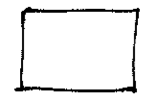
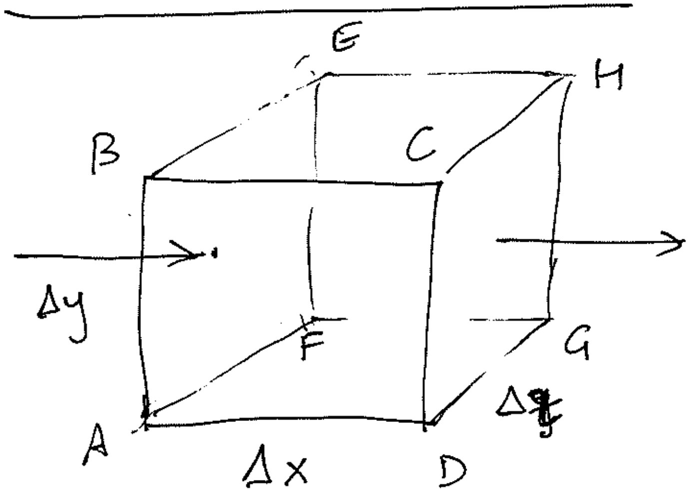
1

Differential Analysis

1) **Mass conservation** 1-D $\longrightarrow$ 



$$\dot{m}_{ABEF} - \dot{m}_{CHGD} = \frac{\partial}{\partial t} m_{cv}$$

$$(\rho u)_x \Delta y \Delta z - (\rho u)_{x+\Delta x} \Delta y \Delta z = \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z)$$

$$(\rho u)_x \Delta y \Delta z - \left[(\rho u)_x \Delta y \Delta z + \frac{\partial}{\partial x}(\rho u) \Delta x \Delta y \Delta z\right] = \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z)$$

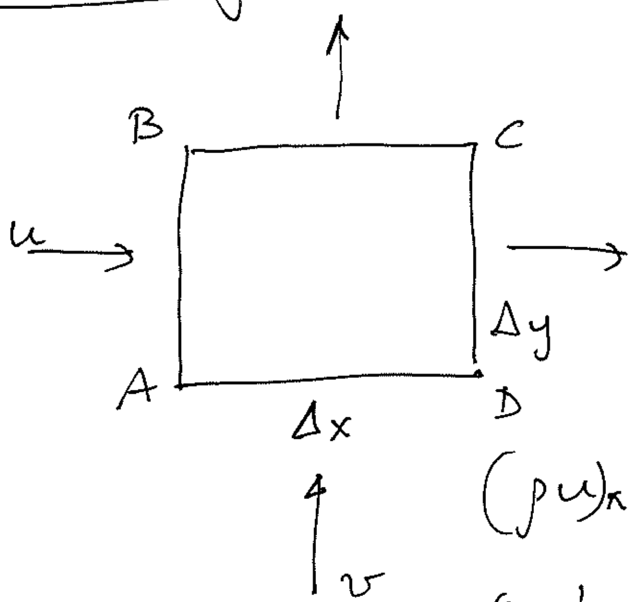
$$-\frac{\partial}{\partial x}(\rho u) = \frac{\partial}{\partial t}(\rho) \Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0} \; *$$

Continuity Equation in 1D

Steady State: $\partial \rho / \partial t = 0 \Rightarrow \frac{\partial}{\partial x}(\rho u) = 0$

Incompressible : $\rho \frac{\partial}{\partial x} u = 0 \Rightarrow \frac{\partial u}{\partial x} = 0 \Rightarrow u$ is constant in x

