow \frac{c^2}{4m^2} < \frac{k}{m}.$

Case III $\quad \left(\frac{c}{2m}\right)^2 = \frac{k}{m}$

14

<u>Case i</u> : $\left(\frac{c}{2m}\right)^2 - \frac{k}{m} > 0 \Rightarrow$ 2 real roots.

$$\lambda_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \; ; \qquad \lambda_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

We have 2 solutions. $x(t) = e^{\lambda_1 t}$; $x(t) = e^{\lambda_2 t}$

<u>general Solution</u> $x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \Rightarrow \dot{x}(t) = \lambda_1 c_1 e^{\lambda_1 t} + \lambda_2 c_2 e^{\lambda_2 t}$

<u>IC.</u> $x(0) = x_0 = c_1 + c_2$

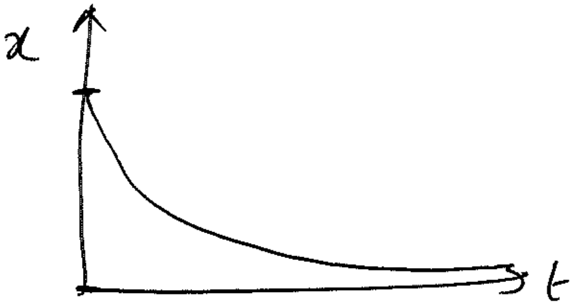
$\dot{x}(0) = \dot{x}_0 = \lambda_1 c_1 + \lambda_2 c_2$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix}$$

$$c_1 = \frac{\begin{vmatrix} x_0 & 1 \\ \dot{x}_0 & \lambda_2 \end{vmatrix}}{\lambda_2 - \lambda_1} = \frac{x_0 \lambda_2 - \dot{x}_0}{\lambda_2 - \lambda_1} \qquad c_2 = \frac{\begin{vmatrix} 1 & x_0 \\ \lambda_1 & \dot{x}_0 \end{vmatrix}}{\lambda_2 - \lambda_1} = \frac{\dot{x}_0 - \lambda_1 x_0}{\lambda_2 - \lambda_1}$$

$$x(t) = \left(\frac{x_0 \lambda_2 - \dot{x}_0}{\lambda_2 - \lambda_1}\right) e^{\left(-\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right)t} + \left(\frac{\dot{x}_0 - \lambda_1 x_0}{\lambda_2 - \lambda_1}\right) e^{\left(-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right)t}$$

15



$\boxed{\text{OVERDAMPED}}$

Case II: $\left(\frac{c}{2m}\right)^2 - \frac{k}{m} < 0 \leftarrow$ $\boxed{\text{UNDERDAMPED}}$

$$\lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = -\frac{c}{2m} \pm \sqrt{(-1)\left(\frac{k}{m} - \left(\frac{c}{2m}\right)^2\right)}$$

$$= -\frac{c}{2m} \pm j\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$$

Define $\omega_d \triangleq \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$

$$\lambda_1 = -\frac{c}{2m} + j\omega_d \quad ; \quad \lambda_2 = -\frac{c}{2m} - j\omega_d .$$

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{+\lambda_2 t} = c_1 e^{-\frac{c}{2m}t} \cdot e^{j\omega_d t} + c_2 e^{-\frac{ct}{2m}} \cdot e^{-j\omega_d t}$$

$$= e^{-\frac{c}{2m}t}\left[\cancel{c_1 \cos}\; c_1 e^{j\omega_d t} + c_2 e^{-j\omega_d t} \right]$$

$$= e^{-\frac{c}{2m}t}\left[c_1 \cos\omega_d t + jc_1 \sin\omega_d t + c_2 \cos\omega_d t - jc_2 \sin\omega_d t \right]$$

$$= e^{-\frac{c}{2m}t}\left[(c_1 + c_2)\cos\omega_d t + (jc_1 - jc_2)\sin\omega_d t \right]$$

Let $A \triangleq c_1 + c_2$; $B \triangleq jc_1 - jc_2$

$$x(t) = e^{-\frac{c}{2m}t}\left[A\cos\omega_d t + B\sin\omega_d t \right]$$

$$\dot{x}(t) = e^{-\frac{c}{2m}t}\left(-A\omega_d \sin\omega_d t + B\omega_d \cos\omega_d t\right) - \frac{c}{2m} e^{-\frac{c}{2m}t}\left[A\cos\omega_d t + B\sin\omega_d t\right]$$

