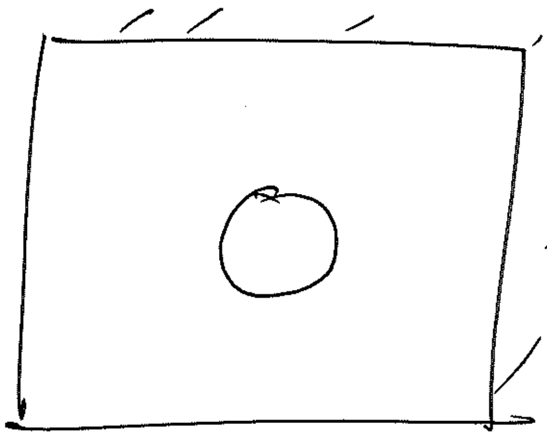
Heat transfer from a solid



Oven

Principle: Conservation of energy

$$\cancel{\dot{E}_{in}}^{\,0} - \cancel{\dot{E}_{out}}^{\,0} + \dot{Q} - \cancel{\dot{W}}^{\,0} + \cancel{\dot{E}_{gen}}^{\,0} = \frac{dE}{dt}$$

$$\dot{Q} = \frac{dE}{dt}$$

m = mass

C_p = Specific heat

A_s = Surface area.

h = heat transfer coefficient

$$E = m C_p T$$

$$\frac{dE}{dt} = \frac{d}{dt}(m C_p T) = m C_p \frac{dT}{dt}$$

$$\dot{Q} = \dot{Q}_{conv} + \dot{Q}_{rad}$$

Ignoring radiation

$$\dot{Q} = \dot{Q}_{conv}$$

$$= h A_s (T - T_o)$$

temp of oven

temp of object

$$m C_p \frac{dT}{dt} = -h A_s (T - T_o)$$

If $T > T_o \Rightarrow \frac{dT}{dt} < 0$

If $T < T_o \Rightarrow \frac{dT}{dt} > 0$

$$m c_p \frac{dT}{dt} = -hA_s(T - T_0) \Rightarrow \frac{dT}{dt} = -k(T - T_0)$$

where $k \triangleq \frac{hA_s}{m c_p}$ $\qquad T(0) = T_i$

$$\frac{dT}{dt} = -k(T - T_0) \Rightarrow \int_{T_i}^{T} \frac{dT}{T - T_0} = \int_0^t -k\,dt$$

$$\Rightarrow \ln(T - T_0)\Big|_{T_i}^{T} = -kt$$

$$\Rightarrow \ln(T - T_0) - \ln(T_i - T_0) = -kt$$

$$\Rightarrow \ln\left(\frac{T - T_0}{T_i - T_0}\right) = -kt$$

$$\Rightarrow \boxed{(T - T_0) = (T_i - T_0)\, e^{-kt}}$$

At $t = 0$ $\quad T(0) - T_0 = T_i - T_0$

$\Rightarrow T(0) = T_i$ ✓

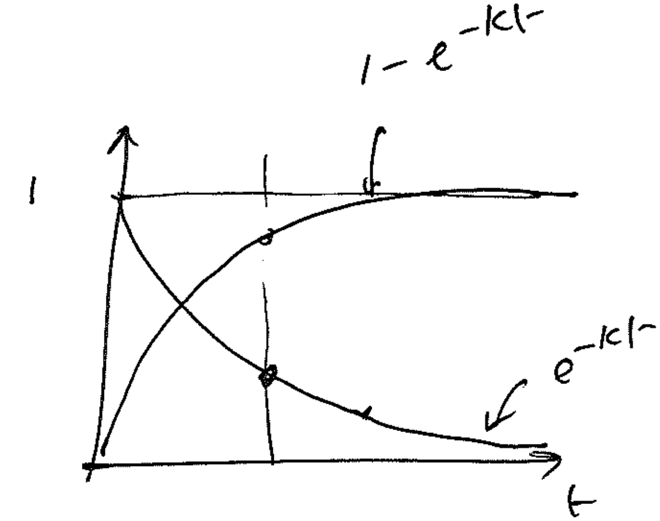
At $t \to \infty$ $\quad T(\infty) - T_0 \to 0$