Continuity in 2D



$$\dot{m}_{AB} - \dot{m}_{CD} + \dot{m}_{AD} - \dot{m}_{CB} = \frac{\partial}{\partial t} m_{cv}$$

$$(\rho u)_x \Delta y \Delta z - (\rho u)_{x+\Delta x} \Delta y \Delta z$$

$$+ (\rho v)_y \Delta x \Delta z - (\rho v)_{y+\Delta y} \Delta x \Delta z = \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z)$$

$$(\rho u)_x \Delta y \Delta z - \left[(\rho u)_x + \frac{\partial}{\partial x}(\rho u)\Delta x\right] \Delta y \Delta z$$

$$+ (\rho v)_y \Delta x \Delta z - \left[(\rho v)_y + \frac{\partial}{\partial y}(\rho v)\Delta y\right] \Delta x \Delta z = \Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t}$$

$$-\frac{\partial}{\partial x}(\rho u) - \frac{\partial}{\partial y}(\rho v) = \frac{\partial \rho}{\partial t}$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$$

Let $\rho \underset{\sim}{u} \triangleq \rho u \hat{i} + \rho v \hat{j}$; $\nabla \triangleq \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}$ (Del operator)

$$\nabla \cdot \rho \underset{\sim}{u} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right) \cdot \left(\rho u \hat{i} + \rho v \hat{j}\right) = \frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \underset{\sim}{u} = 0}$$

$\nabla \cdot \rho \underset{\sim}{u}$ is called the divergence of $\rho \underset{\sim}{u}$

Steady State : $\frac{\partial \rho}{\partial t} = 0$

$$\Rightarrow \quad \nabla \cdot (\rho u) = 0$$

Incompressible : $\rho \nabla \cdot \underset{\sim}{u} = 0$

$$\Rightarrow \quad \nabla \cdot \underset{\sim}{u} = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

In 3D : $\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \underset{\sim}{u} = 0$

$$\rho \underset{\sim}{u} = \rho u \hat{i} + \rho v \hat{j} + \rho w \hat{k}$$

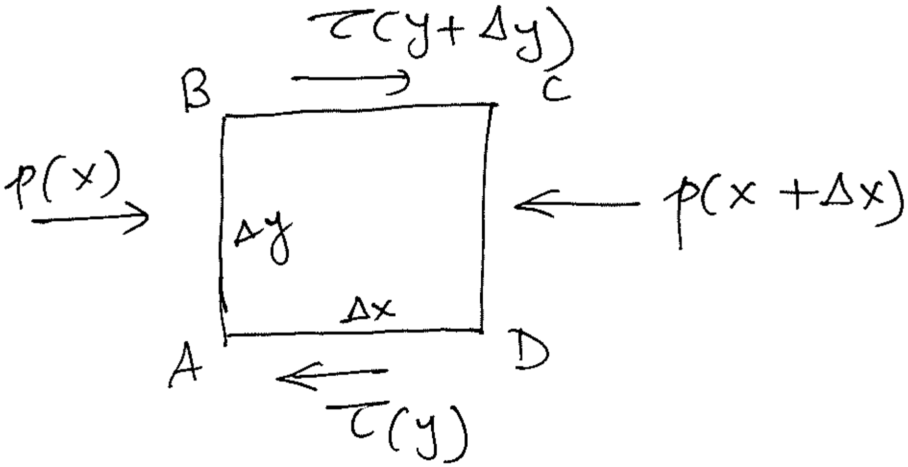
$$\nabla \doteq i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}$$

Incompressible / Steady state

$$\nabla \cdot \underset{\sim}{u} = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Momentum Balance:



$$\mathring{L}_{AB} - \mathring{L}_{CD} + \Sigma F = \frac{\partial}{\partial t} L_{cv}$$

$$\mathring{L}_{AB} = \mathring{m}_{AB}\, u$$
$$= \rho u)(\Delta \cancel{y} \Delta z)\, u$$
$$= (\rho u^2)_x \Delta y \Delta z$$

$$\mathring{L}_{CD} = \mathring{m}_{CD}\, u$$
$$= (\rho u^2)_{x+\Delta x} \Delta y \Delta z$$

$$F_{AB} = p(x)\, \Delta y \Delta z \qquad F_{CD} = -p(x+\Delta x)\, \Delta y \Delta z$$

