

An Algorithm for Solving G-Cubed Models

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This document sets out the algebra involved in solving the G-Cubed class of macroeconomic models, without allowing for optimal policy. Optimal policy is on the todo list. It provides the details needed to implement the two-point boundary solution algorithm described in McKibbin (1987) that builds on the work of Blanchard and Kahn (1980).

1 The non-linear model

The model can be expressed in terms of the following variables:

- the vector of state variables, S_t ;
- the vector of costate or “jump” variables, J_t ;
- the vector, R_t , the subset of endogenous variables that also enter the model in expectations, at time, t of their value in time $t + 1$, $E_t(R_{t+1})$;
- the vector of other endogenous variables, Z_t ; and
- the vector of exogenous variables, X_t .

In non-linear form, these equations can be written:

$$(1) \quad S_{t+1} = \Phi_S(S_{t+1}, E_t(J_{t+1}), R_t, Z_t, E_t(R_{t+1}), S_t, J_t, X_t)$$

$$(2) \quad E_t(J_{t+1}) = \Phi_J(S_{t+1}, E_t(J_{t+1}), R_t, Z_t, E_t(R_{t+1}), S_t, J_t, X_t)$$

$$(3) \quad R_t = \Phi_R(S_{t+1}, E_t(J_{t+1}), R_t, Z_t, E_t(R_{t+1}), S_t, J_t, X_t)$$

$$(4) \quad Z_t = \Phi_Z(S_{t+1}, E_t(J_{t+1}), R_t, Z_t, E_t(R_{t+1}), S_t, J_t, X_t)$$

2 The linear model as a series of differential equations

The model is linearised using a first-order Taylor-series expansion around a point P :

$$P = (\bar{S}_{t+1}, \bar{S}_t, E_t(\bar{J}_{t+1}), \bar{J}_t, \bar{R}_t, E_t(\bar{R}_{t+1}), \bar{Z}_t, \bar{X}_t)$$

In most cases, P is defined as values for the variables in the year prior to the start of model projections.

The point P should be a solution to the model equations. Because the variables, S_{t+1} , J_t , R_t , and Z_t occur on the left and right sides of the equations this is vanishingly unlikely. We need to determine how much of a problem this is by investigating the second derivatives of the Φ functions with respect to those left-hand-side vectors. If those second derivatives are very close to zero, then linearisation using point P will be reasonable.

Adjustments are made to the values selected for variables that are a long way from their equilibrium values, for example, when real interest rates are negative, those negative interest rates will not be used as part of the point around which the model is linearised.

Calculating numerical derivatives at the linearisation point, the model can be approximated by a set of linear differential equations. For A in (S, J, R, Z) , define ϕ_{An} as the matrix of partial derivatives of function $\Phi_A()$ with respect to its n th argument.

To improve readability, define the following notation in time-period t :

$$\begin{aligned} (5) \quad & dS_t = s_t \\ & dJ_t = j_t \\ & dR_t = r_t \\ & dZ_t = z_t \\ & dX_t = x_t \\ & E_t(r_{t+1}) = {}_t r_{t+1} \\ & E_t(j_{t+1}) = {}_t j_{t+1} \end{aligned}$$

The linear model is then the following differential equations.

$$\begin{aligned} (6) \quad & s_{t+1} = \phi_{s1}s_{t+1} + \phi_{s2}{}_t j_{t+1} + \phi_{s3}r_t + \phi_{s4}z_t + \phi_{s5}{}_t r_{t+1} + \phi_{s6}s_t + \phi_{s7}j_t + \phi_{s8}x_t \\ (7) \quad & {}_t j_{t+1} = \phi_{j1}s_{t+1} + \phi_{j2}{}_t j_{t+1} + \phi_{j3}r_t + \phi_{j4}z_t + \phi_{j5}{}_t r_{t+1} + \phi_{j6}s_t + \phi_{j7}j_t + \phi_{j8}x_t \\ (8) \quad & r_t = \phi_{r1}s_{t+1} + \phi_{r2}{}_t j_{t+1} + \phi_{r3}r_t + \phi_{r4}z_t + \phi_{r5}{}_t r_{t+1} + \phi_{r6}s_t + \phi_{r7}j_t + \phi_{r8}x_t \\ (9) \quad & z_t = \phi_{z1}s_{t+1} + \phi_{z2}{}_t j_{t+1} + \phi_{z3}r_t + \phi_{z4}z_t + \phi_{z5}{}_t r_{t+1} + \phi_{z6}s_t + \phi_{z7}j_t + \phi_{z8}x_t \end{aligned}$$

3 The State-Space Form

The State-Space Form (SSF) is obtained by eliminating s_{t+1} , ${}_t j_{t+1}$, r_t , and z_t from the right-hand side of the linear model.

