矢量分析与场论初步

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一、标量场的梯度

1、场的概念(The Concept of Field)

场是用空间位置函数来表征的。在物理学中，经常要研究某种物理量在空间的分布和变化规律。如果物理量是标量，并且空间每一点都对应着该物理量的一个确定数值，则称此空间为标量场。如：电势场、温度场等。如果物理量是矢量，且空间每一点都存在着它的大小和方向，则称此空间为矢量场。如：电场、速度场等。若场中各点物理量不随时间变化，称为稳定场，否则，称为不稳定场。

2、方向导数(Directional Gradient)

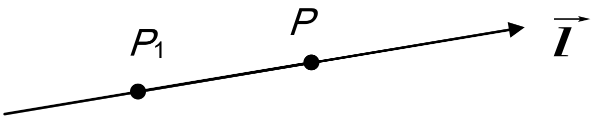
方向导数是标量函数$\varphi(x)$在空间一点沿任意方向$\vec{l}$相对距离的变化率，它的数值与所取$\vec{l}$的方向有关。一般来说，在不同的方向上$\partial\varphi/\partial l|_{P_1}$的值是不同的，但它并不是矢量。如图所示，$\vec{l}$为场中的任意方向，$P_1$是这个方向线上给定的一点，$P_2$为同一线上邻近的一点。$\Delta l$为$p_2$和$p_1$之间的距离，从$p_1$沿$\vec{l}$到$p_2$的增量为$\Delta\varphi=\varphi(p_2)-\varphi(p_1)$若下列极限

$$\lim_{\Delta l\to 0}\frac{\Delta\varphi}{\Delta l}=\lim_{\Delta l\to 0}\frac{\varphi(p_2)-\varphi(p_1)}{\Delta l} \tag{1.1}$$

存在，则该极限值记作$\varphi(\vec{x})$，称之为标量场$\partial\varphi/\partial l|_{P_1}$在$p_1$处沿$\vec{l}$的方向导数。

3.梯度（Gradient）

在某点沿某一确定方向取得$\varphi(\vec{x})$在该点的最大方向导数。



$$\mathrm{grad}\varphi=\nabla\varphi=\frac{\partial\varphi}{\partial n}\hat{\vec{n}} \tag{1.2}$$

$$\frac{\partial\varphi}{\partial l}=\cos\theta\frac{\partial\varphi}{\partial n}=\frac{\partial\varphi}{\partial n}\hat{\vec{n}}\cdot\vec{l}=\mathrm{grad}\varphi\cdot\vec{l} \tag{1.3}$$

4、∇算符（哈密顿算符）(Hamilton Functor)

∇算符既具有微分性质又具有方向性质。在任意方向 $\vec{l}$ 上移动线元距离 dl，φ 的增量 $d\varphi$ 称为方向微分，即

$$d\varphi = \frac{\partial \varphi}{\partial l} dl = \nabla \varphi \cdot d\vec{l} \tag{1.4}$$

显然，任意两点 φ 值差为

$$\varphi_B - \varphi_A = \int_A^B \nabla \varphi \cdot d\vec{l} \tag{1.5}$$

二、矢量场的散度、旋度、高斯定理和斯托克斯定理

1、通量(Fluid)

一个矢量场空间中，在单位时间内，沿着矢量场 $\vec{v}$ 方向通过 $d\vec{s}$ 的流量是 dN，而 dN 是以 ds 为底，以 $v\cos\theta$ 为高的斜柱体的体积，即

$$dN = v\cos\theta ds = \vec{v} \cdot d\vec{s} \tag{1.6}$$

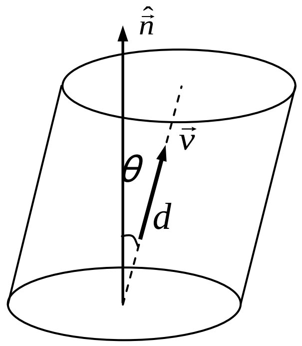
称为矢量 $\vec{v}$ 通过面元 $d\vec{s}$ 的通量。

对于有向曲面 s，总可以将 s 分成许多足够小的面元 $d\vec{s}$，于是通过曲面 s 的通量 N 即为每一面元通量之积

$$N = \iint_s \vec{v} \cdot d\vec{s} \tag{1.7}$$

对于闭合曲面 s，通量 N 为

$$N = \oiint_s \vec{v} \cdot d\vec{s} \tag{1.8}$$



2、散度(Divergence)

