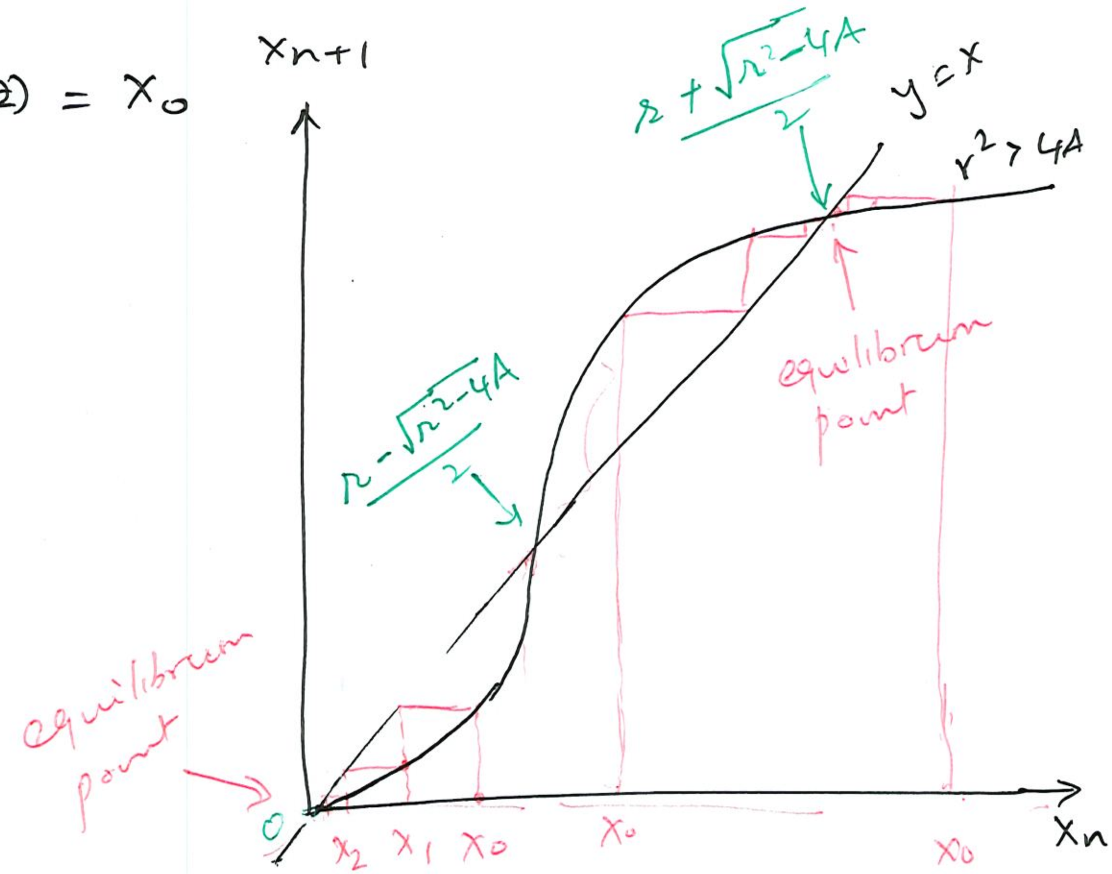
FIRST ORDER DIFFERENCE EQUATIONS

$$x_{n+1} = f(x_n, n)$$

$$x_{n+1} = \frac{r x_n^2}{A + x_n^2} \quad ; \quad x(0) = x_0$$



$\underline{n=0}$ $\quad x_1 = \dfrac{r x_0^2}{A + x_0^2}$

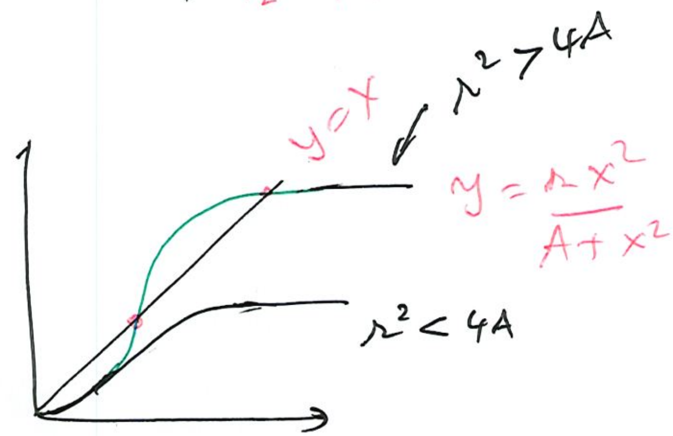
$\underline{n=1}$: $\quad x_2 = \dfrac{r x_1^2}{A + x_1^2}$

$$y = \frac{r x^2}{A + x^2} \qquad r > 0, A > 0$$

For small x $\quad y \approx \dfrac{r x^2}{A}$
$(A >> x^2)$

For large x, $A << x^2$

$$y = \frac{r x^2}{x^2} = r$$



$y = x = \dfrac{r x^2}{A + x^2} \Rightarrow x(A + x^2) = r x^2$

$\Rightarrow x = 0$, ~~A+~~ or

$A + x^2 = r x$

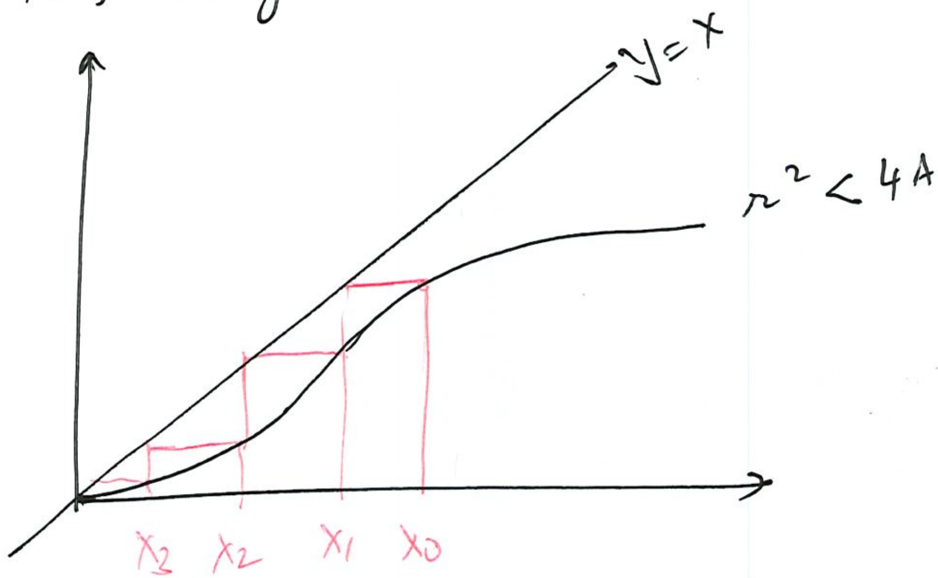
$\Rightarrow x^2 - r x + A = 0$

$$x^2 - rx + A = 0 \Rightarrow x = \frac{r \pm \sqrt{r^2 - 4A}}{2}$$

If $r^2 > 4A$ or $r > \sqrt{2A}$, we have three roots

$$x = 0, \quad \frac{r \pm \sqrt{r^2 - 4A}}{2}$$

If $r^2 < 4A$, only one root $x = 0$



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Continuous population model

Let $N(t)$ be the population at time t

$N(t+\Delta t) - N(t)$ → Change in population over time Δt

$\dfrac{N(t+\Delta t) - N(t)}{\Delta t}$ → rate of population change

$\dfrac{N(t+\Delta t) - N(t)}{\Delta t \quad N(t)}$ → per capita ~~growth~~ change

<u>~~Example~~ Case 1</u>: Constant per capita change.