Working through the necessary steps, first rearrange equation 9 to gather all z_t terms on the left:

$$(10) \quad (I_z - \phi_{z4})z_t = \phi_{z1}s_{t+1} + \phi_{z2} \, {}_tj_{t+1} + \phi_{z3}r_t + \phi_{z5} \, {}_tr_{t+1} + \phi_{z6}s_t + \phi_{z7}j_t + \phi_{z8}x_t$$

Multiplying through by the inverse of $(I_z - \phi_{z4})$ and defining the α coefficient matrices appropriately, we obtain:

$$(11) \quad z_t = \alpha_{z1}s_{t+1} + \alpha_{z2} \, {}_tj_{t+1} + \alpha_{z3}r_t + \alpha_{z5} \, {}_tr_{t+1} + \alpha_{z6}s_t + \alpha_{z7}j_t + \alpha_{z8}x_t$$

Use equation 11, to substitute z_t out of equations 6, 7, and 8, again defining α coefficient matrices appropriately:

$$(12) \quad s_{t+1} = \alpha_{s1}s_{t+1} + \alpha_{s2} \, {}_tj_{t+1} + \alpha_{s3}r_t + \alpha_{s5} \, {}_tr_{t+1} + \alpha_{s6}s_t + \alpha_{s7}j_t + \alpha_{s8}x_t$$

$$(13) \quad {}_tj_{t+1} = \alpha_{j1}s_{t+1} + \alpha_{j2} \, {}_tj_{t+1} + \alpha_{j3}r_t + \alpha_{j5} \, {}_tr_{t+1} + \alpha_{j6}s_t + \alpha_{j7}j_t + \alpha_{j8}x_t$$

$$(14) \quad r_t = \alpha_{r1}s_{t+1} + \alpha_{r2} \, {}_tj_{t+1} + \alpha_{r3}r_t + \alpha_{r5} \, {}_tr_{t+1} + \alpha_{r6}s_t + \alpha_{r7}j_t + \alpha_{r8}x_t$$

Repeat the process to eliminate r_t from the right of the linear model, rearranging equation 14:

$$(15) \quad (I_r - \alpha_{r3})r_t = \alpha_{r1}s_{t+1} + \alpha_{r2} \, {}_tj_{t+1} + \alpha_{r4}z_t + \alpha_{r5} \, {}_tr_{t+1} + \alpha_{r6}s_t + \alpha_{r7}j_t + \alpha_{r8}x_t$$

Multiply through by the inverse of $(I_r - \alpha_{r3})$, to solve for r_t , defining the β coefficient matrices appropriately.

$$(16) \quad r_t = \beta_{r1}s_{t+1} + \beta_{r2} \, {}_tj_{t+1} + \beta_{r5} \, {}_tr_{t+1} + \beta_{r6}s_t + \beta_{r7}j_t + \beta_{r8}x_t$$

Use equation 16, to substitute r_t out of equations 12, 13, and 11, defining the β coefficient matrices appropriately.

$$(17) \quad s_{t+1} = \beta_{s1}s_{t+1} + \beta_{s2} \, {}_tj_{t+1} + \beta_{s5} \, {}_tr_{t+1} + \beta_{s6}s_t + \beta_{s7}j_t + \beta_{s8}x_t$$

$$(18) \quad {}_tj_{t+1} = \beta_{j1}s_{t+1} + \beta_{j2} \, {}_tj_{t+1} + \beta_{j5} \, {}_tr_{t+1} + \beta_{j6}s_t + \beta_{j7}j_t + \beta_{j8}x_t$$

$$(19) \quad z_t = \beta_{z1}s_{t+1} + \beta_{z2} \, {}_tj_{t+1} + \beta_{z5} \, {}_tr_{t+1} + \beta_{z6}s_t + \beta_{z7}j_t + \beta_{z8}x_t$$

Repeat the process to eliminate ${}_t\dot{j}_{t+1}$ from the right of the linear model, rearranging equation 18:

$$(20) \quad (I_j - \beta_{j2}) {}_t\dot{j}_{t+1} = \beta_{j1}s_{t+1} + \beta_{j5} {}_tr_{t+1} + \beta_{j6}s_t + \beta_{j7}\dot{j}_t + \beta_{j8}x_t$$

Multiply through by the inverse of $(I_j - \beta_{j2})$ to solve for ${}_t\dot{j}_{t+1}$, defining the γ coefficient matrices appropriately.

$$(21) \quad {}_t\dot{j}_{t+1} = \gamma_{j1}s_{t+1} + \gamma_{j5} {}_tr_{t+1} + \gamma_{j6}s_t + \gamma_{j7}\dot{j}_t + \gamma_{j8}x_t$$

Use equation 21 to substitute ${}_t\dot{j}_{t+1}$ out of equations 12, 14, and 11, defining the γ coefficient matrices appropriately:

$$(22) \quad s_{t+1} = \gamma_{s1}s_{t+1} + \gamma_{s5} {}_tr_{t+1} + \gamma_{s6}s_t + \gamma_{s7}\dot{j}_t + \gamma_{s8}x_t$$

$$(23) \quad r_t = \gamma_{r1}s_{t+1} + \gamma_{r5} {}_tr_{t+1} + \gamma_{r6}s_t + \gamma_{r7}\dot{j}_t + \gamma_{r8}x_t$$

$$(24) \quad z_t = \gamma_{z1}s_{t+1} + \gamma_{z5} {}_tr_{t+1} + \gamma_{z6}s_t + \gamma_{z7}\dot{j}_t + \gamma_{z8}x_t$$

Repeat the process to eliminate s_{t+1} from the right of the linear model, rearranging equation 22:

$$(25) \quad (I_s - \gamma_{s1})s_{t+1} = \gamma_{s5} {}_tr_{t+1} + \gamma_{s6}s_t + \gamma_{s7}\dot{j}_t + \gamma_{s8}x_t$$