$$F_{AD} = -\tau(y)\, \Delta x \Delta z \qquad F_{BC} = \tau(y+\Delta y)\, \Delta x \Delta z$$

$$\left[(\rho u^2)_x \Delta y \Delta z - (\rho u^2)_{x+\Delta x} \Delta y \Delta z\right] + \left(p(x)\, \Delta y \Delta z - p(x+\Delta x)\, \Delta y \Delta z\right)$$
$$+ \left(\tau(y+\Delta y) \Delta x \Delta z\right) - \tau(y) \Delta x \Delta z) = \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z\, u)$$

$$-\frac{\partial}{\partial x}(\rho u^2)\, \cancel{\Delta x \Delta y \Delta z} - \frac{\partial p}{\partial x}\, \cancel{\Delta x \Delta y \Delta z} + \frac{\partial \tau}{\partial y}\, \cancel{\Delta y \Delta x \Delta z} = \cancel{\Delta x \Delta y \Delta z}\, \frac{\partial}{\partial t}(\rho u)$$

$$-\frac{\partial}{\partial x}(\rho u^2) - \frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y} = \frac{\partial}{\partial t}(\rho u)$$

$$-\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y} = \frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2)$$

$$\frac{\partial}{\partial x}(\rho u^2) = \frac{\partial}{\partial x}(\rho u \cdot u)$$
$$= \rho u \frac{\partial u}{\partial x} + u \frac{\partial (\rho u)}{\partial x}$$

$$\frac{\partial}{\partial t}(\rho u) = \rho \frac{\partial u}{\partial t} + u \frac{\partial \rho}{\partial t}$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) = \rho\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t} + \rho u\frac{\partial u}{\partial x} + u\frac{\partial}{\partial x}(\rho u)$$

$$= \rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + u\underbrace{\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u)\right]}_{0}$$

(By continuity see slide 1)

$$= \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right]$$

$$= \rho\underbrace{\left[\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right]}_{D/Dt} u$$

$$\frac{D}{Dt} \triangleq \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$$ ← Material Derivative/ Substantive Derivative Total Derivative

$$\boxed{\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y}}$$

<u>Newtonian fluid</u> $\tau = \mu\frac{\partial u}{\partial y}$

$$\boxed{\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}}$$ ← Navier Stokes Equation 1D

<u>In 2D</u>

$\underline{X}$: $\cancel{\rho \frac{Du}{}}$ $\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

$\underline{Y}$: $\rho\left[\cancel{\frac{Dv}{Dt}}\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)$$

$$= \frac{\partial}{\partial t} \cancel{+ \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right)\cdot u\hat{i} +}$$

$$= \frac{\partial}{\partial t} + (u\hat{i} + v\hat{j})\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right)$$

$$= \left(\frac{\partial}{\partial t} + \underset{\sim}{u}\cdot\nabla\right)$$

$$\boxed{\rho\frac{D\underset{\sim}{u}}{Dt} = -\nabla p + \mu\nabla^2\underset{\sim}{u}}$$

$$\nabla^2 \underset{\sim}{u} = \nabla\cdot(\nabla u)$$

$$\nabla u = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right)u$$

$$= \left(\hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y}\right)$$

$$\nabla\cdot\nabla u = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right)\cdot\left(\hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

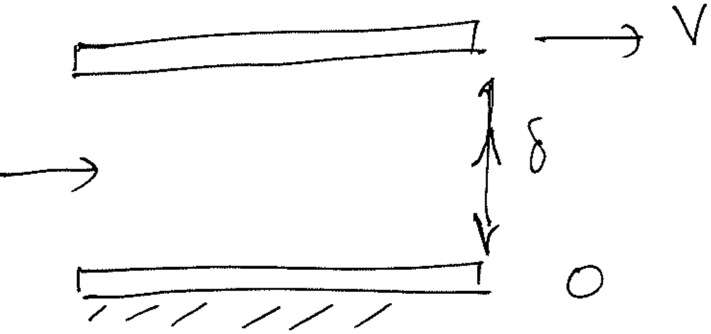
$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$$

$$\rho \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right] = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$$

