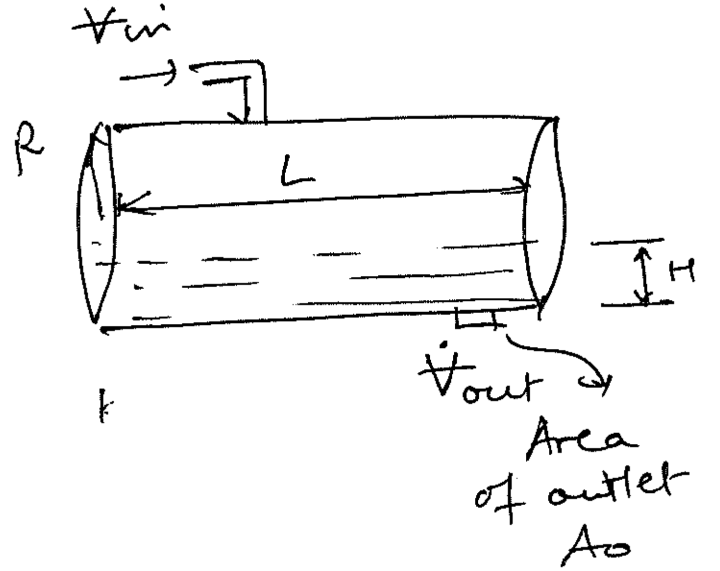


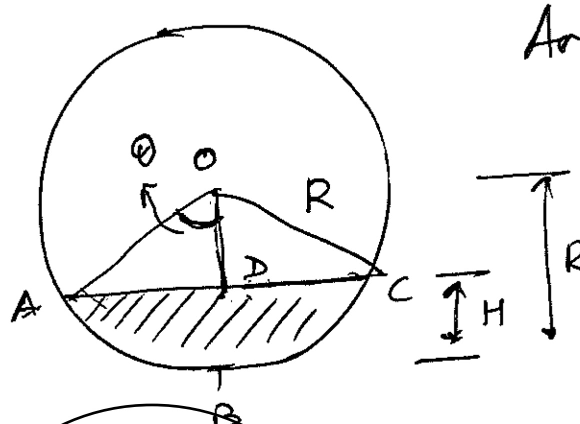
$$\dot{m}_{in} - \dot{m}_{out} = \frac{dm}{dt}$$

$$\rho \dot{\forall}_{in} - \rho \dot{\forall}_{out} = \frac{d}{dt}(\rho \forall)$$

$$\boxed{\frac{d\forall}{dt} = \dot{\forall}_{in} - \dot{\forall}_{out}}$$

$$\dot{\forall}_{out} = A_o V \leftarrow \text{velocity of water}$$

$$= A_o \sqrt{2gH} \quad (V = \sqrt{2gH} \text{ from Bernoulli's equation})$$



Area ABC = Area of sector OABC − Area of Δ OAC

$$= \frac{1}{2} R^2 (2\theta) - \frac{1}{2} (AC)(OD)$$

$$OD = R - H$$

$$AC = 2DC = 2\sqrt{R^2 - (R-H)^2}$$

$$= 2\sqrt{2HR - H^2}$$

$$\theta = \cos^{-1} \frac{OD}{OA} = \cos^{-1} \frac{R-H}{R}$$

Area of sector.

$$2\pi \rightarrow \pi R^2$$

$$\theta \rightarrow \frac{\theta}{2\pi} \pi R^2$$

$$= \frac{1}{2} R^2 \theta$$

$$\forall = \frac{1}{\cancel{2}} R^2 \cancel{2} \cos^{-1} \frac{R-H}{R} L - \left(\frac{1}{\cancel{2}}\right) \cancel{2} \sqrt{2HR-H^2}\,(R-H)\, L$$

$$\frac{d\forall}{dt} \qquad \frac{d}{dt} \cos^{-1}\left(\frac{R-H}{R}\right) = \frac{-1}{\sqrt{1-\frac{(R-H)^2}{R^2}}} \left(-\frac{1}{R}\right) \frac{dH}{dt}$$

$$= \frac{1}{\sqrt{R^2-(R-H)^2} \cdot \frac{1}{R}} \frac{1}{R} \dot{H}$$

$$= \frac{\dot{H}}{\sqrt{2HR-H^2}}$$

$$y = \cos^{-1} x$$

$$x = \cos y$$

$$\frac{dx}{dy} = -\sin y$$

$$= -\sqrt{1-\cos^2 y}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\cos^2 y}}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$\boxed{\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}}$$

$$\frac{d}{dt}\left(\sqrt{2HR-H^2}\,(R-H)\right) = \left(\frac{1}{2\sqrt{2HR-H^2}} (\cancel{2}R-2H)(R-H) + \sqrt{2HR-R^2}\,(-1)\right) \dot{H}$$