Multiply through by the inverse of $(I_s - \gamma_{s1})$, to solve for s_{t+1} , defining the δ coefficient matrices appropriately.

$$(26) \quad s_{t+1} = \delta_{sr} {}_t r_{t+1} + \delta_{ss} s_t + \delta_{sj} j_t + \delta_{sx} x_t$$

Use equation 26 to substitute s_{t+1} out of equations 21, 23, and 24, defining the δ coefficient matrices appropriately. Note that the δ coefficient matrices are referenced throughout the rest of this document. Their subscripts indicate the vectors that they relate, with the first subscript indicating the vector on the left of the equation and the second subscript indicating the vector on the right of the equation. Thus, δ_{sj} is the matrix of SSF coefficients describing the relationship between s_{t+1} on the left and j_t on the right.

$$(27) \quad {}_t j_{t+1} = \delta_{jr} {}_t r_{t+1} + \delta_{js} s_t + \delta_{jj} j_t + \delta_{jx} x_t$$

$$(28) \quad r_t = \delta_{rr} {}_t r_{t+1} + \delta_{rs} s_t + \delta_{rj} j_t + \delta_{rx} x_t$$

$$(29) \quad z_t = \delta_{zr} {}_t r_{t+1} + \delta_{zs} s_t + \delta_{zj} j_t + \delta_{zx} x_t$$

Equations 26, 27, 28, and 29 constitute the SSF.

4 The stable manifold

The stable manifold is the non-exploding path for the costate variables, j_t . It is found using the iterative algorithm described in McKibbin (1987).

4.1 The first iteration to find the stable manifold

The model needs to be solved for paths for all variables from period $t = 0$ to period $t = T$. Use a suitable terminal condition to start the iterative process of computing the transition rule for the costate variable that implies model consistent expectations.

With (linearised) model-consistent expectations, we can slightly simplify notation again, replacing ${}_t r_{t+1}$ and ${}_t j_{t+1}$ with r_{t+1} and j_{t+1} respectively because the model has no sources of uncertainty.

The first iteration yields rules for j_T , r_T in terms of s_T and x_T . These rules must be model consistent with remaining periods. Being the final period, T , a terminal condition is required.

An intuitive terminal condition involves the period-to-period change in the variables being zero from period T to period $T + 1$ so $j_T = j_{T+1}$ and $r_T = r_{T+1}$. This enables elimination of r_t and j_t from the period T SSF equations.

First eliminate $r_{T+1} = r_T$ from the right of equation 28.

$$(30) \quad (I_r - \delta_{rr}) r_T = \delta_{rs} s_T + \delta_{rj} j_T + \delta_{rx} x_T$$

Multiply through by $\Gamma_{rT} = (I_r - \delta_{rr})^{-1}$.

$$(31) \quad r_T = \psi_{rs}s_T + \psi_{rj}j_T + \psi_{rx}x_T$$

The coefficient matrices in equation 31 are defined as follows.

$$(32) \quad \psi_{rs} = \Gamma_{rT}\delta_{rs}$$

$$(33) \quad \psi_{rj} = \Gamma_{rT}\delta_{rj}$$

$$(34) \quad \psi_{rx} = \Gamma_{rT}\delta_{rx}$$

Using the terminal condition again, $r_{T+1} = r_T$ substitute equation 31 for r_{T+1} in equation 27.

$$(35) \quad \begin{aligned} j_{T+1} &= \delta_{jr}(\psi_{rs}s_T + \psi_{rj}j_T + \psi_{rx}x_T) \\ &\quad + \delta_{js}s_T + \delta_{jj}j_T + \delta_{jx}x_T \end{aligned}$$

Given the terminal condition implies $j_{T+1} = j_T$, collect j_T terms on the left.

$$(36) \quad \begin{aligned} (I_j - \delta_{jj} - \delta_{jr}\psi_{rj})j_T &= \delta_{jr}(\psi_{rs}s_T + \psi_{rx}x_T) \\ &\quad + \delta_{js}s_T + \delta_{jx}x_T \end{aligned}$$

Multiply through by $\Gamma_{jT} = (I_j - \delta_{jj} - \delta_{jr}\psi_{rj})^{-1}$ to obtain a rule for j_T in terms of s_T and x_T .

$$(37) \quad j_T = H_{1T}s_T + H_{2T}x_T$$

The coefficient matrices in equation 37 are defined as follows.

$$(38) \quad H_{1T} = \Gamma_{jT}(\delta_{jr}\psi_{rs} + \delta_{js})$$

$$(39) \quad H_{2T} = \Gamma_{jT}(\delta_{jr}\psi_{rx} + \delta_{jx})$$

Replace $j_{T+1} = j_T$ in equation 31 with equation 37.

$$(40) \quad r_T = \psi_{rs}s_T + \psi_{rj}(H_{1T}s_T + H_{2T}x_T) + \psi_{rx}x_T$$

Collecting terms, this can be expressed as a rule for r_T in terms of the state and exogenous variables in period T .

$$(41) \quad r_T = M_{1T}s_T + M_{2T}x_T$$

The coefficient matrices in equation 41 are defined as follows.

$$(42) \quad M_{1T} = \psi_{rs} + \psi_{rj}H_{1T}$$

$$(43) \quad M_{2T} = \psi_{rx} + \psi_{rj}H_{2T}$$

4.2 Further iterations for the stable manifold

Next, derive for the rule for period t assuming a specific form of the rule in period $t + 1$ that includes the rules described in equations 37 and 41 as special cases for the terminal period. This iterative process, continues until the rule is far enough before the terminal period T that the coefficient matrices in the rule for the costate variable do not change with each iteration.

