XIN TAO

CLASSICAL ELECTRODYNAMICS

Contents

讲义说明

2018 版说明: 一个主要变化是重新加入了从分析力学的角度推导 Maxwell 方程的章节，因为这一章节非常有助于从场论的角度理解电动力学，同时也强调了场作为另一种描述物理现象的方式与粒子的同等重要地位。与 2013 年的讲义不同的是，这次的场的分析力学部分为选修章节，不会计入最终成绩，也不会参加正常的考试部分。

2015 版说明：这本讲义主要是我自己讲课时参考所用，其内容主要了参考了 Landau 和 Lifshitz 的《The Classic Theory of Fields》，David J. Griffiths 的《Introduction to Electrodynamics (4th Edition)》。其余部分还参考了 Jackson 的《Classical Electrodynamics》。在这三本书中，Landau 和 Jackson 的书并不太适合一般本科生阅读。这个讲义里对 Landau 的书作了大幅简化，而 Jackson 的书则只是参考了一小部分数学证明和一些个别有意思的章节。正常情况下，我推荐的阅读书是 Griffiths 的电动力学导论。

这个讲义第一版的使用是在 2014 年春季学期，当时采用的单位制是 Gauss 单位制，因为我觉得 Gauss 单位制更适合理论电动力学。但在实行过程中，不少同学向我建议换回国际单位制，以求更好地和之前的课程衔接。因此从 2015 年春季学期开始，我对这个讲义作了比较大幅度的改写，一是增加更多的基础数学知识，二是去除场的分析力学部分，三是将 Gauss 单位制换为 SI 单位制。在这里要特别感谢金泽宇，王贤瞳，和吴昊楠三位同学的协助，他们在改单位制的过程中付出了大量劳动，并且修正了 2014 年讲义中的很多错误。2014 年春季学期班上还有好几位其他同学在 14 年教学过程中也陆续地指出讲义中的错误，但遗憾当时并没有记录名单，在此一并感谢。

1
Introduction to Vector and Tensor Analysis

It is critically important for you to learn basic vector and tensor analysis to understand the electromagnetic field theory. In this chapter, we will explain some basic concepts and practically useful techniques about vector and tensor analysis.

1.1 Vector Analysis

1.1.1 The definition

For most of you, a vector is an object with both *magnitude* and *direction*; e.g., displacement, velocity, etc. This definition is useful and pretty much all you need in 3D space. To understand the vector in higher dimensional or non-Euclidean space; however, this definition is not that intuitative. We introduce here a more general and formal definition about the vector in 3D Euclidean space.

> Any set of three components that transforms in the same manner as the displacement vector when one changes coordinates is called a "vector".

You can see that, the displacement vector is the "model vector" for all vectors. This definition of the vector is very important to understanding 4-vectors in special relativity where laws of phsics is defined in a four dimensional pseudo-Euclidean space called Minkowski space. You can see that some sets of numbers look like a vector, but they do not transform like the model vector (the displacement vector); therefore, they are not vectors.

1.1.2 Einstein Summation Convention

Before we discuss more about vector analysis, we first introduce the Einstein summation convention, a powerful and simple notation. Einstein noticed that in some expressions involving summation over components of vectors, e.g.,

$$\boldsymbol{A} \cdot \boldsymbol{B} = \sum_{i=1}^{3} A_i B_i, \tag{1.1}$$

you can take away the summation sign ($\sum$) without changing the meaning of the expression. Therefore, you can write

$$\boxed{\boldsymbol{A} \cdot \boldsymbol{B} = A_i B_i.} \tag{1.2}$$

Here the repeated index i is called the "*dummy index*" (哑指标), and a dummy index is implicitly summed over. To generalize this convection to more general cases, we need to follow essentially the following three rules:

1. Dummy indices are implicitly summed over.
2. Each index can appear at most twice in any term.
3. Each term must contain identical non-repeated indices, also called free indices (自由指标).

To understand the following rules, we give some examples here. You have already seen rule #1. So for example, if you see $A_i \boldsymbol{e}_i$, it means vector $\boldsymbol{A}$, because

$$A_i \boldsymbol{e}_i = A_1 \boldsymbol{e}_1 + A_2 \boldsymbol{e}_2 + A_3 \boldsymbol{e}_3. \tag{1.3}$$

Rule #2 prohibits expressions like $A_i B_i C_i$, because no vector operations can result in this kind of expression after removing the summation sign, and it can easily produce wrong results.

As an example, we can also use Einstein summation rules to obtain $\boldsymbol{A} \cdot \boldsymbol{B}$ directly using definitions of $\boldsymbol{A}$ and $\boldsymbol{B}$. Suppose that we have two vectors A and $\boldsymbol{B}$ in Cartesian space,

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_i \boldsymbol{e}_i \cdot B_j \boldsymbol{e}_j = A_i B_j \boldsymbol{e}_i \cdot \boldsymbol{e}_j = A_i B_j \delta_{ij} = A_i B_i. \tag{1.4}$$

Here $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Note that we used "i" as the dummy index for $\boldsymbol{A}$ while j as the dummy index for $\boldsymbol{B}$. You cannot write it as $A_i \boldsymbol{e}_i \cdot B_i \boldsymbol{e}_i$, because of the rule #2: the dummy index can appear only twice in a given term. Writing it this way gives you the correct results $A_i B_i$, but that is purely coincidental: in more general cases, $\boldsymbol{e}_i \cdot \boldsymbol{e}_j \neq 0$[1]. For example, let us assume for now that we use a coordinate system where $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = g_{ij}$, then the correct results for $\boldsymbol{A} \cdot \boldsymbol{B} = A_i B_j g_{ij}$, [2] not $A_i B_i$ as you would have by violating rule #2.

Rule #3 is useful when you think about a particular component of the a vector expression. For example, the i_{th} component of $\boldsymbol{A} + \boldsymbol{B}$ is $A_i + B_i$. Here i is the free index in both terms (A_i and B_i). The expression $A_i + B_j$ does not mean anything.

Rules #2 and #3 can be used to identify wrong expressions. For example, $A_i B_i C_i$ cannot be correct because of rule 2: the index i appears three times in this term. Expression $A_i B_j C_j + E_p$ is also wrong, because of rule 3: The non-repeated index in $A_i B_j C_j$ is "i", while in E_p is "p", and they are not identical. On the other hand, $A_i B_j C_j + E_i$ is correct. This expression means the i-th component of the vector $\boldsymbol{A}(\boldsymbol{B} \cdot \boldsymbol{C}) + \boldsymbol{E}$. You have to make yourself familar with the

[1] This kind of coordinate system is non-orthonognal (非正交)

[2] This notation is also problamatic if you are familiar with vector analysis in curvilinear coordinate system; it is used only for illustration of the summation convection. For those of you who are interested, one correct way to write it is $A^i B^j g_{ij}$

Einstein summation convection, as it will be used throughout this class.

The two basic vector operations are the dot and the cross products. For the dot product of two vectors, we already see that it is $\boldsymbol{A} \cdot \boldsymbol{B} = A_i B_i$. For the cross product of two vectors, $\boldsymbol{A} \times \boldsymbol{B}$,

$$\boldsymbol{A} \times \boldsymbol{B} = (A_2 B_3 - A_3 B_2)\boldsymbol{e}_1 + (A_3 B_1 - A_1 B_3)\boldsymbol{e}_2 + (A_1 B_2 - A_2 B_1)\boldsymbol{e}_3.$$

How do you write this using Einstein notation? Let's introduce the Levi-Civita symbol ϵ_{ijk} given by

$$\boxed{\epsilon_{ijk} = \begin{cases} +1 & \text{if } i,j,k \text{ form an even permutation of } 1,2,3. \\ -1 & \text{if } i,j,k \text{ form an odd permutation of } 1,2,3. \\ 0 & \text{otherwise} \end{cases}} \tag{1.5}$$

Hence, $\epsilon_{ijk} = \epsilon_{jki}$, $\epsilon_{ijk} = -\epsilon_{jik}$, and $\epsilon_{iik} = 0$. A very useful identity of ϵ_{ijk} is

$$\boxed{\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km},} \tag{1.6}$$

where $\delta_{jm} = 0$ if $j \neq m$ and 1 if $j = m$. With the Levi-Civita symbol, you can easily see that

$$\boxed{\boldsymbol{A} \times \boldsymbol{B} = \epsilon_{ijk} A_j B_k \boldsymbol{e}_i.} \tag{1.7}$$

1.1.3 The Del Operator

One complicating factor in vector analysis is the del operator, ∇. The gradient of a scalar is ∇T. In Cartesian coordinates,

$$\nabla T = \boldsymbol{e}_1 \frac{\partial T}{\partial x_1} + \boldsymbol{e}_2 \frac{\partial T}{\partial x_2} + \boldsymbol{e}_3 \frac{\partial T}{\partial x_3}, \tag{1.8}$$

where $x_1 \equiv x, x_2 \equiv y, x_3 \equiv z$. With Einstein summation convention,

$$\nabla T = \boldsymbol{e}_i \frac{\partial T}{\partial x_i}. \tag{1.9}$$

Therefore, in Cartesian coordinates,

$$\boxed{\nabla = \boldsymbol{e}_i \frac{\partial}{\partial x_i}.} \tag{1.10}$$

We see that ∇ is a vector operator; it has properties of a vector and an operator at the same time.

Using Equation (1.10), a few expressions related to ∇ can be quickly given. For example, the divergence of a vector,

$$\boxed{\nabla \cdot \boldsymbol{A} = \frac{\partial}{\partial x_i} A_i = \frac{\partial A_i}{\partial x_i}.} \tag{1.11}$$

The curl of a vector $\boldsymbol{A}$ is

$$\boxed{\nabla \times \boldsymbol{A} = \epsilon_{ijk} \boldsymbol{e}_i \frac{\partial}{\partial x_j} A_k = \epsilon_{ijk} \frac{\partial A_k}{\partial x_j} \boldsymbol{e}_i.} \tag{1.12}$$

1.2 Vector Algebra

In this section, I will teach you how to memorize/derive commonly used vector algebra without referring to a handbook. We will use a lot of vector analysis and identities in this class.

$$
\begin{aligned}
\boldsymbol{A} \cdot \boldsymbol{B} \times \boldsymbol{C} &= \boldsymbol{B} \cdot \boldsymbol{C} \times \boldsymbol{A} = \boldsymbol{C} \cdot \boldsymbol{A} \times \boldsymbol{B} \\
&= \boldsymbol{A} \times \boldsymbol{B} \cdot \boldsymbol{C} \\
\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) &= (\boldsymbol{C} \times \boldsymbol{B}) \times \boldsymbol{A} = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C} \\
\nabla(fg) &= f\nabla g + g\nabla f \\
\nabla \cdot (f\boldsymbol{A}) &= f\nabla \cdot \boldsymbol{A} + \boldsymbol{A} \cdot \nabla f \\
\nabla \times (f\boldsymbol{A}) &= f\nabla \times \boldsymbol{A} + \nabla f \times \boldsymbol{A} \\
\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) &= \boldsymbol{B} \cdot \nabla \times \boldsymbol{A} - \boldsymbol{A} \cdot \nabla \times \boldsymbol{B} \\
\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) &= \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A}) \\
&\quad + (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} \\
\nabla \times (\nabla \times \boldsymbol{A}) &= \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} \\
\nabla \cdot \nabla \times \boldsymbol{A} &= 0 \\
\nabla \times \nabla f &= 0 \\
A_\parallel &= \boldsymbol{A} \cdot \frac{\boldsymbol{B}}{B} \equiv \boldsymbol{A} \cdot \hat{\boldsymbol{B}} \\
\boldsymbol{A}_\perp &= -\hat{\boldsymbol{B}} \times (\hat{\boldsymbol{B}} \times \boldsymbol{A})
\end{aligned}
$$

In the last two identities, the direction is w.r.t. $\hat{\boldsymbol{B}}$. And also some useful identities involving $\boldsymbol{r}$, the radial vector:

$$
\begin{aligned}
\nabla \cdot \boldsymbol{r} &= 3 \\
\nabla \times \boldsymbol{r} &= 0 \\
\nabla \boldsymbol{r} &= \mathsf{I} \\
\nabla r &= \frac{\boldsymbol{r}}{r} = \boldsymbol{e_r} \\
\nabla \times [f(r)\boldsymbol{r}] &= 0.
\end{aligned}
$$

We now start from the most basic vector operations and show you how to memorize them.

First, for $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$, you can interchange the dot and cross.

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{A} \times \boldsymbol{B} \cdot \boldsymbol{C}.$$

Or you can cyclically permute the order of vectors, like

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B} \cdot \boldsymbol{C} \times \boldsymbol{A}.$$

Of course, since $\boldsymbol{B} \times \boldsymbol{C} = -\boldsymbol{C} \times \boldsymbol{B}$,

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = -\boldsymbol{A} \cdot \boldsymbol{C} \times \boldsymbol{B}.$$

Second, for $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})$, you can use the middle-outer rule [3].

$$\boldsymbol{A} \times (\underbrace{\boldsymbol{B}}_{\text{middle}} \times \underbrace{\boldsymbol{C}}_{\text{outer}}) = \underbrace{\boldsymbol{B}}_{\text{middle}}(\underbrace{\boldsymbol{A} \cdot \boldsymbol{C}}_{\text{other two dotted}}) - \underbrace{\boldsymbol{C}}_{\text{outer}}(\underbrace{\boldsymbol{B} \cdot \boldsymbol{A}}_{\text{other two dotted}}).$$

[3] *Fundamentals of Plasma Physics* by Paul M. Bellan

$$(\underbrace{\boldsymbol{B}}_{\text{outer}} \times \underbrace{\boldsymbol{C}}_{\text{middle}}) \times \boldsymbol{A} = \underbrace{\boldsymbol{C}}_{\text{middle}}(\underbrace{\boldsymbol{B} \cdot \boldsymbol{A}}_{\text{other two dotted}}) - \underbrace{\boldsymbol{B}}_{\text{outer}}(\underbrace{\boldsymbol{A} \cdot \boldsymbol{C}}_{\text{other two dotted}}).$$

I've found this middle-outer rule quite convenient.

Third, for identities involving ∇, you should treat it as being both a vector and a differential operator. Applying the calculus rule $(ab)' = a'b + ab'$, we know

$$\nabla(fg) = g\nabla f + f\nabla g$$

I find it not difficult to see

$$\begin{aligned} \nabla \cdot (f\boldsymbol{A}) &= f\nabla \cdot \boldsymbol{A} + \boldsymbol{A} \cdot \nabla f \\ \nabla \times (f\boldsymbol{A}) &= f\nabla \times \boldsymbol{A} + \nabla f \times \boldsymbol{A} \end{aligned}$$

The real difficulty most students find in doing vector anlysis is when an expression has complicated identities involving ∇. Here, we introduce two mathematical tricks.

1.2.1 Method 1

This one involves 3 steps; it uses tricks to seperate the operator property and the vector property of the ∇ operator. I'll use $\nabla \times (\boldsymbol{A} \times \boldsymbol{B})$ as an example.

Step 1: Treat ∇ as an operator. Here it applies to both $\boldsymbol{A}$ and $\boldsymbol{B}$. So we add a subscript to ∇ so indicate which vector it operators on.

$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \nabla_{\boldsymbol{A}} \times (\boldsymbol{A} \times \boldsymbol{B}) + \nabla_{\boldsymbol{B}} \times (\boldsymbol{A} \times \boldsymbol{B}).$$

Step 2: Treat $\nabla_{\boldsymbol{A}}$ and $\nabla_{\boldsymbol{B}}$ as different vectors, applying vector rules, and making necessary adjustments to put $\nabla_{\boldsymbol{A}}$ right before $\boldsymbol{A}$ and $\nabla_{\boldsymbol{B}}$ right before $\boldsymbol{B}$. Adjustments should also satisfy vector rules.

$$\begin{aligned} \nabla_{\boldsymbol{A}} \times (\boldsymbol{A} \times \boldsymbol{B}) &= (\nabla_{\boldsymbol{A}} \cdot \boldsymbol{B})\boldsymbol{A} - \boldsymbol{B}(\nabla_{\boldsymbol{A}} \cdot \boldsymbol{A}) \quad \textit{middle-outer rule} \\ &= (\boldsymbol{B} \cdot \nabla_{\boldsymbol{A}})\boldsymbol{A} - \boldsymbol{B}(\nabla_{\boldsymbol{A}} \cdot \boldsymbol{A}) \quad \textit{adjustment} \end{aligned}$$

Similarly, we have

$$\begin{aligned} \nabla_{\boldsymbol{B}} \times (\boldsymbol{A} \times \boldsymbol{B}) &= (\nabla_{\boldsymbol{B}} \cdot \boldsymbol{B})\boldsymbol{A} - (\nabla_{\boldsymbol{B}} \cdot \boldsymbol{A})\boldsymbol{B} \\ &= (\nabla_{\boldsymbol{B}} \cdot \boldsymbol{B})\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla_{\boldsymbol{B}})\boldsymbol{B} \end{aligned}$$

Step 3: Dropping subscripts of ∇, we have

$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) + (\nabla \cdot \boldsymbol{B})\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}$$

Now let's use this trick to derive $\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B})$.

1. $\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \nabla_{\boldsymbol{A}} \cdot (\boldsymbol{A} \times \boldsymbol{B}) + \nabla_{\boldsymbol{B}} \cdot (\boldsymbol{A} \times \boldsymbol{B})$.

2. $\nabla_{\boldsymbol{A}} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \nabla_{\boldsymbol{A}} \times \boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \nabla_{\boldsymbol{A}} \times \boldsymbol{A}$
   $\nabla_{\boldsymbol{B}} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = -\nabla_{\boldsymbol{B}} \times \boldsymbol{B} \cdot \boldsymbol{A} = -\boldsymbol{A} \cdot \nabla_{\boldsymbol{B}} \times \boldsymbol{B}$

3. $\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot \nabla \times \boldsymbol{A} - \boldsymbol{A} \cdot \nabla \times \boldsymbol{B}$

Note: you might ignore adding subscripts after you get familiar with the method.

1.2.2 Method 2

The second trick for vector/tensor analysis uses the property that a vector/tensor identity is valid in all coordinate systems. Using this property, we can perform vector-tensor analysis in the following way:

1. write out the components in Cartesian coordinates

2. do necessary analysis

3. from components, restore the original variables

Usually the last step is the most difficult one, it requires some experience and practice. Also this method is especially powerful when used with the Einstein summation convention.

Example 1: Prove $\nabla\cdot\boldsymbol{r} = 3$ and $\nabla\times\boldsymbol{r} = 0$.

$$\nabla\cdot\boldsymbol{r} = \boldsymbol{e}_i\frac{\partial}{\partial x_i}\cdot x_j\boldsymbol{e}_j = \frac{\partial x_j}{\partial x_i}\delta_{ij} = \frac{\partial x_i}{\partial x_i} = 3,$$
$$\nabla\times\boldsymbol{r} = \boldsymbol{e}_i\epsilon_{ijk}\frac{\partial}{\partial x_j}x_k = \boldsymbol{e}_i\epsilon_{ijk}\delta_{jk} = 0.$$

Example 2: Prove

$$\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\cdot\nabla\times\boldsymbol{B}.$$

Proof:

$$\begin{aligned}
\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) &= \boldsymbol{e}_i\frac{\partial}{\partial x_i}\cdot(\epsilon_{jkl}A_kB_l)\boldsymbol{e}_j = \frac{\partial}{\partial x_i}(\epsilon_{ikl}A_kB_l)\\
&= \epsilon_{ikl}\left(\frac{\partial A_k}{\partial x_i}B_l + A_k\frac{\partial B_l}{\partial x_i}\right) = \epsilon_{ikl}\frac{\partial A_k}{\partial x_i}B_l + \epsilon_{ikl}A_k\frac{\partial B_l}{\partial x_i}\\
&= \epsilon_{lik}\frac{\partial A_k}{\partial x_i}B_l - A_k\epsilon_{kil}\frac{\partial B_l}{\partial x_i}\\
&= (\nabla\times\boldsymbol{A})\cdot\boldsymbol{B} - \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})\\
&= \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})
\end{aligned}$$

Example 3: Prove

$$\nabla\times(\boldsymbol{A}\times\boldsymbol{B}) = (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} - \boldsymbol{B}(\nabla\cdot\boldsymbol{A}) + (\nabla\cdot\boldsymbol{B})\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{B}.$$

Proof:

$$\begin{aligned}
\nabla\times(\boldsymbol{A}\times\boldsymbol{B}) &= \epsilon_{ijk}\frac{\partial}{\partial x_j}(\boldsymbol{A}\times\boldsymbol{B})_k\boldsymbol{e}_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}\epsilon_{klm}A_lB_m\boldsymbol{e}_i\\
&= \epsilon_{ijk}\epsilon_{klm}\left(\frac{\partial A_l}{\partial x_j}B_m + \frac{\partial B_m}{\partial x_j}A_l\right)\boldsymbol{e}_i\\
&= \epsilon_{kij}\epsilon_{klm}\left(\frac{\partial A_l}{\partial x_j}B_m + \frac{\partial B_m}{\partial x_j}A_l\right)\boldsymbol{e}_i\\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\left(\frac{\partial A_l}{\partial x_j}B_m + \frac{\partial B_m}{\partial x_j}A_l\right)\boldsymbol{e}_i
\end{aligned}$$

Take a break and let's continue...

$$
\begin{aligned}
\nabla\times(\boldsymbol{A}\times\boldsymbol{B}) &= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\left(\frac{\partial A_l}{\partial x_j}B_m+\frac{\partial B_m}{\partial x_j}A_l\right)\boldsymbol{e}_i \\
&= \delta_{il}\delta_{jm}\left(\frac{\partial A_l}{\partial x_j}B_m+\frac{\partial B_m}{\partial x_j}A_l\right)\boldsymbol{e}_i \\
&- \delta_{im}\delta_{jl}\left(\frac{\partial A_l}{\partial x_j}B_m+\frac{\partial B_m}{\partial x_j}A_l\right)\boldsymbol{e}_i \\
&= \left(B_j\frac{\partial A_i}{\partial x_j}+A_i\frac{\partial B_j}{\partial x_j}-B_i\frac{\partial A_j}{\partial x_j}-A_j\frac{\partial B_i}{\partial x_j}\right)\boldsymbol{e}_i \\
&= \left(B_j\frac{\partial A_i}{\partial x_j}+\frac{\partial B_j}{\partial x_j}A_i-\frac{\partial A_j}{\partial x_j}B_i-A_j\frac{\partial B_i}{\partial x_j}\right)\boldsymbol{e}_i \\
&= (\boldsymbol{B}\cdot\nabla)\boldsymbol{A}+(\nabla\cdot\boldsymbol{B})\boldsymbol{A}-(\nabla\cdot\boldsymbol{A})\boldsymbol{B}-(\boldsymbol{A}\cdot\nabla)\boldsymbol{B}
\end{aligned}
$$

Practice and master both methods. They are extremely important in the study of classical electrodynamics.

