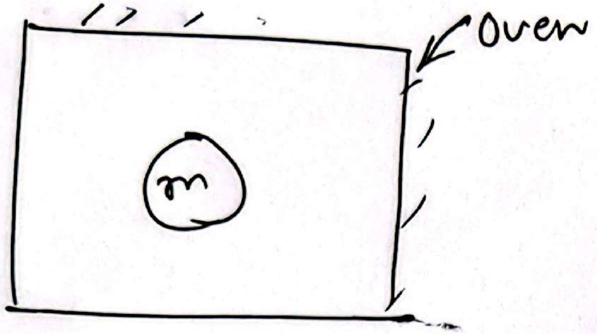
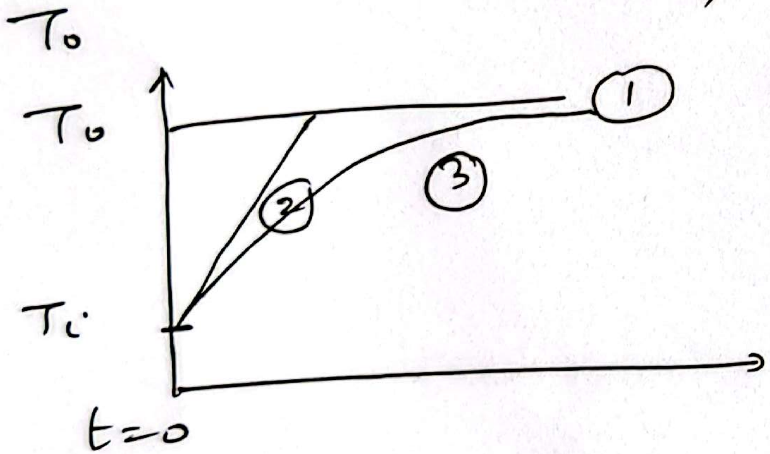


$$\frac{dT}{dt} + kT = kT_o$$

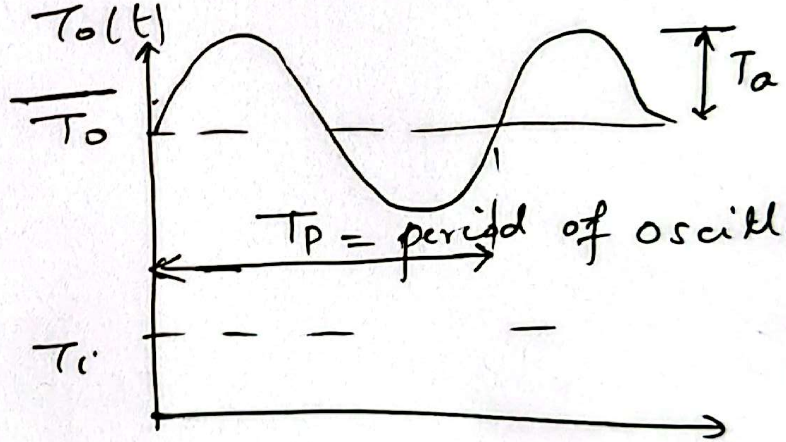
1) T_o constant

2) T_o linearly increasing

3) Exponential function.



4) Sinusoidal T_o



$$T_o(t) = \bar{T}_o + T_a \sin \frac{2\pi}{T_p} t$$

$$\omega_o \triangleq \frac{2\pi}{T_p}$$

$$T_o(t) = \bar{T}_o + T_a \sin \omega_o t$$

$$\frac{dT}{dt} + kT = k(\overline{T}_0 + T_a \sin \omega_0 t) \qquad T(0) = T_i$$

Integrating factors

$$e^{kt}\frac{dT}{dt} + ke^{kt}T = ke^{kt}\overline{T}_0 + kT_a \sin\omega_0 t\, e^{kt}$$

$$\int \frac{d}{dt}(Te^{kt}) = \int_0^t k\overline{T}_0 e^{kt}dt + \int_0^t kT_a e^{kt}\sin\omega_0 t\, dt$$

$$T(t)e^{kt} - T(0)e^{k0} = \cancel{k}\overline{T}_0 \frac{e^{kt}}{\cancel{k}}\Big|_0^t + kT_a \underbrace{\int_0^t e^{kt}\sin\omega_0 t\, dt}_{I}$$