The rules for any time period t from $t = 0$ to $t = T$ involve linear functions of future exogenous variables. For brevity, define these functions as follows:

$$(44) \quad F_{jt} = f_{jt}(x_{t+1}, \dots, x_T)$$

$$(45) \quad F_{rt} = f_{rt}(x_{t+1}, \dots, x_T)$$

$$(46) \quad F_{st} = f_{st}(x_{t+1}, \dots, x_T)$$

If the rules for period t have the same form as the rules for period $t + 1$, a process of induction, back from period T , implies the rules for all time periods from period 0 through to period T and allows calculation of the coefficient matrices in those rules.

The assumed rules for period $t + 1$ are set out below.

$$(47) \quad j_{t+1} = H_{1t+1} s_{t+1} + H_{2t+1} x_{t+1} + F_{jt+1}$$

$$(48) \quad r_{t+1} = M_{1t+1} s_{t+1} + M_{2t+1} x_{t+1} + F_{rt+1}$$

We need to check that the assumed rules in equations 47 and 48 imply the rules for j_t and r_t in equations 49 and 50.

$$(49) \quad j_t = H_{1t} s_t + H_{2t} x_t + F_{jt}$$

$$(50) \quad r_t = M_{1t} s_t + M_{2t} x_t + F_{rt}$$

Along with the SSF, the rules in equations 47 and 48 imply a rule for s_{t+1} . To show this implication, use the SSF equation for the state variables, equation 26, substituting r_{t+1} with the assumed rule for r_{t+1} in equation 48.

$$(51) \quad s_{t+1} = \delta_{sr} (M_{1t+1} s_{t+1} + M_{2t+1} x_{t+1} + F_{rt+1}) + \delta_{ss} s_t + \delta_{sj} j_t + \delta_{sx} x_t$$

Gather all s_{t+1} terms on the left.

$$(52) \quad (I_s - \delta_{sr} M_{1t+1}) s_{t+1} = \delta_{ss} s_t + \delta_{sj} j_t + \delta_{sx} x_t + \delta_{sr} (M_{2t+1} x_{t+1} + F_{rt+1})$$

Multiply through by $\Gamma_{st} = (I_s - \delta_{sr} M_{1t+1})^{-1}$ to obtain the rule for s_{t+1} .

$$(53) \quad s_{t+1} = \tau_{sst} s_t + \tau_{sjt} j_t + \tau_{sxt} x_t + F_{st}$$

The coefficient matrices in equation 53 are defined as follows.

$$(54) \quad \tau_{sst} = \Gamma_{st} \delta_{ss}$$

$$(55) \quad \tau_{sjt} = \Gamma_{st} \delta_{sj}$$

$$(56) \quad \tau_{sxt} = \Gamma_{st} \delta_{sx}$$

The linear function of future exogenous variables is defined as follows:

$$(57) \quad F_{st} = \Gamma_{st} \delta_{sr} (M_{2t+1} x_{t+1} + F_{rt+1})$$

Using the three rules in equations 47, 48, and 53, obtain the coefficient matrices in the rules for j_t , r_t and s_t . Begin by equating the right-hand-sides of the assumed j_{t+1} rule, equation 47, and the SSF for j_{t+1} , equation 27.

$$(58) \quad H_{1t+1} s_{t+1} + H_{2t+1} x_{t+1} + F_{jt+1} = \delta_{jr} r_{t+1} + \delta_{js} s_t + \delta_{jj} j_t + \delta_{jx} x_t$$

Substitute r_{t+1} with the assumed rule for r_{t+1} in equation 48.

$$(59) \quad \begin{aligned} & H_{1t+1} s_{t+1} + H_{2t+1} x_{t+1} + F_{jt+1} \\ &= \delta_{jr} (M_{1t+1} s_{t+1} + M_{2t+1} x_{t+1} + F_{rt+1}) + \delta_{js} s_t + \delta_{jj} j_t + \delta_{jx} x_t \end{aligned}$$

Gather s_{t+1} terms.

$$(60) \quad \begin{aligned} & (H_{1t+1} - \delta_{jr} M_{1t+1}) s_{t+1} + H_{2t+1} x_{t+1} + F_{jt+1} \\ &= \delta_{js} s_t + \delta_{jj} j_t + \delta_{jx} x_t + \delta_{jr} (M_{2t+1} x_{t+1} + F_{rt+1}) \end{aligned}$$

Use the rule for s_{t+1} in equation 53 to eliminate s_{t+1} .

$$(61) \quad \begin{aligned} & (H_{1t+1} - \delta_{jr} M_{1t+1}) (\tau_{sst} s_t + \tau_{sjt} j_t + \tau_{sxt} x_t + F_{st}) + H_{2t+1} x_{t+1} + F_{jt+1} \\ &= \delta_{js} s_t + \delta_{jj} j_t + \delta_{jx} x_t + \delta_{jr} (M_{2t+1} x_{t+1} + F_{rt+1}) \end{aligned}$$

Rearrange to have only terms involving j_t on the left and gather like terms on the right.

