Markov Chains and Transition Matrices: Understanding Customer Movement

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1 Introduction

Markov Chains are mathematical models that describe systems that undergo transitions from one state to another, where the probability of each state depends only on the previous state. These models are widely used in various fields, such as economics, biology, and computer science. In this project, we explore the basic components of Markov Chains with a focus on customer movement between stores as a practical application.

2 Key Concepts and Definitions

2.1 Transition Matrix (Stochastic Matrix)

A **transition matrix** P is a square matrix used to describe the probabilities of moving from one state to another in a Markov chain. Each row represents a current state, and each entry p_{ij} in that row indicates the probability of transitioning to state j from state i. Each row must sum to 1:

$$\sum_j p_{ij} = 1$$

2.2 Regular Transition Matrix

A transition matrix is called **regular** if there exists a positive integer k such that all entries of P^k are positive (i.e., $P^k > 0$). This means it is possible to reach any state from any other state in a finite number of steps, ensuring long-term behavior is stable and predictable.

2.3 Initial Probability Vector

The **initial probability vector** $\vec{v}_0$ is a row or column vector that shows the probability distribution of the system's state at time $t = 0$. For example, if a system starts in state A with certainty, then:

$$\vec{v}_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

2.4 Steady State Vector and Eigenvalue Perspective

Fact: Let P be a stochastic matrix. Then:1 is an eigenvalue of P.
This fact explains why a Markov chain always has a steady state. A **steady state vector** π satisfies:

$$P \cdot \pi = \pi$$

This means π is an **eigenvector** of the transition matrix P, associated with the eigenvalue $\lambda = 1$. This bridges linear algebra with the long-term behavior of Markov chains.

3 Application: Modeling Customer Movement Between Stores

This section demonstrates how Markov chains can be applied to model customer transitions between three retail locations (Stores A, B, and C). The goal is to understand long-term customer distribution and support data-driven business decisions.

3.1 Realistic Example and Transition Matrix

We assume a customer visits one of three stores (A, B, or C). Based on historical data, we construct the following transition matrix P, where each row sums to 1 and represents the probabilities of moving from the current store to the next:

$$P = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

For example, the first row tells us that if a customer is currently at Store A, there is a 60% chance they stay, a 30% chance they go to Store B, and a 10% chance they go to Store C.

3.2 Initial State and Transition

Suppose a customer starts at Store A. The initial probability vector is:

$$\vec{v}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

After one transition:

$$\vec{v}_1 = P \cdot \vec{v}_0 = \begin{bmatrix} 0.6 \\ 0.2 \\ 0.3 \end{bmatrix}$$

We repeat this process (e.g., compute $\vec{v}_2 = P \cdot \vec{v}_1$, etc.) until the vector stabilizes, indicating the **steady state**.

3.3 Finding the Steady State

To compute the steady-state vector π, we solve the system:

$$P \cdot \pi = \pi, \quad \text{with the constraint} \quad \sum \pi_i = 1$$

Solving this, we obtain:

$$\pi = \begin{bmatrix} 0.354 \\ 0.521 \\ 0.125 \end{bmatrix}$$

This means that, in the long run, approximately 35.4% of customers are expected to be in Store A, 52.1% in Store B, and 12.5% in Store C, regardless of their initial location.

3.4 Business Interpretation

This Markov model can help businesses make data-informed decisions. For example, knowing that more than half of all customers will eventually shop in Store B allows managers to optimize inventory, allocate staff more efficiently, or target marketing campaigns at specific locations. Using transition data and steady-state analysis, retailers can better anticipate customer flow and improve operational planning.

4 Conclusion

Markov Chains provide a simple but powerful way to model systems that evolve over time based on fixed probabilities. By examining customer movement between stores, we see how linear algebra tools like matrices and eigenvectors help solve real-world problems and inform decision-making.

References

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