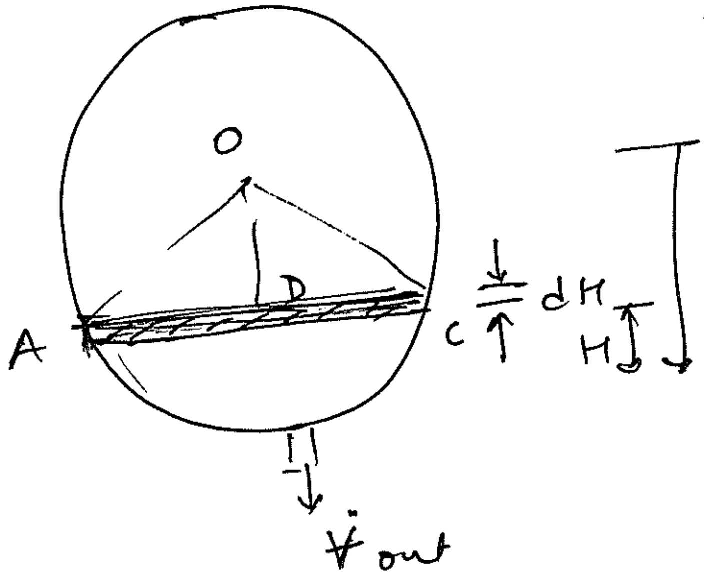
$$= \left(\frac{(R-H)^2}{\sqrt{2HR-H^2}} - \sqrt{2HR-R^2}\right) \dot{H}$$

$$\frac{d\forall}{dt} = \frac{R^2 L \dot{H}}{\sqrt{2RH-H^2}} + \sqrt{2HR-R^2}\, \dot{H} L - \frac{(R-H)^2 \dot{H} L}{\sqrt{2HR-H^2}}$$

1

$$= \frac{R^2 + 2HR - R^2 - (R-H)^2}{\sqrt{2RH-H^2}} \dot{H} L$$

$$= \frac{2(2HR-R^2)\dot{H} L}{\sqrt{2RH-H2}} = 2\sqrt{2RH-H^2}\, \dot{H} L$$



$$d\forall = (AC)\, dH\, L$$

$$AC = 2\,CD$$

$$= 2\sqrt{R^2-(R-H)^2}$$

$$= 2\sqrt{2HR-H^2}$$

$$d\forall = 2\sqrt{2HR-H^2}\, dH\, L$$

$$\frac{d\forall}{dt} = 2\sqrt{2HR-H^2}\,\mathring{H}\, L$$

$$\frac{d\forall}{dt} = \dot{\forall}_{in} - \dot{\forall}_{out} \Rightarrow \boxed{2\sqrt{2HR-H^2}\,\frac{dH}{dt}\,L = \dot{\forall}_{in} - A_o\sqrt{2gH}}$$

let $\dot{\forall}_{in} = 0$

$$2\sqrt{2RH-H^2}\,\frac{dH}{dt}\,L = -A_o\sqrt{2gH}$$

$$\sqrt{\frac{2RH-H^2}{H}}\,dH = -\frac{A_o\sqrt{2g}\,dt}{L2}$$

2

$$\int_{H_o}^{H}\sqrt{2R-H}\,dH = -\frac{A_o}{L}\int_0^t \frac{\sqrt{2g}}{2}\,dt$$

$$\left.\frac{(2R-H)^{3/2}}{-\frac{3}{2}}\right|_{H_0}^{H} = -\frac{A_0}{L}\sqrt{\frac{g}{2}}\; t$$