1.3 Curvilinear coordinates

I do not talk about general vector analysis in curvilinear coordinates in this class; it's quite complicated. If you are interested, see books in my reference list. I will only very briefly review spherical and cylindrical coordinates, since you should have learned this before.

In spherical coordinates $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi)$,

Gradient of a scalar

$$
\nabla f = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\boldsymbol{e}_\phi,
$$

Divergence

$$
\nabla\cdot\boldsymbol{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi},
$$

Curl of a vector

$$
\begin{aligned}
(\nabla\times\boldsymbol{A})_r &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi} \\
(\nabla\times\boldsymbol{A})_\theta &= \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right), \\
(\nabla\times\boldsymbol{A})_\phi &= \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}.
\end{aligned}
$$

Laplacian

$$
\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}.
$$

In cylindrical coordinates $(\boldsymbol{e}_r, \boldsymbol{e}_\phi, \boldsymbol{e}_z)$,

Gradient of a scalar

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\boldsymbol{e}_\phi + \frac{\partial f}{\partial z}\boldsymbol{e}_z,$$

Divergence

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z},$$

Curl of a vector

$$\begin{aligned}(\nabla \times \boldsymbol{A})_r &= \frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\\ (\nabla \times \boldsymbol{A})_\phi &= \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r},\\ (\nabla \times \boldsymbol{A})_z &= \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right) - \frac{1}{r}\frac{\partial A_r}{\partial \phi}.\end{aligned}$$

Laplacian

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}.$$

1.4 *A very brief introduction to tensor*

A tensor may be regarded as the product of two vectors; it has two sets of directions. [4] The simplest tensor is the dyad (并矢), which is just two vectors put together, without a symbol in between. For example, $\boldsymbol{AB}$ is a dyad. It has two sets of directions: one from vector $\boldsymbol{A}$, and one from $\boldsymbol{B}$. In terms of Einstein summation convection,

$$\boldsymbol{AB} = A_i\boldsymbol{e}_i B_j\boldsymbol{e}_j = A_i B_j \boldsymbol{e}_i\boldsymbol{e}_j, \tag{1.13}$$

the component in $\boldsymbol{e}_i\boldsymbol{e}_j$ direction is A_iB_j. Lots of students like to use matrix to represent a tensor. That's OK for the second order tensor, but for more general N-order tensor, it is something not easy to do. So I would like to ask you not to equal a tensor to a matrix, but instead you should use notation similar to Equation (1.13) in tensor operation.

[4] More strictly speaking, we are dealing with second-order tensors in this section.

The basic tensor operation can be introduced via the dot-product between a dyad and a vector [5]. The dot product between a vector and a dyad $\boldsymbol{AB}$ is given by

$$\boxed{\boldsymbol{X} \cdot \boldsymbol{AB} \equiv (\boldsymbol{X} \cdot \boldsymbol{A})\boldsymbol{B},}$$

or

$$\boxed{\boldsymbol{AB} \cdot \boldsymbol{Y} \equiv \boldsymbol{A}(\boldsymbol{B} \cdot \boldsymbol{Y}).}$$

[5] W.D.D'haeseleer, W.N.G.Hltchon, J.D.Callen, and J.L.Shohet, Flux coordinates and magnetic field structure.

Using these rules, we see that,

$$\boldsymbol{X} \cdot \boldsymbol{AB} = X_i \boldsymbol{e}_i \cdot A_j B_k \boldsymbol{e}_j \boldsymbol{e}_k = X_i A_j B_k (\boldsymbol{e}_i \cdot \boldsymbol{e}_j) \boldsymbol{e}_k = X_i A_i B_k \boldsymbol{e}_k.$$

and

$$\boldsymbol{AB} \cdot \boldsymbol{Y} = A_i B_j \boldsymbol{e}_i \boldsymbol{e}_j \cdot Y_k \boldsymbol{e}_k = A_i B_j Y_k \boldsymbol{e}_i (\boldsymbol{e}_j \cdot \boldsymbol{e}_k) = A_i B_k Y_k \boldsymbol{e}_i.$$

You see that in general $\boldsymbol{AB} \neq \boldsymbol{BA}$, since $\boldsymbol{AB} \cdot \boldsymbol{C}$ is a vector in the direction of $\boldsymbol{A}$, while $\boldsymbol{BA} \cdot \boldsymbol{C}$ is a vector in the direction of $\boldsymbol{B}$. In this class, the most common form of tensors is dyad, so you'd better get familiar with this one.

A general second-order tensor is

$$\mathsf{F} = F_{ij} \boldsymbol{e}_i \boldsymbol{e}_j.$$

A general second-order tensor cannot be written as a dyad, but it can be written as a summation of dyads; i.e.,

$$\mathsf{F} = \boldsymbol{ab} + \boldsymbol{cd} + \boldsymbol{ef} + \cdots$$

In 3-D space, you need at least three dyads to represent a general second-order tensor. A vector can be considered to be a first-order tensor, and a scalar is a zeroth-order tensor. It's easy to see that in the component of a tensor, there are two free indices; e.g., F_{ij}. In a vector, there is one; e.g., A_i. In a scalar, there is zero; e.g., C. So the number of free indices in the component of an object equals the rank of the tensor.

> If you define a second-order tensor using transformation properties, then a tensor transforms under coordinate transformations like the products of components of two vectors. As you can see from the definition of a dyad.

The dot-product between a second-order tensor and a vector is a vector. For example,

$$\mathsf{F} \cdot \boldsymbol{A} = F_{ij} \boldsymbol{e}_i \boldsymbol{e}_j \cdot A_k \boldsymbol{e}_k = F_{ij} A_k \boldsymbol{e}_i \delta_{jk} = F_{ij} A_j \boldsymbol{e}_i.$$

So a dot product reduces the order or rank of a tensor by one. The dot-product between a tensor and two vectors gives a scalar, like

$$\boldsymbol{A} \cdot \mathsf{F} \cdot \boldsymbol{B} = A_i \boldsymbol{e}_i \cdot F_{jk} \boldsymbol{e}_j \boldsymbol{e}_k \cdot \boldsymbol{B}_l \boldsymbol{e}_l = A_i F_{jk} B_l \delta_{ij} \delta_{kl} = A_i F_{ik} B_k.$$

This dot product can also be written as

$$\boldsymbol{A} \cdot \mathsf{F} \cdot \boldsymbol{B} \equiv \boldsymbol{BA} : \mathsf{F} \equiv \boldsymbol{F} : \boldsymbol{BA}.$$

A special second-order tensor is the unit tensor I, which has the property that

$$\begin{aligned} \mathsf{I} \cdot \boldsymbol{A} &= \boldsymbol{A} \cdot \mathsf{I} = \boldsymbol{A}, \\ \mathsf{I} : \boldsymbol{AB} &= \boldsymbol{A} \cdot \boldsymbol{B}, \\ \nabla \cdot (\varphi \mathsf{I}) &= \nabla \varphi, \\ \mathsf{I} : \nabla \boldsymbol{B} &= \nabla \cdot \boldsymbol{B}. \end{aligned}$$

Here are two examples involving tensors.
Example 1: Prove that

$$\nabla\cdot(\boldsymbol{f}\boldsymbol{g}\times\boldsymbol{r}) = [\nabla\cdot(\boldsymbol{f}\boldsymbol{g})]\times\boldsymbol{r}+\boldsymbol{g}\times\boldsymbol{f}.$$

Solution:

$$\begin{aligned}\nabla\cdot(\boldsymbol{f}\boldsymbol{g}\times\boldsymbol{r}) &= \boldsymbol{e}_i\frac{\partial}{\partial x_i}\cdot f_k(\boldsymbol{g}\times\boldsymbol{r})_l\boldsymbol{e}_k\boldsymbol{e}_l = \frac{\partial}{\partial x_i}\left[f_i(\boldsymbol{g}\times\boldsymbol{r})_l\right]\boldsymbol{e}_l\\ &= \epsilon_{lkm}\frac{\partial}{\partial x_i}(f_i g_k r_m)\boldsymbol{e}_l\\ &= \epsilon_{lkm}\frac{\partial}{\partial x_i}(f_i g_k)r_m\boldsymbol{e}_l+\epsilon_{lkm}f_i g_k\frac{\partial}{\partial x_i}(r_m)\boldsymbol{e}_l\\ &= \epsilon_{lkm}\left[\nabla\cdot(\boldsymbol{f}\boldsymbol{g})\right]_k r_m\boldsymbol{e}_l+\epsilon_{lkm}g_k\left(\boldsymbol{f}\cdot\nabla\boldsymbol{r}\right)_m\boldsymbol{e}_l\\ &= [\nabla\cdot(\boldsymbol{f}\boldsymbol{g})]\times\boldsymbol{r}+\boldsymbol{g}\times(\boldsymbol{f}\cdot\mathsf{I})\\ &= [\nabla\cdot(\boldsymbol{f}\boldsymbol{g})]\times\boldsymbol{r}+\boldsymbol{g}\times\boldsymbol{f}\end{aligned}$$

Example 2: Prove that

$$\nabla\cdot(\boldsymbol{f}\boldsymbol{g}\boldsymbol{h}) = (\nabla\cdot\boldsymbol{f})\boldsymbol{g}\boldsymbol{h}+(\boldsymbol{f}\cdot\nabla\boldsymbol{g})\boldsymbol{h}+\boldsymbol{g}(\boldsymbol{f}\cdot\nabla\boldsymbol{h}).$$

Solution:

1. $\nabla\cdot(\boldsymbol{f}\boldsymbol{g}\boldsymbol{h}) = \nabla_f\cdot(\boldsymbol{f}\boldsymbol{g}\boldsymbol{h})+\nabla_g\cdot(\boldsymbol{f}\boldsymbol{g}\boldsymbol{h})+\nabla_h\cdot(\boldsymbol{f}\boldsymbol{g}\boldsymbol{h})$
2. $\nabla_f\cdot(\boldsymbol{f}\boldsymbol{g}\boldsymbol{h}) = (\nabla_f\cdot\boldsymbol{f})\boldsymbol{g}\boldsymbol{h}$
   $\nabla_g\cdot(\boldsymbol{f}\boldsymbol{g}\boldsymbol{h}) = (\nabla_g\cdot\boldsymbol{f})\boldsymbol{g}\boldsymbol{h} = (\boldsymbol{f}\cdot\nabla_g)\boldsymbol{g}\boldsymbol{h} = (\boldsymbol{f}\cdot\nabla_g\boldsymbol{g})\boldsymbol{h}$
   $\nabla_h\cdot(\boldsymbol{f}\boldsymbol{g}\boldsymbol{h}) = (\nabla_h\cdot\boldsymbol{f})\boldsymbol{g}\boldsymbol{h} = (\boldsymbol{f}\cdot\nabla_h)\boldsymbol{g}\boldsymbol{h} = \boldsymbol{g}(\boldsymbol{f}\cdot\nabla_h\boldsymbol{h})$
3. $\nabla\cdot(\boldsymbol{f}\boldsymbol{g}\boldsymbol{h}) = (\nabla\cdot\boldsymbol{f})\boldsymbol{g}\boldsymbol{h}+(\boldsymbol{f}\cdot\nabla\boldsymbol{g})\boldsymbol{h}+\boldsymbol{g}(\boldsymbol{f}\cdot\nabla\boldsymbol{h})$

1.5 The Dirac Delta Function

1.5.1 The one-dimensional Dirac Delta Function

The 1D Dirac delta function, $\delta(x)$, is defined as

$$\delta(x) = \begin{cases}0, & \text{if } x\neq 0,\\ \infty, & \text{if } x=0,\end{cases}$$

and

$$\int_{-\infty}^{\infty}\delta(x)\mathrm{d}x = 1.$$

Strictly speaking, $\delta(x)$ is not a function, since its value is not finite at $x=0$. It is a distribution. Note that we also have

$$\int_a^b\delta(x)\mathrm{d}x = 1,$$

as long as $0\in[a,b]$.

Suppose $f(x)$ is a ordinary continuous function, then it immediately follows from the definition of $\delta(x)$ that

$$\boxed{f(x)\delta(x) = f(0)\delta(x).}$$

Correspondingly,

$$\boxed{\int_{-\infty}^{\infty} f(x)\delta(x)\mathrm{d}x = f(0)\int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = f(0).}$$

A slightly more general form of 1D Dirac delta function is of course $\delta(x-a)$,

$$\delta(x-a) = \begin{cases} 0, & \text{if } x \neq a, \\ \infty, & \text{if } x = a, \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(x-a)\mathrm{d}x = 1.$$

And it is straightforward to show that

$$\boxed{\int_{-\infty}^{\infty} f(x)\delta(x-a)\mathrm{d}x = f(a).}$$

We almost always think that expressions involving $\delta(x)$ should be used under an integral sign. In particular, two expressions involving delta functions, $D_1(x)$ and $D_2(x)$, are considered equal if

$$\int_{-\infty}^{\infty} f(x)D_1(x)\mathrm{d}x = \int_{-\infty}^{\infty} f(x)D_2(x)\mathrm{d}x,$$

for all ordinary functions $f(x)$. For example,

$$\boxed{\delta(kx) = \frac{1}{|k|}\delta(x),}$$

from which you can see that $\delta(x)$ is even; i.e., $\delta(-x) = \delta(x)$. Another frequently used expression is

$$\boxed{\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|},}$$

where $g(x)$ is a continuously differentiable function and x_i are all roots of $g(x) = 0$ and of course $g'(x) \neq 0$.

1.5.2 *The three-dimensional Dirac delta function*

The generation of $\delta(x)$ to 3D is very easy.

$$\boxed{\delta(\boldsymbol{r}) = \delta(x)\delta(y)\delta(z),}$$

where $\boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z$ is the displacement vector. You see that

$$\delta(\boldsymbol{r}) = \begin{cases} 0, & \text{if } \boldsymbol{r} \neq 0, \\ \infty, & \text{if } \boldsymbol{r} = 0, \end{cases}$$

and

$$\int_{\text{all space}} \delta(\boldsymbol{r})\mathrm{d}V = 1.$$

Of course, you can have

$$\boxed{\int_{-\infty}^{\infty} f(\boldsymbol{r})\delta(\boldsymbol{r}-\boldsymbol{a})\mathrm{d}\boldsymbol{r} = f(\boldsymbol{a}).}$$

Questions: Now, try to express the charge density $\rho(\boldsymbol{x})$ of an ideal point charge q located at $\boldsymbol{x}_0$ using the Dirac delta function. The charge density satisfies,

$$\int \rho \mathrm{d}V = q.$$

1.6 *Some useful references about vector analysis*

Note: See the course homepage for links to these documents.

1. Curvilinear coordinates and tensors, Chapters 1,2,and 3, *Flux and Coordinates and Magnetic Field Structure*, by D'haeseleer, HItchon, Callen, and Shohet.

2. 向量张量运算符号法 胡友秋，电动力学讲义 (课程主页上可以下载).

3. 张量分析，黄克智，薛明德，陆明万

2
Foundations of Theory of Relativity

2.1 The principle of relativity

In Newtonian mechanics, the Galileo's principle of relativity (PR) is that the laws of mechanics are identical in all interval systems of reference under Galileo transformations. Suppose there are two inertial reference frames K and K'; K' moves with $\boldsymbol{V}$ relative to K. Then

$$\boldsymbol{r} = \boldsymbol{r}' + \boldsymbol{V}t', \tag{2.1}$$

$$t = t'. \tag{2.2}$$

Note that time is absolute in classical mechanics.

It can be shown, however, that Maxwell equations do not satisfy Galileo's principle of relativity under Galileo transformation [1]. There are three possible ways to solve this problem.

[1] Try to prove this for Maxwell Equations in vacuum.

1. The Maxwell equations were wrong. The proper theory of EM was invariant under Galileo transformations.
2. Galileo PR only applied to classical *mechanics*. EM is not mechanics.
3. There exists a general principle of relativity for both classical mechanics and electromagnetism.

Einstein did some really hard thinking and chose 3. He then propose the following two postulates based on experiments done by other people and lots of his own thinking.

> *The principle of relativity.* All the laws of nature are identical in all inertial systems of reference.
>
> *The constancy of the speed of light.* The speed of light (c) is independent of the motion of its source; its numerical value is $c = 2.998 \times 10^{10}\, cm/s$.

It can be immediately shown that time being absolute is not consistent with Einstein's PR. From Galileo's PR, the velocity transforms like

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{V}. \tag{2.3}$$

This equation directly follows from that Δt being invariant. However, this leads to that v can be larger than c, not consistent with Einstein PR.

2.2 *Intervals in spacetime*

2.2.1 *Definition of interval*

For convenience of presentation, we'll first introduce a few concepts.

> *Event*: An event is described by the place it occurred and the time when it occurred.

We also introduce a *fictitious* four-dimensional space; marked by 3 space coordinates and 1 time coordinates. For an idealized particle, an event is defined by three coordinates and the time when the event occurs.

We consider two inertial reference frames K and K', with parallel axes. Suppose K' moves in $\boldsymbol{V}$ relative to K. Now define two events in K system.

- Event 1: sending out a light signal from (x_1, y_1, z_1) at t_1.
- Event 2: the arrival of the signal at (x_2, y_2, z_2) at t_2.

The signal traveled $c\Delta t$, or $(\Delta x^2 + \Delta y^2 + \Delta z^2)^{1/2}$, with definition $\Delta f = f_2 - f_1$. So we have

$$c^2\Delta t^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2) = 0 \tag{2.4}$$

In K', noting that time is not absolute, the coordinates of the same events are

- Event 1, (x_1', y_1', z_1') and t_1'
- Event 2, (x_2', y_2', z_2') and t_2'

Because of the constancy of light speed c, we have in K' system

$$c^2\Delta t'^2 - (\Delta x'^2 + \Delta y'^2 + \Delta z'^2) = 0. \tag{2.5}$$

> If $x_1y_1z_1t_1$ and $x_2y_2z_2t_2$ are the coordinates of any two events, we define Δs by
>
> $$\Delta s = [c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2]^{1/2}, \tag{2.6}$$
>
> and call it the *interval* between these two events.

2.2.2 *The invariance of interval*

From the previous analysis, we reach an important conclusion that *if* $\Delta s = 0$ *in* K*, then* $\Delta s' = 0$ *in any* K'.

To find the relationship between Δs and $\Delta s'$ for $\Delta s \neq 0$, we consider two events infinitely close to each other. In this case, the interval $\mathrm{d}s$ is

$$\mathrm{d}s = [c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2]^{1/2}. \tag{2.7}$$

The form of $\mathrm{d}s$ allows us to regard it as the distance between two world points in the fictitious four-dimensional space. This space is called Minkowski space (axes: x, y, z, and ct), it's pseudo-Euclidean. If Euclidean, the distance would be

$$\Delta l = [c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2]^{1/2}. \tag{2.8}$$

Now the question: What's the relationship between $\mathrm{d}s$ in K and $\mathrm{d}s'$ in K' in general (valid for $\mathrm{d}s \neq 0$)? We have two constraints:

1. If $\mathrm{d}s = 0$, then $\mathrm{d}s' = 0$.
2. $\mathrm{d}s$ and $\mathrm{d}s'$ are infinitesimal of the same order.

From these two conditions, we have

$$\mathrm{d}s = a\mathrm{d}s' \tag{2.9}$$

We now derive the factor a in $\mathrm{d}s = a\mathrm{d}s'$. For an inertial reference frame, space and time are homogeneous.

- If $a = a(t)$, Equation (2.9) violates that time being homogeneous.
- If $a = a(\boldsymbol{X})$, Eq (2.9) depends on the location of the origin of K' in K; it is a violation of space being homogeneous.
- If $a = a(\boldsymbol{V})$, that means Eq (2.9) depends on the direction of K' moving in K; a violation of space being homogeneous.

Therefore we conclude $a = a(|\boldsymbol{V}|)$ or $a = a(V)$, and $\mathrm{d}s = a(V)\mathrm{d}s'$.

To find the value of a, now suppose we have three inertial reference frames K, K_1, K_2, $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are the velocities of K_1 and K_2 relative to K, respectively. Then we have

$$\mathrm{d}s = a(V_1)\mathrm{d}s_1, \tag{2.10}$$
$$\mathrm{d}s = a(V_2)\mathrm{d}s_2, \tag{2.11}$$
$$\mathrm{d}s_1 = a(V_{12})\mathrm{d}s_2. \tag{2.12}$$

From these three equations, we have

$$\frac{a(V_2)}{a(V_1)} = a(V_{12}). \tag{2.13}$$

Note that V_{12} depends on the relative angle between V_1 and V_2, but the left hand side of the above equation does not depends on this angle. Therefore, we can conclude that $a(V_{12})$ must be a constant. And it is easy to see that this constant is 1. Therefore,

$$\boxed{\mathrm{d}s = \mathrm{d}s' \text{ OR } \mathrm{d}s^2 = \mathrm{d}s'^2.} \tag{2.14}$$

The interval between two events is invariant under transformation from one inertial frame to another. This is the mathematical formulation of the invariance of c.

From the invariance of $\mathrm{d}s$, we can immediately reach the following conclusion: If a particle moves with $|\boldsymbol{v}| < c$ in K, then $|\boldsymbol{v}'| < c$ in all other K', because $\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}\boldsymbol{x}^2 = (c^2 - |\boldsymbol{v}|^2)\mathrm{d}t^2$ is an invariant.

2.2.3 *Space-like and time-like intervals*

With the invariance of $\mathrm{d}s$, time is no longer absolute. Statements like "two events occur simultaneously" do not necessarily hold if we transform to another reference frame. Let's now discuss this type of problem.

Our first question is, if two events occur at two different times in K ($\Delta t \neq 0$), can we find a reference frame K' in which $\Delta t' = 0$? Suppose we can find a K' so that $\Delta t' = 0$. From the invariance of Δs^2, we have

$$\Delta s^2 = c^2\Delta t^2 - \Delta\boldsymbol{x}^2 = -\Delta\boldsymbol{x}'^2 < 0. \tag{2.15}$$

Hence if $\Delta s^2 < 0$ or if Δs is imaginary, it is possible to find a reference frame where $\Delta t' = 0$. *Imaginary intervals are said to be space-like.* For space-like intervals, the concepts of "simultaneous", "earlier", and "later" are relative. Note that $c^2\Delta t^2 < \Delta x^2$, meaning that the two events are so separated that no signal can propagate from one point to the other point within Δt.