$$\frac{N(t+\Delta t) - N(t)}{\Delta t \, N(t)} = \underset{(\text{const})}{r}$$

$$\lim_{\Delta t \to 0} \frac{N(t+\Delta t) - N(t)}{\Delta t} \frac{1}{N(t)} = r \implies \frac{dN}{dt} \cdot \frac{1}{N(t)} = r$$

$$\implies \boxed{\frac{dN}{dt} = r\,N(t)}$$

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$$\frac{dN}{dt} = rN(t) \quad ; \quad N(0) = N_0$$

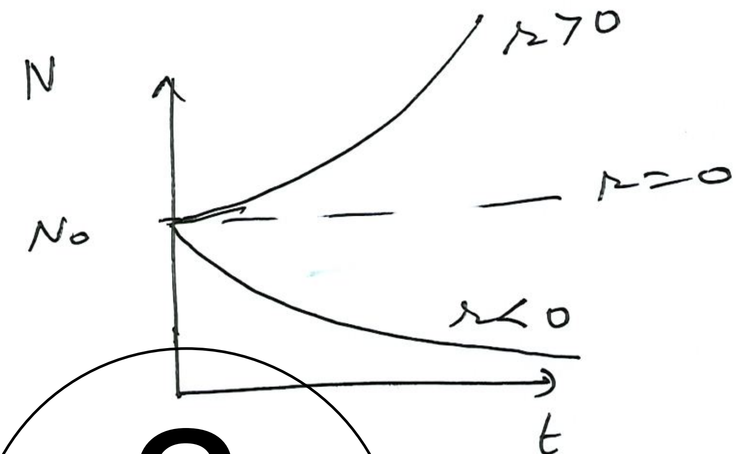
$$\int_{N_0}^{N(t)} \frac{dN}{N} = \int_0^t r\,dt$$

$$= r \int_0^t dt$$

$$\ln N \Big|_{N_0}^{N(t)} = rt \Rightarrow \ln N - \ln N_0 = rt$$

$$\Rightarrow \ln \frac{N}{N_0} = rt$$

$$\Rightarrow \boxed{N(t) = N_0 e^{rt}}$$



$\boxed{r<0}$ $\quad [r] = 1/s \leftarrow$ Rate

$\tau = \frac{1}{|r|} \leftarrow$ time constant

$$N(t) = N_0 e^{-\frac{t}{\tau}}$$

t	$N(t)$
0	N_0
1τ	$N_0 e^{-1} = 0.368 N_0$
2τ	$0.135 N_0$
3τ	$0.05 N_0$
4τ	$0.02 N_0$
5τ	$0.007 N_0$

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<u>Case 2</u>

$$\frac{dN}{dt}\frac{1}{N(t)} = r(N)$$

$$= a - bN$$

$$\frac{dN}{dt} = (a - bN)N. \Rightarrow \int_{N_0}^{N} \frac{dN}{N(a-bN)} = \int_0^t dt$$

$$\frac{1}{N(a-bN)} = \frac{C}{N} + \frac{D}{a-bN}$$

<u>C:</u> $$\frac{\cancel{N}}{\cancel{N}(a-bN)} = \frac{C\cancel{N}}{\cancel{N}} + D\frac{N}{a-bN} \Rightarrow \frac{1}{a-bN} = C + D\frac{N}{a-bN}$$

Set $N=0$ $$C = \frac{1}{a}$$

<u>D:</u> $$\frac{1}{N} = C\frac{a-bN}{N} + D; \text{ Set } N = \frac{a}{b} \Rightarrow \frac{b}{a} = D$$

$$\int \frac{dN}{N(a-bN)} = \int_{N_0}^{N} \frac{1}{a}\frac{dN}{N} + \int_{N_0}^{N_0} \frac{b}{a}\frac{dN}{a-bN}$$

$$\Rightarrow \frac{1}{a}\ln N\Big|_{N_0}^{N} + \frac{b}{a}\frac{\ln(a-bN)}{-b}\Big|_{N_0}^{N}$$