$\Rightarrow T_\infty \to T_0$

$$\frac{T - T_0}{T_i - T_0} = e^{-kt}$$

$$r = \frac{\text{Current temp difference}}{\text{Initial temp difference}}$$

$r(0) = 1$ & $r(t \to \infty) = 0$



$$T(t) = T_0 + (T_i - T_0)\, e^{-kt}$$

$$T(t) = \underline{T_i}\, e^{-kt} + \underline{T_0}\,(1 - e^{-kt})$$

→ weighted average of T_i & T_0
and weights add up to 1

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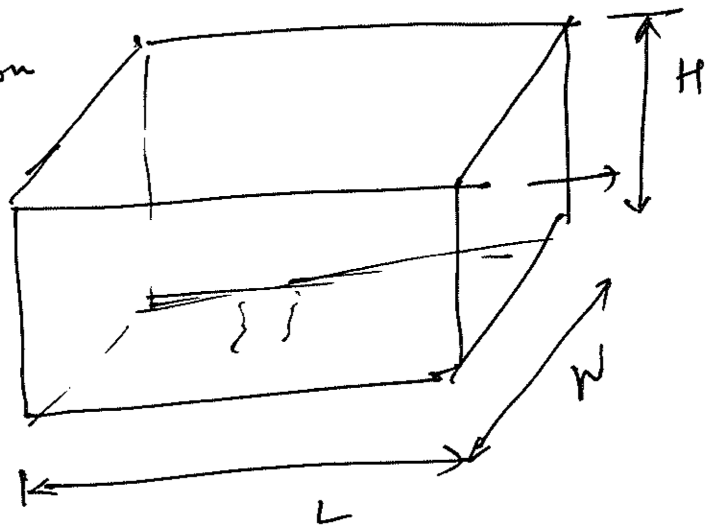
Box Model for modeling pollution in a city

Background concentration C_∞

u →

wind speed



$\dot{q}_g$ = rate of pollutant generation per unit area

Let $C(t)$ be concentration of pollutant (mass ~~unit~~ per unit volume)

Mass Balance for pollutant: $\dot{m}_{in} - \dot{m}_{out} + \dot{m}_{gen} = \frac{dm}{dt}$

$$\dot{m}_{in} = C_\infty \dot{V}_{in} = C_\infty (WH)(u)$$

$$\dot{m}_{generated} = \dot{q}_g LW$$

$$\dot{m}_{destroyed} \sim m \sim CWLH$$

$$\dot{m}_{out} = C\dot{V}_{out} = CWHu$$

$$\dot{m}_{destroyed} = -k_D CWLH$$

$$C_\infty WHu - CWHu + \dot{q}_g LW - k_D CWLH = \frac{d}{dt} C(WLH)$$

Divide through by WLH

$$C_\infty \frac{u}{L} - C\frac{u}{L} + \frac{\dot{q}_g}{H} - k_D C = \frac{dC}{dt}$$

$$\frac{dC}{dt} = -\left(k_D + \frac{u}{L}\right)C + \frac{\dot{q}_g}{H} + C_0\frac{u}{L}$$

$$\boxed{\frac{dC}{dt} + \left(k_D + \frac{u}{L}\right)C = \frac{\dot{q}_g}{H} + C_0\frac{u}{L}} \quad ; \quad \boxed{C(0) = C_0}$$