Following the previous question, it is natural to ask another one: if two events occur at two different times in K ($\Delta t \neq 0$), what's the condition for $\Delta t' \neq 0$ in all K'? From the invariance of Δs^2, we have

$$\Delta s^2 = c^2\Delta t^2 - \Delta\boldsymbol{x}^2 = c^2\Delta t'^2 - \Delta\boldsymbol{x}'^2. \tag{2.16}$$

The minimum value for $c^2\Delta t'^2$ to take is when $\Delta\boldsymbol{x}'^2 = 0$, and

$$\Delta s^2 = c^2\Delta t^2 - \Delta\boldsymbol{x}^2 = c^2\Delta t'^2 > 0. \tag{2.17}$$

Hence if $\Delta s^2 > 0$ or if Δs is real, it is not possible to find a K' so that $\Delta t' = 0$. Note that $c^2\Delta t^2 > \Delta\boldsymbol{x}^2$. *Real intervals are said to be time-like.* Note that for time-like intervals, it is possible to connect two events using a signal with propagation speed less than c, since $\Delta\boldsymbol{x}/\Delta t < c$.

For time-like intervals, the concepts of "future" and "past" are absolute. To see this, let's assume the interval between event 1 and event 2 is time-like, then

$$\Delta s^2 = c^2\Delta t^2 - \Delta\boldsymbol{x}^2 = c^2\Delta t'^2 - \Delta\boldsymbol{x}'^2 > 0. \tag{2.18}$$

Solving for $\Delta t'$ from this equation, we have

$$\Delta t' = \pm|\Delta t|(1 - (\Delta\boldsymbol{x}^2 - \Delta\boldsymbol{x}'^2)/c^2\Delta t^2)^{1/2}. \tag{2.19}$$

Which sign should we take for $\Delta t'$? To see this, we know that $\Delta t' \to \Delta t$ as $\boldsymbol{V} \to 0$, where $\boldsymbol{V}$ is the velocity of K' relative to K.

Hence

$$\Delta t' = \Delta t(1 - (\Delta \boldsymbol{x}^2 - \Delta \boldsymbol{x}'^2)/c^2 \Delta t^2)^{1/2}. \tag{2.20}$$

Therefore for time-like intervals, we must have

1. If $\Delta t > 0$, then $\Delta t' > 0$ in all K'.
2. If $\Delta t < 0$, then $\Delta t' < 0$ in all K'.

That is, the concepts of "future" or "past" are absolute for time-like intervals.

The concept of time-like or space-like intervals are important if two events are causally related. For event 1 to be the reason of event 2, event 1 must occur "before" event 2 in all reference frames; i.e., event 1 is in the "absolute past" of event 2. Therefore the interval must be time-like.

On the other hand, if event 1 is the reason of event 2, a signal has to propagate from event 1 to event 2. The propagation speed of signal is then $\Delta \boldsymbol{x}^2/\Delta t^2$, which is less than c^2, since the interval is time-like. This is the same statement as that *c is the maximum speed of propagation of interaction*.

Finally we point out that because $\mathrm{d}s$ is an absolute concept, the *time-like* or *space-like* property of an interval is also absolute.

2.3 *Proper time*

Since time is not absolute, it is convenient to introduce the concept of *"proper time"*. *Proper time* is the time read by a clock moving with the object; it is normally denoted by τ.

Let's now derive the relationship between $\mathrm{d}t$ observed in laboratory frame K and the proper time $\mathrm{d}\tau$ of the object. In the laboratory frame K, the object moves with a constant velocity $\boldsymbol{v}$. Let the inertial reference frame moving with the object be called K'. From the invariance of $\mathrm{d}s^2$ and $\mathrm{d}\boldsymbol{x}' = 0$ (because the object is still in K'), we have

$$\mathrm{d}s^2 = c^2 \mathrm{d}\tau^2 = c^2 \mathrm{d}t^2 - \mathrm{d}\boldsymbol{x}^2. \tag{2.21}$$

From this equation,

$$\mathrm{d}\tau = \mathrm{d}s/c = \mathrm{d}t\,(1 - \mathrm{d}\boldsymbol{x}^2/c^2\mathrm{d}t^2)^{1/2}. \tag{2.22}$$

But $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{v}$, therefore we have

$$\mathrm{d}\tau = \mathrm{d}t\,(1 - v^2/c^2)^{1/2} \equiv \mathrm{d}t/\gamma, \tag{2.23}$$

where $\gamma = 1/(1-\beta^2)^{1/2}$ with $\beta = v/c$; $\beta \leq 1$ and $\gamma \geq 1$. Note that $\mathrm{d}\tau = \mathrm{d}t/\gamma$; the fact that $\gamma \geq 1$ leads to

$$\mathrm{d}\tau \leq \mathrm{d}t. \tag{2.24}$$

Or by integrating this equation, we have

$$\tau_2 - \tau_1 = \int_{t_1}^{t_2} \mathrm{d}t/\gamma < t_2 - t_1 \tag{2.25}$$

> Conclusion: Moving clocks go more slowly than those at rest.

There are lots of experiments proving this conclusion. Here I list a few:

1. The lifetime of mu-mesons (muons)[2].
   Average lifetime of muons is 2.2×10^6 or about $2\,\mu s$. If they travel at $v \approx c$, the Galilean distance $\approx 600\,m$. Muons can travel from the top of atmosphere to the surface of Earth.

2. A similar example: The lifetime of pions [3].

[2] Ref: Feynman's Lecture on Physics, Vol 1, 15-4.

[3] Ref: Jackson, Classical Electrodynamics, 2nd Edition, pp 520.

On the other hand, the concept is new and confusing. See the following two paradoxes:

1. If K' moves relative to K in $\boldsymbol{V}$, then $\mathrm{d}t' = \mathrm{d}t/\gamma$. From K', however, K moves relative to K' in $-\boldsymbol{V}$. then $\mathrm{d}t = \mathrm{d}t'/\gamma$. How to understand this? [4].

2. The twin paradox [5]: Identical twins; one of them makes a journey into space in a high-speed spaceship and returns home. Which twin should have aged more?

[4] Ref: 郭硕鸿，电动力学，第三版，第204页

[5] Ref: Feynman's Lectures on Physics, Vol 1, 16-2

2.4 *The Lorentz transformation*

With Galileo PR, $\boldsymbol{r} = \boldsymbol{r}' + \boldsymbol{V}t, t = t'$, and we know this is not correct in general. Now time is not absolute, how do we make coordinate transformations? Again we consider two IRF's K and K'; K' moves relative to K in $\boldsymbol{V}$. For an event with coordinate $(t, \boldsymbol{r})$, what's the corresponding coordinate $(t', \boldsymbol{r}')$ in K'?

First, for simplicity, we assume that the two coordinate system have the same origin; i.e., $t = 0, \boldsymbol{r} = 0$ in K corresponds to the point $t' = 0, \boldsymbol{r}' = 0$ in K'. Second, the basic requirement we have is that the transformation should not change any Δs, the distance between any two points in Minkowski space. The Δs in K is

$$\Delta s^2 = c^2 \Delta t^2 - \Delta r^2. \tag{2.26}$$

To make sure that changing coordinate system does not change Δs^2, we make use of the fact that

$$\cosh^2\theta - \sinh^2\theta = 1, \tag{2.27}$$

for any θ. Therefore, we can assume

$$\begin{aligned} c\Delta t &= \Delta s \cosh\theta, \\ \Delta r &= \Delta s \sinh\theta. \end{aligned}$$

in K and in K',

$$\begin{aligned} c\Delta t' &= \Delta s \cosh\theta', \\ \Delta r' &= \Delta s \sinh\theta'. \end{aligned}$$

By parametering $c\Delta t$ and Δr using cosh and sinh, we can always make sure that Δs is an invariant in any coordinate system.

Now we try to find out what is the relation of coordiantes between K and K'. Let's assume $\theta = \theta' + \phi$. Noting $c\Delta t' = \Delta s \cosh\theta'$ and $\Delta r' = \Delta s \sinh\theta'$, we have

$$\begin{aligned} c\Delta t &= \Delta s \cosh(\theta' + \phi) = \Delta s \cosh\theta' \cosh\phi + \Delta s \sinh\theta' \sinh\phi \\ &= c\Delta t' \cosh\phi + \Delta x' \sinh\phi, \end{aligned}$$

$$\begin{aligned} \Delta r &= \Delta s \sinh(\theta' + \phi) = \Delta s \sinh\theta' \cosh\phi + \Delta s \cosh\theta' \sinh\phi \\ &= \Delta r' \cosh\phi + c\Delta t' \sinh\phi. \end{aligned}$$

We rewrite the equations as

$$\Delta r = \Delta r' \cosh\phi + c\Delta t' \sinh\phi, \tag{2.28}$$
$$c\Delta t = \Delta r' \sinh\phi + c\Delta t' \cosh\phi. \tag{2.29}$$

These are the needed equations that relate $(ct', \boldsymbol{r}')$ and $(ct, \boldsymbol{r})$.

To obtain coordinate transformation, let's consider the interval between any event and the origin of the coordinate system. From Equations (2.28) and (2.29), we have

$$r = r' \cosh\phi + ct' \sinh\phi, \tag{2.30}$$
$$ct = r' \sinh\phi + ct' \cosh\phi. \tag{2.31}$$

Starting from the origin of the K' system, we have $r' = 0$. The rotation equations are then

$$r = ct' \sinh\phi, \tag{2.32}$$
$$ct = ct' \cosh\phi. \tag{2.33}$$

We thus have $V/c = \tanh\phi$ with $V = r/t$, the speed of the origin of the K' system observed in K. For simplicity of discussion, let assume that $\boldsymbol{V}$ is in the direction of $\boldsymbol{e}_x$; i.e., K' moves relative to K with $\boldsymbol{V} = V\boldsymbol{e}_x$, then $r = x$, $y = y' = 0, z = z' = 0$, $V_y = V_z = 0$. Now from $V/c = \tanh\phi = (e^\phi - e^{-\phi})/(e^\phi + e^{-\phi})$, we have

$$e^\phi = \sqrt{\frac{V+c}{V-c}}. \tag{2.34}$$

Therefore

$$\sinh\phi = \frac{e^\phi - e^{-\phi}}{2} = \frac{V/c}{\sqrt{1 - V^2/c^2}} \equiv \beta\gamma \tag{2.35}$$
$$\cosh\phi = \frac{e^\phi + e^{-\phi}}{2} = \frac{1}{\sqrt{1 - V^2/c^2}} \equiv \gamma \tag{2.36}$$

Substituting the expressions of $\sinh\phi$ and $\cosh\phi$ into rotation equations, we have,

$$\boxed{\begin{aligned} x &= \frac{x' + Vt'}{\sqrt{1 - V^2/c^2}}, \\ t &= \frac{t' + (V/c^2)x'}{\sqrt{1 - V^2/c^2}}. \\ y &= y', \\ z &= z'. \end{aligned}}$$

These equations are the needed transformations; they are called *Lorentz transformations.* [6]

The inverse transformation can be obtained by noting that K moves relative to K' in $-\boldsymbol{V}$. Therefore we have

$$x' = \frac{x - Vt}{\sqrt{1 - V^2/c^2}}, \tag{2.37}$$

$$t' = \frac{t - (V/c^2)x}{\sqrt{1 - V^2/c^2}}. \tag{2.38}$$

This inverse Lorentz transformation can also be directly obtained by solving for $(t', \boldsymbol{r}')$ from previous equations.

If $V/c \ll 1$ or equivalently $c \to \infty$, $(1 - V^2/c^2)^{1/2} \to 1$. The Lorentz transformation is reduced to the Galileo transformation,

$$x = x' + Vt', \tag{2.39}$$

$$t = t'. \tag{2.40}$$

This means that classical mechanics work well because $c \gg V$, where V means the characteristic speed of our daily life.

From the Lorentz transformation, we derive the proper length, which is the length of an object in an inertial reference frame in which it is at rest. We denote proper length by l_0. Suppose we have a rod parallel to the x'-axis; it's at rest in K' system. In K' system, the length of the rod is $l_0 = |x'_1 - x'_2|$. In K system, to measure the length of the rod, we have x_1 at t_1 and x_2 at t_2, and $t_1 = t_2$. From the inverse Lorentz transformation, we have,

$$x'_1 = \frac{x_1 - Vt_1}{\sqrt{1 - V^2/c^2}}, \tag{2.41}$$

$$x'_2 = \frac{x_2 - Vt_2}{\sqrt{1 - V^2/c^2}}. \tag{2.42}$$

Noting $t_1 = t_2$, we have

$$x'_1 - x'_2 = \frac{x_1 - x_2}{\sqrt{1 - V^2/c^2}}. \tag{2.43}$$

Or $l_0 = l/\sqrt{1 - V^2/c^2}$ or $l = l_0\sqrt{1 - V^2/c^2}$. Since $l < l_0$, this is called *Lorentz contraction*.

Similarly, we can introduce *proper volume,* which is the volume of an object in an IRF in which it is at rest. Since transverse dimensions do not change, we can immediately have

$$V = V_0\sqrt{1 - V^2/c^2}, \tag{2.44}$$

where V_0 is the *proper volume* of the object.

Using the Lorentz transformation, we can re-derive the equation for *proper time*. Suppose a clock to be at rest in K', two events occurred at $\boldsymbol{r}'$ at t'_1 and t'_2. Then $\Delta\tau = t'_2 - t'_1$ and in K,

$$t_1 = \frac{t'_1 + (V/c^2)x'}{\sqrt{1 - V^2/c^2}}, \tag{2.45}$$

$$t_2 = \frac{t'_2 + (V/c^2)x'}{\sqrt{1 - V^2/c^2}}. \tag{2.46}$$

Therefore, $\Delta t = \Delta\tau/\sqrt{1 - V^2/c^2}$, the same as we obtained before.

[6] From the above derivation, you can see that the Lorentz transformation corresponds to the rotation in the 4D Minkowski space involving the t-axis. For example, the above transformation for $\boldsymbol{V} = V\boldsymbol{e}_x$ corresponds to the rotation in the 4D space within the xt plane.

2.5 Transformation of velocities

To completely determine the state of a particle, we need both its space coordinates and velocity. From the Lorentz transformation, we can easily derive the needed formulas for the transformation of velocities.

Suppose a particle has a velocity $\boldsymbol{v}$ in IRF K and $\boldsymbol{v}'$ in IRF K'; K' moves with $\boldsymbol{V} = V\boldsymbol{e}_x$ relative to K. The differentiation of the Lorentz transformation gives

$$\mathrm{d}x = (\mathrm{d}x' + V\mathrm{d}t')/\sqrt{1 - V^2/c^2}, \tag{2.47}$$

$$\mathrm{d}y = \mathrm{d}y', \tag{2.48}$$

$$\mathrm{d}z = \mathrm{d}z', \tag{2.49}$$

$$\mathrm{d}t = (\mathrm{d}t' + V/c^2\mathrm{d}x')/\sqrt{1 - V^2/c^2}. \tag{2.50}$$

The velocity transformation can be obtained easily as $\boldsymbol{v} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$, $\boldsymbol{v}' = \mathrm{d}\boldsymbol{r}'/\mathrm{d}t'$, etc.

The resulting velocity transformations are

$$v_x = \frac{\mathrm{d}x' + V\mathrm{d}t'}{\mathrm{d}t' + (V/c^2)\mathrm{d}x'} = \frac{v'_x + V}{1 + (V/c^2)v'_x} \tag{2.51}$$

$$v_y = \frac{v'_y}{1 + (V/c^2)v'_x}\sqrt{1 - \frac{V^2}{c^2}} \tag{2.52}$$

$$v_z = \frac{v'_z}{1 + (V/c^2)v'_x}\sqrt{1 - \frac{V^2}{c^2}} \tag{2.53}$$

In case of $c \to \infty$, the above equations recovers the familiar Galileo velocity transformation.

2.6 Vectors and Tensors in Spacetime

Minkowski once said: "*Space of itself, and time of itself will sink into mere shadows, and only a kind of union between them shall survive.*" Indeed, from previous introductions, in the framework of the special theory of relativity, time and space are put on equal footing: both time and space are not absolute and are transformed together from one reference system to another reference system.

2.6.1 Four-vectors

It's convenient to discuss time and space together in terms of vectors in a pseudo-Euclidean Minkowski space. For example, the coordinates of an event (ct, x, y, z) can be considered as the components of a *radius four-vector*,

$$x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z. \tag{2.54}$$

The radius four-vector satisfies the Lorentz transformation when changing reference system from K' to K. Note that with this representation, the Lorentz transformation can be written as (easier to remember)

$$x^0 = \gamma(x^{0'} + \beta x^{1'}), \tag{2.55}$$

$$x^1 = \gamma(x^{1'} + \beta x^{0'}), \tag{2.56}$$

where $\gamma = 1/\sqrt{1 - V^2/c^2}$, and $\beta = V/c$.

Like the case in 3D, we model four-vectors after the radius four-vector.

> Definition: any set of four quantities A^0, A^1, A^2, A^3, which transform like the radius four-vector x^α under the change from K' to K is called a four-vector.

This definition essentially means that when changing from K' to K, all four-vectors are transformed using Lorentz transformation. Therefore for K' with axes parallel to K and moves relative to K in $\boldsymbol{V} = V\boldsymbol{e}_x$,

$$A^0 = \frac{A'^0 + (V/c)A'^1}{\sqrt{1 - (V/c)^2}}, \tag{2.57}$$

$$A^1 = \frac{A'^1 + (V/c)A'^0}{\sqrt{1 - (V/c)^2}}, \tag{2.58}$$

and $A^2 = A'^2$ and $A^3 = A'^3$. The component A^0 is called the *time component*, and A^1, A^2, A^3 the *space components*. We sometimes write the four-vector as

$$A^\alpha = (A^0, \boldsymbol{A}), \tag{2.59}$$

where $\boldsymbol{A}$ is a three-dimensional vector. In this course, we use Greek letters $(\alpha, \beta, \cdots)$ to represent $0, 1, 2, 3$, and use Latin letters $(i, j, \cdots)$ to represent $1, 2, 3$. Also we use prefix *four-* to specifically denote variables in the Minkowski space. Ordinary three-dimensional variables will just be called scalars, vectors, or tensors.

The square magnitude of the radius four-vector x^α is defined as

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = s^2, \tag{2.60}$$

which is the square of s, the spacetime interval (*the magnitude of the radius four-vector in Minkowski space*). Similarly, the square magnitude of an arbitrary four-vector A^α is

$$(A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2. \tag{2.61}$$

To more conveniently represent the dot-product between four-vectors, we introduce two kinds of components of four-vectors:

1. The contravariant (逆变) components, denoted by A^α.

2. The covariant (协变) components, denoted by A_α.

so that A^α and A_α are related by

$$A_0 = A^0,\ A_1 = -A^1,\ A_2 = -A^2,\ A_3 = -A^3. \tag{2.62}$$

By using contravariant and covariant components, the square magnitude of a four-vector A^α can now be compactly written as

$$\sum_{\alpha=0}^{3} A^\alpha A_\alpha = A^0 A_0 + A^1 A_1 + A^2 A_2 + A^3 A_3 \equiv A^\alpha A_\alpha. \tag{2.63}$$

The dot-product between two four-vectors A^α and B^α is $A^\alpha B_\alpha$ or $A_\alpha B^\alpha$. In these expressions, we have used the Einstein summation rule. Another example, the square magnitude of the radius four-vector, the interval, can be written as $s^2 = x_\alpha x^\alpha$. Note the locations of indices in these expressions.

There are simple ways to memorize the location of sub/super scripts:

1. 中文：上逆下协

2. English: There is a "r" in both "contra-" (contra-variant) and "super-" (super-script).

To convert between the contravariant and the covariant components, one needs to use the metric coefficients $g^{\alpha\beta}$. From the definitions of contra- and co-variant components above, you should be able to see that for the four-vectors in Minkowski space,

$$(g_{\alpha\beta}) = (g^{\alpha\beta}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{2.64}$$

Therefore we can convert between A^α and A_α by

$$A_\alpha = g_{\alpha\beta} A^\beta \text{ or } A^\alpha = g^{\alpha\beta} A_\beta. \tag{2.65}$$

The metric tensor g plays an very important role in the general theory of relativity (not covered by this course).

In your textbook, the authors represented the Minkowski space in a different way. Let $x_0 = ct$, $x_1 = \mathrm{i}x$, $x_2 = \mathrm{i}y$, and $x_3 = \mathrm{i}z$, then

$$s^2 = (x_0)^2 + (x_1)^2 + (x_2)^2 + (x_3)^2. \tag{2.66}$$

There are two reasons why I wouldn't use this in my class:

1. Hard to understand why one would need the contra- and co-variant components of a vector.

2. Hard to generalize this to the General Theory of Relativity (the metric g, *curved* or *flat* spacetime).

The concept of contra- and co-variant components of a vector is important in using curvilinear coordinates. Curvilinear coordinates are very convenient in certain cases (like fusion and space plasmas, general relativity, etc). For more information about curvilinear coordinates, see my reference list near the end of the previous Chapter.

Here are a few rules regarding contra- and co-variant components of four-vectors.

1. The dot product between two vectors is $A^\alpha B_\alpha$ or $A_\alpha B^\alpha$.

2. The squared length of A^α is $A_\alpha A^\alpha$ or $A^\alpha A_\alpha$.

3. You can convert between A^α and A_α by $A^\alpha = g^{\alpha\beta} A_\beta$ and $A_\alpha = g_{\alpha\beta} A^\beta$.

Regarding Einstein notation of four-vectors and four-tensors, here are two simple rules that you should keep in mind.

1. For a dummy index, one index must be superscript and one must be subscript; e.g., $A_\alpha B^\alpha$. Terms like $A_\alpha B_\alpha$ do not make sense, as you can see from the definitions of contra and co-variant components of a four-vector given above.

2. Equations do not change the position of a free index, like $A_\alpha = g_{\alpha\beta} A^\beta$. Note the location of the free index α.

2.6.2 *Four-scalars*

Definition: Like 3D case, we define four-scalars as invariants under the transformation of coordinates, here the Lorentz transformation.

