1) <u>Difference Equations</u>

2) Ordinary Differential Equations (ODES)

- Initial Value
  - 1st order
    - linear
    - non-linear
  - 2nd order
    - linear
    - non-linear
  - Systems of ODEs
- Boundary Value

0

Difference Equations

Let $B(n)$ be the balance in a bank account at ~~time~~ ~~in~~ time period n

$$B(n+1) = B(n) + iB(n) + d - w$$

(deposits) (withdrawals)

$$B(n+1) = B(n)(1+i) + (d-w) \leftarrow \boxed{1^{st} \text{ order difference equation}}$$

1. Setting $B(n+1) = B(n)$

$$\cancel{B(n)} = \cancel{B(n)} + B(n)i + (d-w)$$

$$B(n) = -\left(\frac{d-w}{i}\right) \leftarrow \text{Equilibrium}$$

2. Let $d = w$

$$B(n+1) = B(n)(1+i) \quad ; \quad B(0) = B_0$$

$n=0$: $B(0+1) = B(0)(1+i) \Rightarrow B(1) = B_0(1+i)$

$n=1$: $B(1+1) = B(1)(1+i) \Rightarrow B(2) = (B_0(1+i))(1+i) = B_0(1+i)^2$

$n=2$: $B(2+1) = B(2)(1+i) \Rightarrow B(3) = (B_0(1+i)^2)(1+i) = B_0(1+i)^3$

$$\boxed{B(n) = B_0(1+i)^n}$$

(3) $w = 0$, $d =$ constant (3)

$$B(n+1) = B(n)(1+i) + d \quad ; \quad B(0) = B_0$$

<u>$n=0$</u> $B(0+1) = B(0)(1+i) + d \Rightarrow B(1) = B_0(1+i) + d$

<u>$n=1$</u> $B(1+1) = B(1)(1+i) + d \Rightarrow B(2) = \left(B_0(1+i) + d\right)(1+i) + d$

$$= B_0(1+i)^2 + d(1+i) + d$$

<u>$n=2$</u> $B(2+1) = B(2)(1+i) + d \Rightarrow B(3) = \left(B_0(1+i)^2 + d(1+i) + d\right)(1+i) + d$

$$= B_0(1+i)^3 + d(1+i)^2 + d(1+i) + d$$

$$B(4) = B_0(1+i)^4 + d(1+i)^3 + d(1+i)^2 + d(1+i) + d$$

$$B(n) = B_0(1+i)^n + d\sum_{j=0}^{j=n-1}(1+i)^j$$

$$S \triangleq \sum_{j=0}^{j=n-1}(1+i)^j \qquad \text{Define } \alpha \triangleq (1+i) \Rightarrow S = \sum_{0}^{n-1}\alpha^j$$

$$S = 1 + \alpha + \alpha^2 + \dots \alpha^{n-1} \rightarrow (1)$$

$$\alpha S = \alpha + \alpha^2 + \alpha^3 + \dots \alpha^{n-1} + \alpha^n \rightarrow (2)$$

Subtract (1) from (2) $\cancel{\alpha S}$ $\alpha S - S = \alpha^n - 1 \Rightarrow S = \dfrac{\alpha^n - 1}{\alpha - 1}$

Re-sub $\alpha = 1+i \Rightarrow S = \dfrac{(1+i)^n - 1}{1+i-1} = \dfrac{(1+i)^n - 1}{i}$

2

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Difference Equations

Let $B(n)$ be the balance in a bank account at ~~the~~ ~~in~~ time period n

$$B(n+1) = B(n) + i\,B(n) + \underset{(\text{deposits})}{d} - \underset{(\text{withdrawals})}{w}$$

$$B(n+1) = B(n)(1+i) + (d-w) \leftarrow \boxed{\text{1st order difference equation}}$$

1. Setting $B(n+1) = B(n)$

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$$\boxed{B(n) = B_0(1+i)^n}$$