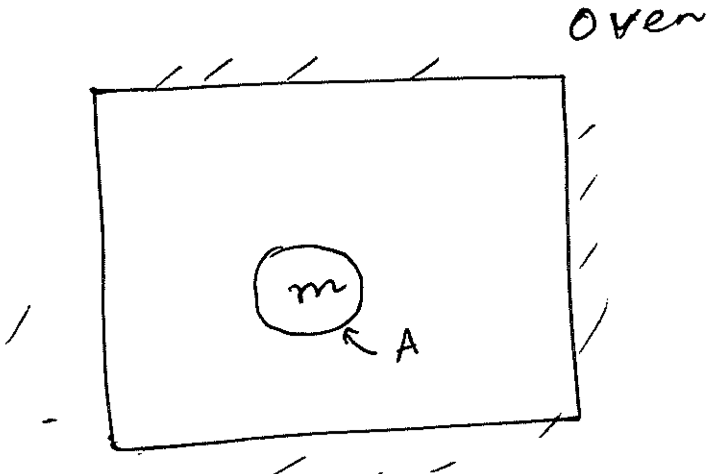
$$(2R-H)^{3/2} - (2R-H_0)^{3/2} = \frac{3}{2} A_0 \sqrt{\frac{g}{2}}\; t$$

$$(2R-H)^{3/2} = (2R-H_0)^{3/2} + \frac{3}{2} A_0 \sqrt{\frac{g}{2}}\; t$$

$$\boxed{H(t) = 2R - \left((2R_{\cancel{0}} - H_0)^{3/2} + \frac{3}{2} A_0 \sqrt{\frac{g}{2}}\; t\right)^{2/3}}$$

3

Heat Transfer



m is made of material of specific heat C_p

$\dot{E}_{in} = 0$, $\dot{E}_{out} = 0$, $\dot{Q}$ ✓, $\dot{W} = 0$, $\dot{E}_{generated} = 0$

$\dot{Q}$

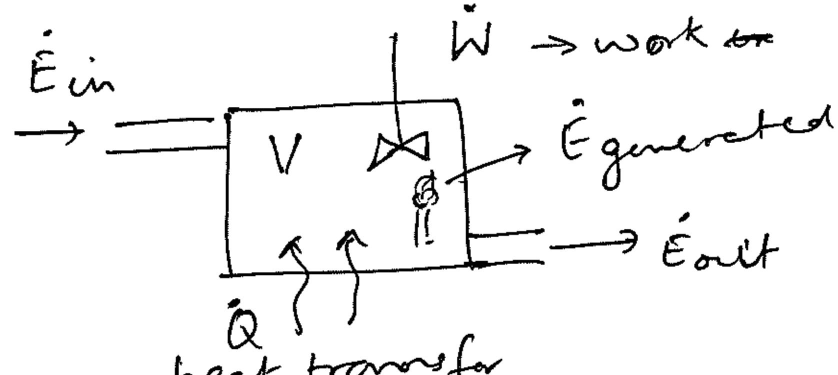
$\dot{Q}_{conv}$

$h A (T - T_0)$ — $(T - T_0)$: temp difference; A: surface area; h: heat transfer convective coefficient

$\dot{Q}_{rad} = \sigma A \epsilon (T^4 - T_0^4)$ — ϵ: emissivity

σ: Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8}\ W m^{-2} K^{-4}$

Conservation of energy



$$\dot{E}_{in} - \dot{E}_{out} + \dot{Q} + \dot{W} + \dot{E}_{generated} = \frac{dE}{dt}$$

$$\Downarrow$$

$$\dot{Q} = \frac{dE}{dt}$$

$$\frac{dE}{dt} = \frac{d}{dt}(m C_p T)$$

$$= m C_p \frac{dT}{dt}$$

5b

$\cancel{mc_pd}$ $\quad \frac{dE}{dt} = \dot{Q}$

$$mc_p \frac{dT}{dt} = -\left(hA(T-T_0) + \sigma A \epsilon (T^4 - T_0^4)\right)$$

$$\frac{dT}{dt} = -\frac{1}{mc_p}\left(hA(T-T_0) + \sigma A \epsilon (T^4 - T_0^4)\right)$$