设封闭曲面 s 所包围的体积为 ΔV，则

$$\oiint_s \vec{A} \cdot d\vec{s} / \Delta V \tag{1.9}$$

就是矢量场 $\vec{A}(\vec{x})$ 在 ΔV 中单位体积的<u>**平均通量**</u>，或者**平均发散量**。当闭合曲面 s 及其所包围的体积 ΔV 向其内某点 $M(\vec{x})$ 收缩时，若平均发散量的极限值存在，便记作

$$\mathrm{div}\vec{A} = \nabla \cdot \vec{A} = \lim_{\Delta V \to 0} \frac{\oiint_s \vec{A} \cdot d\vec{s}}{\Delta V} \tag{1.10}$$

称为矢量场 $\vec{A}(\vec{x})$ 在该点的散度(div 是 divergence 的缩写)。

散度的重要性在于，可用表征空间各点矢量场发散的强弱程度，当

$\operatorname{div}\vec{A} > 0$，表示该点有散发通量的正源；当 $\operatorname{div}\vec{A} < 0$，表示该点有吸收通量的负源；当 $\operatorname{div}\vec{A} = 0$，表示该点为无源场。

3、高斯定理(Gauss's Theorem)

$$\oiint_s \vec{A}\cdot d\vec{s} = \int_V \nabla\cdot\vec{A}dV \tag{1.11}$$

它能把一个闭合曲面的面积分转为对该曲面所包围体积的体积分，反之亦然。

4、矢量场的环流(The Circumfluence of Vector's Field)

在数学上，将矢量场 $\vec{A}(\vec{x})$ 沿一条有向闭合曲线 L（即取定了线正方向的闭合曲线）的线积分

$$c = \oint_L \vec{A}\cdot d\vec{l} \tag{1.12}$$

称为 $\vec{A}$ 沿该曲线 L 的循环量或环流量。

5、旋度(Rotation)

设想将闭合曲线缩小到其内某一点附近，那么以闭合曲线 L 为界的面积 ΔS 逐渐缩小，$\oint_L \vec{A}\cdot d\vec{l}$ 也将逐渐减小，一般说来，这两者的比值有一极限值，记作

$$\lim_{\Delta s\to 0}\frac{\oint_L \vec{A}\cdot d\vec{l}}{\Delta s} \tag{1.13}$$

即单位面积平均环流的极限。它与闭合曲线的形状无关，但显然依赖于以闭合曲线为界的面积法线方向 $\hat{\vec{n}}$，且通常L的正方向与 $\hat{\vec{n}}$ 规定要构成<u>右手螺旋法则</u>，为此定义

$$\operatorname{rot}\vec{A} = \nabla\times\vec{A} = \lim_{\Delta s\to 0}\frac{\oint_L \vec{A}\cdot d\vec{l}}{\Delta s}\hat{\vec{n}} \tag{1.14}$$

称为矢量场 $\vec{A}(\vec{x})$ 的旋度（rot 是 rotation 缩写）。

旋度的重要性在于，可用以表征矢量在某点附近各方向上环流强弱的程度，如果场中处处 $\operatorname{rot}\vec{A} = 0$ 称为无旋场。

6、斯托克斯定理（Stoke's Theorem）

$$\oint_L \vec{A}\cdot d\vec{s} = \iint_s (\nabla\times\vec{A})\cdot d\vec{s} \tag{1.15}$$

它能把对任意闭合曲线边界的线积分转换为该闭合曲线为界的任意曲面的面积分，反之亦然。

7、度量系数(Measurement Coefficents)