4

(3) $w = 0$, $d =$ constant (3)

$$B(n+1) = B(n)(1+i) + d \quad ; \quad B(0) = B_0$$

$\underline{n=0}$ $\quad B(0+1) = B(0)(1+i) + d \Rightarrow B(1) = B_0(1+i) + d$

$\underline{n=1}$ $\quad B(1+1) = B(1)(1+i) + d \Rightarrow B(2) = (B_0(1+i) + d)(1+i) + d$

$$= B_0(1+i)^2 + d(1+i) + d$$

$\underline{n=2}$ $\quad B(2+1) = B(2)(1+i) + d \Rightarrow B(3) = (B_0(1+i)^2 + d(1+i) + d)(1+i) + d$

$$= B_0(1+i)^3 + d(1+i)^2 + d(1+i) + d$$

$$B(4) = B_0(1+i)^4 + d(1+i)^3 + d(1+i)^2 + d(1+i) + d$$

$$B(n) = B_0(1+i)^n + d \sum_{j=0}^{j=n-1} (1+i)^j$$

$$S \triangleq \sum_{j=0}^{j=n-1} (1+i)^j \qquad \text{Define } \alpha \triangleq (1+i) \Rightarrow S = \sum_{0}^{n-1} \alpha^j$$

$$S = 1 + \alpha + \alpha^2 + \dots \alpha^{n-1} \rightarrow (1)$$

$$\alpha S = \alpha + \alpha^2 + \alpha^3 + \dots \alpha^{n-1} + \alpha^n \rightarrow (2)$$

Subtract (1) from (2) $\quad \cancel{\alpha S} \; \alpha S - S = \alpha^n - 1 \Rightarrow S = \dfrac{\alpha^n - 1}{\alpha - 1}$

Re-sub $\alpha = 1+i \Rightarrow S = (\cancel{1+i})^n \dfrac{(1+i)^n - 1}{1+i-1} = \dfrac{(1+i)^n - 1}{i}$

5

$$\boxed{B(n) = B_0(1+i)^n + d\,\frac{(1+i)^n - 1}{i}}$$

Ex ~~At~~ College is supposed to cost \$100K in 2042.
How much money should be deposited each year
in a fund that bears 5% interest per year?
Assume zero starting balance.

$B(18) = 100$ $\quad B_0 = 0$

$i = 5\%$ $\quad n = 18$ years

$$B(18) = 0 + d\,\frac{(1+i)^n - 1}{i}$$

$$100 = d\,\frac{(1.05)^{18} - 1}{0.05}$$

$$\Rightarrow \boxed{d = 3554/\text{year}}$$

Population Model

Let $P(n)$ be the population at time n

$$P(n+1) = P(n) + (b - d)P(n) + I - E$$

#births, #deaths, immigration, emigration.

= Let $r \triangleq b - d$ $\quad$ $M \triangleq I - E$ net migration.