$$(62) \quad \begin{aligned} & (\delta_{jj} - (H_{1t+1} - \delta_{jr} M_{1t+1}) \tau_{sjt}) j_t \\ &= ((H_{1t+1} - \delta_{jr} M_{1t+1}) \tau_{sst} - \delta_{js}) s_t \\ &+ ((H_{1t+1} - \delta_{jr} M_{1t+1}) \tau_{sxt} - \delta_{jx}) x_t \\ &+ H_{2t+1} x_{t+1} + F_{jt+1} + (H_{1t+1} - \delta_{jr} M_{1t+1}) F_{st} - \delta_{jr} (M_{2t+1} x_{t+1} + F_{rt+1}) \end{aligned}$$

Multiply through by $\Gamma_{jt} = (\delta_{jj} - (H_{1t+1} - \delta_{jr} M_{1t+1}) \tau_{sjt})^{-1}$ to obtain the rule for j_t in equation 49.

The implied coefficient matrix definitions are:

$$(63) \quad H_{1t} = \Gamma_{jt} ((H_{1t+1} - \delta_{jr} M_{1t+1}) \tau_{sst} - \delta_{js})$$

$$(64) \quad H_{2t} = \Gamma_{jt} ((H_{1t+1} - \delta_{jr} M_{1t+1}) \tau_{sxt} - \delta_{jx})$$

The implied function of future exogenous variables in the rule for j_t is:

$$(65) \quad F_{jt} = \Gamma_{jt} (H_{2t+1} x_{t+1} + F_{jt+1} + (H_{1t+1} - \delta_{jr} M_{1t+1}) F_{st} - \delta_{jr} (M_{2t+1} x_{t+1} + F_{rt+1}))$$

Next derive the rule for r_t in equation 50.

To derive the rule for r_t , replace r_{t+1} with the assumed rule for r_{t+1} in the SSF equation for r_t , equation 28.

$$(66) \quad r_t = \delta_{rr} (M_{1t+1}s_{t+1} + M_{2t+1}x_{t+1} + F_{rt+1}) + \delta_{rs}s_t + \delta_{rj}j_t + \delta_{rx}x_t$$

Substitute for s_{t+1} with the rule in equation 53.

$$(67) \quad r_t = \delta_{rr} (M_{1t+1}(\tau_{sst}s_t + \tau_{sjt}j_t + \tau_{sxt}x_t + F_{st}) + M_{2t+1}x_{t+1} + F_{rt+1}) + \delta_{rs}s_t + \delta_{rj}j_t + \delta_{rx}x_t$$

Gather like terms on the right.

$$(68) \quad \begin{aligned} r_t = & (\delta_{rs} + \delta_{rr}M_{1t+1}\tau_{sst})s_t \\ & + (\delta_{rj} + \delta_{rr}M_{1t+1}\tau_{sjt})j_t \\ & + (\delta_{rx} + \delta_{rr}M_{1t+1}\tau_{sxt})x_t \\ & + \delta_{rr}(M_{1t+1}F_{st} + M_{2t+1}x_{t+1} + F_{rt+1}) \end{aligned}$$

Replace j_t with its rule in equation 49 to obtain the required form for the rule for r_t .

$$(69) \quad \begin{aligned} r_t = & (\delta_{rs} + \delta_{rr}M_{1t+1}\tau_{sst})s_t \\ & + (\delta_{rj} + \delta_{rr}M_{1t+1}\tau_{sjt})(H_{1t}s_t + H_{2t}x_t + F_{jt}) \\ & + (\delta_{rx} + \delta_{rr}M_{1t+1}\tau_{sxt})x_t \\ & + \delta_{rr}(M_{1t+1}F_{st} + M_{2t+1}x_{t+1} + F_{rt+1}) \end{aligned}$$

Gathering like terms on the right produces the rule for r_t in equation 50 as required.

$$(70) \quad \begin{aligned} r_t = & (\delta_{rs} + \delta_{rr}M_{1t+1}\tau_{sst} + (\delta_{rj} + \delta_{rr}M_{1t+1}\tau_{sjt})H_{1t})s_t \\ & + (\delta_{rx} + \delta_{rr}M_{1t+1}\tau_{sxt} + (\delta_{rj} + \delta_{rr}M_{1t+1}\tau_{sjt})H_{2t})x_t \\ & + \delta_{rr}(M_{1t+1}F_{st} + M_{2t+1}x_{t+1} + F_{rt+1}) + (\delta_{rj} + \delta_{rr}M_{1t+1}\tau_{sjt})F_{jt} \end{aligned}$$

The implied coefficient matrix definitions are:

$$(71) \quad M_{1t} = \delta_{rs} + \delta_{rr}M_{1t+1}(\tau_{sst} + \tau_{sjt}H_{1t}) + \delta_{rj}H_{1t}$$

$$(72) \quad M_{2t} = \delta_{rx} + \delta_{rr}M_{1t+1}(\tau_{sxt} + \tau_{sjt}H_{2t}) + \delta_{rj}H_{2t}$$

The implied function of future exogenous variables in the rule for s_t is:

$$(73) \quad F_{rt} = \delta_{rr}(M_{1t+1}F_{st} + M_{2t+1}x_{t+1} + F_{rt+1}) + (\delta_{rj} + \delta_{rr}M_{1t+1}\tau_{sjt})F_{jt}$$

This result confirms that the rules for r_t and j_t , have the form specified in 50 and 49 assuming the form for the rules for r_{t+1} and j_{t+1} in equations 47 and 48. Given that the rules for j_T and r_T are special cases of the form specified in equations 47 and 48, we have proved that the rules have the form in equations 50 and 49, for all time periods $t = 0$ through to $t = T$.