设 x,y,z 是某点的笛卡尔坐标，x_1,x_2,x_3 是这点的正交曲线坐标，长度元的平方表示为

$$dl^2 = dx^2 + dy^2 + dz^2 = h_1^2 dx_1^2 + h_2^2 dx_2^2 + h_3^2 dx_3^2 \tag{1.16}$$

其中

$$h_i = \sqrt{(\frac{\partial x}{\partial x_i})^2 + (\frac{\partial y}{\partial x_i})^2 + (\frac{\partial z}{\partial x_i})^2} \qquad (i = 1,2,3) \tag{1.17}$$

称**<u>度量系数</u>**（或**<u>拉梅系数</u>**），正交坐标系完全由三个拉梅系数h_1, h_2, h_3 来描述。

8、哈密顿算符∇、梯度、散度、旋度及拉普拉斯算符∇^2在正交曲线坐标系下的一般表达式(The General Expression of Hamilton Operator, Gradient, Divergence, Rotation and Laplace Operator in Orthogonal Curvilinear Coordinates)

$$\begin{aligned}
\nabla &= \vec{e}_1 \frac{1}{h_1}\frac{\partial}{\partial x_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial}{\partial x_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial}{\partial x_3} \\
\nabla\varphi &= \vec{e}_1 \frac{1}{h_1}\frac{\partial\varphi}{\partial x_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial\varphi}{\partial x_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial\varphi}{\partial x_3} \\
\nabla\cdot\vec{A} &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3A_1) + \frac{\partial}{\partial x_2}(h_3h_1A_2) + \frac{\partial}{\partial x_3}(h_2h_1A_3)\right]
\end{aligned} \tag{1.18}$$

$$\nabla\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\vec{e}_1 & h_2\vec{e}_2 & h_3\vec{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_2} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

$$\begin{aligned}
&= \frac{\vec{e}_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3A_3) - \frac{\partial}{\partial x_3}(h_2A_2)\right] + \frac{\vec{e}_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1A_1) - \frac{\partial}{\partial x_1}(h_3A_3)\right] \\
&+ \frac{\vec{e}_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2A_2) - \frac{\partial}{\partial x_2}(h_1A_1)\right]
\end{aligned} \tag{1.19}$$

$$\nabla^2\varphi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(\frac{h_2h_3}{h_1}\frac{\partial\varphi}{\partial x_1}) + \frac{\partial}{\partial x_2}(\frac{h_3h_1}{h_2}\frac{\partial\varphi}{\partial x_2}) + \frac{\partial}{\partial x_3}(\frac{h_1h_2}{h_3}\frac{\partial\varphi}{\partial x_3})\right] \tag{1.20}$$

其中$\vec{e}_1,\vec{e}_2,\vec{e}_3$为正交曲线坐标系的基矢；$\varphi = \varphi(x_1, x_2, x_3)$是一个标量函数；$\vec{A} = \vec{A}(x_1, x_2, x_3) = A_1\vec{e}_1 + A_2\vec{e}_2 + A_3\vec{e}_3$是一个矢量函数，只有在笛卡尔坐标系中，$\nabla^2\vec{A} = (\nabla^2\vec{A})_1\vec{e}_1 + (\nabla^2\vec{A})_2\vec{e}_2 + (\nabla^2\vec{A})_3\vec{e}_3$，在其它正交坐标系中

$$(\nabla^2\vec{A})_i \neq \nabla^2\vec{A}_i \tag{1.21}$$

9、不同坐标系中的微分表达式(Difference Expression in Different Coordinates)

a) 笛卡尔坐标

x_1=x， x_2=y， x_3= z　　h_1=1，h_2=1，h_3=1

$$\nabla = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z} \tag{1.22}$$

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \tag{1.23}$$

$$\nabla^2 \vec{A} = (\nabla^2 A_x)\vec{e}_x + (\nabla^2 A_y)\vec{e}_y + (\nabla^2 A_z)\vec{e}_z$$

