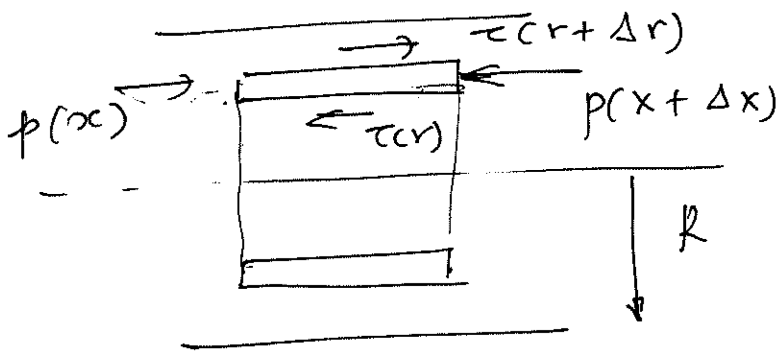
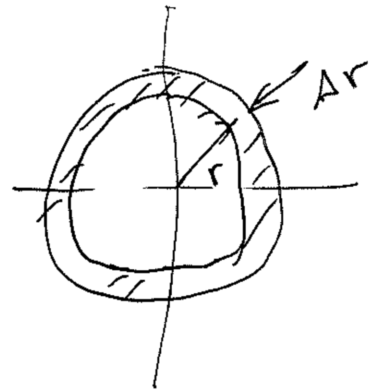
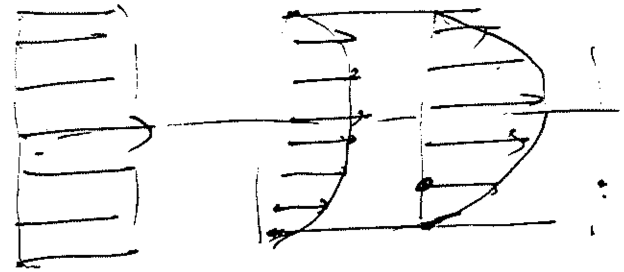
Pipe flow







Steady State

$$\dot{L}_{in} - \dot{L}_{out} + \sum F = \frac{\partial \cancel{L_{cv}}^{\,0}}{\partial t}$$

$$\dot{L}_{in} - \dot{L}_{out} = 0 \Rightarrow \text{no velocity gradient in } x\text{-direction}$$

(i.e. fully developed)

$$p(x)\, 2\pi r \Delta r - p(x+\Delta x)\, 2\pi r \Delta r - [\tau(r) A(r)] + [\tau(r+\Delta r) A(r+\Delta r)]$$

$$- (\tau A)_r + (\tau A)_{r+\Delta r} = 0$$

$$p(x)\, 2\pi r \Delta r - \frac{\partial p}{\partial x}\Delta x\, 2\pi r \Delta r - p(x)\, 2\pi r \Delta r - (\tau A)_r + (\tau A)_r + \frac{\partial (\tau A)}{\partial r}\Delta r = 0$$

$$-\frac{\partial p}{\partial x} 2\pi r \Delta r \Delta x + \frac{\partial}{\partial r}(\tau A)\,\Delta r = 0 \qquad A = 2\pi r \Delta x$$

$$-\frac{\partial p}{\partial x} \cancel{2\pi} r \cancel{\Delta r} \cancel{\Delta x} + \frac{\partial}{\partial r}\left(\cancel{2\pi} r \cancel{\Delta x}\, \tau\right) \cancel{\Delta r} = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial r}(r\tau) = \frac{\partial p}{\partial x} r$$

$$\tau = \mu \frac{du}{dr}$$

$$\frac{\partial}{\partial r}\left(r \mu \frac{\partial u}{\partial r} \right) = \frac{dp}{dx} r$$

$$\cancel{\left(r \mu \frac{du}{dr} \right.} = 1 \qquad \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r} \right) = \frac{1}{\mu} \frac{dp}{dx} r$$

$$r \frac{\partial u}{\partial r} = \frac{1}{\mu} \frac{dp}{dx} \frac{r^2}{2} + c_1$$

$$\frac{\partial u}{\partial r} = \frac{1}{\mu} \frac{dp}{dx} \frac{r}{2} + \frac{c_1}{r}$$