We have also derived the relationship between the coefficient matrices in the rules for all time periods and have derived the rules for the state variables in all time periods and the relationship between the coefficient matrices in those rules and the coefficient matrices in the SSF and the rules for j and r .

4.2.1 A rule for r_{t+1} in terms of s_t

A rule for r_{t+1} in terms of s_t and exogenous variables is helpful when generating model projections. It also underpins an alternative way of deriving the rule for r_t in terms of s_t and exogenous variables.

To obtain the rule expressing r_{t+1} in terms of s_t and exogenous variables, replace s_{t+1} in equation 48 with the right side of equation 53.

$$(74) \quad r_{t+1} = M_{1t+1} (\tau_{sst}s_t + \tau_{sjt}j_t + \tau_{sxt}x_t + F_{st}) + M_{2t+1} x_{t+1} + F_{rt+1}$$

Eliminate j_t using equation 49.

$$(75) \quad r_{t+1} = M_{1t+1} (\tau_{sst}s_t + \tau_{sjt} (H_{1t} s_t + H_{2t} x_t + F_{jt}) + \tau_{sxt}x_t + F_{st}) + M_{2t+1} x_{t+1} + F_{rt+1}$$

Collect like terms on the right.

$$(76) \quad \begin{aligned} r_{t+1} = & M_{1t+1} (\tau_{sst} + \tau_{sjt}H_{1t}) s_t \\ & + M_{1t+1} (\tau_{sxt} + \tau_{sjt}H_{2t}) x_t \\ & + M_{1t+1} (F_{st} + \tau_{sjt}F_{jt}) + M_{2t+1}x_{t+1} + F_{rt+1} \end{aligned}$$

Define the following coefficient matrices.

$$(77) \quad \mu_{1t} = M_{1t+1} (\tau_{sst} + \tau_{sjt}H_{1t})$$

$$(78) \quad \mu_{2t} = M_{1t+1} (\tau_{sxt} + \tau_{sjt}H_{2t})$$

Define the following function of time t and future exogenous variables.

$$(79) \quad G_{rt} = M_{1t+1} (F_{st} + \tau_{sjt}F_{jt}) + M_{2t+1}x_{t+1} + F_{rt+1}$$

Using these definitions, we can express a the rule for r_{t+1} in terms of s_t and x_t and future exogenous variables.

$$(80) \quad r_{t+1} = \mu_{1t}s_t + \mu_{2t}x_t + G_{rt}$$

The rule for r_t then obtains by eliminating ${}_tr_{t+1}$ and j_t from the SSF equation for r_t , equation 28, using and use equations equation 80 and 49 respectively.

$$(81) \quad r_t = \delta_{rr} (\mu_{1t}s_t + \mu_{2t}x_t + G_{rt}) + \delta_{rs}s_t + \delta_{rj} (H_{1t}s_t + H_{2t}x_t + F_{jt}) + \delta_{rx}x_t$$

Collect like terms on the right to obtain the required rule for r_t .

$$(82) \quad r_t = (\delta_{rs} + \delta_{rr}\mu_{1t} + \delta_{rj}H_{1t}) s_t + (\delta_{rx} + \delta_{rr}\mu_{2t} + \delta_{rj}H_{2t}) x_t + \delta_{rr}G_{rt} + \delta_{rj}F_{jt}$$

The implied coefficient matrix definitions are:

$$(83) \quad M_{1t} = (\delta_{rs} + \delta_{rr}\mu_{1t} + \delta_{rj}H_{1t})$$

$$(84) \quad M_{2t} = (\delta_{rx} + \delta_{rr}\mu_{2t} + \delta_{rj}H_{2t})$$

It can be shown that these definitions are equivalent to the definitions in equations 71 and 72.

The implied function of future exogenous variables in the rule for s_t is:

$$(85) \quad F_{rt} = \delta_{rr}G_{rt} + \delta_{rj}F_{jt}$$

This definition is also equivalent to that in equation 73.

4.3 Convergence of the stable manifold

Convergence is deemed to have happened when the matrices, H_{1t} , H_{2t} , M_{1t} and M_{2t} stabilise to be independent of the time period, H_1 , H_2 , M_1 and M_2 .

As shown by Blanchard and Kahn (1980), the stable manifold is only unique if *matrix?* has exactly one eigenvalue outside the unit circle for each costate variable in the model. Otherwise the solution is not unique. It would be helpful to be explicit about how the linearised model in this paper maps to the linear model in 1a of Blanchard and Kahn (1980) so that we can determine how to assess uniqueness based on the J matrix (J is from the Jordan canonical form of the full linear model coefficient matrix) eigenvalues when we form the linear model.

What can we say about whether convergence is guaranteed? Is there a connection between uniqueness and a guarantee of convergence?

4.4 State dynamics and system stability

To check that the stable manifold does imply stability of the system, check the eigenvalues of the matrix in the rule relating the state in period $t+1$ to the state in period t , obtained after substituting the rules j_t and r_{t+1} back into the SSF equation for the state variables. All of those eigenvalues must lie inside the unit circle if the system is stable.