b)圆柱坐标系

坐标变量:x_1=r， x_2=φ，x_3=z，与笛卡儿坐标的关系：x=$rcos$， y=$rsin\varphi$，z= z

拉梅系数:h_1=1，h_2=r， h_3=1

$$\nabla = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{\partial}{r\partial \phi} + \vec{e}_z \frac{\partial}{\partial z} \tag{1.24}$$

$$\nabla u = \vec{e}_r \frac{\partial u}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial u}{\partial \phi} + \vec{e}_z \frac{\partial u}{\partial z}$$

$$\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \tag{1.25}$$

$$\nabla \times \vec{A} = \begin{vmatrix} \frac{1}{r}\vec{e}_r & \vec{e}_\phi & \frac{1}{r}\vec{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix}$$

$$= (\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z})\vec{e}_r + (\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r})\vec{e}_\phi + \left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial \phi}\right]\vec{e}_z$$

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\nabla^2 \vec{A} = (\nabla^2 \vec{A})_r \vec{e}_r + (\nabla^2 \vec{A})_\phi \vec{e}_\phi + (\nabla^2 \vec{A})_z \vec{e}_z \tag{1.26}$$

c)球坐标系

坐标变量： $x_1 = r$，$x_2 = \theta$，$x_3 = \phi$

与笛卡儿坐标的关系： $x = r\sin\theta\cos\phi$， $y = r\sin\theta\sin\phi$， $z = r\cos\theta$

拉梅系数：$h_1=1$，$h_2=r$， $h_3=r\sin\theta$

$$\nabla=\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\nabla u=\vec{e}_r\frac{\partial u}{\partial r}+\vec{e}_\theta\frac{1}{r}\frac{\partial u}{\partial\theta}+\vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial u}{\partial\phi} \tag{1.27}$$

$$\nabla\cdot\vec{A}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta)+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\vec{A}=\begin{vmatrix}\vec{e}_r\dfrac{1}{r^2\sin\theta} & \vec{e}_\theta\dfrac{1}{r\sin\theta} & \vec{e}_\phi\dfrac{1}{r}\\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi}\\ A_r & rA_\theta & r\sin\theta A_\phi\end{vmatrix}$$

$$=\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi)-\frac{\partial A_\theta}{\partial\phi}\right]\vec{e}_r+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{\partial}{\partial r}(rA_\phi)\right]\vec{e}_\theta+\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}-\right]\vec{e}_\phi \tag{1.28}$$

$$\nabla^2u=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial u}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2u}{\partial\phi^2} \tag{1.29}$$

$$\nabla^2\vec{A}=(\nabla^2\vec{A})_r\vec{e}_r+(\nabla^2\vec{A})_\theta\vec{e}_\theta+(\nabla^2\vec{A})_\phi\vec{e}_\phi$$

其中

$$(\nabla^2\vec{A})_r=\nabla^2\vec{A}_r-\frac{2}{r^2}\left[A_r+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta)+\frac{1}{\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right] \tag{1.30}$$

$$\begin{aligned}(\nabla^2\vec{A})_\theta&=\nabla^2\vec{A}_\theta+\frac{2}{r^2}(\frac{\partial A_r}{\partial\theta}-\frac{A_\theta}{2\sin^2\theta}-\frac{\cos\theta}{\sin^2\theta}\frac{\partial A_\phi}{\partial\phi})\\(\nabla^2\vec{A})_\phi&=\nabla^2\vec{A}_\phi+\frac{2}{r^2\sin\theta}(\frac{\partial A_r}{\partial\phi}+\mathrm{ctg}\theta\frac{\partial A_\theta}{\partial\phi}-\frac{A_\phi}{2\sin\theta})\end{aligned} \tag{1.31}$$