$$u(r) = \frac{1}{\mu} \frac{dp}{dx} \frac{r^2}{4} + c_1 \ln r + c_2$$

$$u(r = R) = 0 \qquad \& \qquad \tau = \mu \frac{du}{dr}\bigg|_{r=0} = 0$$

$$\cancel{0 = c_2} \qquad c_2 = -\frac{1}{4\mu} \frac{dp}{dx} \frac{R^2}{4}$$

$$\cancel{c_1 = 0} \qquad c_1 = 0$$

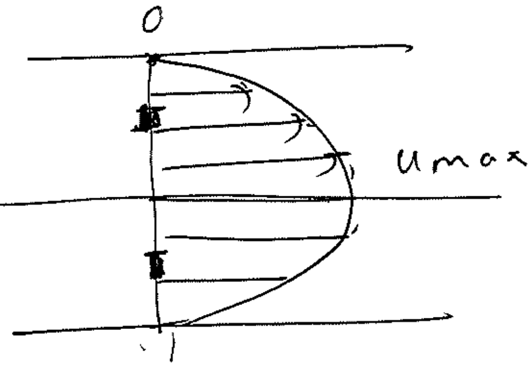
$$u(r) = \frac{1}{\mu} \frac{dp}{dx} \frac{r^2}{4} - \frac{1}{4\mu} \frac{dp}{dx} \frac{R^2}{4}$$

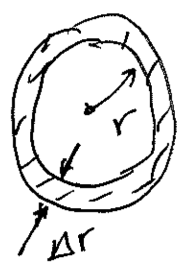
$$= \frac{1}{4\mu} \left(-\frac{dp}{dx} \right) \left(R^2 - r^2 \right)$$

$$\boxed{u(r) = \frac{1}{4\mu} \left(-\frac{dp}{dx} \right) R^2 \left[1 - \frac{r^2}{R^2} \right]}$$

Let $u_{max} \triangleq \frac{1}{4\mu}\left(-\frac{dp}{dx}\right) R^2$

$$\boxed{u(r) = u_{max}\left[1 - \frac{r^2}{R^2}\right]}$$





$$Q = \int dQ = \int_0^R (2\pi r\, dr)\, u(r)$$

$$= 2\pi u_{max} \int_0^R r\left(1 - \frac{r^2}{R^2}\right) dr$$

$$= 2\pi u_{max} \int_0^R \left(r - \frac{r^3}{R^2}\right) dr$$

$$= 2\pi u_{max} \left(\frac{R^2}{2} - \frac{R^4}{4R^2}\right)$$

$$= 2\pi u_{max} \frac{R^2}{4}$$

$$= \frac{\pi}{2} u_{max} R^2$$

Average velocity $\bar{u}$

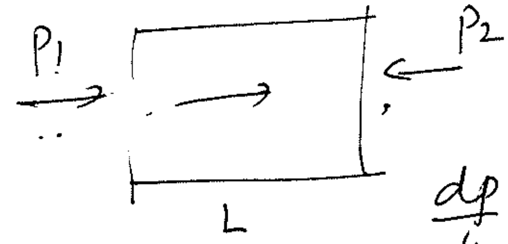
$$Q = A\bar{u} = \frac{\pi}{2} R^2 u_{max}$$

$$= \frac{A u_{max}}{2}$$

$$\bar{u} = \frac{u_{max}}{2}$$

$$u_{max} = 2\bar{u}$$

$$u_{max} = \frac{1}{4\mu}\left(-\frac{dp}{dx}\right) R^2$$



$$\frac{dp}{dx} = \frac{p_2 - p_1}{L}$$

$$-\frac{dp}{dx} = \frac{p_1 - p_2}{L}$$

$$-\frac{dp}{dx} = 4\mu \frac{u_{max}}{R^2}$$

$$\frac{p_1 - p_2}{L} = 4\mu \frac{2\bar{u}}{R^2}$$

$$= \frac{4\mu \, 2\bar{u}}{(D/2)^2}$$

$$= 32 \frac{\mu \bar{u}}{D^2}$$

$$\frac{p_1 - p_2}{L} = 32 \frac{\mu \bar{u}^2}{\bar{u} D \quad D}$$

$$= 32 \frac{\mu}{\bar{u} D} \frac{L}{D} \bar{u}^2$$

Multiply by 2 and divide by ρ

$$\frac{p_1 - p_2}{\rho} = 64 \frac{\mu}{\rho \bar{u} D} \frac{L}{D} \frac{\bar{u}^2}{2}$$