$$P(n+1) = P(n) + rP(n) + M$$

$$= P(n)(1+r) + M$$

(1) Let $M = 0$; $P(0) = P_0$

$$\boxed{P(n+1) = P(n)(1+r)}$$

$P(1) = P(0)(1+r) \Rightarrow P(1) = P_0(1+r)$

$P(2) = P(1)(1+r) \Rightarrow P(2) = P_0(1+r)^2$

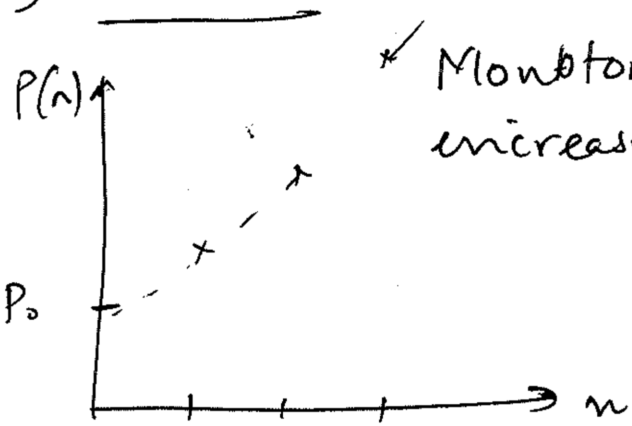
$\vdots$

$$\boxed{P(n) = P_0(1+r)^n}$$

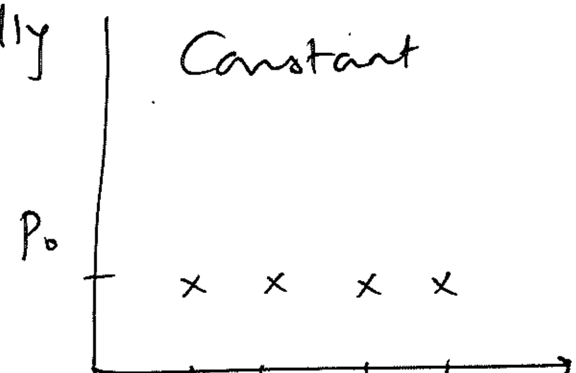
let $\alpha \triangleq 1+r$

$P(n) = P_0(1+r)^n = P_0\alpha^n$

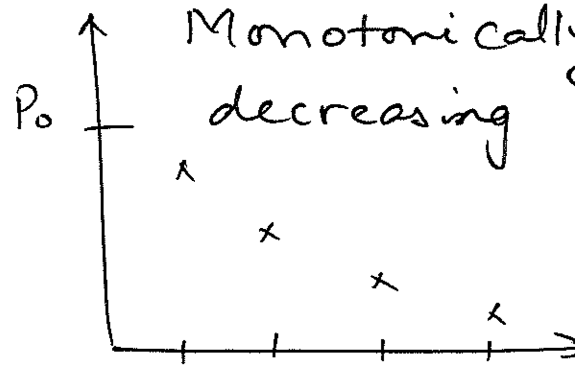
1) $\alpha > 1$ Monotonically increasing