To implement this stability test and to obtain a rule for the state dynamics, it is useful to write the rule for the state variables as a function of state variables in the previous period and exogenous variables.

To do so, substitute the rule for j_t , equation 49, and the rule for r_{t+1} , equation 80, into the SSF equation for the state variables, equation 26 where t is a sufficient number of periods before T for the rules to be expressed in terms of H_1 , H_2 , M_1 and M_2 .

$$(86) \quad s_{t+1} = \delta_{sr} (\mu_{1t}s_t + \mu_{2t}x_t + G_{rt}) + \delta_{ss}s_t + \delta_{sj} (H_1s_t + H_2x_t + F_{jt}) + \delta_{sx}x_t$$

Collect like terms on the right.

$$(87) \quad s_{t+1} = (\delta_{ss} + \delta_{sr}\mu_{1t} + \delta_{sj}H_1) s_t + (\delta_{sx} + \delta_{sr}\mu_{2t} + \delta_{sj}H_2) x_t + \delta_{sr}G_{rt} + \delta_{sj}F_{jt}$$

Define the following coefficient matrices, N_1 and N_2 .

$$(88) \quad N_1 = \Delta (\delta_{ss} + \delta_{sj}H_1)$$

$$(89) \quad N_2 = \Delta (\delta_{sx} + \delta_{sj}H_2)$$

Define the function of future exogenous variables G_{st} :

$$(90) \quad G_{st} = \Delta (\delta_{sr} (M_2x_{t+1} + F_{rt+1}) + \delta_{sj}F_{jt})$$

The state transition equation can then be written as:

$$(91) \quad s_{t+1} = N_1s_t + N_2x_t + G_{st}$$

The matrix to consider when assessing stability is N_1 , the state transition matrix. Stability requires all eigenvalues of this matrix to be inside the unit circle.

5 Generating a baseline in the G-Cubed model

There are two different concepts of a base year used in the G-Cubed model. The first is the base year for calibration to which the model parameters that aren't already estimated outside the model, are calibrated. This year is currently 2011. The other is the projection base year, the year from which projections are made. For example, the projection base year is currently 2018.

Because G-Cubed is an intertemporal model, it is necessary to calculate a baseline, or “business as usual”, solution before the model can be used for policy simulations. In order to do so we follow a 3-step procedure which includes specifying exogenous variables such as population and productivity growth rates, solving the model for primitive baseline projection, adjusting key constants to match to the stable manifold.

The first step is to make assumptions about the future course of key exogenous variables. We take the underlying long-run rate of world labour supply growth plus productivity growth to be 1.8 percent per annum, and take the long-run real interest rate to be 4 percent. We also assume that tax rates and the shares of government spending devoted to each commodity remain unchanged. In addition to the above

assumptions at an aggregate level, we also assume for sectoral productivity (or technological) growth rates by sector and by country.

Given these exogenous inputs for sectoral productivity growth and population growth, we then solve the model with endogenous growth. The drivers of endogenous growth include capital accumulation, sectoral demand for other inputs of energy and materials, etc. Additionally, critical to the nature and scale of growth across countries are these plus the underlying assumptions that financial capital flows to where the return is highest, physical capital is sector specific in the short run, labour can flow freely across sectors within a country but not between countries and that international trade in goods and financial capital is possible subject to existing tax structures and trade restrictions.

After solving the model for primitive baseline projection, we face a problem of benchmarking the base year of the projection to the actual data we observe in the database. In other words, after solving the model from the projection base year (i.e. 2018) to 2100 it is unlikely that the model will reproduce the base year's historical data. In particular, it is unlikely that the costate (or jumping) variables based on current and expected future paths of the exogenous variables in the model will equal the actual values of those variables in the database.

This problem arises in all intertemporal models and is not unique to the G-Cubed. Most intertemporal models either assume that the model starts in steady state or modify the actual data to be a solution of the model. We assume that the model at current stage is not in steady state and we adjust the model for it to match to the actual data. To implement this we have developed a technique that makes the initial year uniquely on the stable manifold of the model. We adjust some key constants in the model so that the base year is a solution of the model - not in steady state but along the stable manifold transitioning towards a steady state.

Note that the intertemporal constants adjust costate variables and some other variables.

After all the process above, we obtain a baseline projection of the model transitioning on the stable manifold towards the steady state with the actual database in projection base year replicated by the model and the dynamic adjustment after the base year based on inherited dynamics, initial conditions and future projection of exogenous variables. Then we are able to run scenarios using the model by solving the model for each period after 2018 given any shocks to variables, shocks to information sets (announcements about future policies), or changes in initial conditions.

The detailed assumptions and procedures will be discussed in the following subsections.

5.1 GDP scaling for the linear model

The real variables in the model are normalised in effective labour units to prevent them from changing over time, in the steady state, because of population growth or productivity growth. Effective labour units are approximated by real GDP so the real variables for each region are divided by that region's real GDP in the same year.

5.2 Projections for the exogenous variables

In addition to the assumptions about constant aggregate variables such as the long-run rate of world labour supply growth plus productivity growth, the long-run real interest rate, tax rates and the shares of government spending devoted to each commodity, we also assume for sectoral productivity (or technological) growth rates by sector and by country. These assumptions, along with the initial dynamics from the year before the projection base year to the projection base year, constitute key inputs for the baseline projections.