16

$$x(0) = x_0 = A$$

$$\dot{x}(0) = B\omega_d - \frac{c}{2m} A = \dot{x}_0 \Rightarrow B\omega_d = \dot{x}_0 - \frac{c}{2m} x_0$$

$$\Rightarrow B = \left(\frac{\dot{x}_0}{\omega_d} - \frac{c\, x_0}{2m\omega_d} \right)$$

$$x(t) = e^{-\frac{c}{2m}t} \left[\underbrace{x_0}_{P\cos\phi} \cos\omega_d t + \underbrace{\left(\frac{\dot{x}_0}{\omega_d} - \frac{c}{2m}\frac{x_0}{\omega_d} \right)}_{P\sin\phi} \sin\omega_d t \right]$$

$$P\cos\phi = x_0$$

$$P\sin\phi = \frac{(\dot{x}_0 - c/2m\, x_0)}{\omega_d}$$

$$P = \sqrt{x_0^2 + \frac{(\dot{x}_0 - c/2m\, x_0)^2}{\omega_d^2}} \qquad \phi = \tan^{-1}\left(\frac{\dot{x}_0 - c/2m\, x_0}{\omega_d} \right) \frac{1}{x_0}$$

$$\boxed{x(t) = e^{-c/2m\, t} P \cos(\omega_d t - \phi)}$$

17

Case III $\quad \frac{c^2}{4m^2} = \frac{k}{m} \leftarrow$ CRITICALLY DAMPED

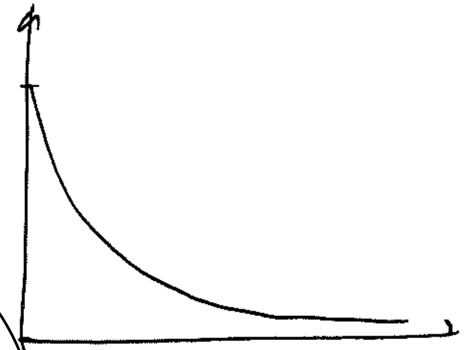
$$\cancel{\lambda_1} \quad \lambda = -\frac{c}{2m} \pm \sqrt{\cancelto{0}{\frac{c^2}{4m^2} - \frac{k}{m}}} \Rightarrow \lambda_1, \lambda_2 = -\frac{c}{2m} \leftarrow \text{Repeated roots.}$$

General Solution $\quad x(t) = (c_1 + c_2 t)\, e^{-\frac{c}{2m}t} \Rightarrow \dot{x}(t) = c_2 e^{-c/2m\, t} - \frac{c}{2m} e^{-c/2m\, t}(c_1 + c_2 t)$

$$x(0) = x_0 = c_1$$

$$\dot{x}(0) = \dot{x}_0 = c_2 - \frac{c}{2m} c_1 \Rightarrow c_2 = \dot{x}_0 + \frac{c}{2m} x_0$$

$$x(t) = \left(x_0 + \left(\dot{x}_0 + \frac{c}{2m} x_0\right) t\right) e^{-\frac{c}{2m}t}.$$



$\ddot{x} + 5\dot{x} + 4x = 0 \qquad x(0) = 1; \quad \dot{x}(0) = 0$

Let $x = e^{\lambda t} \Rightarrow \dot{x}(t) = \lambda e^{\lambda t} \Rightarrow \ddot{x}(t) = \lambda^2 e^{\lambda t}$

$\lambda^2 + 5\lambda + 4 = 0 \Rightarrow \lambda^2 + 4\lambda + \lambda + 4 = 0 \Rightarrow (\lambda+4)(\lambda+1) = 0$

$\Rightarrow \underbrace{\lambda = -4, \quad \lambda = -1}_{\text{Real roots.}}$

$x(t) = c_1 e^{-t} + c_2 e^{-4t}$ ← general solution.

$\cancel{\dot{x}(t) =} \quad \dot{x}(t) = -c_1 e^{-t} - 4c_2 e^{-4t}$

$x(0) = 1 = c_1 + c_2$

$\dot{x}(0) = 0 = -c_1 - 4c_2$

$\Rightarrow \cancel{c_4} \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$c_1 = \dfrac{\begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix}}{3} = +\dfrac{4}{3} \qquad c_2 = \dfrac{\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix}}{3} = +\dfrac{1}{3}$

$$\boxed{x(t) = \frac{4}{3} e^{-t} + \frac{1}{3} e^{-4t}}$$

$$\ddot{x} + 5\underset{x_2}{\underset{\downarrow}{\dot{x}}} + 4\underset{x_1}{\underset{\downarrow}{x}} = 0$$

$$x_2 = \dot{x} = \dot{x}_1 \Rightarrow \dot{x}_1 = x_2$$

Since $x_2 = \dot{x} \Rightarrow \dot{x}_2 = \ddot{x}$

$$\dot{x}_2 + 5x_2 + 4x_1 = 0 \Rightarrow \dot{x}_2 = -4x_1 - 5x_2$$

$$\boxed{\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -4x_1 - 5x_2\end{aligned}} \Rightarrow \begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -4 & -5\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$$