<u>Steady State:</u> $\partial u / \partial t = 0$

<u>Uniform flow:</u> $\frac{\partial u}{\partial x} = 0$

$$-\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} = 0$$



$$\mu \frac{d^2 u}{dy^2} = \frac{dp}{dx}$$

$$\mu \frac{du}{dy} = \frac{dp}{dx} y + c_1$$

$$\mu u = \frac{dp}{dx} \frac{y^2}{2} + c_1 y + c_2$$

$u(y=0) = 0 \Rightarrow 0 = 0 + c_1(0) + c_2 \Rightarrow c_2 = 0$

$u(y=\delta) = V \Rightarrow \mu V = \frac{dp}{dx} \frac{\delta^2}{2} + c_1 \delta \Rightarrow c_1 = \mu \frac{V}{\delta} - \frac{dp}{dx} \frac{\delta}{2}$

$$\mu u(y) = \frac{dp}{dx}\frac{y^2}{2} + \left(\mu\frac{V}{\delta} - \frac{dp}{dx}\frac{\delta}{2}\right)y$$

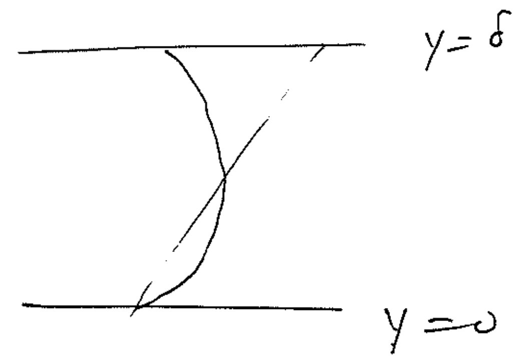
$$u(y) = \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - \delta y\right) + \frac{V}{\delta}y$$

$$= \frac{1}{2\mu}\left(-\frac{dp}{dx}\right)\left(\delta y - y^2\right) + \frac{V}{\delta}y$$

$$= \frac{1}{2\mu}\left(-\frac{dp}{dx}\right)\delta^2\left[\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right] + \left(\frac{V}{\delta}\right)y$$