$$= \frac{64}{Re} \frac{L}{D} \frac{\bar{u}^2}{2} \implies \frac{p_1 - p_2}{\rho g} = \left(\frac{64}{Re}\right) \frac{L}{D} \frac{\bar{u}^2}{2g}$$

head loss $\quad f \frac{L}{D} \frac{\bar{u}^2}{2g}$

Wall shear stress

$$\tau = \mu \frac{du}{dr}\Big|_{r=R}$$

$$u(r) = u_{max}\left[1 - \frac{r^2}{R^2}\right]$$

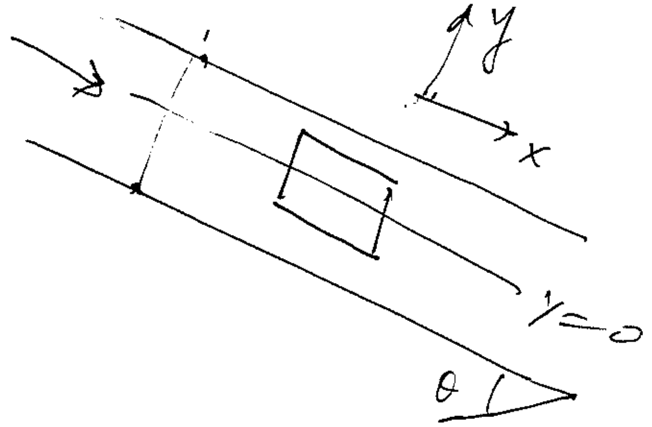
$$\frac{du}{dr} = u_{max}\left[-\frac{2r}{R^2}\right]\Big|_{R}$$

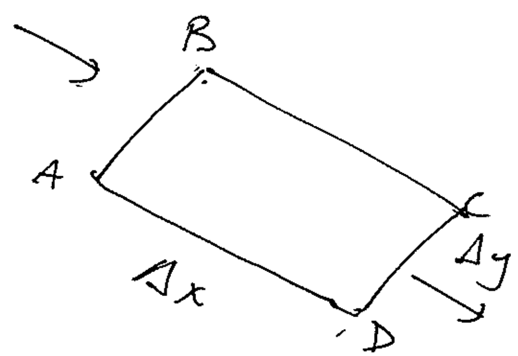
$$\frac{du}{dr}\Big|_{r=R} = u_{max}\left[-\frac{2}{R}\right]$$

$$\tau = \mu\, u_{max} \frac{(-2)}{R}$$

Wall friction loss (force) $= \tau A$

$$= \left(\mu\, u_{max} \frac{2}{R}\right)(2\pi R)\, L$$





$$\dot{m} = \rho A V$$

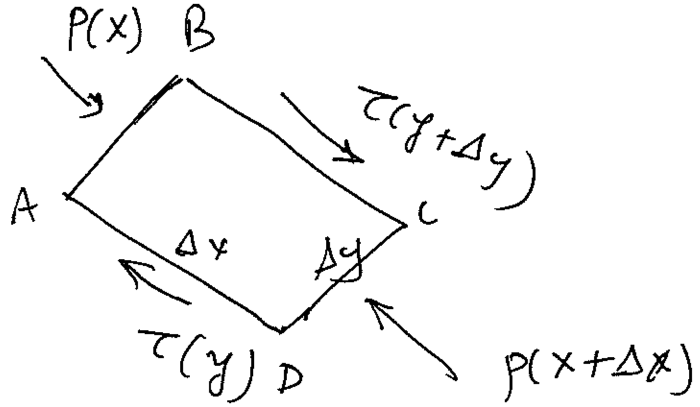
$$\dot{m}_{AB} - \dot{m}_{CD} = \frac{\partial}{\partial t} m_{cv}$$

$$(\rho u)_x \Delta y \Delta z - (\rho u)_{x+\Delta x} \Delta y \Delta z = \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z)$$

$$(\rho u \cancel{x})_x \Delta y \Delta z - \left[(\rho u)_x \Delta y \Delta z + \frac{\partial}{\partial x}(\rho u) \Delta x \Delta y \Delta z\right] = \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z)$$