Because they are invariant under Lorentz transformations, the four-scalars play important roles in the study of special relativity and electrodynamics. You should learn how to construct four-scalars from four-vectors and four-tensors. For example, like the 3D case, the dot product between two four-vectors is a four-scalar. In the Minkowski space, the dot product between two four-vectors A^α and B^α is $A^\alpha B_\alpha$ or $A_\alpha B^\alpha$, and it is a four-scalar. *Note that there is no free index in this term.* Example: the interval $s^2 = x_\alpha x^\alpha$ is a four-scalar. We can "construct" more invariants after we learn more four-vectors.

2.7 *Four-tensors*

Definition: A *second*-order *four*-tensor is a set of 16 quantities $F^{\alpha\beta}$, which under coordinate transformations, transform like the products of components of two four-vectors.

A four-tensor can be written in different forms: $F^{\alpha\beta}$, $F_{\alpha\beta}$, $F^{\alpha}_{\cdot\beta}$, and $F_{\alpha}^{\cdot\beta}$. Different positions of indices represent different kind of components of a four-tensor. In case of a four-vector, there are two different kinds of components (co- and contra-variant components); in the case of a four-tensor, there are four different kinds of components. Again one use the metric tensor g to convert between different kinds of representations of a four-tensor. One just needs to remember the rules of Einstein notation, and you can immediately figure out that,

$$F_{\mu\nu} = g_{\mu\alpha} F^{\alpha\beta} g_{\beta\nu} \tag{2.67}$$

$$F_{\mu}^{\cdot\beta} = g_{\mu\alpha} F^{\alpha\beta} \tag{2.68}$$

$$\cdots\cdots \tag{2.69}$$

Noting that in the special relativity,

$$g_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = \beta = 0 \\ -1 & \text{if } \alpha = \beta \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.70}$$

From $F_{\mu\nu} = g_{\mu\alpha} F^{\alpha\beta} g_{\beta\nu}$, we have the following simple rules:

1. raising or lowering index $(1, 2, 3)$ changes the sign,
2. raising or lowering index (0) does not change the sign.

Example: $F_{00} = F^{00}$, $F_{01} = -F^{01}$, $F_{11} = F^{11}$, $F_0^{\cdot 0} = F^{00}$, $F_0^{\cdot 1} = F^{01}$, $F_0^{\cdot 1} = -F_{01}$, $F_1^{\cdot 0} = -F^{10}$, $\cdots\cdots$

From four-vectors A^α and B^α, we know $A^\alpha A_\alpha$ and $A^\alpha B_\alpha$ are four-scalars; i.e., they are invariants under Lorentz transformations. The trick to construct 4-scalars from 4-vectors/tensors is that all indices are dummy indices; there is no free index in the final expression. Similarly, we can also construct four-scalars from 4-tensors. For example, for $F^{\alpha\beta}$, we know that

$$F_{\alpha\beta} F^{\beta\alpha} = \text{inv.} \tag{2.71}$$

Also it's possible to construct four-scalars from four-vectors and four-tensors, like $A_\alpha B_\beta F^{\beta\alpha}$ is a four-scalar, since there is no free index in the final expression. This operation is called *contraction*.

2.7.1 *Basic four-vector/tensor differential calculus.*

Let's now introduce four-vector calculus. The four-gradient of a scalar ϕ is the four-vector

$$\frac{\partial \phi}{\partial x^\alpha} = \left(\frac{1}{c} \frac{\partial \phi}{\partial t}, \nabla \phi \right) \tag{2.72}$$

From the location of the index, we know they are the covariant components of a four-vector. To make this more obvious, some people write the four-gradient of ϕ as,

$$\frac{\partial \phi}{\partial x^\alpha} \equiv \partial_\alpha \phi.$$

Similarly, you might have the contra-variant components of a four-vector,

$$\frac{\partial \phi}{\partial x_\alpha} \equiv \partial^\alpha \phi.$$

The 4-divergence of a 4-vector $A^\alpha = (A^0, \boldsymbol{A})$ is a four-scalar,

$$\frac{\partial A^\alpha}{\partial x^\alpha} = \partial_\alpha A^\alpha = \partial^\alpha A_\alpha = \frac{1}{c} \frac{\partial A^0}{\partial t} + \nabla \cdot \boldsymbol{A}. \tag{2.73}$$

It's a scalar under coordinate transformation.

2.8 *Four-velocity and four-acceleration*

Using the knowledge of four vectors, we can construct the four-velocity by $u^\alpha = \mathrm{d}x^\alpha/\mathrm{d}\tau$, where $\mathrm{d}\tau = \mathrm{d}t/\gamma$,

$$u^0 = \frac{c\mathrm{d}t}{\mathrm{d}t}\gamma = \gamma c, \tag{2.74}$$

$$u^1 = \frac{\mathrm{d}x}{\mathrm{d}t}\gamma = \gamma v_x, \tag{2.75}$$

$$u^2 = \gamma v_y, \tag{2.76}$$

$$u^3 = \gamma v_z. \tag{2.77}$$

Therefore, the four-velocity is

$$u^\alpha = (\gamma c, \gamma v_x, \gamma v_y, \gamma v_z) = (\gamma c, \gamma \boldsymbol{v}). \tag{2.78}$$

The contraction of the four-velocity is, noting that $\mathrm{d}\tau = \mathrm{d}s/c$,

$$u^\alpha u_\alpha = \frac{\mathrm{d}x^\alpha \mathrm{d}x_\alpha}{\mathrm{d}\tau \mathrm{d}\tau} = \frac{\mathrm{d}s^2}{\mathrm{d}s^2/c^2} = c^2, \tag{2.79}$$

or it can be calculated directly by

$$u^\alpha u_\alpha = \gamma^2(c^2 - v^2) = c^2, \tag{2.80}$$

which is a scalar and an invariant.

One can construct the four-acceleration similarly by $w^\alpha = \mathrm{d}u^\alpha/\mathrm{d}\tau$, Without calculating the components of the four-acceleration, one can show that

$$u_\alpha w^\alpha = 0. \tag{2.81}$$

This shows that the four-acceleration is always "perpendicular" to the four-velocity.

3
Relativistic Dynamics

3.1 A brief review of Lagrangian mechanics

We'll use some Lagrangian mechanics in this class, so let me first briefly give it a review, using materials from Landau's book "Mechanics".

In classical mechanics, the action is defined by the integral

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t) \mathrm{d}t. \tag{3.1}$$

Suppose $q(t_1) = q^{(1)}$ and $q(t_2) = q^{(2)}$, there could be infinite number of paths connecting t_1 and t_2. It is required that the actual path minimizes S; i.e.,

$$\delta S = 0. \tag{3.2}$$

This is called *the principle of least action*, with L the Lagrangian.

To obtain equations of motion from the principle of least action, note that if $q = q(t)$ is the actual path, then for a given variation of q,

$$q(t) + \delta q(t), \tag{3.3}$$

the change in S is

$$\delta S = \int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta \dot{q}, t) \mathrm{d}t - \int_{t_1}^{t_2} L(q, \dot{q}, t) \mathrm{d}t = 0. \tag{3.4}$$

The constraints we have are

$$\delta q(t_1) = 0, \qquad \delta q(t_2) = 0, \tag{3.5}$$

since $q(t_1) = q^{(1)}$ and $q(t_2) = q^{(2)}$ are given.

Keeping only the leading (first-order) terms in δS, we have

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) \mathrm{d}t = 0. \tag{3.6}$$

Noting that $\delta \dot{q}$ is, because of the variation in q,

$$\delta \dot{q} = \frac{\mathrm{d}}{\mathrm{d}t} (q + \delta q) - \frac{\mathrm{d}}{\mathrm{d}t} (q) = \frac{\mathrm{d}}{\mathrm{d}t} \delta q. \tag{3.7}$$

Therefore,

$$\delta S = \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]\Bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}\right)\delta q \mathrm{d}t = 0. \tag{3.8}$$

Because $\delta q(t_1) = 0$ and $\delta q(t_2) = 0$, we have

$$\delta S = \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}\right)\delta q \mathrm{d}t = 0, \tag{3.9}$$

Since δS must vanish for all δq, we have

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} = 0, \tag{3.10}$$

these equations are called *Lagrange's equations*. If we know L of a system, then we can write down its equations of motion. Note that the Lagrangian is not uniquely determined for a given system. For if we have two Lagrangians $L'(q, \dot{q}, t)$ and $L(q, \dot{q}, t)$, and

$$L' = L + \frac{\mathrm{d}}{\mathrm{d}t} f(q, t), \tag{3.11}$$

then

$$S' = \int_{t_1}^{t_2} L'\mathrm{d}t = \int_{t_1}^{t_2} L\mathrm{d}t + \int_{t_1}^{t_2} \frac{\mathrm{d}f}{\mathrm{d}t}\mathrm{d}t = S + f(q^{(2)}, t) - f(q^{(1)}, t). \tag{3.12}$$

Therefore $\delta S'$ and δS would lead to the same equations of motion.

Now we apply the concept to solving a free body problem in classical mechanics. To consider mechanical problems, it's necessary to choose a reference frame. In principle, we can choose any reference system we like. However if we choose one where time or space is inhomogeneous, then the description of nature laws would become unnecessarily difficult. The simplest one is thus the so-called *inertial reference frame* so that space is homogeneous and isotropic,and time is homogeneous. In such a frame, a free body would move with a constant velocity or is at rest forever.

Let's find the the Lagrangian for a free particle in an inertial reference system. Note that time is homogeneous so L does not depend on t. Space is homogeneous so L does not depend on $\boldsymbol{r}$. Therefore, $L = L(\boldsymbol{v})$. Space is isotropic, therefore L must not depend on the direction of $\boldsymbol{v}$. *Conclusion:* $L = L(|\boldsymbol{v}|)$ *or* $L = L(v^2)$. Applying the Lagrange's equation (3.10), noting that $\partial L/\partial \boldsymbol{v}$ must be a function of $\boldsymbol{v}$ only, the equation of motion is

$$\dot{\boldsymbol{v}} = 0. \tag{3.13}$$

This equation states that a free body in an inertial reference frame moves with a constant velocity. Of course, this velocity can equal 0, which means the free body is at rest all the time. This is *the laws of inertia*.

To find the form of $L(v^2)$, we need to use Galileo's principle of relativity. Consider two IRF's K and K', K moves in ϵ relative to K'.

$$\boldsymbol{v}' = \boldsymbol{v} + \epsilon. \tag{3.14}$$

Because of Galileo's PR,

$$L' = L(v'^2) = L(v^2 + 2\boldsymbol{v}\cdot\epsilon + \epsilon^2), \tag{3.15}$$

the leading terms of the L' are

$$L(v'^2) = L(v^2) + \frac{\partial L}{\partial v^2} 2\boldsymbol{v}\cdot\epsilon \tag{3.16}$$

We know from Galileo's PR that L' and L should lead to the same equations of motion, therefore,

$$L' = L + \frac{\mathrm{d}}{\mathrm{d}t} f(\boldsymbol{r}, t), \tag{3.17}$$

where $f(q,t)$ is some function of $\boldsymbol{r}$ and t, Hence

$$\frac{\partial L}{\partial v^2} 2\boldsymbol{v}\cdot\epsilon = \frac{\mathrm{d}}{\mathrm{d}t} f(\boldsymbol{r}, t) = \frac{\partial f}{\partial t} + \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\cdot\nabla f(\boldsymbol{r}, t). \tag{3.18}$$

Because $\boldsymbol{v} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$, we must have

$$\frac{\partial L}{\partial v^2} 2\epsilon = \nabla f(\boldsymbol{r}, t). \tag{3.19}$$

The LHS is a function of v only, while the RHS is a function of t and $\boldsymbol{r}$ only; therefore

$$\frac{\partial L}{\partial v^2} = \alpha, \tag{3.20}$$

where α is some constant. From

$$\frac{\partial L}{\partial v^2} = \alpha, \tag{3.21}$$

we have for a free particle

$$L = \alpha v^2. \tag{3.22}$$

Normally we write $\alpha = m/2$, where m is called *mass*, and

$$L = \frac{1}{2} m v^2. \tag{3.23}$$

> From previous derivations, we have learned that either $\mathrm{d}S$ is an invariant under Galileo transformation, or $\mathrm{d}S$ and $\mathrm{d}S'$ differ by a $\mathrm{d}f(\boldsymbol{r}, t)$. We will use this property of action in relativistic dynamics. Of course, there the action is an invariant under the Lorentz transformation (a four-scalar)

The Lagrangian for a particle moving in a given external field can be obtained by adding to $L = mv^2/2$ a quantity describing the interaction between the field and the particle,

$$L = \frac{1}{2} m v^2 - U(\boldsymbol{r}, t), \tag{3.24}$$

so that the equations of motion become

$$m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\nabla U. \tag{3.25}$$

Note that now the total action S becomes

$$\boxed{S = S_p + S_f = \int_{t_1}^{t_2} \left(\frac{1}{2} m v^2 - U(\boldsymbol{r}, t) \right) \mathrm{d}t.} \tag{3.26}$$

Using Lagrangian mechanics, it is very convenient to discuss conservation properties. Here we discuss the conserved quantities from the isotropy and homogeneity of space and time.

If time is homogeneous, then for a closed system $L = L(q_i, \dot{q}_i)$. The total time derivative of L is

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \dot{q}_i \frac{\partial L}{\partial q_i} + \frac{\mathrm{d}\dot{q}_i}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} = \dot{q}_i \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_i} \right) + \frac{\mathrm{d}\dot{q}_i}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} \tag{3.27}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} \left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) \tag{3.28}$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} \left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \right) = 0 \tag{3.29}$$

We define total energy E of the system by

$$\boxed{E \equiv \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L.} \tag{3.30}$$

Therefore, if time is homogeneous, E is a constant.

> Equation (3.30) defines the energy of the system. If, however, we consider a free particle, then it is the definition of energy for the particle. We will use this to define particle energy in relativistic dynamics.

If space is homogeneous; i.e., if we replace $\boldsymbol{r}_i$ by $\boldsymbol{r}_i + \boldsymbol{\epsilon}$, then

$$\delta L = \sum_i \frac{\partial L}{\partial \boldsymbol{r}_i} \cdot \delta \boldsymbol{r}_i = \boldsymbol{\epsilon} \cdot \sum_i \frac{\partial L}{\partial \boldsymbol{r}_i} = 0. \tag{3.31}$$

Therefore

$$\sum_i \frac{\partial L}{\partial \boldsymbol{r}_i} = 0. \tag{3.32}$$

From Lagrange's equation,

$$\sum_i \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \boldsymbol{v}_i} = \frac{\mathrm{d}}{\mathrm{d}t} \sum_i \frac{\partial L}{\partial \boldsymbol{v}_i} = 0. \tag{3.33}$$

If space is homogeneous, for a closed system,

$$\boxed{\boldsymbol{P} \equiv \sum_i \frac{\partial L}{\partial \boldsymbol{v}_i}} \tag{3.34}$$

remains constant. This is called the momentum of the system

> Similarly, we will use this to define the momentum of a free particle in relativistic dynamics.

3.2 Relativistic action for a free particle

I'll now apply what I've introduced to obtain the relativistic action for a free particle,

$$S = \int_a^b \mathrm{d}S. \tag{3.35}$$

The relativistic action must satisfy

1. $\mathrm{d}S$ is invariant under Lorentz transformation, or differed by a $\mathrm{d}f(x^\alpha)$ when transformed from K' to K.

2. It should recover non-relativistic dynamics if $c \to \infty$ ($v/c \to 0$).

From the previous chapter, we know that for a free particle, the event interval $\mathrm{d}s$ is an invariant. Try

$$S = \int_a^b \alpha \mathrm{d}s, \tag{3.36}$$

where α is a 4-scalar. Let's see whether S can recover non-relativistic dynamics if $c \to \infty$. If it can, then it suits our need and can be used as the relativistic action for a free particle.

To recover the non-relativistic dynamics, we first write S in the normal 3D form; i.e., $S = \int L\mathrm{d}t$ form. Because $\mathrm{d}s = c\mathrm{d}\tau = c\sqrt{1 - v^2/c^2}\mathrm{d}t$, we write

$$S = \int_a^b \alpha \mathrm{d}s = \int_a^b \alpha c \sqrt{1 - \frac{v^2}{c^2}}\mathrm{d}t. \tag{3.37}$$

From this form of S, we have the Lagrangian

$$L = \alpha c \sqrt{1 - \frac{v^2}{c^2}}. \tag{3.38}$$

In case of $c \to \infty$, we have

$$L = \alpha c - \frac{\alpha v^2}{2c}. \tag{3.39}$$

Therefore, if $\alpha = -mc$, then S satisfies the two constraints.

The relativistic action for a free particle is then

$$\boxed{S = -mc \int_a^b \mathrm{d}s.} \tag{3.40}$$

And the corresponding Lagrangian is

$$\boxed{L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}},} \tag{3.41}$$

or $L = -mc^2/\gamma$, with $\gamma = 1/\sqrt{1 - v^2/c^2}$.

3.3 Energy and momentum

The momentum of a particle is given by

$$\boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}}. \tag{3.42}$$

Using $L = -mc^2\sqrt{1 - v^2/c^2}$, we have

$$\boldsymbol{p} = -mc^2 \frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\left(-\frac{1}{c^2}\right) 2\boldsymbol{v},$$

or

$$\boxed{\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1 - v^2/c^2}} = \gamma m\boldsymbol{v}.}$$

If $c \to \infty$, $\boldsymbol{p} = m\boldsymbol{v}$, the non-relativistic (NR) momentum.

The energy of a particle is defined from L by

$$\mathcal{E} = \boldsymbol{v} \cdot \frac{\partial L}{\partial \boldsymbol{v}} - L = \boldsymbol{p} \cdot \boldsymbol{v} - L. \tag{3.43}$$

Using $\boldsymbol{p} = \gamma m\boldsymbol{v}$, we have

$$\boxed{\mathcal{E} = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \gamma mc^2} \tag{3.44}$$

If $v \to 0$, $\mathcal{E} \to mc^2$, the rest energy of the particle. In case of $c \to \infty$, $\mathcal{E} = mc^2 + \frac{1}{2}mv^2$.

This definition of energy leads to that mass is not a conserved quantity anymore in relativistic mechanics. If we have a body, with mass m, consisting of n particles. In classical mechanics, we have $m = \sum_i m_i$. In relativistic mechanics, however, the energy of a body at least contains

1. the rest energies of its constituent particles $\sum_i m_i c^2$
2. the kinetic energy of particles
3. the interaction energy
4. $\cdots\cdots$

therefore, $mc^2 \neq \sum_i m_i c^2$ and $m \neq \sum_i m_i$. *Only the law of conservation of energy holds.*

The energy and momentum are closely related. Note that the 4-velocity we introduced is $u^\alpha = (\gamma c, \gamma \boldsymbol{v})$. Multiplying u^α by m, a 4-scalar, we have the four-momentum vector

$$p^\alpha = mu^\alpha = (\gamma mc, \gamma m\boldsymbol{v}) = \left(\frac{\mathcal{E}}{c}, \boldsymbol{p}\right). \tag{3.45}$$

Since p^α is a four-vector, we can apply the knowledge we know about four-vector. First, we can immediately obtain their transformation equation, since any four-vector transforms like the radius four-vector x^α; i.e.,

$$\mathcal{E} = \frac{\mathcal{E}' + Vp'_x}{\sqrt{1 - V^2/c^2}}, \tag{3.46}$$

$$p_x = \frac{p'_x + (V/c^2)\mathcal{E}'}{\sqrt{1 - V^2/c^2}}, \tag{3.47}$$

$$p_y = p'_y, \tag{3.48}$$

$$p_z = p'_z. \tag{3.49}$$

Here $(p_x, p_y, p_z) = \boldsymbol{p}$. Also the dot product of a four-vector with itself is a Lorentz invariant. Since four-momentum $p^\alpha = mu^\alpha$, we immediately have

$$p_\alpha p^\alpha = m^2c^2. \tag{3.50}$$

Substituting $p_\alpha = (\mathcal{E}/c, -\boldsymbol{p})$ and $p^\alpha = (\mathcal{E}/c, \boldsymbol{p})$, we have

$$\frac{\mathcal{E}^2}{c^2} - p^2 = m^2c^2, \tag{3.51}$$

or

$$\boxed{\mathcal{E}^2 = p^2c^2 + m^2c^4, \text{ and } \boldsymbol{p} = \frac{\mathcal{E}}{c^2}\boldsymbol{v}.} \tag{3.52}$$

this is the energy-momentum relation. If $v \to c$, both $\boldsymbol{p}$ and $\mathcal{E}$ become infinite unless $m \to 0$. Actually, for a photon ($m = 0$), we have

$$p = \frac{\mathcal{E}}{c}. \tag{3.53}$$

This equation is also useful for the case $\mathcal{E} \gg \mathcal{E}_0$, then $p \approx \mathcal{E}/c$.

3.4 The Covariant Equation of Motion

We can derive the equations of motion in covariant form by viewing S as a function in the Minkowski space [1],

[1] Question: What's the benefit of doing this?

$$S = -mc\int_a^b \mathrm{d}s = -mc\int_a^b \sqrt{\mathrm{d}x_\alpha \mathrm{d}x^\alpha}. \tag{3.54}$$

Then if we vary the actual path x^α by δx^α,

$$\delta S = -mc\,\delta\int_a^b \mathrm{d}s = -mc\,\delta\int_a^b \sqrt{\mathrm{d}x_\alpha \mathrm{d}x^\alpha} = 0, \tag{3.55}$$

from *the principle of least action*.