Let $f(T) = hA(T-T_0) + \sigma \epsilon A (T^4 - T_0^4)$

Linearize $f(T)$ about $T = T_0$

$$f(T) = \cancelto{0}{f(T_0)} + f'(T_0)(T-T_0)$$

$$f'(T_0) = hA + \sigma \epsilon A 4T^3 \Big|_{T=T_0}$$

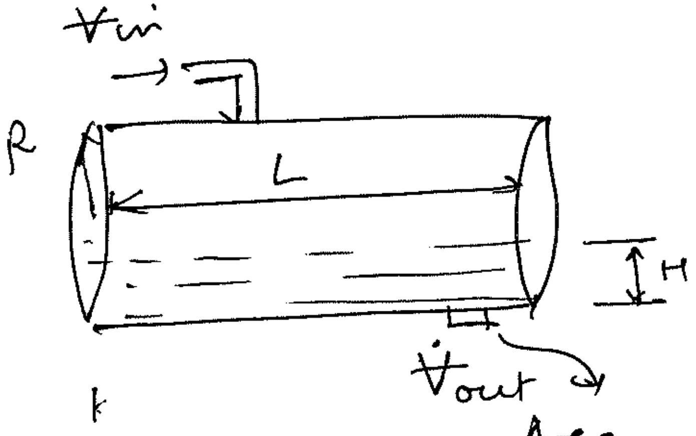
$$= hA + 4\sigma\epsilon A T_0^3$$

$$f(T) \approx (hA + 4\sigma\epsilon A T_0^3)(T-T_0)$$

$$\frac{dT}{dt} = \frac{-1}{mc_p}(hA + 4\sigma\epsilon A T_0^3)(T-T_0)$$

5

Let $k \triangleq \dfrac{hA + 4\sigma\epsilon A T_0^3}{mc_p}$



$$\dot{m}_{in} - \dot{m}_{out} = \frac{dm}{dt}$$

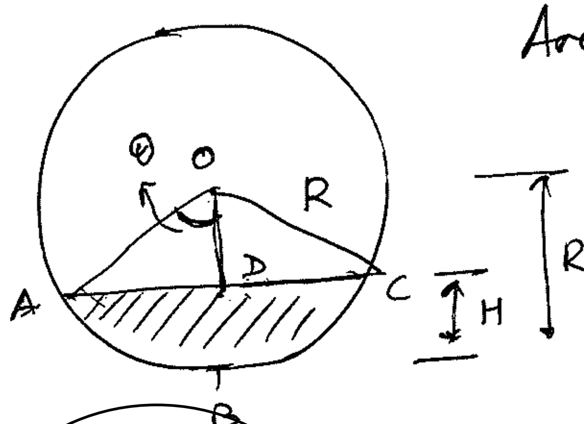
$$\rho \dot{V}_{in} - \rho \dot{V}_{out} = \frac{d}{dt}(\rho V)$$

$$\boxed{\frac{dV}{dt} = \dot{V}_{in} - \dot{V}_{out}}$$

$\dot{V}_{out} = A_o V$ ← velocity of water

$= A_o\sqrt{2gH}$ ($V = \sqrt{2gH}$ from Bernoulli's equation)

LONG APPROACH



Area ABC = Area of sector OABC − Area of Δ OAC

$$= \frac{1}{2}R^2(2\theta) - \frac{1}{2}(AC)(OD)$$

$OD = R - H$

$AC = 2DC = 2\sqrt{R^2 - (R-H)^2}$

$= 2\sqrt{2HR - H^2}$

$$\theta = \cos^{-1}\frac{OD}{OA} = \cos^{-1}\frac{R-H}{R}$$

Area of sector:

$2\pi \rightarrow \pi R^2$

$\theta \rightarrow \frac{\theta}{2\pi}\pi R^2$

$= \frac{1}{2}R^2\theta$

6

$$V = \frac{1}{\cancel{2}} R^2 \cancel{2} \cos^{-1}\frac{R-H}{R}\, L - \left(\frac{1}{\cancel{2}}\right) \cancel{2} \sqrt{2HR-H^2}\,(R-H)\, L$$ (2)

$$\frac{dV}{dt} \quad \frac{d}{dt}\cos^{-1}\left(\frac{R-H}{R}\right) = \frac{-1}{\sqrt{1-\frac{(R-H)^2}{R^2}}}\left(-\frac{1}{R}\right)\frac{dH}{dt}$$