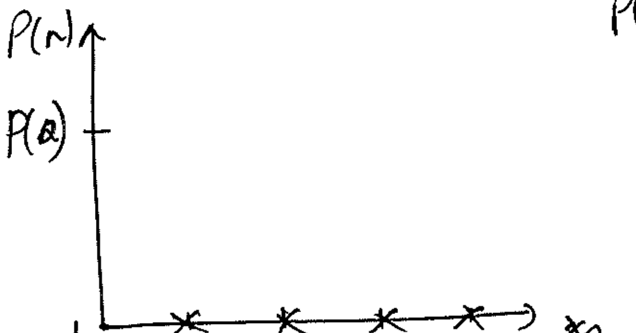
2) $\alpha = 1$ Constant



3) $0 < \alpha < 1$ Monotonically decreasing



4) $\alpha = 0$



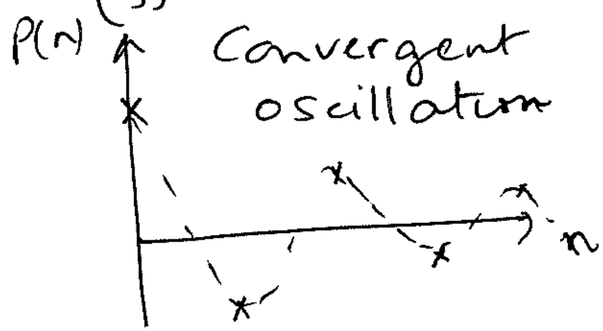
$\alpha = -1/2$

$P(1) = P_0(-1/2)$

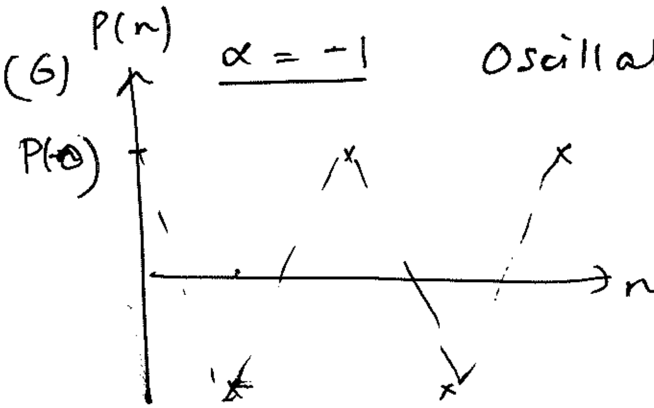
$P(2) = P_0(-1/2)^2$

$P(3) = P_0(-1/2)^3$

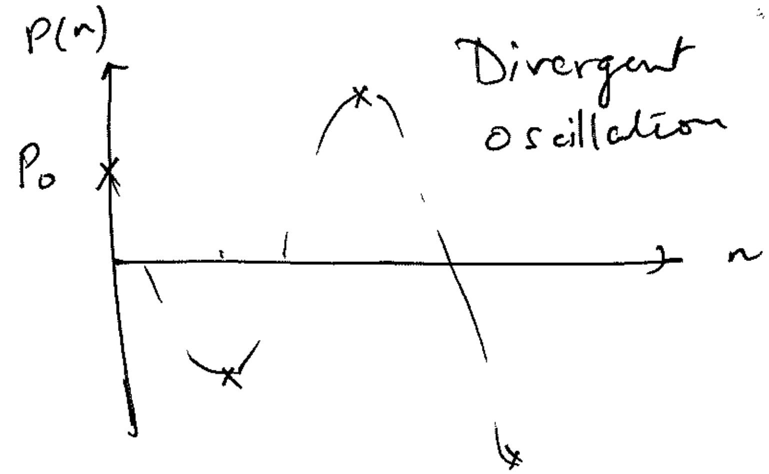
(5) $-1 < \alpha < 0$ Convergent oscillation