Let's first calculate $\delta\sqrt{\mathrm{d}x_\alpha \mathrm{d}x^\alpha}$.

$$\delta\sqrt{\mathrm{d}x_\alpha \mathrm{d}x^\alpha} = \frac{1}{2\mathrm{d}s}\,\delta(\mathrm{d}x_\alpha \mathrm{d}x^\alpha) = \frac{1}{2\mathrm{d}s}\,(\delta\mathrm{d}x_\alpha \mathrm{d}x^\alpha + \mathrm{d}x_\alpha \delta\mathrm{d}x^\alpha) \tag{3.56}$$

Note that we can switch the upper- and lower- indices,

$$\delta\mathrm{d}x_\alpha \mathrm{d}x^\alpha = \delta\mathrm{d}x^\alpha \mathrm{d}x_\alpha. \tag{3.57}$$

Therefore, $\delta\sqrt{\mathrm{d}x_\alpha \mathrm{d}x^\alpha} = \mathrm{d}x_\alpha \delta\mathrm{d}x^\alpha/\mathrm{d}s$. Noting that $\mathrm{d}s = c\mathrm{d}\tau$, $\mathrm{d}x_\alpha/\mathrm{d}s = u_\alpha/c$, and

$$\delta\sqrt{\mathrm{d}x_\alpha \mathrm{d}x^\alpha} = u_\alpha \delta\mathrm{d}x^\alpha/c = u_\alpha \mathrm{d}\delta x^\alpha/c. \tag{3.58}$$

With $\delta\sqrt{\mathrm{d}x_\alpha \mathrm{d}x^\alpha} = u_\alpha \mathrm{d}\delta x^\alpha/c$, we have

$$\delta S = -m\int_a^b u_\alpha \mathrm{d}\delta x^\alpha = -mu_\alpha \delta x^\alpha|_a^b + m\int_a^b \delta x^\alpha \frac{\mathrm{d}u_\alpha}{\mathrm{d}s}\mathrm{d}s. \tag{3.59}$$

If we want to find the actual path, then we fix $x^\alpha(a)$ and $x^\alpha(b)$,

$$\delta S = 0 \Rightarrow \frac{\mathrm{d}u^\alpha}{\mathrm{d}s} = 0 \tag{3.60}$$

or the four-velocity u^α is constant.

4
Charges in a Given Electromagnetic Field

4.1 Four-potentials of a field

As in classical mechanics, the total action for a particle in a given field is

$$S = S_p + S_{pf} \tag{4.1}$$

Here $S_p = -mc \int \mathrm{d}s$ is the action for the particle, and S_{pf} is the action characterizing the interaction. Experiments suggest that the interaction is determined by the charge q, and the four-vector potential A_α characterizing the field.

To make the integral S_{pf} a Lorentz invariant, we can construct

$$S_{pf} = -q \int_a^b A_\alpha \mathrm{d}x^\alpha, \tag{4.2}$$

where $A^\alpha = (\phi/c, \boldsymbol{A})$ or $A_\alpha = (\phi/c, -\boldsymbol{A})$. Therefore the total action is

$$S = \int_a^b \left(-mc\mathrm{d}s - qA_\alpha \mathrm{d}x^\alpha\right). \tag{4.3}$$

In 3D coordinates,

$$S_{pf} = \int_a^b \left(q\,\boldsymbol{A} \cdot \mathrm{d}\boldsymbol{r} - q\phi \mathrm{d}t\right) \tag{4.4}$$

$$= \int_a^b \left(q\,\boldsymbol{A} \cdot \boldsymbol{v} - q\phi\right) \mathrm{d}t, \tag{4.5}$$

and

$$S = \int_{t_1}^{t_2} \left(-\frac{mc^2}{\gamma} + q\,\boldsymbol{A} \cdot \boldsymbol{v} - q\phi\right) \mathrm{d}t. \tag{4.6}$$

The total Lagrangian is [1]

$$\boxed{L = \underbrace{-\frac{mc^2}{\gamma}}_{\text{free particle}} + \underbrace{q(\boldsymbol{A} \cdot \boldsymbol{v} - \phi)}_{\text{interaction with the field}}\ .} \tag{4.7}$$

[1] Find the Hamiltonian and the canonoical momentum ¶ using this Lagrangian.

4.2 Equations of motion of a charge in a field

We derive here the equations of motion of a charge. For simplicity, we assume that the charge is small, so its effects on the field can be neglected. The field potential do not depend on the position and velocity of the charge. The Lagrange's equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \boldsymbol{v}}\right) - \frac{\partial L}{\partial \boldsymbol{r}} = 0 \tag{4.8}$$

From the total Lagrangian,

$$\frac{\partial L}{\partial \boldsymbol{r}} = \nabla L = q\nabla(\boldsymbol{A}\cdot\boldsymbol{v}) - q\nabla\phi. \tag{4.9}$$

Noting that

$$\nabla(\boldsymbol{A}\cdot\boldsymbol{v}) = \underbrace{(\boldsymbol{A}\cdot\nabla)\boldsymbol{v}}_{=0} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{A} + \boldsymbol{v}\times(\nabla\times\boldsymbol{A}) + \underbrace{\boldsymbol{A}\times(\nabla\times\boldsymbol{v})}_{=0}. \tag{4.10}$$

On the other hand,

$$\frac{\partial L}{\partial \boldsymbol{v}} = \boldsymbol{p} + q\boldsymbol{A}. \tag{4.11}$$

Substituting Equations (4.9)-(4.11) into the Lagrange's equation (4.8) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{p} + q\boldsymbol{A}) = q(\boldsymbol{v}\cdot\nabla)\boldsymbol{A} + q\boldsymbol{v}\times(\nabla\times\boldsymbol{A}) - q\nabla\phi \tag{4.12}$$

However,

$$q\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A} = q\left[\frac{\partial \boldsymbol{A}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{A}\right]. \tag{4.13}$$

Correspondingly,

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \underbrace{-q\frac{\partial \boldsymbol{A}}{\partial t} - q\nabla\phi}_{\text{no dependence on } \boldsymbol{v}} + q\boldsymbol{v}\times(\nabla\times\boldsymbol{A}). \tag{4.14}$$

The right hand side of the equation can be divided into two parts; one depends on velocity $\boldsymbol{v}$ and the other not. If we define the *electric field intensity* to be

$$\boxed{\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi.} \tag{4.15}$$

Similarly, we define the *magnetic field intensity* to be

$$\boxed{\boldsymbol{B} = \nabla\times\boldsymbol{A}.} \tag{4.16}$$

Then the equations of motion of a charge can be simply written as

$$\boxed{\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = q\boldsymbol{E} + q\boldsymbol{v}\times\boldsymbol{B}.} \tag{4.17}$$

This is the *Lorentz force* felt by a particle in a given field.

4.3 *Gauge invariance*

Note that we can describe the field in two ways. We can use A^α as in the action S_{pf}. Or as in the equations of motion, we can use only $\boldsymbol{E}$ and $\boldsymbol{B}$,

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B}. \tag{4.18}$$

However, a given $\boldsymbol{E}$ and $\boldsymbol{B}$ do not allow the unique determination of $\boldsymbol{A}$ and ϕ. For example, it's easy to show that the transformation

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla f \tag{4.19}$$

$$\phi' = \phi - \frac{\partial f}{\partial t} \tag{4.20}$$

do not change $\boldsymbol{E}$ and $\boldsymbol{B}$. Therefore all equations must be invariant under the transformation of the potentials; this invariance is called *gauge invariance* (规范不变性). This is the first (and maybe the last) gauge invariance you have ever seen and it is a very important property of fields in more advanced field theories describing other types of interactions.

The gauge invariance can also be clearly seen from the action,

$$S_{pf} = -q \int_a^b A_\alpha \mathrm{d}x^\alpha, \tag{4.21}$$

If we replace

$$A_\alpha \to A_\alpha - \frac{\partial f}{\partial x^\alpha}, \tag{4.22}$$

where f is a function of t and $\boldsymbol{r}$, then

$$S_{pf} = -q \int_a^b A_\alpha \mathrm{d}x^\alpha + q \int_a^b \frac{\partial f}{\partial x^\alpha} \mathrm{d}x^\alpha \tag{4.23}$$

$$= -q \int_a^b A_\alpha \mathrm{d}x^\alpha + q \int_a^b \mathrm{d}f. \tag{4.24}$$

The total differential $\mathrm{d}f$ has no effect on the equations of motion.

The gauge invariance allows one to choose *one*[2] auxiliary condition for $\boldsymbol{A}$ and ϕ. Two popular gauges are Coulomb gauge and Lorenz gauge. One is

[2] This can be seen from that a scalar function is involved in Equation (4.22).

$$\boxed{\nabla \cdot \boldsymbol{A} = 0, \text{ the Coulomb gauge}} \tag{4.25}$$

This gauge is popular in some problems. The Lorenz gauge is very popular in theoretical analysis. In the Lorenz gauge, we require

$$\boxed{\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0. \text{ the Lorenz gauge}} \tag{4.26}$$

or in 4D form

$$\boxed{\frac{\partial A^\alpha}{\partial x^\alpha} = 0.} \tag{4.27}$$

Note that if $\boldsymbol{A}$ and ϕ satisfy the Lorenz gauge in one inertial reference frame, then they satisfy the Lorenz gauge in all other inertial reference frames.

4.4 *Static electromagnetic fields*

If potentials do not have time dependence, then

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \tag{4.28}$$

$$\boldsymbol{E} = -\nabla \phi. \tag{4.29}$$

Therefore, ϕ determines $\boldsymbol{E}$, while $\boldsymbol{A}$ determines $\boldsymbol{B}$ for static electromagnetic fields. In case of static fields, we can add to ϕ an arbitrary constant. Usually, we choose $\phi(\infty) = 0$. Also it is obvious that $\boldsymbol{E}$ and $\boldsymbol{B}$ fields are completely decoupled in case of static fields.

For a closed system in a static EM field, we have

$$\frac{\partial L}{\partial t} = 0. \tag{4.30}$$

From Mechanics, we know that the energy is conserved,

$$\mathcal{E} = \boldsymbol{v} \cdot \frac{\partial L}{\partial \boldsymbol{v}} - L = \underbrace{\gamma m c^2}_{\text{kinetic}} + \underbrace{q\phi}_{\text{potential}} = \text{const.} \tag{4.31}$$

If the field is constant and uniform, then simple forms of $\boldsymbol{A}$ and ϕ can be obtained. From $\boldsymbol{E} = -\nabla\phi$,

$$\phi = -\boldsymbol{E} \cdot \boldsymbol{r}. \tag{4.32}$$

From $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, we can choose $\boldsymbol{A}$ to be

$$\boldsymbol{A} = \frac{1}{2} \boldsymbol{B} \times \boldsymbol{r}. \tag{4.33}$$

The choices of ϕ and $\boldsymbol{A}$ can be easily proved by

$$-\nabla\phi = \nabla(\boldsymbol{E} \cdot \boldsymbol{r}) = \boldsymbol{E} \cdot \nabla \boldsymbol{r} = \boldsymbol{E} \cdot \mathsf{I} = \boldsymbol{E}, \tag{4.34}$$

$$\nabla \times \boldsymbol{A} = \frac{1}{2} \nabla \times (\boldsymbol{B} \times \boldsymbol{r}) = \frac{1}{2} \left[\boldsymbol{B} \nabla \cdot \boldsymbol{r} - (\boldsymbol{B} \cdot \nabla) \boldsymbol{r} \right] = \boldsymbol{B}. \tag{4.35}$$

Note that we have used the fact that $\boldsymbol{E}$ and $\boldsymbol{B}$ are uniform in the proof.

4.5 *Motion in uniform static EM fields*

First, let's consider the motion in a uniform static $\boldsymbol{E}$ field. The equations of motion are

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = q\boldsymbol{E}. \tag{4.36}$$

Assuming $\boldsymbol{E} = E\boldsymbol{e}_x$, $p_x(t = 0) = 0$, we have

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = qE \Rightarrow p_x = qEt \tag{4.37}$$

$$\frac{\mathrm{d}p_y}{\mathrm{d}t} = 0 \Rightarrow p_y = \text{const} \tag{4.38}$$

$$\frac{\mathrm{d}p_z}{\mathrm{d}t} = 0 \Rightarrow p_z = \text{const} \tag{4.39}$$

Assuming $p_y = p_0$, and $p_z = 0$, then $p_x = qEt, p_y = p_0$.

The kinetic energy [3] of the particle is

$$\mathcal{E}_k = \sqrt{m^2c^4 + p^2c^2} = \sqrt{\mathcal{E}_0^2 + (qEct)^2}, \tag{4.40}$$

where

$$\mathcal{E}_0 = m^2c^4 + p_y^2c^2 + p_z^2c^2 = m^2c^4 + p_0^2c^2, \tag{4.41}$$

is the energy at $t = 0$.

To find the trajectory of the particle, we need to solve

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{v} = \frac{\boldsymbol{p}}{\gamma m} = \frac{\boldsymbol{p}c^2}{\mathcal{E}_k}. \tag{4.42}$$

The components of the equation are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{qEc^2t}{\sqrt{\mathcal{E}_0^2 + (qEct)^2}}, \tag{4.43}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{p_0c^2}{\sqrt{\mathcal{E}_0^2 + (qEct)^2}}. \tag{4.44}$$

Note that as $t \to \infty$, $v_x \to c$ and $v_y \to 0$. Integrate $\mathrm{d}x/\mathrm{d}t$ and $\mathrm{d}y/\mathrm{d}t$, assuming $\boldsymbol{r}(t = 0) = 0$,

$$x = \frac{1}{qE}\sqrt{\mathcal{E}_0^2 + (qEct)^2}, \tag{4.45}$$

$$y = \frac{p_0c}{qE}\sinh^{-1}\left(\frac{qEct}{\mathcal{E}_0}\right), \tag{4.46}$$

From $y = y(t)$, we obtain $t = t(y)$. From $x = x(t)$, we have

$$x = \frac{\mathcal{E}_0}{qE}\cosh\frac{qEy}{p_0c}. \tag{4.47}$$

Now let's consider the motion of a particle in a uniform static $\boldsymbol{B}$ field, assuming $\boldsymbol{B} = B\boldsymbol{e}_z$. The equations of motion are

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = q\boldsymbol{v} \times \boldsymbol{B}. \tag{4.48}$$

Using $\boldsymbol{p} = \gamma m\boldsymbol{v}$ that γ is constant in a $\boldsymbol{B}$ field,

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = q\boldsymbol{v} \times \frac{\boldsymbol{B}}{\gamma m} = \boldsymbol{v} \times \boldsymbol{\Omega}, \tag{4.49}$$

where $\boldsymbol{\Omega} = q\boldsymbol{B}/\gamma m = \Omega\boldsymbol{e}_z$. The components of $\mathrm{d}\boldsymbol{v}/\mathrm{d}t$ are

$$\dot{v}_x = \Omega v_y, \qquad \dot{v}_y = -\Omega v_x, \qquad \dot{v}_z = 0, \tag{4.50}$$

from which $v_z = v_{z0}$.

To solve for $\boldsymbol{v}$, let $\widehat{v} = v_x + \mathrm{i}v_y$, then from

$$\dot{v}_x = \Omega v_y, \tag{4.51}$$

$$\mathrm{i}\dot{v}_y = -\Omega\mathrm{i}v_x, \tag{4.52}$$

[3] Here "kinetic energy" means γmc^2, as opposed to the potential energy $q\phi$. See Equation (4.31)

we have $\dot{\hat{v}} = -i\Omega\hat{v}$, from which $\hat{v} = \hat{v}_0\, e^{-\mathrm{i}\Omega t}$. Here $\hat{v}_0 = v_{\perp 0} e^{-\mathrm{i}\alpha}$ is a complex constant with α a constant. From $\hat{v} = v_{\perp 0}\, e^{-\mathrm{i}(\Omega t+\alpha)}$ and $\hat{v} = v_x + \mathrm{i} v_y$, we have

$$v_x = v_{\perp 0} \cos(\Omega t + \alpha), \tag{4.53}$$

$$v_y = -v_{\perp 0} \sin(\Omega t + \alpha), \tag{4.54}$$

It's clear that $v_\perp = \sqrt{v_x^2 + v_y^2} = \text{const} = v_{\perp 0}$. We see from v_x and v_y that Ω is the angular frequency.

The trajectory can be obtain by integrating $\boldsymbol{r} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$, which gives

$$x = x_0 + \rho \sin(\Omega t + \alpha), \tag{4.55}$$

$$y = y_0 + \rho \cos(\Omega t + \alpha), \tag{4.56}$$

$$z = z_0 + v_{z0} t. \tag{4.57}$$

Here $\rho = v_\perp/\Omega$ is the gyro-radius (or $|\rho|$). In case of $v/c \to 0$, $\gamma \to 1$, $\Omega \to qB/m$.

To make life more complicated, let's now consider the motion of a charge in a static uniform $\boldsymbol{E}$ and $\boldsymbol{B}$ field. We consider only non-relativistic case here ($v \ll c$). Assume $\boldsymbol{B} = B\boldsymbol{e}_z$, $\boldsymbol{E} = E_y \boldsymbol{e}_y + E_z \boldsymbol{e}_z$. For a non-relativistic particle, $\boldsymbol{p} = m\boldsymbol{v}$.

$$m\dot{\boldsymbol{v}} = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B}. \tag{4.58}$$

The components of the equation are

$$m\dot{v}_x = q v_y B, \tag{4.59}$$

$$m\dot{v}_y = qE_y - q v_x B, \tag{4.60}$$

$$m\dot{v}_z = qE_z, \tag{4.61}$$

from which we immediately have $v_z = qE_z t/m + v_{z0}$.

To solve for $\boldsymbol{v}$, we again let $\hat{v} = v_x + \mathrm{i} v_y$. Then

$$\frac{\mathrm{d}\hat{v}}{\mathrm{d}t} + \mathrm{i}\Omega\hat{v} = \mathrm{i}\frac{qE_y}{m}. \tag{4.62}$$

The general solution of the equation is

$$\hat{v} = \hat{v}_0 e^{-\mathrm{i}\Omega t} + \frac{qE_y}{m\Omega}. \tag{4.63}$$

Or letting $\hat{v}_0 = v_{\perp 0} e^{-\mathrm{i}\alpha}$,

$$v_x = v_{\perp 0} \cos(\Omega t + \alpha) + \frac{qE_y}{m\Omega} = v_{\perp 0} \cos(\Omega t + \alpha) + \frac{E_y}{B}, \tag{4.64}$$

$$v_y = -v_{\perp 0} \sin(\Omega t + \alpha). \tag{4.65}$$

The x-velocity consists of two parts,

$$v_x = \underbrace{v_{\perp 0} \cos(\Omega t + \alpha)}_{\text{oscillatory}} + \underbrace{E_y/B}_{\text{non-oscillatory}} = \tilde{v}_x + \bar{v}_x \tag{4.66}$$

For cases like this, it's frequently useful to average over the fast oscillation to obtain the average motion by

$$\overline{v}_x = \frac{1}{T}\int_t^{t+T} v_x(t')\mathrm{d}t' = \frac{1}{T}\int_t^{t+T} v_{\perp 0}\cos(\Omega t'+\alpha)\mathrm{d}t' + E_y/B, \tag{4.67}$$

or $\overline{v}_x = E_y/B$. Note that $\overline{v}_y = 0$. The motion in x-y plane is the very famous $\boldsymbol{E}\times\boldsymbol{B}$ drift. In vector form, the drift velocity is $\boldsymbol{E}\times\boldsymbol{B}/B^2$.

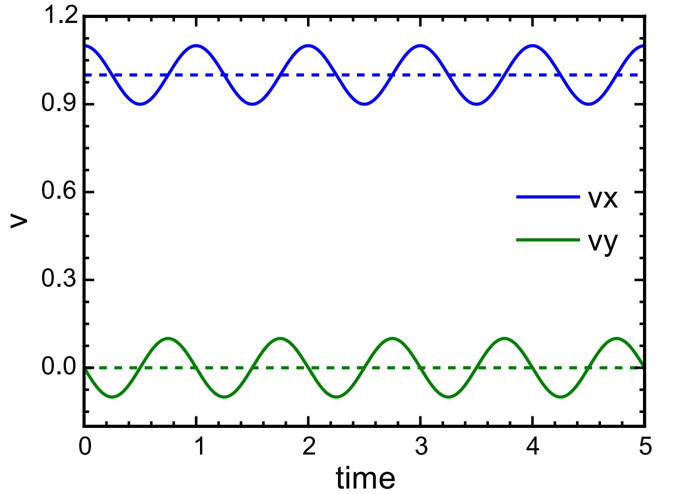


Fig: v_x and v_y from Equations (4.64) and (4.65).

To find the trajectory, we integrate v_x, v_y and v_z and

$$x = \rho\sin(\Omega t+\alpha) + v_{\boldsymbol{E}}t + x_0 \tag{4.68}$$
$$y = \rho[\cos(\Omega t+\alpha) - \cos\alpha] + y_0 \tag{4.69}$$
$$z = \frac{qE_z}{2m}t^2 + v_{z0}t + z_0 \tag{4.70}$$

Here $v_{\boldsymbol{E}} = E_y/B$ is the $\boldsymbol{E}\times\boldsymbol{B}$ drift velocity.

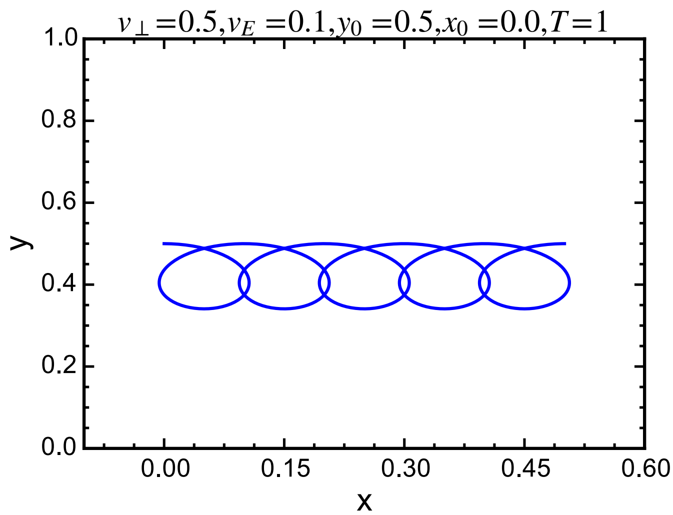


Fig: Trajectories in the x-y plane for different $v_{\boldsymbol{E}}$ and $v_\perp = \rho\Omega$.

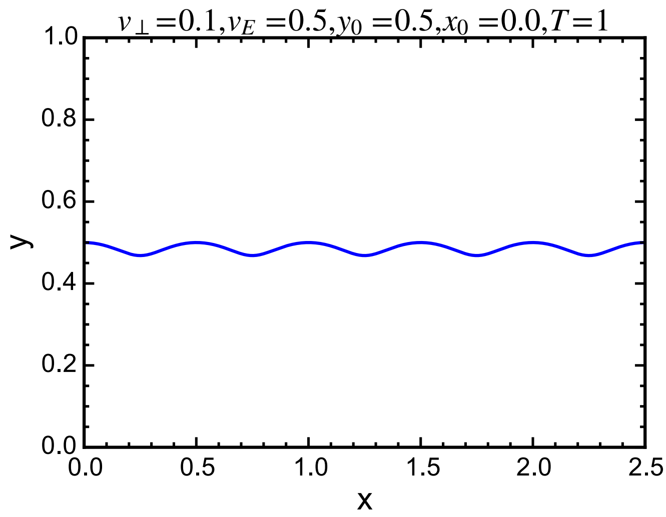


Fig: Trajectories in the x-y plane for different v_E and $v_\perp = \rho\Omega$.