$$= \frac{1}{\sqrt{R^2-(R-H)^2}\cdot\frac{1}{R}}\,\frac{1}{R}\,\dot{H}$$

$$= \frac{\dot{H}}{\sqrt{2HR-H^2}}$$

$$\frac{d}{dt}\left(\sqrt{2HR-H^2}\,(R-H)\right) = \left(\frac{1}{2\sqrt{2HR-H^2}}(2R-2H)(R-H) + \sqrt{2HR-R^2}\,(-1)\right)\dot{H}$$

$$= \left(\frac{(R-H)^2}{\sqrt{2HR-H^2}} - \sqrt{2HR-R^2}\right)\dot{H}$$

$$\frac{dV}{dt} = \frac{R^2 L \dot{H}}{\sqrt{2RH-H^2}} + \sqrt{2HR-R^2}\,\dot{H} L - \frac{(R-H)^2}{\sqrt{2HR-H^2}}\dot{H} L$$

$$= \frac{R^2 + 2HR - R^2 - (R-H)^2}{\sqrt{2RH-H^2}}\,\dot{H} L$$

$$= \frac{2(2HR-R^2)}{\sqrt{2RH-H^2}}\,\dot{H} L = 2\sqrt{2RH-H^2}\,\dot{H} L$$

$y = \cos^{-1}x$

$x = \cos y$

$\frac{dx}{dy} = -\sin y$

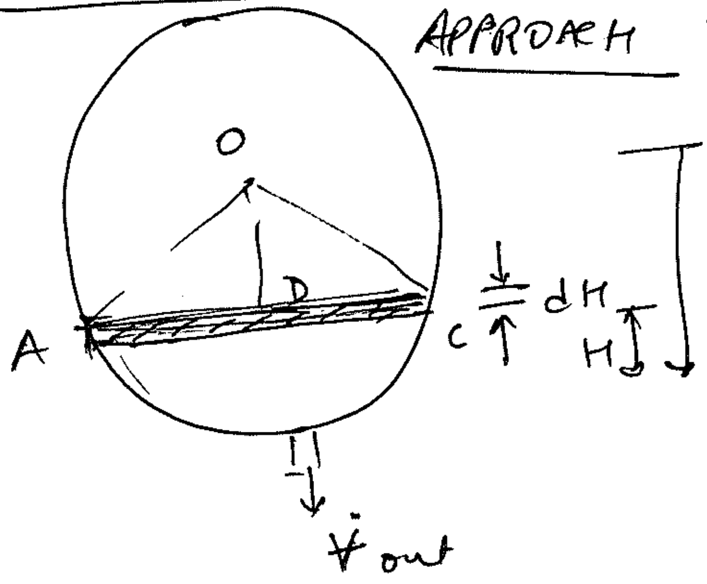
$= -\sqrt{1-\cos^2 y}$

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\cos^2 y}}$

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$

$$\boxed{\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}}$$

ALTERNATE SHORT APPROACH



$$d\forall = (AC)\, dH\, L$$

$$AC = 2\, CD$$

$$= 2\sqrt{R^2 - (R-H)^2}$$

$$= 2\sqrt{2HR - H^2}$$

$$d\forall = 2\sqrt{2HR - H^2}\, dH\, L$$

$$\frac{d\forall}{dt} = 2\sqrt{2HR - H^2}\, \dot{H}\, L$$

$$\frac{d\forall}{dt} = \dot{\forall}_{in} - \dot{\forall}_{out} \Rightarrow \boxed{2\sqrt{2HR - H^2}\,\frac{dH}{dt}\, L = \dot{\forall}_{in} - A_o\sqrt{2gH}}$$

Let $\dot{\forall}_{in} = 0$

$$2\sqrt{2RH - H^2}\,\frac{dH}{dt}\, L = -A_o\sqrt{2gH}$$

$$\sqrt{\frac{2RH - H^2}{H}}\, dH = -\frac{A_o\sqrt{2g}\,dt}{L\,2}$$

$$\int_{H_o}^{H} \sqrt{2R - H}\; dH = -\frac{A_o}{L}\int_0^t \frac{\sqrt{2g}}{2}\, dt$$