IC

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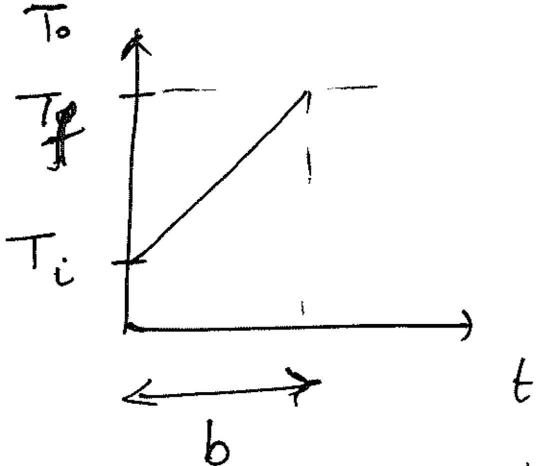
$$\frac{dT}{dt} = -k(T-T_0) \Rightarrow \frac{dT}{dt} + kT = kT_0 \qquad T(0) = T_i$$



$$T_0(t) = \left(\frac{T_f - T_i}{b}\right) t + T_i$$

$$\boxed{\frac{dT}{dt} + kT = kT_i + k\left(\frac{T_f - T_i}{b}\right) t \;;\quad T(0) = T_i}$$

Not separable.

$$\frac{dT}{dt} = f(T,t) = f_1(T)\, f_2(t)$$

<u>Undetermined coefficients</u>:

<u>Homogenous Solution</u>: $\frac{dT}{dt} + kT = 0 \Rightarrow T(t) = e^{-kt}$

<u>Particular Solution</u> Let $T_p(t) = At + B \Rightarrow \frac{dT_p}{dt} = A$

$$A + k(At + B) = kT_i + k\left(\frac{T_f - T_i}{b}\right) t$$

Equate the constant terms and linear term.

$$kB + A = kT_i \;; \quad Ak = k\left(\frac{T_f - T_i}{b}\right) \Rightarrow A = \frac{T_f - T_i}{b}.$$

$$kB + \frac{T_f - T_i}{b} = kT_i \Rightarrow B = T_i - \frac{T_f - T_i}{bk}$$

Particular Solution: $T_p(t) = \left(\frac{T_f - T_i}{b}\right) t + \left(T_i - \frac{T_f - T_i}{bk}\right)$

General Solution. $T(t) = ce^{-kt} + \left(\frac{T_f - T_i}{b}\right)t + \left(T_i - \frac{T_f - T_i}{bk}\right)$

Apply IC $\quad T(0) = T_i$

$$\cancel{T_i} = c + 0 + \left(\cancel{T_i} - \frac{T_f - T_i}{bk}\right) \Rightarrow c = \frac{T_f - T_i}{bk}$$

$$T(t) = \frac{T_f - T_i}{bk} e^{-kt} - \frac{T_f - T_i}{bk} + T_i + \frac{T_f - T_i}{b} t$$

$$= \frac{T_f - T_i}{bk}\left(e^{-kt} - 1\right) + T_i + \left(\frac{T_f - T_i}{b}\right) t$$

~~t=0~~ $T(0) = \cancel{T_i}\left(-\frac{T_f - T_i}{bk}\right) + T_i + \left(\frac{T_f - T_i}{b}\right)($

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$$\frac{dT}{dt} + kT = kT_i + k\left(\frac{T_f - T_i}{b}\right) t$$

$$\int_0^t t\,dt = \int_0^t u\,du = \int_0^t x\,dx$$ 8

$$e^{kt}\frac{dT}{dt} + ke^{kt} = kT_i e^{kt} + k\,\frac{T_f - T_i}{b}\,t e^{kt}$$

$$\frac{d}{dt}\left(Te^{kt}\right) = \text{"}$$

$$\int d\left(Te^{kt}\right) = \int_0^t kT_i e^{kt'}dt' + k\,\frac{T_f - T_i}{b}\int_0^t t e^{kt}\,dt$$

$$= kT_i \frac{e^{kt'}}{k}\Big|_0^t + k\left(\frac{T_f - T_i}{b}\right)\left[\frac{te^{kt}}{k} - \frac{1}{k^2}\left(e^{kt}-1\right)\right]$$

$$Te^{kt} - T(0) = T_i\left(e^{kt}-1\right) + \left(\frac{T_f - T_i}{b}\right)\left[te^{kt} - \frac{1}{k}\left(e^{kt}-1\right)\right]$$