补充知识：

$$\frac{\partial\vec{e}_1}{\partial q_1}=-\frac{\vec{e}_2}{H_2}\frac{\partial H_1}{\partial q_2}-\frac{\vec{e}_3}{H_3}\frac{\partial H_1}{\partial q_3}$$

$$\frac{\partial\vec{e}_1}{\partial q_2}=\frac{\vec{e}_2}{H_1}\frac{\partial H_2}{\partial q_1}$$

$$\frac{\partial\vec{e}_1}{\partial q_3}=\frac{\vec{e}_3}{H_1}\frac{\partial H_3}{\partial q_1}$$

$$\frac{\partial\vec{e}_2}{\partial q_1}=\frac{\vec{e}_1}{H_2}\frac{\partial H_1}{\partial q_2}$$

$$\frac{\partial\vec{e}_2}{\partial q_2}=-\frac{\vec{e}_3}{H_3}\frac{\partial H_2}{\partial q_3}-\frac{\vec{e}_1}{H_1}\frac{\partial H_2}{\partial q_1}$$

$$\frac{\partial\vec{e}_2}{\partial q_3}=\frac{\vec{e}_3}{H_2}\frac{\partial H_3}{\partial q_2}$$

$$\frac{\partial\vec{e}_3}{\partial q_1}=\frac{\vec{e}_1}{H_3}\frac{\partial H_1}{\partial q_3}$$

$$\frac{\partial\vec{e}_3}{\partial q_2}=\frac{\vec{e}_2}{H_3}\frac{\partial H_2}{\partial q_3}$$

$$\frac{\partial\vec{e}_3}{\partial q_3}=-\frac{\vec{e}_1}{H_1}\frac{\partial H_3}{\partial q_1}-\frac{\vec{e}_2}{H_2}\frac{\partial H_3}{\partial q_2}$$

三、二阶微分算符　格林定理

1、一阶微分运算(First-order Difference Calculation)

a) 设 $r=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$ 为源点 $\vec{x}'$ 与场 $\vec{x}$ 之间的距离，r 的方向规定为由源点指向场点，试分别对场点和源点求标量场r 的梯度。

$$\begin{aligned}\nabla r &= \vec{e}_x\frac{\partial r}{\partial x}+\vec{e}_y\frac{\partial r}{\partial y}+\vec{e}_z\frac{\partial r}{\partial z}\\ &=\vec{e}_x\frac{(x-x')}{r}+\vec{e}_y\frac{(y-y')}{r}+\vec{e}_z\frac{(z-z')}{r}\\ &=\frac{1}{r}\left[\vec{e}_x(x-x')+\vec{e}_y(y-y')+\vec{e}_z(z-z')\right]=\frac{\vec{r}}{r}=\hat{\vec{r}}\end{aligned}\tag{1.32}$$

$$\begin{aligned}\nabla' r &= \vec{e}_x\frac{\partial r}{\partial x'}+\vec{e}_y\frac{\partial r}{\partial y'}+\vec{e}_z\frac{\partial r}{\partial z'}\\ &=-\vec{e}_x\frac{(x-x')}{r}--\vec{e}_y\frac{(y-y')}{r}--\vec{e}_z\frac{(z-z')}{r}\\ &=-\frac{\vec{r}}{r}=-\hat{\vec{r}}=-\nabla r\end{aligned}\tag{1.33}$$

b) 设u是空间坐标x, y, z的函数，证明

$$\nabla f(u)=\frac{df}{du}\nabla u\tag{1.34}$$

证：这是求复合函数的导数（梯度），按复合函数微分法则，有

$$\begin{aligned}\nabla f(u) &= \vec{e}_x\frac{\partial f(u)}{\partial x}+\vec{e}_y\frac{\partial f(u)}{\partial y}+\vec{e}_z\frac{\partial f(u)}{\partial z}\\ &=\vec{e}_x\frac{df(u)}{du}\frac{\partial u}{\partial x}+\vec{e}_y\frac{df(u)}{du}\frac{\partial u}{\partial y}+\vec{e}_z\frac{df(u)}{du}\frac{\partial u}{\partial z}\\ &=\frac{df(u)}{du}(\vec{e}_x\frac{\partial u}{\partial x}+\vec{e}_y\frac{\partial u}{\partial y}=\vec{e}_z\frac{\partial u}{\partial z})=\frac{df(u)}{du}\nabla u\end{aligned}$$