$$T(t)e^{kt} - T_i = \overline{T}_0(e^{kt}-1) + kT_a I$$

$$I = \int_0^t \underbrace{e^{kt}}_{u}\underbrace{\sin\omega_0 t\, dt}_{dv} = -e^{kt}\frac{\cos\omega_0 t}{\omega_0}\Big|_0^t + \int_0^t \underbrace{\frac{\cos\omega_0 t}{\omega_0}}_{v}\underbrace{ke^{kt}}_{u}dt$$

$$= \frac{-e^{kt}\cos\omega_0 t + 1}{\omega_0} + \frac{k}{\omega_0}\int e^{kt}\cos\omega_0 t\, dt$$

$$= \frac{-e^{kt}\cos\omega_0 t + 1}{\omega_0} + \frac{k}{\omega_0}\left(e^{kt}\frac{\sin\omega_0 t}{\omega_0}\Big|_0^t - \int \frac{\sin\omega_0 t}{\omega_0} ke^{kt} \right.$$

$$= \frac{-e^{kt}\cos\omega_0 t + 1}{\omega_0} + \frac{k}{\omega_0}\left[\frac{e^{kt}\sin\omega_0 t}{\omega_0} - \frac{k}{\omega_0} I \right]$$

$$I\left[1+\frac{k^2}{\omega_0^2}\right] = \frac{-e^{+kt}\cos\omega_0 t + 1}{\omega_0} + \frac{k}{\omega_0^2} e^{kt}\sin\omega_0 t$$

$$I = \frac{\omega_0}{k^2+\omega_0^2}\left(-e^{kt}\cos\omega_0 t + 1\right) + \frac{k}{k^2+\omega_0^2} e^{kt}\sin\omega_0 t$$

$$T(t)e^{kt} - T_i = \bar{T}_0 e^{kt} - \bar{T}_0 + \frac{k T_a \omega_0}{k^2+\omega_0^2}\left(-e^{kt}\cos\omega_0 t + 1\right) + \frac{k^2}{k^2+\omega_0^2} e^{kt}\sin\omega_0 t$$

$$T(t) = T_0 + T_i e^{-kt} - \bar{T}_0 e^{-kt} + \frac{k T_a \omega_0}{k^2+\omega_0^2} e^{-kt} + k\,\frac{k\sin\omega_0 t - \omega_0\cos\omega_0 t}{k^2+\omega_0^2}$$

Method of undetermined coefficients

$$\frac{dT}{dt} + kT = kT_0 + kT_a \sin\omega_0 t$$

Homogenous: $\frac{dT}{dt} + kT = 0 \Rightarrow T_H(t) = ce^{-kt}$

Particular Solution: $T_p(t) = A + B\cos\omega_0 t + C\sin\omega_0 t$

$$\frac{dT_p}{dt} = 0 - \omega_0 B\sin\omega_0 t + C\omega_0\cos\omega_0 t$$

$$-\omega_0 B\sin\omega_0 t + C\omega_0\cos\omega_0 t + k(A + B\cos\omega_0 t + C\sin\omega_0 t)$$
$$= kT_0 + kT_a\sin\omega_0 t$$

Constant: $kA = kT_0 \Rightarrow A = T_0$

Sine/cosine:

sin: $-B\omega_0 + kC = kT_a$ ✓

cos. $C\omega_0 + kB = 0$ ✓

$$\begin{bmatrix} -\omega_0 & k \\ k & \omega_0 \end{bmatrix}\begin{bmatrix} B \\ C \end{bmatrix} = \begin{bmatrix} kT_a \\ 0 \end{bmatrix}$$

$$B = \frac{\begin{vmatrix} kT_a & k \\ 0 & \omega_0 \end{vmatrix}}{\begin{vmatrix} -\omega_0 & k \\ k & \omega_0 \end{vmatrix}} = \frac{kT_a\omega_0}{-(k^2+\omega_0^2)} =$$

$$C = \frac{\begin{vmatrix} -\omega_0 & kT_a \\ k & 0 \end{vmatrix}}{\begin{vmatrix} -\omega_0 & k \\ k & \omega_0 \end{vmatrix}} = \frac{-k^2T_a}{-(k^2+\omega_0^2)} = \frac{k^2T_a}{k^2+\omega_0^2}$$