$$\int_0^t \underbrace{t}_{u}\,\underbrace{e^{kt}\,dt}_{dv} = \frac{te^{kt}}{k}\Big|_0^t - \int_0^t \frac{e^{kt}}{k}\,dt$$

$$= \frac{te^{kt}}{k} - \frac{1}{k}\,\frac{e^{kt}}{k}\Big|_0^t$$

$$= \frac{te^{kt}}{k} - \frac{1}{k^2}\left(e^{kt}-1\right)$$

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$$Te^{kt} = \cancel{T_i} \quad T_i e^{kt} + \frac{T_f - T_i}{b} t e^{kt} - \frac{T_f - T_i}{bk}(e^{kt} - 1)$$

Multiply be e^{-kt}

$$T(t) = T_i + \frac{T_f - T_i}{b} t \, \cancel{e} - \frac{T_f - T_i}{bk}(1 - e^{-kt})$$

$$T(t) = \frac{T_f - T_i}{bk}(e^{-kt} - 1) + T_i + \left(\frac{T_f - T_i}{b}\right) t$$

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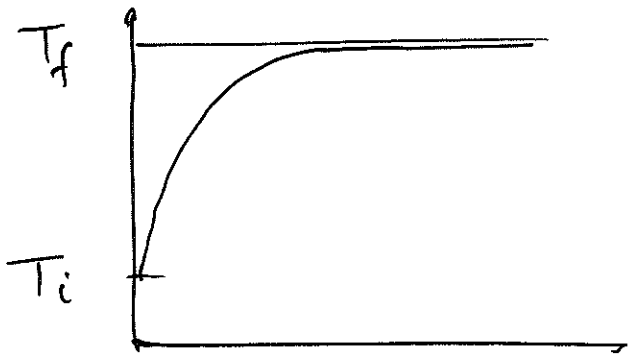

$$T_0 = T_i e^{-\beta t} + \cancel{T_f\ T_0}\ T_f(1-e^{-\beta t})$$

$$T_0 = T_f \cancel{+} (T_i - T_f) e^{-\beta t}$$

$$\frac{dT}{dt} + kT = k(T_0)$$

$$= k\left(T_f + (T_i - T_f) e^{-\beta t}\right)$$

<u>Integrating factors.</u>

$$e^{kt}\frac{dT}{dt} + kTe^{kt} = ke^{kt} T_f + k(T_i - T_f) e^{(k-\beta)t}$$

$$\frac{d}{dt}(Te^{kt}) = \quad ''$$

$$\int d(Te^{kt}) = \int_0^t ke^{kt} T_f \, dt + \int_0^t k(T_i - T_f)\, e^{(k-\beta)t} \, dt$$

$$T(t)e^{kt} - T(0) = kT_f \frac{e^{kt}}{k}\Big|_0^t + k(T_i - T_f)\frac{e^{(k-\beta)t}}{k-\beta}\Big|_0^t$$

$$T(t)e^{kt} = T_i + T_f(e^{kt} - 1) + \frac{k}{k-\beta}(T_i - T_f)\left[e^{(k-\beta)t} - 1\right]$$

$$T(t) = \underline{T_i e^{-kt}} + T_f\left(1-e^{-kt}\right) + \frac{k}{k-\beta}\left(T_i - T_f\right)\left[e^{-\beta t} - e^{-kt}\right] \quad (1)$$

<u>Case I</u> $\quad k >> \beta$

$$T(t) \approx T_f + \frac{k}{k}\left(T_i - T_f\right)\left[e^{-\beta t} - 0\right]$$

$$= T_f + \left(T_i - T_f\right) e^{-\beta t}$$

<u>Case II</u> $\quad k << \beta$

$$T(t) = T_i e^{-kt} + T_f\left(1-e^{-kt}\right) + \frac{k}{-\beta}\left(T_i - T_f\right)\left[0 - e^{-kt}\right]$$

$$= T_i e^{-kt} + T_f\left(1-e^{-kt}\right) + \frac{k}{\beta}\underbrace{\left(T_i - T_f\right)}_{<0} e^{-kt}$$