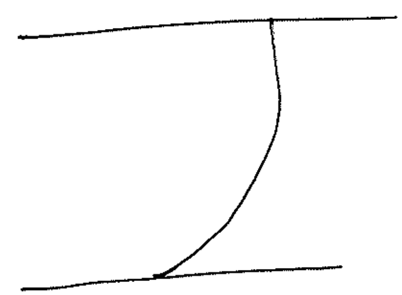
When $y = 0$; $u(0) = 0$

$y = \delta$; $u(\delta) = V$

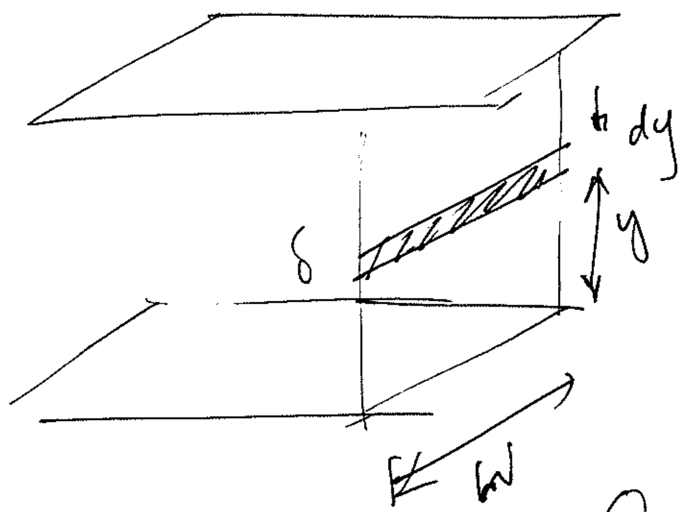


$$\tau(y=0) = \mu\frac{\partial u}{\partial y}\bigg|_{y=0}$$

$$\tau(y=\delta) = \mu\frac{\partial u}{\partial y}\bigg|_{y=\delta}$$



Flow Rate of fluid between the plates



$$Q = \int_\delta dQ = \int (dy\, W)\, u(y)$$

$$Q = W \int_0^\delta \left(\frac{1}{2\mu}\left(-\frac{dp}{dx}\right)\delta^2 \left[\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] + \frac{V}{\delta}\, y \right) dy$$

Let $\eta = \frac{y}{\delta} \Rightarrow d\eta = \frac{dy}{\delta} \Rightarrow dy = \delta\, d\eta$

$$Q = W \int_0^1 \left(\frac{1}{2\mu}\left(-\frac{dp}{dx}\right)\delta^2 \left[\eta - \eta^2\right] + V\eta \right)\delta\, d\eta$$

$$= W\left(\frac{1}{2\mu}\left(-\frac{dp}{dx}\right)\delta^3 \left[\frac{\eta^2}{2} - \frac{\eta^3}{3}\right]_0^1 + \frac{V\eta^2}{2}\Big|_0^1 \delta \right)$$

$$= W\left[\frac{1}{2\mu}\left(-\frac{dp}{dx}\right)\delta^3 \frac{1}{6} + \frac{V\delta}{2} \right]$$

$$(u_{avg})\, A = Q$$

$$(u_{avg})\, W\delta = W\left[\frac{1}{2\mu}\left(-\frac{dp}{dx}\right)\frac{\delta^3}{6} + \frac{V\delta}{2} \right]$$

$$\boxed{u_{avg} = \frac{1}{2\mu}\left(-\frac{dp}{dx}\right)\frac{\delta^2}{6} + \frac{V}{2}}$$

Pressure gradient required to obtain a flow rate Q

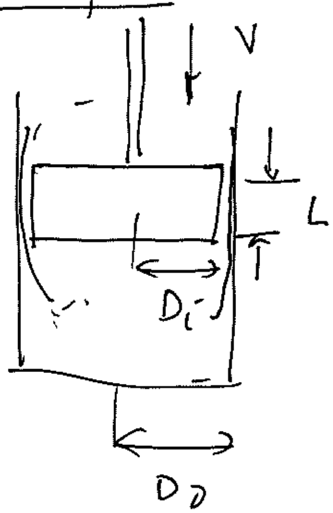
Since $$Q = W\left[\frac{1}{2\mu}\left(-\frac{dp}{dx}\right)\frac{\delta^3}{6} + \frac{V}{2}\delta\right]$$

$$\left(\frac{dp}{dx}\right) = \left(-\frac{Q}{W} + \frac{V}{2}\delta\right)\frac{6}{\delta^3}\,2\mu$$

$$\frac{Q}{W} = \frac{1}{2\mu}\left(-\frac{dp}{dx}\right)\frac{\delta^3}{6} + \frac{V}{2}\delta$$

$$\frac{1}{2\mu}\left(\frac{dp}{dx}\right)\frac{\delta^3}{6} = \frac{V}{2}\delta - \frac{Q}{W}$$