$$-\frac{\partial}{\partial x}(\rho u) = \frac{\partial}{\partial t}(\rho)$$

$$\Rightarrow \quad \boxed{\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\rho u = 0}$$



$$\dot{L}_{AB} - \dot{L}_{CD} + \Sigma F = \frac{\partial L_{cv}}{\partial t}$$

$$\dot{L}_{AB} = \dot{m}_{AB}\, u(x) = (\rho u)_x \Delta y \Delta z\ u = (\rho u^2)_x \Delta y \Delta z$$

$$\dot{L}_{CD} = \dot{m}_{CD}\, u(x+\Delta x) = (\rho u^2)_{x+\Delta x} \Delta y \Delta z$$

$$\Sigma F = p(x)\Delta y \Delta z - p(x+\Delta x)\Delta y \Delta z + \tau(y+\Delta y)\Delta x \Delta z - \tau(y)\Delta x \Delta z + (\Delta m) g \sin\theta$$

$$\Delta m = \rho \Delta x \Delta y \Delta z$$

$$L_{cv} = \Delta m\, u = \rho \Delta x \Delta y \Delta z\, u$$

$$(\rho u^2)_x \Delta y \Delta z - (\rho u^2)_{x+\Delta x} \Delta y \Delta z + p(x)\Delta y \Delta z - p(x+\Delta x)\Delta y \Delta z + \tau(y+\Delta y)\Delta x \Delta z - \tau(y)\Delta x \Delta z + \rho \Delta x \Delta y \Delta z\, g\sin\theta = \frac{\partial}{\partial t}(\rho u \Delta x \Delta y \Delta z)$$

$$-\frac{\partial}{\partial x}(\rho u^2) - \frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y} + \rho g \sin\theta = \frac{\partial}{\partial t}(\rho u)$$

$$\frac{\partial (\rho u)}{\partial t} + \frac{\partial}{\partial x}(\rho u^2) = -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y} + \rho g \sin\theta.$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) = \rho \frac{\partial u}{\partial t} + u \frac{\partial \rho}{\partial t} + \rho u \frac{\partial u}{\partial x} + u \frac{\partial \rho u}{\partial x}$$

$$\frac{\partial}{\partial x}(u \cdot \rho u) = \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right] + u \underbrace{\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho u\right]}_{=0}$$

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right] = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) + \rho g \sin\theta$$

Define $\frac{D}{Dt} \triangleq \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$

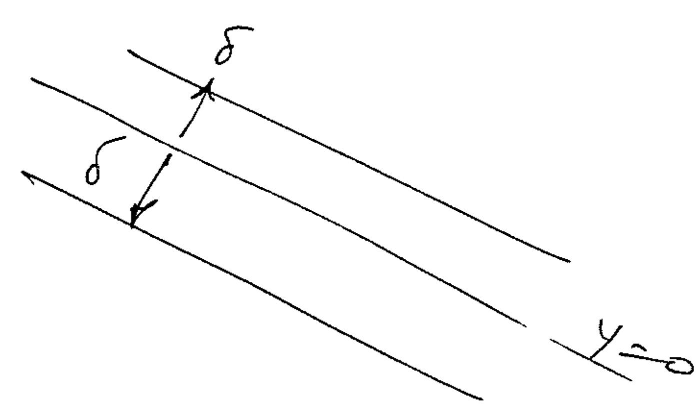
(Material Derivative)

$$\boxed{\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + \rho g \sin\theta}$$

Steady state $\frac{\partial}{\partial t} = 0$

Uniform flow $\frac{\partial}{\partial x} = 0$

(Fully developed flow)



~~To~~ τ @ upper plate $= \mu \left.\frac{du}{dy}\right|_{y=\delta}$

$\cancel{\frac{du}{dy}} =$ $\quad u = u_m\left(1 - \left(\frac{y}{\delta}\right)^2\right)$

$$\frac{du}{dy} = u_m\left(-\frac{2y}{\delta^2}\right)$$

$$\left.\frac{du}{dy}\right|_{y=\delta} = -u_m\frac{2}{\delta}$$

τ @ upper plate $= -\mu u_m \frac{2}{\delta}$

$= -\mu \frac{3\bar{u}}{2}\frac{2}{\delta}$ $\quad$ Since $u_m = \frac{3}{2}\bar{u}$

$= -3\frac{\mu\bar{u}}{\delta}$