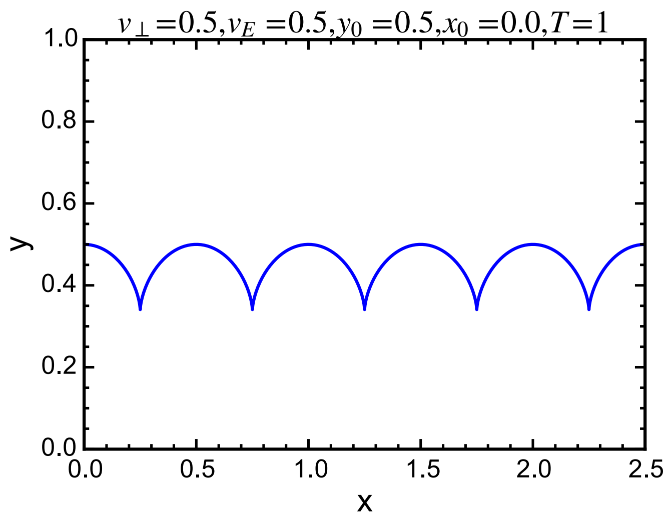


Fig: Trajectories in the x-y plane for different v_E and $v_\perp = \rho\Omega$.

4.6 *The electromagnetic field tensor*

To obtain the Lorentz equation of motion in covariant form, we consider the action in four-dimensional form. Recall the action of a particle in a given field is

$$S = \int_a^b (-mc\mathrm{d}s - qA_\alpha \mathrm{d}x^\alpha) \tag{4.71}$$

The principle of least action leads to

$$\delta S = \delta \int_a^b (-mc\mathrm{d}s - qA_\alpha \mathrm{d}x^\alpha) = 0. \tag{4.72}$$

Recall that $\delta \mathrm{d}s = \mathrm{d}x_\alpha \mathrm{d}\delta x^\alpha / \mathrm{d}s = u_\alpha \mathrm{d}\delta x^\alpha / c$,

$$\delta S = -\int_a^b (mu_\alpha \mathrm{d}\delta x^\alpha + qA_\alpha \mathrm{d}\delta x^\alpha + q\delta A_\alpha \mathrm{d}x^\alpha) = 0. \tag{4.73}$$

Integrating by parts, δS becomes

$$\begin{aligned}\delta S = &\int_a^b (m\delta x^\alpha \mathrm{d}u_\alpha + q\delta x^\alpha \mathrm{d}A_\alpha - q\delta A_\alpha \mathrm{d}x^\alpha) \\ &- [mu_\alpha + qA_\alpha]\, \delta x^\alpha|_a^b\,. \end{aligned} \tag{4.74}$$

Because $\delta x^\alpha(a) = \delta x^\alpha(b) = 0$,

$$\delta S = \int_a^b (m\delta x^\alpha \mathrm{d}u_\alpha + q\delta x^\alpha \mathrm{d}A_\alpha - q\delta A_\alpha \mathrm{d}x^\alpha). \tag{4.75}$$

Noting $\mathrm{d}A_\alpha = (\partial A_\alpha/\partial x^\beta)\mathrm{d}x^\beta$ and $\delta A_\alpha = (\partial A_\alpha/\partial x^\beta)\delta x^\beta$,

$$\begin{aligned}\delta S &= \int_a^b \left(m\mathrm{d}u_\alpha \delta x^\alpha + q\frac{\partial A_\alpha}{\partial x^\beta}\mathrm{d}x^\beta \delta x^\alpha - q\frac{\partial A_\alpha}{\partial x^\beta}\delta x^\beta \mathrm{d}x^\alpha \right) && (4.76)\\ &= \int_a^b \left(m\mathrm{d}u_\alpha \delta x^\alpha + q\frac{\partial A_\alpha}{\partial x^\beta}\mathrm{d}x^\beta \delta x^\alpha - q\frac{\partial A_\beta}{\partial x^\alpha}\delta x^\alpha \mathrm{d}x^\beta \right) && (4.77)\\ &= \int_a^b \left[m\frac{\mathrm{d}u_\alpha}{\mathrm{d}\tau} - q\left(\frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}\right) u^\beta \right] \mathrm{d}\tau \delta x^\alpha. && (4.78)\end{aligned}$$

If we define a four-tensor $F^{\alpha\beta}$ by

$$\boxed{F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta},} \tag{4.79}$$

then

$$\delta S = \int_a^b \left[m\frac{\mathrm{d}u_\alpha}{\mathrm{d}\tau} - qF_{\alpha\beta}u^\beta \right] \mathrm{d}\tau \delta x^\alpha. \tag{4.80}$$

The tensor $F_{\alpha\beta}$ is called the *electromagnetic field tensor*; the tensor is anti-symmetric,

$$F_{\alpha\beta} = -F_{\beta\alpha}. \tag{4.81}$$

To obtain the equations of motion, we apply the principle of least action. From $\delta S = 0$,

$$\boxed{m\frac{\mathrm{d}u_\alpha}{\mathrm{d}\tau} = qF_{\alpha\beta}u^\beta.} \tag{4.82}$$

We can also move indices up or down,

$$m\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} = qF^{\alpha\beta}u_\beta. \tag{4.83}$$

Equations (4.82) or (4.83) are the covariant form of the equations of motion of a charge in a given field.

Now let's find out the components of the electromagnetic field tensor $F_{\alpha\beta}$; they can be obtained by noting that $A_\alpha = (\phi/c, -\boldsymbol{A})$,

$$\text{e.g.,}\quad F_{10} = \frac{\partial A_0}{\partial x^1} - \frac{\partial A_1}{\partial x^0} = \frac{1}{c}\frac{\partial \phi}{\partial x} + \frac{1}{c}\frac{\partial A_x}{\partial t} = -E_x/c. \tag{4.84}$$

In SI units,

$$F_{\alpha\beta} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{bmatrix}. \tag{4.85}$$

From $F_{\alpha\beta}$, we can immediately obtain

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix}, \tag{4.86}$$

using the rule

1. Moving the time index (0) $\Rightarrow$ No change in sign.
2. Moving space indices $(1, 2, 3)$ $\Rightarrow$ change sign.

Using $F^{\alpha\beta}$, we obtain components of the covariant equations of motion, Equation (4.83), rewritten here

$$\mathrm{d}u^\alpha/\mathrm{d}\tau = \frac{q}{m} F^{\alpha\beta} u_\beta. \tag{4.87}$$

For example, the time component ($\alpha = 0$) is

$$m\frac{\mathrm{d}u^0}{\mathrm{d}\tau} = qF^{0\beta}u_\beta = q\left(-\frac{\boldsymbol{E}}{c}\right)\cdot(-\gamma\boldsymbol{v}). \tag{4.88}$$

Noting that $u^0 = \gamma c$ and $\mathrm{d}\tau = \mathrm{d}t/\gamma$, we have

$$\gamma m \frac{\mathrm{d}\gamma c}{\mathrm{d}t} = \gamma\frac{q}{c}\boldsymbol{E}\cdot\boldsymbol{v}, \tag{4.89}$$

$$\text{or} \quad \frac{\mathrm{d}\mathcal{E}_k}{\mathrm{d}t} = q\boldsymbol{E}\cdot\boldsymbol{v}, \quad \text{with } \mathcal{E} = \gamma mc^2 \tag{4.90}$$

The space components of $\mathrm{d}u^\alpha/\mathrm{d}t$ can be obtained similarly; they are just the Lorentz equations of motion (Equation (4.17)).

4.7 *Lorentz transformation of the EM fields*

4.7.1 *Rules of transformation*

The Lorentz transform of potentials ϕ and $\boldsymbol{A}$ can be obtained easily, because $A^\alpha = (\phi/c, \boldsymbol{A})$ is a four-vector. Any 4-vector transforms like x^α; therefore

$$\phi = \frac{\phi' + VA'_x}{\sqrt{1 - V^2/c^2}}, \tag{4.91}$$

$$A_x = \frac{A'_x + (V/c^2)\phi'}{\sqrt{1 - V^2/c^2}}, \tag{4.92}$$

$$A_y = A'_y, \tag{4.93}$$

$$A_z = A'_z. \tag{4.94}$$

Of course, this is for K' move with $\boldsymbol{V} = V\boldsymbol{e}_x$ in K.

The transformation of $\boldsymbol{E}$ and $\boldsymbol{B}$ can be obtained from the transformation of components of $F^{\alpha\beta}$. For the $\boldsymbol{E}$ field,

$$E_x = E'_x, \qquad E_y = \frac{E'_y + VB'_z}{\sqrt{1 - V^2/c^2}} \qquad E_z = \frac{E'_z - VB'_y}{\sqrt{1 - V^2/c^2}}. \tag{4.95}$$

For the $\boldsymbol{B}$ field,

$$B_x = B'_x, \quad B_y = \frac{B'_y - (V/c^2)E'_z}{\sqrt{1 - V^2/c^2}} \quad B_z = \frac{B'_z + (V/c^2)E'_y}{\sqrt{1 - V^2/c^2}}. \tag{4.96}$$

The non-relativistic case $V/c \ll 1$ is very important,

$$E_x = E'_x, \quad E_y = E'_y + VB'_z, \qquad E_z = E'_z - VB'_y. \tag{4.97}$$

$$B_x = B'_x, \quad B_y = B'_y - (V/c^2)E'_z, \quad B_z = B'_z + (V/c^2)E'_y. \tag{4.98}$$

Or in vector form,

$$\boldsymbol{E} = \boldsymbol{E}' + \boldsymbol{B}' \times \boldsymbol{V}, \qquad \boldsymbol{B} = \boldsymbol{B}' - \frac{1}{c^2}\boldsymbol{E}' \times \boldsymbol{V}. \tag{4.99}$$

$$\text{or } \boldsymbol{E} = \boldsymbol{E}' + c\boldsymbol{B}' \times \boldsymbol{\beta}, \qquad \boldsymbol{B} = \boldsymbol{B}' - (\boldsymbol{E}'/c) \times \boldsymbol{\beta}, \tag{4.100}$$

with $\boldsymbol{\beta} = \boldsymbol{V}/c$.

Note that if $\boldsymbol{B}' = 0$ in K' system, then in K system,

$$\boldsymbol{B} = \boldsymbol{\beta} \times (\boldsymbol{E}/c). \tag{4.101}$$

If $\boldsymbol{E}' = 0$ in K' system, then in the K system,

$$\boldsymbol{E} = c\boldsymbol{B} \times \boldsymbol{\beta}. \tag{4.102}$$

In both cases, $\boldsymbol{E} \perp \boldsymbol{B}$ in K. This point can be obtained more clearly by considering the invariants of $\boldsymbol{E}$ and $\boldsymbol{B}$.

4.7.2 *Lorentz invariants of electromagnetic fields*

From the field tensor $F^{\alpha\beta}$, we can obtain two important and useful invariants by constructing four-scalars. First, we can easily see that $F_{\alpha\beta}F^{\beta\alpha} = \text{inv}$. Because $F_{\alpha\beta}$ is anti-symmetric, normally we write it as

$$F_{\alpha\beta}F^{\alpha\beta} = \text{ inv}, \tag{4.103}$$

which leads to

$$\boxed{c^2B^2 - E^2 = \text{inv}.} \tag{4.104}$$

To construct the second invariant, we first introduce $e^{\alpha\beta\mu\nu}$ *the completely anti-symmetric unit tensor of fourth-order.* The tensor $e^{\alpha\beta\mu\nu}$ has the following properties,

1. $e^{\alpha\beta\mu\nu}$ changes sign if interchanging any pair of indices
2. $e^{\alpha\beta\mu\nu} = 0$ if any two indices are the same
3. normally we choose $e^{0123} = 1$.

You can see that $e^{\alpha\beta\mu\nu}$ is the 4D version of Levi-Civita symbol ϵ_{ijk}. An extra note: $e^{\alpha\beta\mu\nu}$ is actually a *pseudo-tensor*. Read Pg 17-18 of Landau's book about this. Basically a pseudo-tensor transforms similarly as real tensors in rotations of coordinates. They differ only in those transformation that cannot be reduced to rotation of coordinates. Since Lorentz transformation is a rotation of four-dimensional space, we can use pseudo-tensors to construct Lorentz invariants.

Using $e^{\alpha\beta\mu\nu}$, we can construct the following invariant,

$$F_{\alpha\beta}e^{\beta\alpha\mu\nu}F_{\nu\mu} = \text{inv}. \tag{4.105}$$

Because $e^{\alpha\beta\mu\nu}$ and $F_{\alpha\beta}$ are anti-symmetric, we write

$$e^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu} = \text{inv}, \tag{4.106}$$

which leads to

$$\boxed{\boldsymbol{E}\cdot\boldsymbol{B} = \text{inv}.} \tag{4.107}$$

In short summary, the two important invariants of the field are

$$\boxed{\begin{aligned} c^2B^2 - E^2 &= \text{inv} \\ \boldsymbol{E}\cdot\boldsymbol{B} &= \text{inv} \end{aligned}}$$

Using two invariants, we draw the following conclusions:

1. If $E > cB$ in K, then $E > cB$ in all other K', and vice versa.
2. If $\boldsymbol{E} \perp \boldsymbol{B}$ in K, then we can find a K' so that
   - $\boldsymbol{E} = 0$ if $c^2B^2 - E^2 > 0$.
   - $\boldsymbol{B} = 0$ if $c^2B^2 - E^2 < 0$.
3. If $\boldsymbol{E} \perp \boldsymbol{B}$ in K, then $\boldsymbol{E}' \perp \boldsymbol{B}'$ in all other K' if $E' \neq 0$ and $B' \neq 0$.

4.7.3 *Fields of a uniformly moving charge*

As an example of the above transformation rules, we use them to obtain the electromagnetic fields of a uniformly moving charge. Since the velocity of the charge is constant, we analyze the problem in two steps:

1. Calculate the field in the reference system K' where the charge is at rest.
2. Make a coordinate transformation to the observation reference system K.

The problem in step 1 is purely electrostatic, and the problem in step 2 is just a coordinate transformation. So this is a very easy problem in principle.

In K' system, let a static point charge q be located at the origin $x' = y' = z' = 0$. The potentials of the charge q in K' is

$$\phi' = \frac{q}{4\pi\epsilon_0 r'} \quad \text{and} \quad \boldsymbol{A}' = 0. \tag{4.108}$$

This equation finishes step 1. Note that we use potentials here [4] to be consistent with the case in the next section, where the source for generating the field is general.

[4] For example, you can obtain $\boldsymbol{E}'$ and $\boldsymbol{B}'$ first, and then coordinate transform to obtain $\boldsymbol{E}$ and $\boldsymbol{B}$.

We know how to transformation ϕ' and $\boldsymbol{A}'$ to ϕ and $\boldsymbol{A}$ in K.

$$\phi/c = \gamma(\phi'/c + \beta A'_x) = \frac{\phi'(\boldsymbol{r}')/c}{\sqrt{1-v^2/c^2}}, \tag{4.109}$$

$$A_x = \gamma(A'_x + \beta\phi'/c) = \frac{(v/c^2)\phi'(\boldsymbol{r}')}{\sqrt{1-v^2/c^2}}, \tag{4.110}$$

$$A_y = A'_y = 0, \tag{4.111}$$

$$A_z = A'_z = 0, \tag{4.112}$$

since $A^\alpha = (\phi/c, \boldsymbol{A})$ is a four-vector. This looks pretty easy, except that currently $\phi = \phi(\boldsymbol{r}')$ and $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{r}')$. We need one more step to express ϕ and $\boldsymbol{A}$ in terms of $\boldsymbol{r}$ instead of $\boldsymbol{r}'$, because the ∇ operator in K system is w.r.t. $\boldsymbol{r}$. To do this, we know that $r' = \sqrt{x'^2 + y'^2 + z'^2}$ and

$$x' = \frac{x - vt}{\sqrt{1-v^2/c^2}}, \tag{4.113}$$

$$y' = y, \tag{4.114}$$

$$z' = z. \tag{4.115}$$

Therefore

$$r' = \sqrt{\frac{(x-vt)^2}{1-v^2/c^2} + y^2 + z^2}. \tag{4.116}$$

Substituting $r' = r'(x,y,z)$ in Equation (4.116) into ϕ and $\boldsymbol{A}$ in Equations (4.109)-(4.112) gives

$$\phi = \frac{q}{4\pi\epsilon_0 r'\sqrt{1-v^2/c^2}} \tag{4.117}$$

$$= \frac{q}{4\pi\epsilon_0\sqrt{(x-vt)^2 + (1-v^2/c^2)(y^2+z^2)}} \tag{4.118}$$

$$\equiv \frac{q}{4\pi\epsilon_0 r^*}, \tag{4.119}$$

$$A_x = \frac{v}{c^2}\phi, \tag{4.120}$$

$$A_y = A_z = 0. \tag{4.121}$$

From ϕ and $\boldsymbol{A}$, we can directly obtain $\boldsymbol{E}$ and $\boldsymbol{B}$.

Note that $r^* = \sqrt{(x-vt)^2 + (1-v^2/c^2)(y^2+z^2)}$,

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} = -\nabla\phi - \frac{v}{c^2}\frac{\partial\phi}{\partial t}\boldsymbol{e}_x. \tag{4.122}$$

Noting $\phi = q/4\pi\epsilon_0 r^*$, we have

$$\frac{\partial\phi}{\partial x} = -\frac{q}{4\pi\epsilon_0 r^{*2}}\frac{\partial r^*}{\partial x} = -\frac{q}{4\pi\epsilon_0 r^{*3}}(x-vt) \tag{4.123}$$

$$\frac{\partial\phi}{\partial y} = -\frac{q}{4\pi\epsilon_0 r^{*2}}\frac{\partial r^*}{\partial y} = -\frac{q}{4\pi\epsilon_0 r^{*3}}\left(1-\frac{v^2}{c^2}\right)y \tag{4.124}$$

$$\frac{\partial\phi}{\partial z} = -\frac{q}{4\pi\epsilon_0 r^{*2}}\frac{\partial r^*}{\partial z} = -\frac{q}{4\pi\epsilon_0 r^{*3}}\left(1-\frac{v^2}{c^2}\right)z \tag{4.125}$$

$$\frac{\partial\phi}{\partial t} = -\frac{q}{4\pi\epsilon_0 r^{*2}}\frac{\partial r^*}{\partial t} = -\frac{q}{4\pi\epsilon_0 r^{*3}}(x-vt)(-v) \tag{4.126}$$

Combining Equations (4.123)-(4.126) and substituting them to Equation (4.122) leads to

$$\boldsymbol{E} = -\nabla\phi - \frac{v}{c^2}\frac{\partial\phi}{\partial t}\boldsymbol{e}_x \tag{4.127}$$

$$= \frac{q}{4\pi\epsilon_0 r^{*3}}(x - vt)\boldsymbol{e}_x + \frac{q}{4\pi\epsilon_0 r^{*3}}\left(1 - \frac{v^2}{c^2}\right) y\boldsymbol{e}_y + \frac{q}{4\pi\epsilon_0 r^{*3}}\left(1 - \frac{v^2}{c^2}\right) z\boldsymbol{e}_z - \frac{v^2}{c^2}\frac{q}{4\pi\epsilon_0 r^{*3}}(x - vt)\boldsymbol{e}_x \tag{4.128}$$

$$= \frac{q}{4\pi\epsilon_0 r^{*3}}\left(1 - \frac{v^2}{c^2}\right)\left[(x - vt)\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z\right] \tag{4.129}$$

$$= \left(1 - \frac{v^2}{c^2}\right)\frac{q}{4\pi\epsilon_0 r^{*3}}\boldsymbol{R}, \tag{4.130}$$

where $\boldsymbol{R} \equiv (x - vt)\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z$ is radius vector from the charge position at t to the field point (x, y, z) in K, because the charge trajectory is given by $\boldsymbol{x} = vt\boldsymbol{e}_x$.

To find the magnetic field $\boldsymbol{B}$, we note that $\boldsymbol{B}' = 0$ in K', hence in K system

$$\boldsymbol{B} = \frac{\boldsymbol{v}}{c^2} \times \boldsymbol{E}. \tag{4.131}$$

Let's now discuss the electric field $\boldsymbol{E}$, since $\boldsymbol{B}$ is simple if we know $\boldsymbol{E}$. Introducing angle θ, which is the angle between $\boldsymbol{R}$ and the direction of motion ($\boldsymbol{e}_x$ in our case), then

$$R_\parallel = x - vt, \tag{4.132}$$

$$R_\perp = y^2 + z^2 = R^2 \sin^2\theta, \tag{4.133}$$

$$r^{*2} = R_\parallel^2 + \left(1 - \frac{v^2}{c^2}\right) R_\perp^2 = R^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right). \tag{4.134}$$

Using Equations (4.132)-(4.134), we rewrite $\boldsymbol{E}$ in Equation (4.130) using θ and $\boldsymbol{R}$ as

$$\boldsymbol{E} = \left(1 - \frac{v^2}{c^2}\right)\frac{q\boldsymbol{R}}{4\pi\epsilon_0 R^3\left[1 - (v^2/c^2)\sin^2\theta\right]^{3/2}} \tag{4.135}$$

$$= \frac{q\boldsymbol{R}}{4\pi\epsilon_0 R^3}\frac{1 - \beta^2}{(1 - \beta^2\sin^2\theta)^{3/2}}. \tag{4.136}$$

For a fixed distance $\boldsymbol{R}$, the value of the field $\boldsymbol{E}$ maximize at $\theta = \pi/2$ (perpendicular direction),

$$E_\perp = E_{\theta=\pi/2} = \frac{q}{4\pi\epsilon_0 R^2}(1 - \beta^2)^{-1/2} = \frac{q}{4\pi\epsilon_0 R^2}\gamma, \tag{4.137}$$

and minimizes at parallel directions ($\theta = 0$ or π),

$$E_\parallel = E_{\theta=0} = E_{\theta=\pi} = \frac{q}{4\pi\epsilon_0 R^2}(1 - \beta^2) = \frac{1}{\gamma^2}\frac{q}{4\pi\epsilon_0 R^2}. \tag{4.138}$$

So as velocity increases, $E_\parallel$ decreases and $E_\perp$ increases. Or the field is contracted in the direction of motion. If $v \to c$, then $E_\parallel \to 0$ and $E_\perp \to \infty$, because

$$\boldsymbol{E} = \frac{q\boldsymbol{R}}{4\pi\epsilon_0 R^3}\frac{1 - \beta^2}{(1 - \beta^2\sin^2\theta)^{3/2}}. \tag{4.139}$$

As $\beta \to 1$, the denominator $1 - \beta^2 \sin^2\theta \to 0$ within $\Delta\theta$. The $\boldsymbol{E}$ field at a given distance from the source is large only within a narrow range of angles.