c)设$\vec{r}=\vec{e}_x(x-x')+\vec{e}_y(y-y')+\vec{e}_z(z-z')=\vec{x}-\vec{x}'$ 求$\nabla\cdot\vec{r}$和$\nabla'\cdot\vec{r}$？

d)设 u 是空间坐标 x,y,z 的函数，证明$\nabla\cdot\vec{A}(u)=\nabla u\cdot\frac{d\vec{A}}{du}$.

$$\begin{aligned}\nabla\cdot\vec{A}(u) &= \frac{\partial A_x(u)}{\partial x}+\frac{\partial A_y(u)}{\partial y}+\frac{\partial A_z(u)}{\partial z}\\ &=\frac{dA_x(u)}{du}\frac{\partial u}{\partial x}+\frac{dA_y(u)}{du}\frac{\partial u}{\partial y}+\frac{dA_z(u)}{du}\frac{\partial u}{\partial z}\\ &=\frac{d\vec{A}(u)}{du}\cdot\nabla u=\nabla u\cdot\frac{d\vec{A}(u)}{du}\end{aligned}$$

e)设u是空间坐标x, y, z的函数，证明

$$\nabla\times\vec{A}(u)=\nabla u\times\frac{d\vec{A}(u)}{du}$$

$$\begin{aligned}\nabla\times\vec{A}(u)&=\vec{e}_x\left(\frac{\partial A_z(u)}{\partial y}-\frac{\partial A_y(u)}{\partial z}\right)+\vec{e}_y\left(\frac{\partial A_x(u)}{\partial z}-\frac{\partial A_z(u)}{\partial x}\right)+\vec{e}_z\left(\frac{\partial A_y(u)}{\partial x}-\frac{\partial A_x(u)}{\partial y}\right)\\&=\vec{e}_x\left(\frac{dA_z(u)}{du}\frac{\partial u}{\partial y}-\frac{dA_y(u)}{du}\frac{\partial u}{\partial z}\right)+\vec{e}_z\left(\frac{dA_y(u)}{du}\frac{\partial u}{\partial x}-\frac{dA_x(u)}{du}\frac{\partial u}{\partial y}\right)\\&=\begin{vmatrix}\vec{e}_x&\vec{e}_y&\vec{e}_z\\\frac{\partial u}{\partial x}&\frac{\partial u}{\partial y}&\frac{\partial u}{\partial z}\\\frac{dA_x(u)}{du}&\frac{dA_y(u)}{du}&\frac{dA_z(u)}{du}\end{vmatrix}=\nabla u\times\frac{d\vec{A}(u)}{du}\end{aligned}$$

2、二阶微分运算(Calculation of Two-order Difference)

将算符∇作用于梯度、散度和旋度，则称为二阶微分运算，设$\varphi(\vec{x})$为标量场，$\vec{g}(\vec{x})$，$\vec{f}(\vec{x})$为矢量场。

并假设φ和$\vec{g}$，$\vec{f}$的分量具有所需要的阶的连续微商，则不难得到：

（1）标量场的梯度必为无旋场　　$\nabla\times(\nabla\varphi)=0$　　(1.35)

（2）矢量场的旋度必为无散场　　$\nabla\cdot(\nabla\times\vec{g})=0$　　(1.36)

（3）无旋场可表示一个标量场的梯度　　若$\nabla\times\vec{g}=0$，　则$\vec{g}=\nabla\varphi$　　(1.37)

（4）无散场可表示一个矢量场的旋度　　若$\nabla\cdot\vec{g}=0$，　则$\vec{g}=\nabla\times\vec{f}$　　(1.38)

（5）标量场的梯度的散度为

$$\nabla\cdot(\nabla\varphi)=\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial z}\right)=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\qquad(1.39)$$