3

Particular Solution

$$T_p(t) = \bar{T}_0 + \frac{k^2 T_a}{k^2+\omega_0^2} \sin\omega_0 t - \frac{k\omega_0 T_a}{k^2+\omega_0^2} \cos\omega_0 t$$

general Solution: $\cancel{y}$ $T(t) = T_H(t) + T_p(t)$

$$\therefore T(t) = ce^{-kt} + \bar{T}_0 + \frac{k^2 T_a \sin\omega_0 t}{k^2+\omega_0^2} - \frac{k\omega_0 T_a \cos\omega_0 t}{k^2+\omega_0^2}$$

$$T(0) = T_i$$

$$T(0) = T_i = c + \bar{T}_0 - \frac{k\omega_0 T_a}{k^2+\omega_0^2} \Rightarrow c = (T_i - \bar{T}_0) + \frac{k\omega_0}{k^2+\omega_0^2} T_a$$

$$T(t) = \left(\left(T_i - \bar{T}_0\right) \cancel{+} \frac{k\omega_0}{k^2+\omega_0^2} T_a\right) e^{-kt} + T_a k \frac{k\sin\omega_0 t - \omega_0 \cos\omega_0 t}{k^2+\omega_0^2} + \bar{T}_0$$

4

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad ; \quad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\frac{dT}{dt} + kT = k\left(\bar{T}_0 + T_a e^{j\omega_0 t}\right)$$

Integrating factors

$$e^{kt}\frac{dT}{dt} + ke^{kt}T = k\bar{T}_0 e^{kt} + kT_a e^{kt} e^{j\omega_0 t}$$

$$\int d\left(Te^{kt}\right) = k\bar{T}_0\int_0^t e^{kt}\,dt + kT_a\int_0^t e^{(k+j\omega_0)t}\,dt$$

$$T(t)e^{kt} - T(0) = \cancel{k}\bar{T}_0 \frac{e^{kt}}{\cancel{k}}\Big|_0^t + kT_a \frac{e^{(k+j\omega_0)t}}{k+j\omega_0}\Big|_0^t$$

$$T(t)e^{kt} - T_i = \bar{T}_0\left(e^{kt} - 1\right) + \frac{k}{k+j\omega_0}T_a\left[e^{kt}.e^{j\omega_0 t} - 1\right]$$

$$T(t) = T_i e^{-kt} + \bar{T}_0\left(1 - e^{-kt}\right) + \frac{k}{k+j\omega_0}T_a\left[e^{j\omega_0 t} - e^{-kt}\right]$$

$$kT_a \frac{e^{j\omega_0 t}}{(k+j\omega_0)} - \frac{e^{-kt}}{(k+j\omega_0)}$$

5

$$\frac{1}{k+j\omega_0} = \frac{1}{(k+j\omega_0)(k-j\omega_0)}\,(k-j\omega_0) = \frac{k-j\omega_0}{k^2+\omega_0^2}$$

$$\frac{e^{j\omega_0 t}}{k+j\omega_0} = \frac{\cos\omega_0 t + j\sin\omega_0 t}{k+j\omega_0}\;\frac{k-j\omega_0}{k+j\omega_0}$$

$$= \frac{(k\cos\omega_0 t + \omega_0\sin\omega_0 t) + (k\sin\omega_0 t - \omega_0\cos\omega_0 t)\,j}{k^2+\omega_0^2}$$

Retaining imaginary parts

$$T(t) = T_i\, e^{-kt} + \bar{T}_0 (1-e^{-kt}) + e^{-kt}\, \frac{k\, T_a (+\omega_0)}{k^2+\omega_0^2} + \frac{kT_a\,(k\sin\omega_0 t - \omega_0\cos\omega_0 t)}{k^2+\omega_0^2}$$

$t \Rightarrow \infty$

$$T(t) = \bar{T}_0 + \frac{kT_a}{k^2+\omega_0^2}\,(k\sin\omega_0 t - \omega_0\cos\omega_0 t)$$

$$\frac{C\sin\phi}{C\cos\phi} = \frac{\omega_0}{k} \Rightarrow \phi = \tan^{-1}\frac{\omega_0}{k}$$

Let $k = C\cos\phi$, $\omega_0 = C\sin\phi$ $\Bigg\} \Rightarrow$ $k^2 = C^2\cos^2\phi$, $\cos^2 = C^2\sin^2\phi$, $k^2+\omega_0^2 = C^2$

$$k\sin\omega_0 t - \omega_0\cos\omega_0 t = C\sin\omega_0 t\cos\phi - C\sin\phi\cos\omega_0 t$$

$$= C(\sin\omega_0 t\cos\phi - \sin\phi\cos\omega_0 t)$$

$$= C\sin(\omega_0 t - \phi)$$

$$= \sqrt{k^2+\omega_0^2}\,\sin(\omega_0 t - \phi) \quad \text{where } \phi = \tan^{-1}\frac{\omega_0}{k}$$