5
The Electromagnetic Field Equations

5.1 Electrodynamics before Maxwell

From your class on electromagnetism, you should have learned the following laws

$$
\begin{aligned}
&\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0, && \text{Gauss's law.}\\
&\nabla \cdot \boldsymbol{B} = 0, && \text{no name}\\
&\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, && \text{Faraday's law}\\
&\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}. && \text{Ampere's law}
\end{aligned}
$$

There are inconsistencies in these equations that cannot not be resolved by these equations themselves. Taking the divergence of Ampere's law, you will get

$$\nabla \cdot (\nabla \times \boldsymbol{B}) = \mu_0 \nabla \cdot \boldsymbol{j}.$$

While

$$\nabla \cdot (\nabla \times \boldsymbol{B}) = 0,$$

is always correct,

$$\nabla \cdot \boldsymbol{j} = 0,$$

only holds if the current is steady. This implies that Ampere's law only holds for static case.

5.2 The charge conservation law

To see how Maxwell fixed these laws, we need to first describe the conservation of charge. We first prove the equation of continuity in a more traditional way, then we express it using the current density 4-vector.

The conservation of charge means

$$\frac{\partial}{\partial t} \int \rho \mathrm{d}V = -\oint \rho \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S}, \tag{5.1}$$

i.e., the time rate change of the total charge in volume V equals minus the total amount of charge leaving the system. Using $\boldsymbol{j} = \rho\boldsymbol{v}$,

$$\frac{\partial}{\partial t}\int \rho \mathrm{d}V = \int \frac{\partial \rho}{\partial t}\mathrm{d}V = -\oint \boldsymbol{j}\cdot \mathrm{d}\boldsymbol{S} = -\int_V \nabla\cdot \boldsymbol{j}\mathrm{d}V. \tag{5.2}$$

Since this integration must hold for any volume V, the above equation leads to the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \nabla\cdot \boldsymbol{j} = 0. \tag{5.3}$$

This equation of continuity can also be directly proved using the definition of ρ. For simplicity, we note that the charge density ρ for a single particle[1] is defined as, From this equation, we see

$$\rho(t, \boldsymbol{r}) = q\delta[\boldsymbol{r} - \boldsymbol{r}_0(t)]. \tag{5.4}$$

[1] How do you derive the equation of continuity for a system of particles? Hint: Consider what is the appropriate definition of $\boldsymbol{v}$.

Then

$$\frac{\partial \rho}{\partial t} = q\frac{\partial \delta}{\partial \boldsymbol{r}_0}\cdot\frac{\mathrm{d}\boldsymbol{r}_0}{\mathrm{d}t} = q\left(-\frac{\partial \delta}{\partial \boldsymbol{r}}\right)\cdot \boldsymbol{v} = -\nabla\cdot(\rho\boldsymbol{v}),$$

since $\nabla\cdot \boldsymbol{v} = 0$. Therefore, with $\boldsymbol{j} \equiv \rho\boldsymbol{v}$,

$$\frac{\partial \rho}{\partial t} + \nabla\cdot \boldsymbol{j} = 0.$$

Now we introduce the current four-vector. While the charge q is a Lorentz invariant, ρ is not, since $\mathrm{d}q = \rho\mathrm{d}V$ and $\mathrm{d}V$ is not an invariant. Multiplying $\mathrm{d}q = \rho\mathrm{d}V$ by $\mathrm{d}x^\alpha$ leads to

$$\underbrace{\mathrm{d}q\mathrm{d}x^\alpha}_{\text{four-vector}} = \rho\mathrm{d}V\mathrm{d}x^\alpha = \underbrace{\mathrm{d}V\mathrm{d}t}_{\text{four-scalar}}\ \rho\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}. \tag{5.5}$$

Therefore $\rho\mathrm{d}x^\alpha/\mathrm{d}t$ must be a four-vector, we define it to be the current density four-vector; i.e.,

$$j^\alpha = \rho\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}, \tag{5.6}$$

or $j^\alpha = (\rho c, \boldsymbol{j})$ with $\boldsymbol{j} = \rho\boldsymbol{v}$ the normal current density vector you're familiar with.

Since j^α is a four-vector, its divergence is a four-scalar; i.e.,

$$\frac{\partial j^\alpha}{\partial x^\alpha} = \text{inv}. \tag{5.7}$$

Writing this out with $j^\alpha = (\rho c, \boldsymbol{j})$ and using the equation of continuity, we have

$$\boxed{\frac{\partial j^\alpha}{\partial x^\alpha} = \frac{\partial \rho}{\partial t} + \nabla\cdot \boldsymbol{j} = 0.} \tag{5.8}$$

That is the charge conservation law or the equation of continuity can be expressed as that the the-divergence of the current density 4-vector is 0 (the current density 4-vector is divergence-free).

5.3 How Maxwell fixed Ampere's law

Ampere's law gives

$$\nabla\cdot(\nabla\times\boldsymbol{B}) = \mu_0\nabla\cdot\boldsymbol{j},$$

Because the left hand side (LHS) is always right, one might try to fix this by adding a term to the right hand side (RHS). Well, we've just learned the equation of continuity, therefore, we can fix Ampere's law using [2]

[2] Note that in actual history, Maxwell added the extra term for a reason based on ether, a model which is now considered to be incorrect.

$$\nabla\cdot(\nabla\times\boldsymbol{B}) = \mu_0\left(\nabla\cdot\boldsymbol{j} + \frac{\partial\rho}{\partial t}\right).$$

Considering Gauss's law that $\nabla\cdot\boldsymbol{E} = \rho/\epsilon_0$, the above equation is written as

$$\nabla\cdot(\nabla\times\boldsymbol{B}) = \mu_0\nabla\cdot\boldsymbol{j} + \mu_0\epsilon_0\frac{\partial}{\partial t}\nabla\cdot\boldsymbol{E} = \nabla\cdot\left(\mu_0\boldsymbol{j} + \mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\right).$$

Therefore, the Ampere's law can be fixed to be

$$\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{j} + \mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}.$$

The new extra term is called by Maxwell the *displacement current*,

$$\boldsymbol{j}_d = \epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}.$$

This new term means that

> A changing electric field induces a magnetic field.

Now the new electrodynamics has a certain kind of symmetry in it.

Of course, theoretical convenience and consistency are only suggestive. The confirmation of Maxwell's law came later with Hertz's experiments on electromagnetic waves after Maxwell's death.

5.4 Maxwell's Equations

The whole set of Maxwell's equations are

$\nabla\cdot\boldsymbol{E} = \rho/\epsilon_0,$	Gauss's law.
$\nabla\cdot\boldsymbol{B} = 0,$	no name
$\nabla\times\boldsymbol{E} = -\dfrac{\partial\boldsymbol{B}}{\partial t},$	Faraday's law
$\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{j} + \mu_0\epsilon_0\dfrac{\partial\boldsymbol{E}}{\partial t}.$	Ampere's law

These are the fundamental equations of the EM field theory. The Maxwell equations and the Lorentz force law,

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}),$$

are the entire content of the theoretical classical electrodynamics.

*5.5 *Deriving Maxwell equations using the principle of least action*

5.5.1 *The homogeneous Maxwell equations*

From definitions of $\boldsymbol{E}$ and $\boldsymbol{B}$ using potentials ϕ and $\boldsymbol{A}$,

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \tag{5.9}$$

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}, \tag{5.10}$$

we have

$$\nabla \cdot \boldsymbol{B} = 0, \tag{5.11}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}. \tag{5.12}$$

These are the two homogeneous Maxwell equations.

5.5.2 *The action of the electromagnetic field*

To obtain the other two equations of fields, we introduce the action of the electromagnetic field. Previously we considered the action for charges in a given field

$$S = \underbrace{S_p}_{\text{free particle}} + \underbrace{S_{pf}}_{\text{interaction}} . \tag{5.13}$$

The action for the whole system is, by adding S_f to the total action S,

$$S = \underbrace{S_p}_{\text{free particle}} + \underbrace{S_{pf}}_{\text{interaction}} + \underbrace{S_f}_{\text{field only}} . \tag{5.14}$$

To derive the field action S_f, we note the following three constraints,

1. The field satisfies the principle of supposition. Equations are linear in $\boldsymbol{E}$ and $\boldsymbol{B}$; therefore, S_f must be quadratic in $\boldsymbol{E}$ and $\boldsymbol{B}$; i.e., $\sim E^2$ or B^2.

2. S_f must be a 4-scalar.

Considering these constraints, other people have figured out $S_f \propto \int F_{\alpha\beta}F^{\alpha\beta}\mathrm{d}\Omega$. In SI units,

$$S_f = -\frac{1}{4\mu_0}\int F_{\alpha\beta}F^{\alpha\beta}\mathrm{d}\Omega, \quad \text{where } \mathrm{d}\Omega = c\mathrm{d}t\mathrm{d}V \tag{5.15}$$

In 3D form, since $F_{\alpha\beta}F^{\alpha\beta} = 2(B^2 - E^2/c^2)$,

$$S_f = \frac{c}{2\mu_0}\int (E^2/c^2 - B^2)\mathrm{d}V\mathrm{d}t, \tag{5.16}$$

or the Lagrangian for the field is

$$L_f = \frac{c}{2\mu_0}\int (E^2/c^2 - B^2)\mathrm{d}V. \tag{5.17}$$

Putting all pieces together, the total action is

$$S = -\sum mc\int \mathrm{d}s - \sum\int qA_\alpha \mathrm{d}x^\alpha - \frac{1}{4\mu_0 c}\int F_{\alpha\beta}F^{\alpha\beta}\mathrm{d}\Omega, \tag{5.18}$$

where the summation is over all particles. Note that now charges are not assumed to be small, so fields include external fields and the field produced by the particles themselves.

Using the charge density 4-vector j^α, the interaction action S_{pf} becomes

$$\begin{aligned} S_{pf} &= -\sum\int qA_\alpha \mathrm{d}x^\alpha = -\int \rho A_\alpha \mathrm{d}V\mathrm{d}x^\alpha \\ &= -\int \rho A_\alpha \mathrm{d}V\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\mathrm{d}t \\ &= -\frac{1}{c}\int j^\alpha A_\alpha \mathrm{d}\Omega. \end{aligned}$$

Substituting this form of S_{pf} into the Equation (5.18)

$$\boxed{S = -\sum mc\int \mathrm{d}s - \frac{1}{c}\int j^\alpha A_\alpha \mathrm{d}\Omega - \frac{1}{4\mu_0 c}\int F_{\alpha\beta}F^{\alpha\beta}\mathrm{d}\Omega.}$$

This is the total action expressed using the current density 4-vector.

5.5.3 *The inhomogeneous Maxwell equations*

In this subsection, we use the principle of least action to determine equations obeyed by fields. Deriving field equations using the principle of least action is similar to deriving particle equations. Recall that to obtain particle trajectory, we vary $x^\alpha \to x^\alpha + \delta x^\alpha$, and then we obtain equations satisfied by x^α. Now we want to obtain equations obeyed by field A^α, we do it in the following three steps.

1. Particles move along their actual path
2. Vary $A^\alpha \to A^\alpha + \delta A^\alpha$.
3. The field equations can be obtained from $\delta S = 0$.

First, noting that particles all move along their actual path ($\delta S_p = 0$), the variation of action due to δA^α is

$$\delta S = \underbrace{\delta S_p}_{=0} - \frac{1}{c}\,\delta\int A_\alpha j^\alpha \mathrm{d}\Omega - \frac{1}{4\mu_0 c}\delta\int F_{\alpha\beta}F^{\alpha\beta}\mathrm{d}\Omega. \tag{5.19}$$

Since particle move along their actual trajectories, $\delta j^\alpha = 0$,

$$\delta S = -\frac{1}{c}\int \delta A_\alpha j^\alpha \mathrm{d}\Omega - \frac{1}{4\mu_0 c}\delta\int F_{\alpha\beta}F^{\alpha\beta}\mathrm{d}\Omega. \tag{5.20}$$

The integrand of δS_{pf} is in the form of $(\cdots)\delta A_\alpha$, so we do not need to do anything about it. We just need to write the integrand of δS_f in a similar form.

Now let's calculate

$$\delta S_f = -\frac{1}{4\mu_0 c}\int \delta(F_{\alpha\beta}F^{\alpha\beta})\mathrm{d}\Omega. \tag{5.21}$$

The variation of the action of field is

$$\delta S_f = -\frac{1}{4\mu_0 c}\int \delta(F_{\alpha\beta}F^{\alpha\beta})\mathrm{d}\Omega \tag{5.22}$$

$$= -\frac{1}{4\mu_0 c}\int (\delta F_{\alpha\beta}F^{\alpha\beta} + F_{\alpha\beta}\delta F^{\alpha\beta})\mathrm{d}\Omega \tag{5.23}$$

$$= -\frac{1}{2\mu_0 c}\int F^{\alpha\beta}\delta F_{\alpha\beta}\mathrm{d}\Omega. \tag{5.24}$$

Substituting

$$F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta} \Rightarrow \delta F_{\alpha\beta} = \frac{\partial \delta A_\beta}{\partial x^\alpha} - \frac{\partial \delta A_\alpha}{\partial x^\beta} \tag{5.25}$$

into Equation (5.24) leads to

$$\delta S_f = -\frac{1}{2\mu_0 c}\int F^{\alpha\beta}\delta F_{\alpha\beta}\mathrm{d}\Omega \tag{5.26}$$

$$= -\frac{1}{2\mu_0 c}\int \left(F^{\alpha\beta}\frac{\partial \delta A_\beta}{\partial x^\alpha} - F^{\alpha\beta}\frac{\partial \delta A_\alpha}{\partial x^\beta}\right)\mathrm{d}\Omega \tag{5.27}$$

$$= -\frac{1}{2\mu_0 c}\int \left(F^{\alpha\beta}\frac{\partial \delta A_\beta}{\partial x^\alpha} - F^{\beta\alpha}\frac{\partial \delta A_\beta}{\partial x^\alpha}\right)\mathrm{d}\Omega \tag{5.28}$$

$$= -\frac{1}{2\mu_0 c}\int \left(F^{\alpha\beta}\frac{\partial \delta A_\beta}{\partial x^\alpha} + F^{\alpha\beta}\frac{\partial \delta A_\beta}{\partial x^\alpha}\right)\mathrm{d}\Omega \tag{5.29}$$

$$= -\frac{1}{\mu_0 c}\int F^{\alpha\beta}\frac{\partial \delta A_\beta}{\partial x^\alpha}\mathrm{d}\Omega, \tag{5.30}$$

where we have used $F^{\alpha\beta} = -F^{\beta\alpha}$. Perform the integration in Equation (5.30) by parts,

$$\delta S_f = -\frac{1}{\mu_0 c}\int F^{\alpha\beta}\frac{\partial \delta A_\beta}{\partial x^\alpha}\mathrm{d}\Omega \tag{5.31}$$

$$= -\frac{1}{\mu_0 c}\left[\int \frac{\partial}{\partial x^\alpha}\left(F^{\alpha\beta}\delta A_\beta\right)\mathrm{d}\Omega - \int \frac{\partial F^{\alpha\beta}}{\partial x^\alpha}\delta A_\beta \mathrm{d}\Omega\right]. \tag{5.32}$$

The first term can be proved to be 0 using Gauss's theorem in four-dimensional space,

$$\int \frac{\partial}{\partial x^\alpha}\left(F^{\alpha\beta}\delta A_\beta\right)\mathrm{d}\Omega = \int F^{\alpha\beta}\delta A_\beta \mathrm{d}S_\alpha = 0, \tag{5.33}$$

because

$$\delta A_\beta = 0 \text{ for } t = t_0 \text{ and } t = t_1. \tag{5.34}$$

$$F^{\alpha\beta} = 0 \text{ for } \boldsymbol{r} = \infty. \tag{5.35}$$

Therefore Equation (5.32) becomes

$$\delta S_f = \frac{1}{\mu_0 c}\int \frac{\partial F^{\alpha\beta}}{\partial x^\alpha}\delta A_\beta \mathrm{d}\Omega. \tag{5.36}$$

The variation of the total action is $\delta S = \delta S_{pf} + \delta S_f$,

$$\delta S = \int \left(-\frac{1}{c} j^\beta + \frac{1}{\mu_0 c} \frac{\partial F^{\alpha\beta}}{\partial x^\alpha} \right) \delta A_\beta \mathrm{d}\Omega \tag{5.37}$$

$$= -\int \left(\frac{1}{c} j^\beta + \frac{1}{\mu_0 c} \frac{\partial F^{\beta\alpha}}{\partial x^\alpha} \right) \delta A_\beta \mathrm{d}\Omega. \tag{5.38}$$

The principle of least action $\delta S = 0$ gives

$$\frac{\partial F^{\beta\alpha}}{\partial x^\alpha} = -\mu_0 j^\beta. \tag{5.39}$$

The components of these equations are

$$-\frac{1}{c}\frac{\partial E_x}{\partial x} - \frac{1}{c}\frac{\partial E_y}{\partial y} - \frac{1}{c}\frac{\partial E_z}{\partial z} = -\mu_0 \rho c \tag{5.40}$$

$$\frac{1}{c^2}\frac{\partial E_x}{\partial t} - \frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z} = -\mu_0 j_x \tag{5.41}$$

$$\frac{1}{c^2}\frac{\partial E_y}{\partial t} + \frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z} = -\mu_0 j_y \tag{5.42}$$

$$\frac{1}{c^2}\frac{\partial E_z}{\partial t} - \frac{\partial B_y}{\partial x} + \frac{\partial B_x}{\partial y} = -\mu_0 j_z \tag{5.43}$$

In vector form, these equations give, using $c^2 = 1/\mu_0\epsilon_0$,

$$\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0, \tag{5.44}$$

$$\nabla \times \boldsymbol{B} = \mu_0\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \mu_0 \boldsymbol{j}. \tag{5.45}$$

the inhomogeneous Maxwell equations.

5.6 *Integral forms of Maxwell equations*

We now obtain useful laws by integrating the differential Maxwell equations. First,

$$\int_V \nabla \cdot \boldsymbol{B}\, \mathrm{d}V = \oint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}. \quad \textit{Gauss's theorem} \tag{5.46}$$

The integral $\int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$ is the flux through surface $\boldsymbol{S}$,

$$\nabla \cdot \boldsymbol{B} = 0 \Rightarrow \oint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 0, \tag{5.47}$$

means

> the flux of $\boldsymbol{B}$ through any closed surface is 0.

We now integrate the $\nabla \times \boldsymbol{E}$ equation. First,

$$\int_S \nabla \times \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \oint_l \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}. \quad \textit{Stokes' theorem} \tag{5.48}$$

From the differential equation,

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{5.49}$$

we have the conclusion

$$\int_S \nabla \times \boldsymbol{E} \cdot \, \mathrm{d}\boldsymbol{S} = \oint_l \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\frac{\partial}{\partial t} \int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = -\frac{\partial \Phi}{\partial t}. \tag{5.50}$$

In the equation

$$\oint_l \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\frac{\partial \Phi}{\partial t}, \tag{5.51}$$

the two terms mean

1. Φ: the magnetic flux through surface $\boldsymbol{S}$.
2. $\oint_l \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}$: the electromotive force in the contour.

> The electromotive force thus equals minus the time rate change of Φ.

Third, the integral form of Gauss's law is

$$\int \nabla \cdot \boldsymbol{E} \mathrm{d}V = \frac{1}{\epsilon_0} \int \rho \mathrm{d}V, \tag{5.52}$$

$$\Rightarrow \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{1}{\epsilon_0} \int \rho \mathrm{d}V = Q/\epsilon_0. \tag{5.53}$$

> The flux of $\boldsymbol{E}$ through a closed surface equals $1/\epsilon_0$ times the total charge inside the volume.

The integral form of Ampere's law is

$$\int \nabla \times \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} + \mu_0 \int \boldsymbol{j} \cdot \mathrm{d}\S \tag{5.54}$$

$$\Rightarrow \oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 \int \left(\boldsymbol{j} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right) \cdot \mathrm{d}\boldsymbol{S} \tag{5.55}$$

> The circulation of $\boldsymbol{B}$ around any contour equals μ_0 times the sum of the displacement current and the true current through the surface.

5.7 *Energy density and energy flux of the EM field*

Now we discuss some conservation properties of electromagnetic field regarding their energy and momentum. Recall that field is similar to a continuous media, therefore, when talking about field energy and momentum, we almost always talk about *energy density* and *momentum density* in the context of classical electrodynamics.

> Field carries energy; and the total energy is conserved.

To figure out the energy density of the field, we see that the work done by EM field on a charge q in interval $\mathrm{d}t$ is

$$\mathrm{d}\varepsilon_k = \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{l} = q(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{v}\mathrm{d}t = q\boldsymbol{E}\cdot\boldsymbol{v}\mathrm{d}t \tag{5.56}$$

If we consider a continuous distribution of charge,

$$\frac{\mathrm{d}\varepsilon_k}{\mathrm{d}t} = \int \rho\boldsymbol{E}\cdot\boldsymbol{v}\mathrm{d}V = \int \boldsymbol{j}\cdot\boldsymbol{E}\mathrm{d}V. \tag{5.57}$$

Through conservation of energy, $\mathrm{d}\varepsilon_k$ must equal minus the energy change of the field in time $\mathrm{d}t$.