(6) $\alpha = -1$ Oscillation



(7) ~~α>1~~ $\alpha < -1$ Divergent oscillation



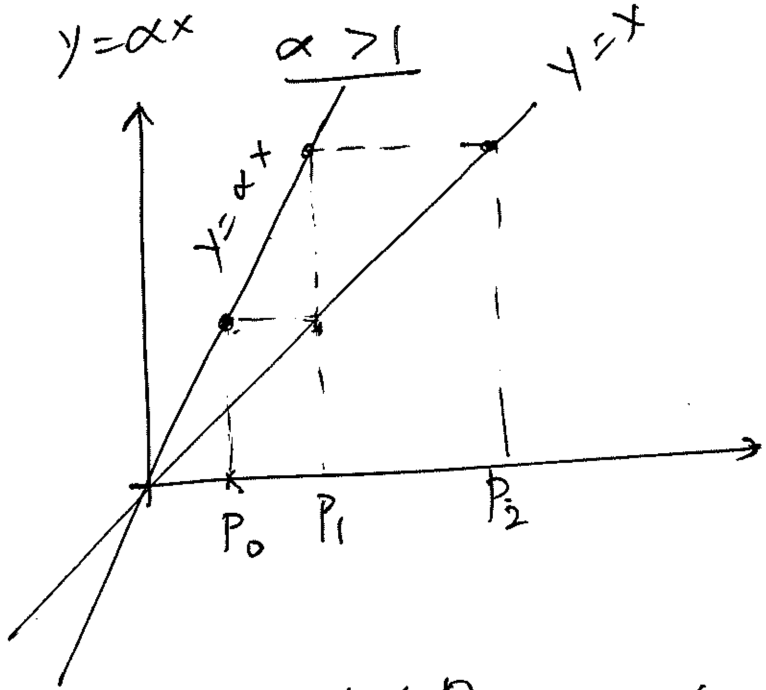
Cobweb method

$$\underbrace{P(n+1)} = \alpha P(n)$$

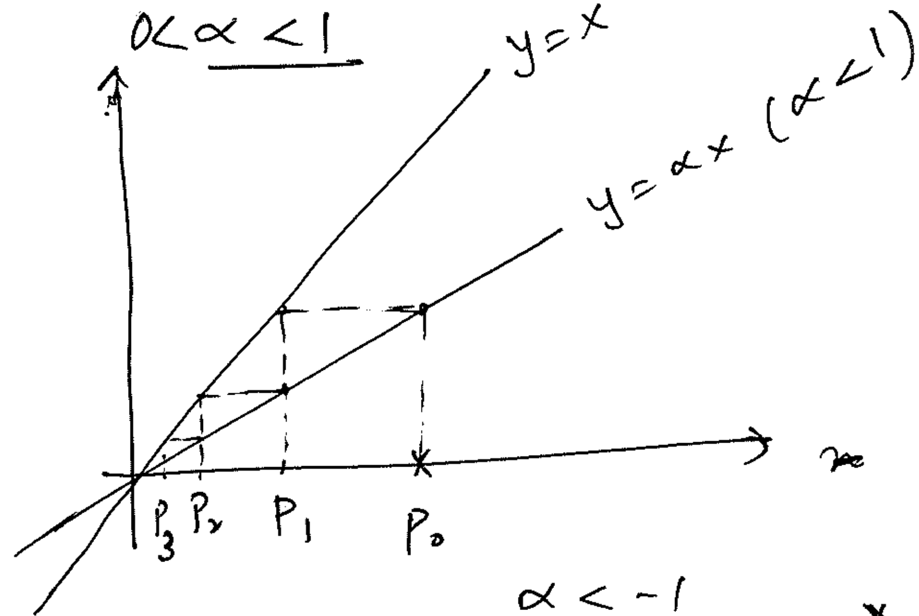
Relabelling

$$y_{n+1} = \alpha y_n$$

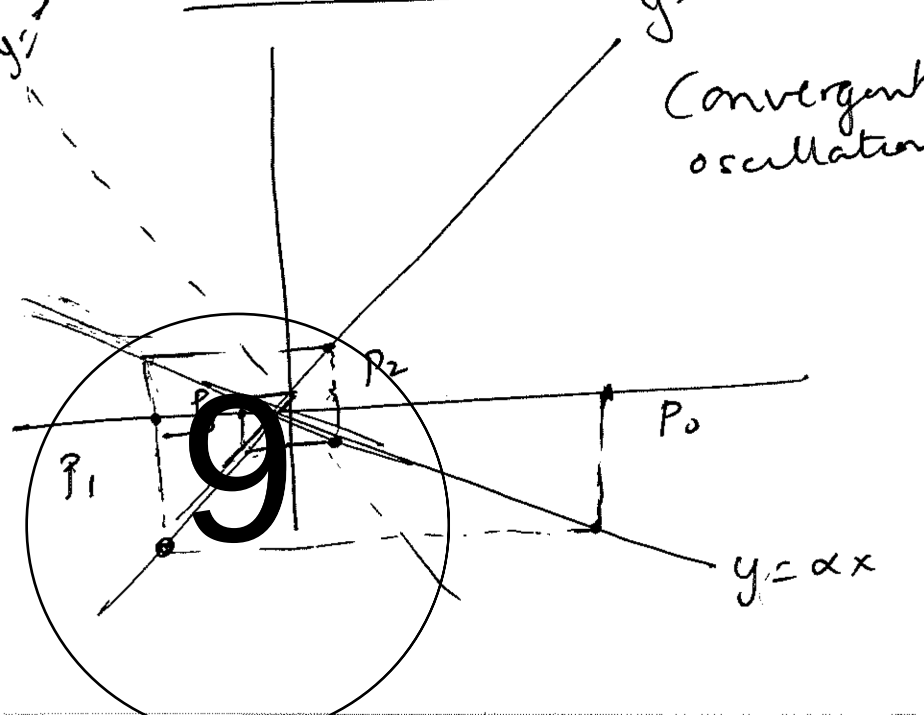
$\alpha > 1$



$0 < \alpha < 1$

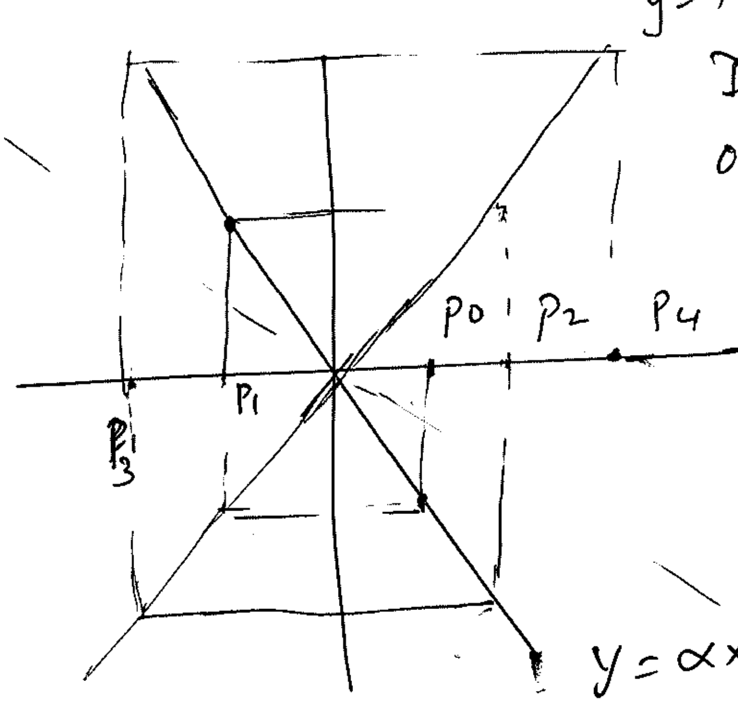


$-1 < \alpha < 0$



Convergent oscillation

$\alpha < -1$



Divergent oscillation.