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$$\Rightarrow \frac{1}{a}\ln\frac{N}{N_0} - \frac{1}{a}\ln\frac{a-bN}{a-bN_0} = t$$

$$\ln\left(\frac{\frac{N}{N_0}}{\frac{a-bN}{a-bN_0}}\right) = at$$

$$\Rightarrow \ln\left[\left(\frac{N}{a-bN}\right)\left(\frac{a-bN_0}{N_0}\right)\right] = at$$

$$\left(\frac{N}{a-bN}\right) = \underbrace{\frac{N_0}{(a-bN_0)}}_{\alpha} e^{at} \Rightarrow \frac{N}{a-bN} = \alpha e^{at}$$

$$\Rightarrow N = a\alpha e^{at} - b\alpha e^{at} N$$

$$\Rightarrow N(1 + b\alpha e^{at}) = a\alpha e^{at}$$

$$\Rightarrow \boxed{N(t) = \frac{a\alpha e^{at}}{1 + b\alpha e^{at}}}$$

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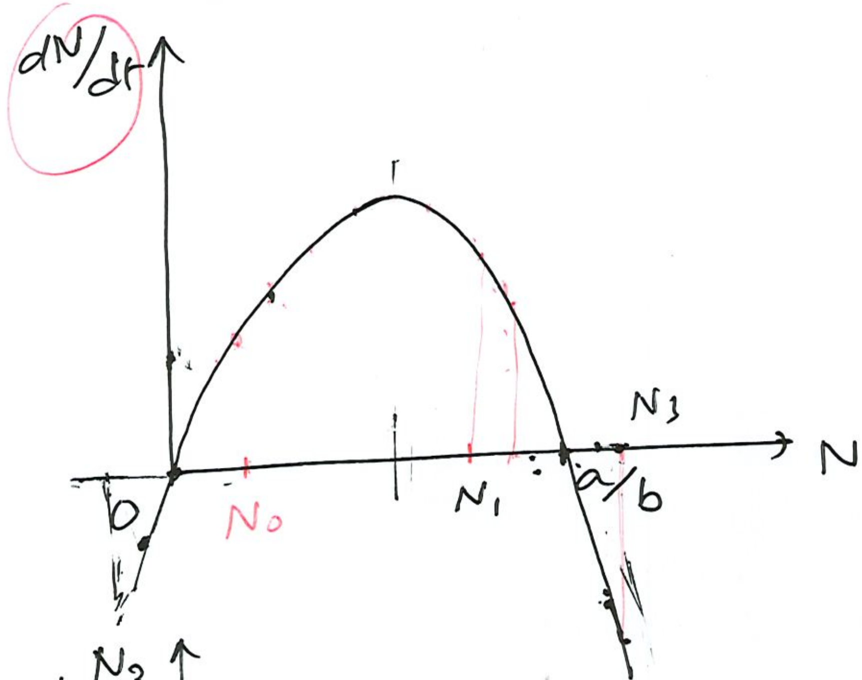
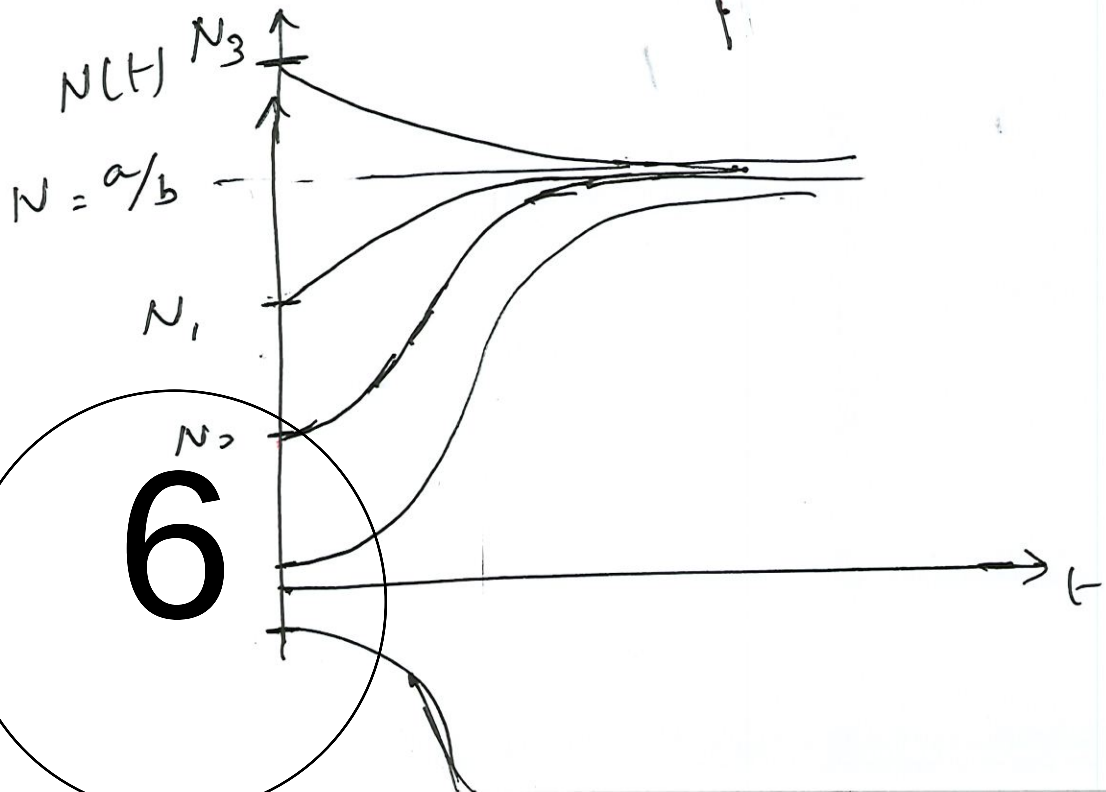
$$\frac{dN}{dt} = (a - bN)N \qquad a, b > 0$$