Let
$$\left.\begin{aligned} x_1 &= X_1 e^{\lambda t} \\ x_2 &= X_2 e^{\lambda t}\end{aligned}\right\}$$
We need to solve for λ, X_1 and X_2

$$\dot{x}_1 = X_1 \lambda e^{\lambda t} \; ; \quad \dot{x}_2 = X_2 \lambda e^{\lambda t}$$

$$\begin{bmatrix}x_1 \\ x_2\end{bmatrix} = \begin{bmatrix}X_1 \\ X_2\end{bmatrix} e^{\lambda t} \; ; \quad \begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}X_1 \\ X_2\end{bmatrix} \lambda e^{\lambda t}$$

$$\lambda \cancel{e^{\lambda t}} \begin{bmatrix}X_1 \\ X_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -4 & -5\end{bmatrix}\begin{bmatrix}X_1 \\ X_2\end{bmatrix} \cancel{e^{\lambda t}}$$

$$\lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \cancel{\left(\begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} - \{\lambda\}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$ax = 0$$

$$\frac{a}{a}x = \frac{0}{a}$$

$$\left(\begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\lambda & 1 \\ -4 & -5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to be ~~non trivial~~ non-trivial

$$\begin{vmatrix} -\lambda & 1 \\ -4 & -5-\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(-5-\lambda) + 4 = 0$$

$$\Rightarrow \lambda^2 + 5\lambda + 4 = 0 \leftarrow \text{characteristic polynomial}$$

$$\Rightarrow (\lambda + 4)(\lambda + 1) = 0$$

$$\Rightarrow \lambda = -4, \lambda = -1 \leftarrow \text{Eigenvalues of matrix } A$$

21

$\underline{\lambda = -1}$ $\quad \begin{bmatrix} 1 & 1 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 + x_2 = 0$

Let $x_1 = 1 \Rightarrow x_2 = -1$

$\underline{\text{Solution}}$: $\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$

$\underline{\lambda = -4}$ $\quad \begin{bmatrix} +4 & 1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 4x_1 + x_2 = 0$

$\Rightarrow$ Let $x_1 = 1 \Rightarrow x_2 = -4.$

Solution : $\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t}$

$\underline{\text{general Solution}}$ $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t}$

$\underline{\text{Initial conditions}}$: $\quad x(0) = x_0 \Rightarrow x_1(0) = \overset{=1}{x_0} \Rightarrow \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix}$

$\dot{x}(0) = \dot{x}_0 \qquad x_2(0) = \overset{=0}{\dot{x}_0}$

22

$\begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$c_1 = 4/3 \qquad c_2 = +1/3$

$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{4}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \frac{1}{3} \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} \Rightarrow x_1(t) = \frac{4}{3} e^{-t} + \frac{1}{3} e^{-4t}$

$$A = \begin{bmatrix} 0 & 1 \\ -4 & 5 \end{bmatrix} = V \Lambda V^{-1}$$

$$= \cancel{\tfrac{1}{3}} \begin{bmatrix} 1 & 1 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ -1 & -1 \end{bmatrix} \frac{1}{3}$$ ← Diagonal Decomposition of A.

$$V^{-1} = -\frac{1}{3} \begin{bmatrix} -4 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4/3 & 1/3 \\ -1/3 & -1/3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 4 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\dot{x} = Ax$$

Let $x = Vy$

$$V\dot{y} = AVy \Rightarrow \dot{y} = (V^{-1} A V) y = \Lambda y$$

if $A = V \Lambda V^{-1} \Rightarrow \Lambda = V^{-1} A V$

$$V^{-1} A V = \underbrace{V^{-1} V}_{I} \Lambda \underbrace{V^{-1} V}_{I}$$

23

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow \begin{array}{l} \dot{y}_1 = -y_1 \Rightarrow y_1 = c_1 e^{-t} \\ \dot{y}_2 = -4y_2 \Rightarrow y_2 = c_2 e^{-4t} \end{array}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{-4t} \end{bmatrix}$$