（6）矢量场的旋度的旋度为　　$\nabla\times(\nabla\times\vec{g})=\nabla(\nabla\cdot\vec{g})-\nabla^2\vec{g}$　　(1.40)

3、▽运算于乘积(Calculation of Multiplication with ▽)

（1）$\nabla\times(\nabla\varphi)=0$

$$\nabla\times(\nabla\varphi)=\begin{pmatrix}\vec{e}_x & \vec{e}_y & \vec{e}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \dfrac{\partial\varphi}{\partial x} & \dfrac{\partial\varphi}{\partial y} & \dfrac{\partial\varphi}{\partial z}\end{pmatrix}$$

$$=\vec{e}_x\left(\frac{\partial^2\varphi}{\partial y\partial z}-\frac{\partial^2\varphi}{\partial z\partial y}\right)+\vec{e}_y\left(\frac{\partial^2\varphi}{\partial z\partial x}-\frac{\partial^2\varphi}{\partial x\partial z}\right)+\vec{e}_z\left(\frac{\partial^2\varphi}{\partial x\partial y}-\frac{\partial^2\varphi}{\partial y\partial x}\right)$$

$$=0$$

(2) $\nabla\cdot(\nabla\times\vec{g})=0$

$$\nabla\cdot(\nabla\times\vec{g})=\left(\vec{e}_x\frac{\partial}{\partial x}+\vec{e}_y\frac{\partial}{\partial y}+\vec{e}_z\frac{\partial}{\partial z}\right)\cdot\begin{pmatrix}\vec{e}_x & \vec{e}_y & \vec{e}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ g_x & g_y & g_z\end{pmatrix}$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial g_z}{\partial y}-\frac{\partial g_y}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial g_x}{\partial z}-\frac{\partial g_z}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial g_y}{\partial x}-\frac{\partial g_x}{\partial y}\right)$$

$$=\frac{\partial^2 g_z}{\partial x\partial y}-\frac{\partial^2 g_y}{\partial x\partial z}+\frac{\partial^2 g_x}{\partial y\partial z}-\frac{\partial^2 g_z}{\partial z\partial x}+\frac{\partial^2 g_y}{\partial z\partial x}-\frac{\partial^2 g_x}{\partial z\partial y}$$

$$=0$$

(3) $\nabla(\varphi\psi)=\psi\nabla\varphi+\varphi\nabla\psi$

$$\nabla(\varphi\psi)=\vec{e}_x\frac{\partial}{\partial x}(\varphi\psi)+\vec{e}_y\frac{\partial}{\partial y}(\varphi\psi)+\vec{e}_z\frac{\partial}{\partial z}(\varphi\psi)$$

$$=\vec{e}_x(\psi\frac{\partial\varphi}{\partial x}+\varphi\frac{\partial\psi}{\partial x})+\vec{e}_y(\psi\frac{\partial\varphi}{\partial y}+\varphi\frac{\partial\psi}{\partial y})+\vec{e}_z(\psi\frac{\partial\varphi}{\partial z}+\varphi\frac{\partial\psi}{\partial z})$$

$$=\psi(\vec{e}_x\frac{\partial\varphi}{\partial x}+\vec{e}_y\frac{\partial\varphi}{\partial y}+\vec{e}_z\frac{\partial\varphi}{\partial z})+\varphi\ (\vec{e}_x\frac{\partial\psi}{\partial x}+\vec{e}_y\frac{\partial\psi}{\partial y}+\vec{e}_z\frac{\partial\psi}{\partial z})$$

$$=\psi\nabla\varphi+\varphi\nabla\psi$$

(4) $\nabla\cdot(\varphi\vec{g})=\varphi\nabla\cdot\vec{g}+\nabla\varphi\cdot\vec{g}$

$$\nabla\cdot(\varphi\vec{g})=(\nabla_\varphi+\nabla_{\vec{g}})\cdot(\varphi\vec{g})$$