The default projection for each exogenous variable is its value in the base year. This default projection is adjusted to capture various factors, including, importantly, productivity growth, population growth, and improvements in energy efficiency over the projection years. Productivity growth and population growth

are summed to obtain the growth in effective labour productivity. (the way the model works in terms of AL - effective labour - should be expanded on somewhere - perhaps here).

In projecting productivity growth, we follow the approach suggested by Barro (1991) and Barro (2015). Over long periods, Barro estimates that the average catchup rate of individual countries to the worldwide productivity frontier is 2% per year.

Spell out the inputs that we load from modprod.csv. What are the parameters and how do we estimate them.

The Barro catchup model is built into the G-cubed's sectoral catchup rate, which is defined in equation 92.

$$(92) \quad \frac{p_{r,t}}{p_{u,t}} = 1 + (1 - c_{r,t}) \left(\frac{p_{r,t-1}}{p_{u,t-1}} - 1 \right)$$

where p stands for productivity level, r for region (u is one specific region for the US), t for time period (which is year for the G-Cubed). $c_{r,t}$ is the catchup rate for country r in year t , p_r/p_u is the productivity ratio between country r and the US, $1 - \frac{p_r}{p_u}$ is the gap. Thus, each year the gap shrinks by the catchup rate.

The initial productivity gap in each sector of each region determines the subsequent sectoral productivity growth rate. We base estimates of the gap on the 2018 Groningen Growth and Development database, (Bolt et al., 2018), estimating the initial productivity level in each sector of each region in the model, and then taking the ratio of the initial productivity to the equivalent sector in the United States, which we assume to have the highest effective labour productivity among all regions.

Besides the initial gaps, other factors can also affect the catchup rates, e.g. we expect that regions will catch up more quickly to the frontier due to economic reforms (e.g., China) or more slowly to the frontier due to institutional rigidities (e.g., Russia). Thus, we vary the catchup rate over time. The calibration of the catchup rate attempts to replicate recent growth experiences of each country and region in the model.

NOT SURE THIS PARAGRAPH IS NEEDED: The sectoral productivity growth is the exogenous drivers of sector growth for each country. The growth in the capital stock in each sector of each region is determined endogenously within the model. Given these exogenous inputs for sectoral productivity growth and population growth, we then solve the model with the other drivers of endogenous growth.

Population growth projections are based on the data available from ..., summing across countries to obtain regional growth rates, using ??? formula for aggregation (see the code provided by Larry Liu).

What variables in the model are adjusted by the effective labour productivity growth and how do those variables enter into the model equations?

Energy efficiency improvements are based on the assumption that energy efficiency improves by 1% per annum, except in the electricity distribution sector. (Is there anything more to say about this? What deviations from this 1% rule exist in the aeei CSV file? Why are they there?).

What variables in the model are adjusted by the energy efficiency improvements and how do those variables enter into the model equations?

5.3 Solving the model for primitive baseline projection

The drivers of endogenous growth include capital accumulation, sectoral demand for other inputs of energy and materials, etc. Additionally, critical to the nature and scale of growth across countries are these plus the underlying assumptions that financial capital flows to where the return is highest, physical capital is sector specific in the short run, labour can flow freely across sectors within a country but not between countries and that international trade in goods and financial capital is possible subject to existing tax structures and trade restrictions.

5.4 Adjusting key constants to match to the stable manifold

Upon solving the model, we obtain primitive baseline projection in which the base year results should not match to the actual data. Uniquely in the G-Cubed, we adjust some key constants in the model instead of actual data, such that the costate variables will equal the actual values of themselves in the base year. Once the costate variables align to the actual data, we know that the base year of projection is uniquely on the stable manifold of the model.

(Lucid: around here, not sure whether this is correct) (Geoff: I will draft this section having done the implementation of that part of the model. For now, focus on the exogenous variable projections).

To implement this, we developed a technique in which we adjust some key constants in the model so that the base year is a solution of the model - not in steady state but along the stable manifold transitioning towards a steady state.

The technique involves adding a set of constants, one for each costate variable, to the model's costate equations (arbitrage equations). For example, the constants for Tobin's q for each sector in each country are added to the arbitrage equation for each sector's q . Similarly, constants for each real exchange rate are added to the interest arbitrage equation for each country, and a constant for human wealth is added to the equation for human wealth.

To calculate the constants we use Newton's Method to find a set of values that will make the model's costate variables in 2018 exactly equal their 2018 historical values. We assume the constants remain unchanged over time and therefore they do slightly change the steady state of the model. After the constants have been determined, the model will reproduce the base year exactly given the state variables inherited from 2017 and the assumed future paths of all exogenous variables. Many of the intertemporal constants can be interpreted as risk premia: sectoral equity risk premia, country risk premia, human capital risk premia.

One additional problem is to solve for both real and nominal interest rates consistently since the real interest rate is the nominal interest rate from the money market equilibrium less the ex ante expected inflation rate. To produce the expected inflation rate implicit in historical data for 2018 we add a constant to the equation for nominal wages in each country.

Completing the above steps, with the initial conditions, future projection of exogenous variables, base year projections matched to actual data and the dynamic adjustment post base year, we obtain the baseline projection transitioning on the stable manifold towards the steady state .

5.5 Initialising the value of state variables

To be completed.

5.6 Initialising the value of endogenous variables

To be completed.

5.7 Initialising the value of costate variables

To be completed.

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