$$= aN - bN^2$$





At $N = N_0$, $\frac{dN}{dt} > 0$

<u>Equilibrium value</u>

$$\frac{dN}{dt} = 0$$

$$(a - bN)N = 0$$

$$N = 0 \text{ or } N = \frac{a}{b}$$

$N = 0 \longrightarrow$ unstable equilibrium point

$N = \frac{a}{b} \longleftarrow$ Stable equilibrium.

Linearization about equilibrium points

Taylor series of a function $f(x)$ about $x = a$ is

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \dots$$

Linear approximation of $f(x)$ about $x = a$ is

$$f(x) \simeq f(a) + f'(a)(x-a)$$

$$\frac{dN}{dt} = f(N) \quad \text{where} \quad f(N) = (a - bN)N = aN - bN^2$$

$$f'(N) = a - 2bN$$

About $N = 0$:

$$f(0) = 0 \;; \quad f'(0) = a$$

$$f(N) = \cancel{f(0)}^{\,0} + \underbrace{f'(0)(N-0)}_{aN} \Rightarrow f(N) = aN.$$

$$\frac{dN}{dt} = aN \Rightarrow N(t) = N_0 e^{at}$$

$N(t)$ increases with t

$\Rightarrow$ unstable.

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About $N = a/b$ $\quad f\left(\frac{a}{b}\right) = 0 \quad f'\left(\frac{a}{b}\right) = a - (2b)\frac{a}{b}$

$$= -a$$

$$f(N) = \cancelto{0}{f\left(\frac{a}{b}\right)} + f'\left(\frac{a}{b}\right)\left(N - \frac{a}{b}\right)$$

$$f(N) = -a\left(N - \frac{a}{b}\right)$$

$$\frac{dN}{dt} \approx -a\left(N - \frac{a}{b}\right)$$

Let $\tilde{N} = N - \frac{a}{b} \Rightarrow \frac{d\tilde{N}}{dt} = \frac{dN}{dt}$

$$\frac{d\tilde{N}}{dt} = -a\tilde{N} \Rightarrow \tilde{N}(t) = \tilde{N}_0 e^{-at}$$