$$=\nabla_\varphi\cdot(\varphi\vec{g})+\nabla_{\vec{g}}\cdot(\varphi\vec{g})$$

$$=\nabla_\varphi\varphi\cdot\vec{g}+\varphi\nabla_{\vec{g}}\cdot\vec{g}$$

$$=\nabla\varphi\cdot\vec{g}+\varphi\nabla\cdot\vec{g}$$

(5) $\nabla\times(\varphi\vec{g})=\varphi\nabla\times\vec{g}+\nabla\varphi\times\vec{g}$

$$\begin{aligned}\nabla\times(\varphi\vec{g}) &= (\nabla_{\varphi}+\nabla_{\vec{g}})\times(\varphi\vec{g}) \\ &= \nabla_{\varphi}\times(\varphi\vec{g})+\nabla_{\vec{g}}\times(\varphi\vec{g}) \\ &= \nabla_{\varphi}\varphi\times\vec{g}+\varphi\nabla_{\vec{g}}\times\vec{g} \\ &= \nabla\varphi\times\vec{g}+\varphi\nabla\times\vec{g}\end{aligned}$$

(6) $\nabla\cdot(\vec{g}\times\vec{f}) = \vec{f}\cdot(\nabla\times\vec{g})-\vec{g}\cdot(\nabla\times\vec{f})$

$$\begin{aligned}\nabla\cdot(\vec{g}\times\vec{f}) &= (\nabla_{\vec{g}}+\nabla_{\vec{f}})\cdot(\vec{g}\times\vec{f}) \\ &= \nabla_{\vec{g}}\cdot(\vec{g}\times\vec{f})+\nabla_{\vec{f}}\cdot(\vec{g}\times\vec{f})\end{aligned}$$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$$

$$\nabla_{\vec{g}}\cdot(\vec{g}\times\vec{f}) = \vec{f}\cdot(\nabla_{\vec{g}}\times\vec{g}) = \vec{f}\cdot(\nabla\times\vec{g})$$

$$\begin{aligned}\nabla_{\vec{f}}\cdot(\vec{g}\times\vec{f}) &= -\nabla_{\vec{f}}\cdot(\vec{f}\times\vec{g}) = -\vec{g}\cdot(\nabla_{\vec{f}}\times\vec{f}) \\ &= -\vec{g}\cdot(\nabla\times\vec{f})\end{aligned}$$

$$\nabla\cdot(\vec{g}\times\vec{f}) = \vec{f}\cdot(\nabla\times\vec{g})-\vec{g}\cdot(\nabla\times\vec{f})$$

(7) $\nabla\times(\vec{g}\times\vec{f}) = (\vec{f}\cdot\nabla)\vec{g}+(\nabla\cdot\vec{f})\vec{g}-(\vec{g}\cdot\nabla)\vec{f}-(\nabla\cdot\vec{g})\vec{f}$

$$\begin{aligned}\nabla\times(\vec{g}\times\vec{f}) &= (\nabla_{\vec{g}}+\nabla_{\vec{f}})\times(\vec{g}\times\vec{f}) \\ &= \nabla_{\vec{g}}\times(\vec{g}\times\vec{f})+\nabla_{\vec{f}}\times(\vec{g}\times\vec{f})\end{aligned}$$

$$\begin{aligned}\nabla\times(\vec{g}\times\vec{f}) &= -\nabla_{\vec{g}}\times(\vec{f}\times\vec{g})+\nabla_{\vec{f}}\times(\vec{g}\times\vec{f}) \\ &= -(\nabla_{\vec{g}}\cdot\vec{g})\vec{f}+(\vec{f}\cdot\nabla_{\vec{g}})\vec{g}+(\nabla_{\vec{f}}\cdot\vec{f})\vec{g}-(\vec{g}\cdot\nabla_{\vec{f}})\vec{f} \\ &= -(\nabla\cdot\vec{g})\vec{f}+(\vec{f}\cdot\nabla)\vec{g}+(\nabla\cdot\vec{f})\vec{g}-(\vec{g}\cdot\nabla)\vec{f}\end{aligned}$$