

## Introduction

This coursework covers material in the first three weeks of the module. It is divided into three parts which roughly will correspond to the material in each week; however you can carry out the assignment in the time that suits you, with all parts being handed in as a single submission. The assignments will be closely related to the support in the lab classes which you should attend to get help with completing them.

## Tasks

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**Week 1:** 10. January

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### 1. Solving Underdetermined Problems (30%)

As explained in lectures the linear equation  $x_1 + 2x_2 = 5$  is used to introduce the concept of minimum norm solutions of underdetermined problems:

$$\text{minimize } \Phi = \sum_i |x_i|^p, \quad \text{subject to } Ax = b,$$

where in our case,  $A = \begin{pmatrix} 1 & 2 \end{pmatrix}$ ,  $b = 5$ .

- a.) Write a function of two variables,  $x$  and  $p$ , where  $x$  is a vector of length 2 and  $p$  is a scalar; this function will compute the value of  $\Phi$  as given above.
- b.) Use library functions to compute solutions of the above optimization problem, for  $p = 1, 1.5, 2, 2.5, 3, 3.5, 4$ .
- c.) Plot the solutions you have obtained as points on a 2D graph together with the line representing the constraint equation  $x_1 + 2x_2 = 5$ . The result should look like Figure 1.
- d.) Another solution method is to use the *Moore-Penrose generalised inverse*

$$A^\dagger := A^T(AA^T)^{-1},$$

and then apply it to get  $x_{\text{MP}} = A^\dagger b$ . Implement this solution and plot it on the same graph as in the previous question. What value of  $p$  does this correspond to and why?

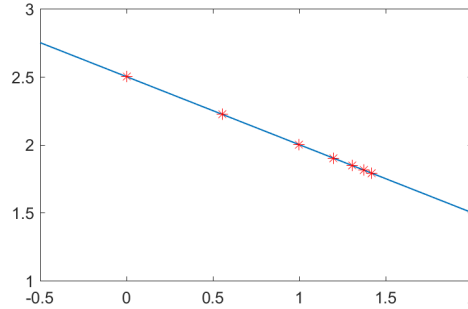


Figure 1: Example for a line plot for Exercise 1

In parts 2 and 3 of the assignment we will implement discrete convolution of a 1D function by an explicit matrix vector multiplication, analyse its behaviour and compare to other methods for performing convolution.

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**Week 2:** 17. January

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## 2. Singular Value Decomposition (30%)

In this part we analyse the matrix representation of convolution using Singular Value Decomposition (SVD).

- a.) Set up a spatial grid on the interval  $[-1, 1]$  in  $n$  equally spaced steps of size  $\delta n$ . The grid represents the values  $[x_1, \dots, x_n]$  with  $x_i = -1 + (i-1)\delta n$  for  $i = 1, \dots, n$  and  $\delta n = 2/(n-1)$ . Make sure that  $x_1 = -1$  and  $x_n = 1$ .
- b.) Create a vector of values of the Gaussian function centred at  $\mu = 0$  with  $\sigma = 0.2$ , given by

$$G(x) = \frac{\delta n}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

You should evaluate this function at the grid points you create in part a).

- c.) Create the convolution matrix of size  $n \times n$  with entries

$$A_{i,j} = G(x_i - x_j) = \frac{\delta n}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x_i - x_j)^2}{2\sigma^2}\right) \quad (1)$$

- d.) Plot the matrix  $A$  as an image, where  $A$  is considered as a 2D-array for the case  $n = 100$ .
- e.) Compute the SVD of matrix  $A$  using library functions. You should obtain three matrices  $U$ ,  $W$ ,  $V$ , where  $W$  is a diagonal matrix of same size as  $A$  containing the singular values. Verify that the equation  $A = UWV^T$  is satisfied.

- f.) Compute the pseudoinverse  $A^\dagger$  of  $A$  by using the formula  $A^\dagger = VW^\dagger U^\top$  as given in lectures. This requires you to create a method for constructing  $W^\dagger$ . For the case  $n = 10$ , check that this has the property  $WW^\dagger = W^\dagger W = Id_n$  where  $Id_n$  is the  $n \times n$  Identity matrix. Check also that  $AA^\dagger = A^\dagger A = Id_n$ .
- g.) Repeat the last two steps for  $n = 20$ . What do you observe? Choose  $n = 100$  again and plot the first 9 columns of  $V$ , the last 9 columns of  $V$ , and the singular values on a logarithmic scale, i.e.  $\log(\text{diag}(W))$ .

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**Week 3:** 24. January

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### 3. Convolutions and Fourier transform (40%)

In the following we want to examine the convolution of a 1D signal; we define a step function on the interval  $[-1, 1]$  by

$$f(x) = \chi_{(-0.95, -0.6]}(x) + 0.2\chi_{(-0.6, -0.2]}(x) - 0.5\chi_{(-0.2, 0.2]}(x) + 0.7\chi_{(0.4, 0.6]}(x) - 0.7\chi_{(0.6, 1]}(x),$$

where the characteristic function of an interval  $(a, b]$  is defined as

$$\chi_{(a, b]}(x) = \begin{cases} 1 & \text{for } a < x \leq b \\ 0 & \text{otherwise.} \end{cases}$$

Here's a plot of  $f$  in Figure 2.

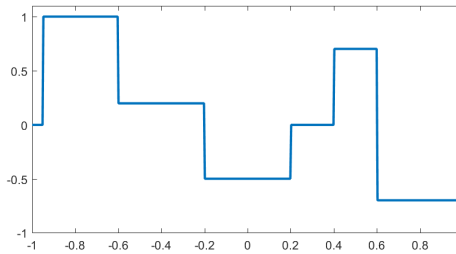


Figure 2: The step function  $f$

- a.) Create a function for  $f$  as above and plot it on a grid on the interval  $[-1, 1]$ . Choose a sufficiently large number of grid points  $n$  to resolve the jumps.
- b.) Compute the matrix  $A$  as in part 2 for  $\sigma = 0.05, 0.1, 0.2$  and plot the singular values.
- c.) Verify that the plot of the singular values follows (half) a Gaussian function and determine the variance of this Gaussian in each case.
- d.) Perform the convolution of the function  $f$  with the the matrix  $A$  (by matrix multiplication) for all three choices of  $\sigma$  and plot the result.

- e.) Since convolution is equivalent to multiplication in Fourier space, perform convolution by multiplication in Fourier space for the three choices of  $\sigma$  and plot the result (remember to take the inverse Fourier transform); comment on any differences that you observe.
- f.) Repeat the convolution with the matrix  $A$  using periodic boundary conditions when assembling  $A$ .

## Report

Write one report for all 3 parts. Explain your method and present your results and figures. Make sure that you provide an answer to all questions. The total length of the report would normally be between 6-10 pages. Submit your report using Moodle. Code can be uploaded separately.