$\Rightarrow \tilde{N}(t)$ decreases to 0

$\Rightarrow N - \frac{a}{b} \to 0$

$\Rightarrow N \to \frac{a}{b} \to$ stable equilibrium point

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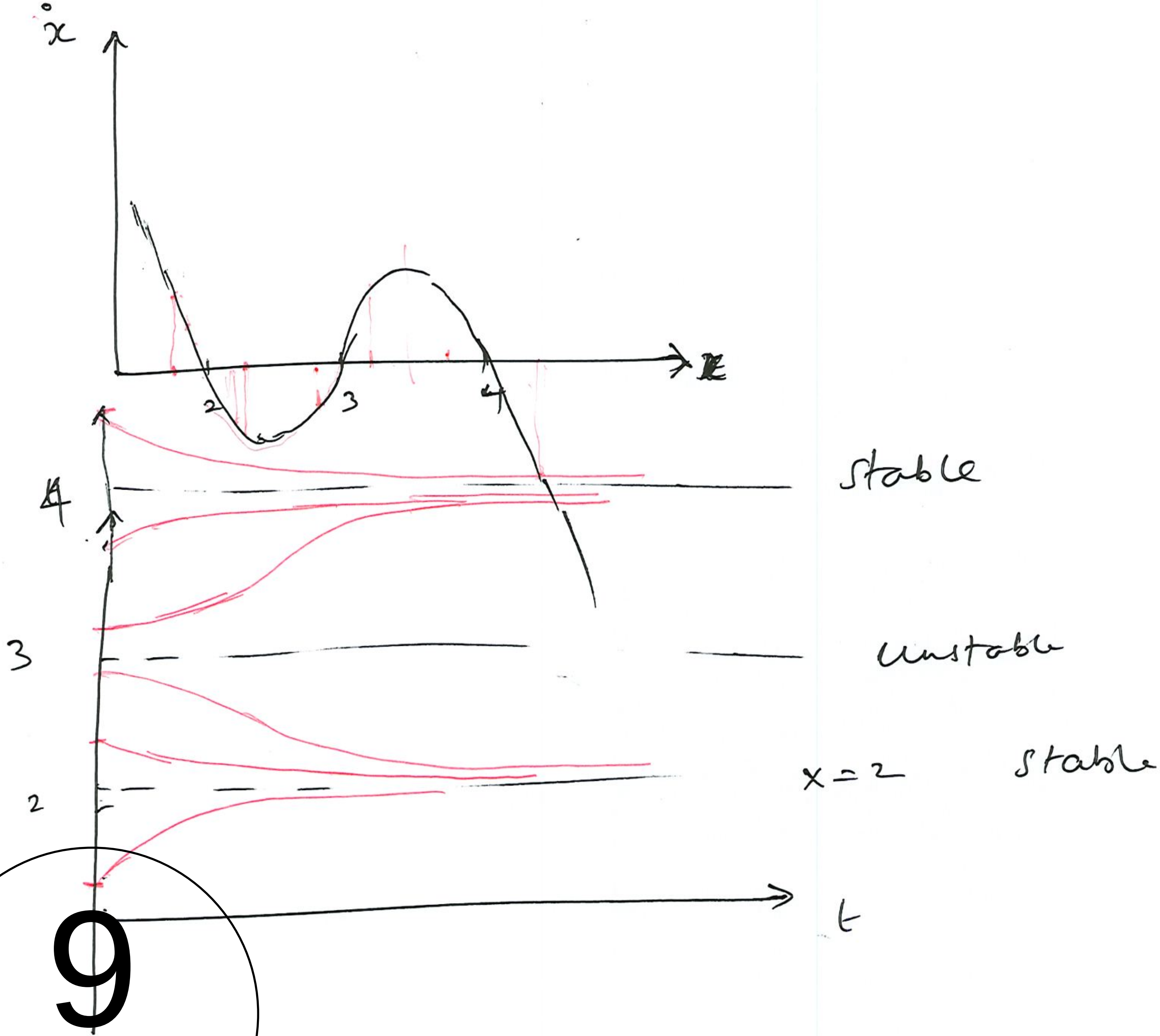
$\dot{x} = (x-2)(3-x)(x-4)$



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