8

$$\left.\frac{(2R-H)^{3/2}}{-\frac{3}{2}}\right|_{H_0}^{H} = -\frac{A_0}{L}\sqrt{\frac{g}{2}}\; t$$

$$(2R-H)^{3/2} - (2R-H_0)^{3/2} = \frac{3}{2}A_0\sqrt{\frac{g}{2}}\; t$$

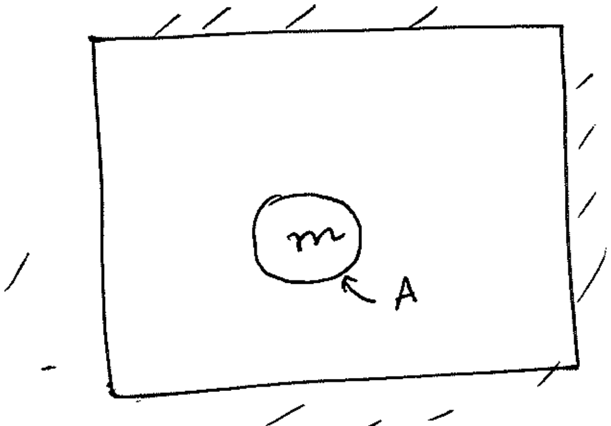
$$(2R-H)^{3/2} = (2R-H_0)^{3/2} + \frac{3}{2}A_0\sqrt{\frac{g}{2}}\; t$$

$$\boxed{H(t) = 2R - \left((2R-H_0)^{3/2} + \frac{3}{2}A_0\sqrt{\frac{g}{2}}\; t\right)^{2/3}}$$

9

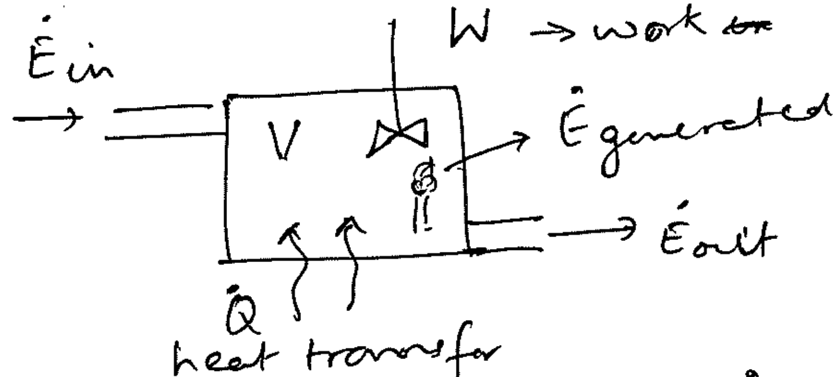
Heat Transfer

oven



m is made of material of specific heat C_p

Conservation of energy



$$\dot E_{in} - \dot E_{out} + \dot Q + \dot W + \dot E_{generated} = \frac{dE}{dt}$$

$$\Downarrow$$

$\dot E_{in} = 0$, $\dot E_{out} = 0$,
$\dot Q$ ✓, $\dot W = 0$, $\dot E_{generated} = 0$

$$\dot Q = \frac{dE}{dt}$$

$$\frac{dE}{dt} = \frac{d}{dt}(m C_p T) = m C_p \frac{dT}{dt}$$

$\dot Q$

$\dot Q_{conv} = \underbrace{h}_{\text{heat transfer convective coefficient}} \underbrace{A}_{\text{Surface area}} \underbrace{(T - T_0)}_{\text{temp difference}}$

$\dot Q_{rad} = \sigma A \epsilon (T^4 - T_0^4)$

ϵ → emissivity

σ → Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8}\ W m^{-2} K^{-4}$

~~mcpd~~ $\frac{dE}{dt} = \dot{Q}$ (5b)

$$mc_p \frac{dT}{dt} = -\left(hA(T-T_0) + \sigma A \epsilon (T^4 - T_0^4)\right)$$

$$\frac{dT}{dt} = -\frac{1}{mc_p}\left(hA(T-T_0) + \sigma A \epsilon (T^4 - T_0^4)\right)$$

Let $f(T) = hA(T-T_0) + \sigma \epsilon A (T^4 - T_0^4)$

Linearize $f(T)$ about $T = T_0$

$$f(T) = \cancelto{0}{f(T_0)} + f'(T_0)(T-T_0)$$

$$f'(T_0) = hA + \sigma \epsilon A 4T^3 \Big|_{T=T_0}$$

$$= hA + 4\sigma \epsilon A T_0^3$$

$$f(T) \approx (hA + 4\sigma\epsilon A T_0^3)(T - T_0)$$

$$\frac{dT}{dt} = -\frac{1}{mc_p}(hA + 4\sigma\epsilon A T_0^3)(T - T_0)$$

11

let $k \triangleq \frac{hA + 4\sigma\epsilon A T_0^3}{mc_p}$

$$\boxed{\frac{dT}{dt} = -k(T - T_0) \qquad T(0) = T_i \text{ initial temperature,}}$$