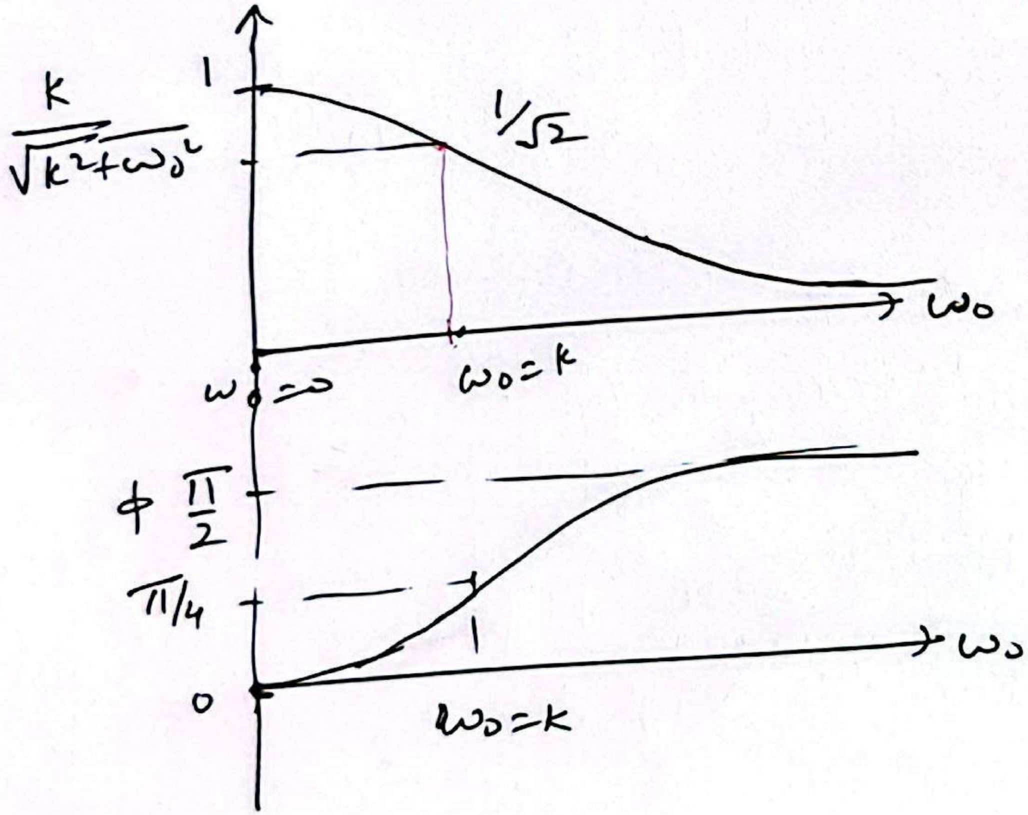
6

$$T(t) = \bar{T}_0 + \frac{k T_a}{k^2 + \omega_0^2} \sqrt{k^2 + \omega_0^2} \sin(\omega_0 t - \phi)$$

$$= \bar{T}_0 + \frac{k T_a}{\sqrt{k^2 + \omega_0^2}} \sin(\omega_0 t - \phi)$$

$$T_0(t) = \bar{T}_0 + T_a \sin(\omega_0 t \cancel{-\phi}) \longrightarrow T(t) = \bar{T}_0 + T_a \frac{k}{\sqrt{k^2 + \omega_0^2}} \sin(\omega_0 t - \phi)$$

where $\phi = \tan^{-1} \frac{\omega_0}{k}$



$$\frac{k}{\sqrt{k^2 + \omega_0^2}} \xrightarrow{\omega_0 \to \infty} \sim \frac{k}{\sqrt{\omega_0^2}} \sim \frac{k}{\omega_0} \to 0$$

$\phi - \omega_0$

$\phi = \tan^{-1} \frac{\omega_0}{k}$

$\omega_0 \to 0 \qquad \phi \to 0$

$\omega_0 \to \infty \qquad \phi \to$

$k \quad \omega_0 = k \qquad \phi = \pi/4$

7