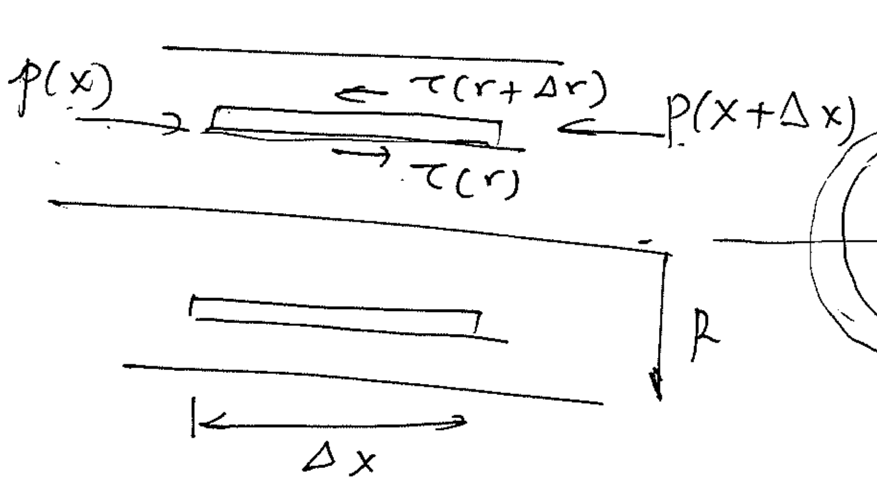
Dashpot

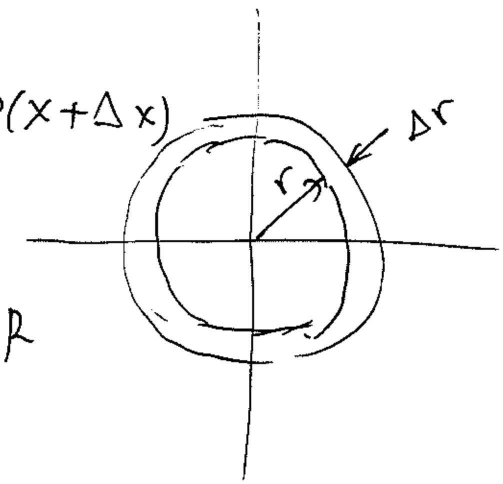


$$\delta = \frac{D_o - D_i}{2}$$

$$\frac{dp}{dx} = \frac{P_2 - P_1}{L}$$

Flow in pipes





Steady and incompressible flow.

$\Sigma F_x = 0.$

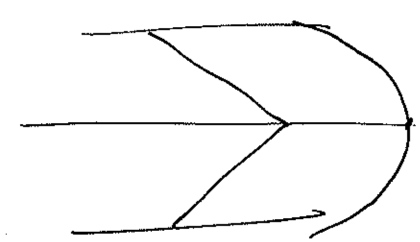
$$p(x)\, 2\pi r \Delta r - p(x+\Delta x)\, 2\pi r \Delta r + \left(\tau(r) A(r)\right) - \left(\tau(r) A(r)\right)_{r+\Delta r} = 0$$

$$-\frac{\partial p}{\partial x} 2\pi r \Delta r \Delta x - \frac{\partial}{\partial r}(\tau A)\Delta r = 0$$

$A = 2\pi r \Delta x$

$$-\frac{\partial p}{\partial x} 2\pi r \Delta r \Delta x - \frac{\partial}{\partial r}\left(2\pi r \Delta x\, \tau(r)\right)\Delta r = 0$$

$$-\frac{\partial p}{\partial x} r - \frac{\partial}{\partial r}\left(\tau(r)\, r\right) = 0$$



$$\frac{\partial}{\partial r}\left(\tau(r)\, r\right) = -\frac{\partial p}{\partial x} r$$

$$\frac{\partial}{\partial r}\left(-\mu \frac{\partial \mu}{\partial r} r\right) = -\frac{\partial p}{\partial x} r$$

$$\Rightarrow \boxed{\frac{\partial}{\partial r}\left(\mu r \frac{\partial u}{\partial r}\right) = \frac{\partial p}{\partial x} r \;;\quad u(R) = 0 \quad \left.\frac{\partial u}{\partial r}\right|_{r=0} = 0}$$

1) Solve for u

2) & Determine Q

3) Determine u_{avg}

4) Derive : $\Delta p = f \frac{L}{D} \frac{V^2}{2}$ where $f = \frac{64}{Re}$