6

$\Rightarrow$ Let T_0 be a constant

Separation of variables

$$\int \frac{dT}{T - T_0} = -\int k\, dt$$

$$\Rightarrow \quad \ln(T - T_0)\Big|_{T_i}^{T(t)} = -kt\Big|_0^t$$

$$\Rightarrow \quad \ln(T(t) - T_0) - \ln(T_i - T_0) = -kt$$

$$\Rightarrow \quad \ln\left(\frac{T(t) - T_0}{T_i - T_0}\right) = -kt$$

$$\Rightarrow \quad T(t) - T_0 = e^{-kt}(T_i - T_0)$$

$$\Rightarrow \quad \boxed{T(t) = T_0 + e^{-kt}(T_i - T_0)}$$

12

$$T(t) = T_0 + e^{-kt}(T_i - T_0)$$

$$\frac{dT}{dt} = -k(T - T_0) \Rightarrow \frac{dT}{dt} + kT_0 = kT_0 \qquad T(0) = T_i \qquad ?$$

<u>Integrating factors</u>

$$\boxed{\frac{d}{dt}\left(T(t)e^{kt}\right) = \frac{dT}{dt}e^{kt} + T k e^{kt}}$$

$$e^{kt}\frac{dT}{dt} + e^{kt} k T_{\cancel{0}} = kT_0 e^{kt}$$

$$\frac{d}{dt}\left(T(t)e^{kt}\right) = kT_0 e^{kt}$$

$$\int_{T_i,\, t=0}^{T,\, t} d\left(T(t)e^{kt}\right) = \int_0^t kT_0 e^{kt}\, dt$$

$$T(t)e^{kt} - T_i e^{k0} = \cancel{k} T_0 \frac{e^{kt}}{\cancel{k}}\Big|_0^t$$

$$T(t)e^{kt} - T_i = T_0\left(e^{kt} - 1\right)$$

$$T(t)e^{kt} = (T_i - T_0) + T_0 e^{kt}$$

$$\boxed{T(t) = T_0 + (T_i - T_0)e^{-kt}}$$

Method of undetermined coefficients

8

$$\frac{dT}{dt} + kT = kT_o \qquad T(0) = T_i$$

$$T(t) = T_h(t) + T_p(t)$$

Homogeneous

$$\frac{dT}{dt} + kT = 0 \implies T(t) = e^{-kt}$$

Particular Solution — RHS is a constant

$\therefore$ Choose $T_p(t) = A$ (Some constant to be determined)

$$\frac{dT_p(t)}{dt} = 0$$

$$0 + kA = kT_o \implies A = T_o \implies T_p(t) = T_o$$

$$T(t) = ce^{-kt} + T_o$$

IC $\quad T(0) = T_i = c + T_o \implies c = T_i - T_o$

$$\boxed{T(t) = T_o + (T_i - T_o)e^{-kt}}$$

14

1) $$\underbrace{T(t) - T_o}_{\text{Difference at a given time } t} = \underbrace{(T_i - T_o)}_{\text{Initial difference}} e^{-kt}$$

For $t = 0$ $\quad e^{-kt} = 1 \Rightarrow T(0) - T_o = T_i - T_o$

$t \to \infty$ $\quad e^{-kt} \to 0 \Rightarrow \cancel{T(t)}\ T(\infty) - T_o = 0$

$\Rightarrow T(\infty) \to T_o$

2) $$T(t) = T_o + (T_i - T_o) e^{-kt}$$

$[k] = 1/s$

$$\tau = \frac{1}{k}$$

↳ Time constant

$$k = \frac{hA + 4\sigma\epsilon A T_o^3}{mC_p} = \frac{hA}{mC_p} + \frac{4\sigma\epsilon A T_o^3}{mC_p}$$

1st term $k = \dfrac{hA}{mC_p} \Rightarrow \tau = \dfrac{mC_p}{hA}$