To see the energy variation of the field, we express $\boldsymbol{j}\cdot\boldsymbol{E}$ using field quantities only. The assumption here is that the only other way of changing energy is through the field.

$$\begin{aligned}
\boldsymbol{j}\cdot\boldsymbol{E} &= \frac{1}{\mu_0}\left(\nabla\times\boldsymbol{B} - \mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\right)\cdot\boldsymbol{E} \\
&= \frac{1}{\mu_0}\nabla\times\boldsymbol{B}\cdot\boldsymbol{E} - \epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\cdot\boldsymbol{E} \\
&= \frac{1}{\mu_0}\left(\nabla\times\boldsymbol{B}\cdot\boldsymbol{E} - \nabla\times\boldsymbol{E}\cdot\boldsymbol{B}\right) + \frac{1}{\mu_0}\nabla\times\boldsymbol{E}\cdot\boldsymbol{B} - \frac{\epsilon_0}{2}\frac{\partial E^2}{\partial t} \\
&= -\frac{1}{\mu_0}\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B}) - \frac{1}{2\mu_0}\frac{\partial B^2}{\partial t} - \frac{\epsilon_0}{2}\frac{\partial E^2}{\partial t} \\
&= -\frac{1}{\mu_0}\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B}) - \frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right).
\end{aligned}$$

Reorganizing the equation as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right) = -\boldsymbol{j}\cdot\boldsymbol{E} - \nabla\cdot\left(\frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B}\right). \tag{5.58}$$

Denoting the Poynting flux vector by

$$\boxed{\boldsymbol{P} = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B},} \tag{5.59}$$

the equation can be written as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right) + \boldsymbol{j}\cdot\boldsymbol{E} = -\nabla\cdot\boldsymbol{P}. \tag{5.60}$$

To see what the equation (5.60) means, we apply $\int \mathrm{d}V$ to it.

$$\int \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right)\mathrm{d}V + \int \boldsymbol{j}\cdot\boldsymbol{E}\mathrm{d}V = -\int \nabla\cdot\boldsymbol{P}\mathrm{d}V.$$

or using Gauss' theorem

$$\int \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right)\mathrm{d}V + \int \boldsymbol{j}\cdot\boldsymbol{E}\mathrm{d}V = -\int \boldsymbol{P}\cdot\mathrm{d}\boldsymbol{S} \tag{5.61}$$

If we extend the integration to all space, the field is 0 at ∞, Equation (5.61) becomes

$$\int \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right)\mathrm{d}V + \int \boldsymbol{j}\cdot\boldsymbol{E}\mathrm{d}V = 0. \tag{5.62}$$

Noting that

$$\frac{\mathrm{d}\varepsilon_k}{\mathrm{d}t} = \int \boldsymbol{j} \cdot \boldsymbol{E} \mathrm{d}V, \tag{5.63}$$

Equation (5.61) then becomes

$$\frac{\partial}{\partial t}\left(\underbrace{\int \left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right)\mathrm{d}V}_{\text{field energy}} + \underbrace{\varepsilon_k}_{\text{particle kinetic energy}} \right) = 0, \tag{5.64}$$

from which we conclude that

$$\boxed{w = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)} \tag{5.65}$$

is the field energy density. Yes, the field has energy.

Expressed using w, Equation (5.61) is

$$\frac{\partial}{\partial t}\left(\int w \mathrm{d}V + \varepsilon_k\right) = -\oint \boldsymbol{P} \cdot \mathrm{d}\boldsymbol{S}. \tag{5.66}$$

The left hand side (LHS) is the total energy variation inside a given finite volume, thus $\boldsymbol{P}$, the Poynting flux, is the field energy flux density – the amount of energy passing through unit area per unit time.

5.8 *Momentum density and the Maxwell stress tensor*

We derive the momentum of the field in a similar way as that of the field energy density. That is,

Field carries momentum; the total momentum is conserved.

The time rate change of momentum of a particle q,

$$\frac{\mathrm{d}\boldsymbol{p}_m}{\mathrm{d}t} = \boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}). \tag{5.67}$$

The subscript "m" denotes "mechanical" quantity. If we consider a continuous distribution of charges, then

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{p}_m}{\mathrm{d}t} &= \int \rho(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})\mathrm{d}V \\ &= \int (\rho\boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B})\,\mathrm{d}V.\end{aligned} \tag{5.68}$$

We now express ρ and $\boldsymbol{j}$ using $\boldsymbol{E}$ and $\boldsymbol{B}$, and the purpose is of course to express the final equation in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{p}_m + \boldsymbol{p}_f\right) + \int \nabla \cdot \mathsf{G}\mathrm{d}V = 0, \tag{5.69}$$

where $\boldsymbol{p}_f$ is the field momentum, and G is the field momentum flux, which is a tensor (since momentum is a vector).

We now express the right hand side of Equation (5.69) using field quantities only. From Maxwell equations,

$$\rho = \epsilon_0 \nabla \cdot \boldsymbol{E} \quad \text{and} \quad \boldsymbol{j} = \frac{1}{\mu_0}\left(\nabla \times \boldsymbol{B} - \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right) \tag{5.70}$$

The first term $\rho \boldsymbol{E} = \epsilon_0 \boldsymbol{E}(\nabla \cdot \boldsymbol{E})$. The second term is

$$\boldsymbol{j} \times \boldsymbol{B} = \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \times \boldsymbol{B}. \tag{5.71}$$

Using

$$\boldsymbol{B} \times \frac{\partial \boldsymbol{E}}{\partial t} = -\frac{\partial}{\partial t}(\boldsymbol{E} \times \boldsymbol{B}) + \boldsymbol{E} \times \frac{\partial \boldsymbol{B}}{\partial t}, \tag{5.72}$$

and adding $\boldsymbol{B}(\nabla \cdot \boldsymbol{B})$, which is 0, to equation (5.68) gives,

$$\begin{aligned} &\rho \boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B} = -\epsilon_0 \frac{\partial}{\partial t}(\boldsymbol{E} \times \boldsymbol{B}) \\ &+ \left[\epsilon_0 \boldsymbol{E}(\nabla \cdot \boldsymbol{E}) + \frac{1}{\mu_0}\boldsymbol{B}(\nabla \cdot \boldsymbol{B}) - \epsilon_0 \boldsymbol{E} \times (\nabla \times \boldsymbol{E}) - \frac{1}{\mu_0}\boldsymbol{B} \times (\nabla \times \boldsymbol{B})\right]. \end{aligned} \tag{5.73}$$

Substituting Equation (5.73) into equation (5.68) gives

$$\begin{aligned} &\frac{\mathrm{d}\boldsymbol{p}_m}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\int \epsilon_0 \boldsymbol{E} \times \boldsymbol{B} \mathrm{d}V\right) = \\ &\int \left[\epsilon_0 \boldsymbol{E}(\nabla \cdot \boldsymbol{E}) + \frac{1}{\mu_0}\boldsymbol{B}(\nabla \cdot \boldsymbol{B}) - \epsilon_0 \boldsymbol{E} \times (\nabla \times \boldsymbol{E}) - \frac{1}{\mu_0}\boldsymbol{B} \times (\nabla \times \boldsymbol{B})\right] \mathrm{d}V \end{aligned} \tag{5.74}$$

Define

$$\boxed{\boldsymbol{g} = \epsilon_0 \boldsymbol{E} \times \boldsymbol{B},} \tag{5.75}$$

which will be shown later to be the field momentum density, and

$$\boldsymbol{p}_f = \int \boldsymbol{g} \mathrm{d}V, \tag{5.76}$$

which is the total field momentum in the chosen volume, then Equation (5.74) is

$$\begin{aligned} &\frac{\mathrm{d}\boldsymbol{p}_m}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{p}_f}{\mathrm{d}t} = \\ &\int \left[\epsilon_0 \boldsymbol{E}(\nabla \cdot \boldsymbol{E}) + \frac{1}{\mu_0}\boldsymbol{B}(\nabla \cdot \boldsymbol{B}) - \epsilon_0 \boldsymbol{E} \times (\nabla \times \boldsymbol{E}) - \frac{1}{\mu_0}\boldsymbol{B} \times (\nabla \times \boldsymbol{B})\right] \mathrm{d}V, \end{aligned} \tag{5.77}$$

Let's now figure out what's in the right hand side. Recall that in Homework 2, you have proved

$$\boldsymbol{E} \times (\nabla \times \boldsymbol{E}) = (\nabla \cdot \boldsymbol{E})\boldsymbol{E} - \nabla \cdot \left(\boldsymbol{E}\boldsymbol{E} - \frac{1}{2}E^2 \mathsf{I}\right), \tag{5.78}$$

so

$$\boldsymbol{E}(\nabla \cdot \boldsymbol{E}) - \boldsymbol{E} \times (\nabla \times \boldsymbol{E}) = \nabla \cdot \left(\boldsymbol{E}\boldsymbol{E} - \frac{1}{2}E^2 \mathsf{I}\right) \tag{5.79}$$

similarly,

$$\boldsymbol{B}(\nabla\cdot\boldsymbol{B}) - \boldsymbol{B}\times(\nabla\times\boldsymbol{B}) = \nabla\cdot\left(\boldsymbol{BB} - \frac{1}{2}B^2\mathsf{I}\right) \tag{5.80}$$

Define the Maxwell stress tensor T to be

$$\boxed{\mathsf{T} = \left[\epsilon_0\boldsymbol{EE} + \frac{1}{\mu_0}\boldsymbol{BB} - \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\mathsf{I}\right],} \tag{5.81}$$

Equation (5.77) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{p}_m + \boldsymbol{p}_f\right) = \int \nabla\cdot\mathsf{T}\mathrm{d}V = \oint \mathrm{d}\boldsymbol{S}\cdot\mathsf{T} = \oint \mathsf{T}\cdot\mathrm{d}\boldsymbol{S}, \tag{5.82}$$

because T is symmetric. Or written in the form of Equation (5.69),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{p}_m + \boldsymbol{p}_f\right) + \int \nabla\cdot\mathsf{G}\,\mathrm{d}V = 0, \tag{5.83}$$

where $\mathsf{G} = -\mathsf{T}$ is the field momentum flux tensor.

To actually understand the meaning of $\boldsymbol{p}_f$ and T (or G), let $V \to \infty$. Then $\mathsf{T} = 0$ since field is 0 at infinity. Equation (5.82) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{p}_m + \boldsymbol{p}_f\right) = 0. \tag{5.84}$$

From the conservation of momentum, $\boldsymbol{p}_f$ thus represents the momentum of the field.

To see the meaning of T or G, let V be finite, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{p}_m + \boldsymbol{p}_f\right) = \oint \mathsf{T}\cdot\mathrm{d}\boldsymbol{S} = -\oint \mathsf{G}\cdot\mathrm{d}\boldsymbol{S}. \tag{5.85}$$

Therefore T represents the field momentum flux flowing into the volume through surface. Or G, the field momentum flux tensor, represents the field momentum flux flowing out of the volume through surface.

In a Cartesian coordinate system, let the normal direction of $\mathrm{d}\boldsymbol{S}$ to be $\boldsymbol{n} = (n_1, n_2, n_3)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{p}_m + \boldsymbol{p}_f\right)_i = \oint T_{ij}n_j\mathrm{d}S \tag{5.86}$$

Therefore, $T_{ij}n_j$ is the ith component of the momentum per unit area per unit time into the volume. On the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{p}_m + \boldsymbol{p}_f\right) = \text{force on the field and particles}, \tag{5.87}$$

$T_{ij}n_j$ is the ith component of the force per unit area through S. If the field is static, then $\boldsymbol{p}_f$ is constant,

$$\frac{\mathrm{d}\boldsymbol{p}_m}{\mathrm{d}t} = \oint \mathsf{T}\cdot\mathrm{d}S \tag{5.88}$$

$$\text{or}\quad \frac{\mathrm{d}(\boldsymbol{p}_m)_i}{\mathrm{d}t} = \oint T_{ij}n_j\mathrm{d}S \tag{5.89}$$

On the other hand, if $\boldsymbol{p}_m$ is constant, then

$$\frac{\mathrm{d}\boldsymbol{p}_f}{\mathrm{d}t} = \int \frac{\partial \boldsymbol{g}}{\partial t}\mathrm{d}V = \int \nabla\cdot \mathsf{T}\mathrm{d}V \tag{5.90}$$

or

$$\frac{\partial \boldsymbol{g}}{\partial t} = \nabla\cdot \mathsf{T} = -\nabla\cdot \mathsf{G}, \tag{5.91}$$

or

$$\frac{\partial \boldsymbol{g}}{\partial t} + \nabla\cdot \mathsf{G} = 0, \tag{5.92}$$

further proving that the field momentum flux is G.

6
General Electrostatics

6.1 General considerations

The Maxwell equations are,

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{6.1}$$

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}, \tag{6.2}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \mu_0 \boldsymbol{j}, \tag{6.3}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{6.4}$$

In this chapter, we consider static EM fields; i.e.,

$$\frac{\partial}{\partial t} = 0. \tag{6.5}$$

The corresponding static EM field equations are

$$\nabla \times \boldsymbol{E} = 0 \tag{6.6}$$

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}, \tag{6.7}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}, \tag{6.8}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{6.9}$$

Notes that

$$\boldsymbol{E} = -\nabla \phi - \frac{\partial \boldsymbol{A}}{\partial t} = -\nabla \phi,$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}.$$

That is, E is completely determined using ϕ, and $\boldsymbol{E}$ and $\boldsymbol{B}$ are decoupled. So we separate static electromagnetic fields into two parts: one about the electrostatic field and the other about magnetostatic field. In this chapter, we discuss electrostatics.

6.2 Electrostatic fields

In case of a electrostatic field, $\boldsymbol{E} = -\nabla \phi$. Using ϕ can significantly simplify analysis, since it's a scalar. One exception is if the problem possesses some symmetry, then you can apply Gauss's theorem to obtain $\boldsymbol{E}$.

The Coulomb's law expressed using ϕ is

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \Rightarrow \nabla^2 \phi = -\frac{\rho}{\epsilon_0}. \tag{6.10}$$

If $\rho = 0$ (e.g., vacuum), the equation becomes

$$\nabla^2 \phi = 0. \quad \text{(The Laplace equation.)} \tag{6.11}$$

$\boldsymbol{E}$ and ϕ apparently satisfy the principle of superposition.

We first calculate the electrostatic field of a point charge. The ES field of a distribution of charges can be obtained from the principle of superposition. This is an example of applying Gauss's law to a problem with symmetry. For a point charge, because of symmetry, $\boldsymbol{E} = E(r)\boldsymbol{e}_r$, where r is the distance to the charge, and $\boldsymbol{e}_r = \boldsymbol{r}/r$. From the Coulomb's law,

$$\int \nabla \cdot \boldsymbol{E} \mathrm{d}V = \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \int \nabla \cdot \boldsymbol{E} \mathrm{d}V = \int \frac{\rho}{\epsilon_0} \mathrm{d}V = \frac{q}{\epsilon_0}, \tag{6.12}$$

where we have used $\rho = q\delta(\boldsymbol{r})$. Choose the volume to be a sphere centering at the charge,

$$\oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = 4\pi r^2 E = \frac{q}{\epsilon_0}. \tag{6.13}$$

Therefore, the ES field of a point charge is

$$\boxed{\boldsymbol{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \boldsymbol{e}_r.}$$

The potential of the field is clearly

$$\boxed{\phi = \frac{1}{4\pi\epsilon_0} \frac{q}{r},}$$

since

$$-\nabla \phi = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \nabla r = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \boldsymbol{e}_r = \boldsymbol{E}. \tag{6.14}$$

Using this expression of ϕ, we can obtain an useful identify for $\delta(\boldsymbol{r})$. Putting $\phi = q/4\pi\epsilon_0 r$ into $\nabla^2 \phi = -\rho/\epsilon_0$, for a point charge at $\boldsymbol{r} = 0$, $\rho = q\delta(\boldsymbol{r})$, therefore,

$$\nabla^2 \left(\frac{q}{4\pi\epsilon_0 r} \right) = -\frac{q\delta(r)}{\epsilon_0} \tag{6.15}$$

or

$$\nabla^2 \left(\frac{1}{r} \right) = -4\pi\delta(r) \tag{6.16}$$

6.3 *Electrostatic field multipole moments*

One electrostatic problem we frequently encountered is to calculate the potential or the $\boldsymbol{E}$ field of a system of charges. Normally the calculation is tedious and is best done using a computer. However, in lots of cases, we are only interested in the field far far away from the charges. For example, we might be interested in the field produced by a group of molecules at a distance much larger compared to the spatial scale of system molecules. In this case, we can use Taylor expansion to obtain leading fields of the charge system. This technique and the corresponding fields are introduced in this section.

Let's introduce a coordinate system where the origin is within the system of charges, the radius vector of charge a is $\boldsymbol{x}'_a$. The ES potential at an arbitrary point $\boldsymbol{x}$ is

$$\phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\sum_a \frac{q_a}{|\boldsymbol{x}-\boldsymbol{x}'_a|}. \tag{6.17}$$

We now consider field at large distances $|\boldsymbol{x}| \gg |\boldsymbol{x}'_a|$; i.e., at distances large compared to the dimensions of the system. The method is to expand ϕ using the small parameter $|\boldsymbol{x}'_a|/r$, with $r \equiv |\boldsymbol{x}|$, and find out the dominant terms. We will also drop script "a" with understanding that the summation is over all charges.

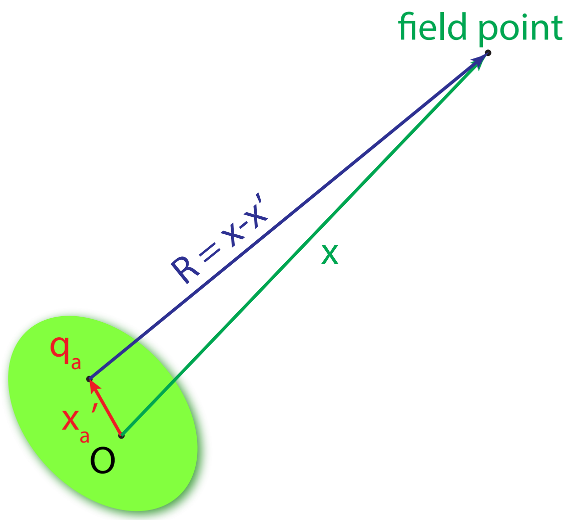


Illustration of the coordinates used in this chapter.

Using $|\boldsymbol{x}'| \ll |\boldsymbol{x}|$, we can Taylor expand a function $f(\boldsymbol{x}-\boldsymbol{x}')$ by

$$f(\boldsymbol{x}-\boldsymbol{x}') = f(\boldsymbol{x}) - \boldsymbol{x}'\cdot\nabla f(\boldsymbol{x}) + \frac{1}{2}x'_i x'_j \frac{\partial^2}{\partial x_i \partial x_j} f(\boldsymbol{x}) + \cdots \tag{6.18}$$

In our case, $f(\boldsymbol{x}-\boldsymbol{x}') = \phi(\boldsymbol{x}-\boldsymbol{x}')$, and

$$\phi = \frac{1}{4\pi\epsilon_0}\sum_a \frac{q}{|\boldsymbol{x}-\boldsymbol{x}'|} = \phi^{(0)} + \phi^{(1)} + \phi^{(2)} + \cdots, \tag{6.19}$$

where $\phi^{(i)}$ denotes the ith order term. If you compare, e.g., $\phi^{(1)}$ and $\phi^{(0)}$,

$$\frac{\phi^{(1)}}{\phi^{(0)}} \sim \frac{\boldsymbol{x}'\cdot\nabla\phi}{\phi} \sim \frac{|\boldsymbol{x}'|\phi/|\boldsymbol{x}|}{\phi} \sim \frac{|\boldsymbol{x}'|}{|\boldsymbol{x}|} \ll 1. \tag{6.20}$$

If we denote $\epsilon \equiv |x'|/|x|$, then $\phi^{(1)}/\phi^{(0)} \sim \mathcal{O}(\epsilon)$. Similarly, you can show that $\phi^{(2)}/\phi^{(0)} \sim \mathcal{O}(\epsilon^2)$.

Applying Equation (6.18) to Equation (6.19), you can immediately arrive at the conclusion that

$$\phi^{(0)} = \phi(x) = \frac{1}{4\pi\epsilon_0}\sum_a \frac{q}{r}, \quad \text{with } r \equiv |x|, \tag{6.21}$$

$$\phi^{(1)} = -\frac{1}{4\pi\epsilon_0}\sum_a q x'_a \cdot \nabla\frac{1}{r}, \tag{6.22}$$

$$\phi^{(2)} = \frac{1}{8\pi\epsilon_0}\sum_a q x'_i x'_j \frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right). \tag{6.23}$$

We'll now treat these terms one by one.

The 0^{th} order term $\phi^{(0)}$, normally called the *monopole* term, is

$$\boxed{\phi^{(0)} = \phi(x) = \frac{1}{4\pi\epsilon_0}\frac{\sum_i q_i}{r}.} \tag{6.24}$$

This is equivalent to ignoring the internal distribution of charges and treating them as a single point charge with $Q = \sum_a q$. Apparently $\phi^{(0)}$ dominates unless $Q = 0$, which is common, e.g., when calculating fields produced by water molecules. The 0^{th} term is also consistent with our intuition: if we are far far away from a distribution of charges, we can approximate the charges by a point charge.

If $Q = 0$ ($\phi^{(0)} = 0$), we then need higher order terms $\phi^{(1)}$, or even $\phi^{(2)}$ if $\phi^{(1)} = 0$, to better approximate ϕ. Let's first discuss $\phi^{(1)}$. Introducing the electrostatic dipole moment $\boldsymbol{p}$,

$$\boxed{\boldsymbol{p} = \sum_a q\boldsymbol{x}',} \tag{6.25}$$

we can write $\phi^{(1)}$, the electrostatic dipole potential, as

$$\boxed{\phi^{(1)} = -\frac{1}{4\pi\epsilon_0}\sum_a q\boldsymbol{x}' \cdot \nabla\frac{1}{r} = -\frac{\boldsymbol{p}}{4\pi\epsilon_0}\cdot\nabla\frac{1}{r} = \frac{\boldsymbol{p}\cdot\boldsymbol{x}}{4\pi\epsilon_0 r^3}.} \tag{6.26}$$

This is the potential due to the dipole moment of the charges.

The field $\boldsymbol{E} = -\nabla\phi$. Note here $\nabla = \partial/\partial\boldsymbol{x}$; i.e., with respect to the field coordinate, therefore the dipole moment $\boldsymbol{p}$ is treated as a constant when we perform ∇. Performing the gradient operation,

$$\boldsymbol{E} = -\nabla\frac{\boldsymbol{p}\cdot\boldsymbol{x}}{4\pi\epsilon_0 r^3} = -\frac{1}{4\pi\epsilon_0 r^3}\nabla(\boldsymbol{p}\cdot\boldsymbol{x}) - \frac{1}{4\pi\epsilon_0}(\boldsymbol{p}\cdot\boldsymbol{x})\nabla\frac{1}{r^3} \tag{6.27}$$

$$= -\frac{1}{4\pi\epsilon_0 r^3}\boldsymbol{p}\cdot\nabla\boldsymbol{x} - \frac{\boldsymbol{p}\cdot\boldsymbol{x}}{4\pi\epsilon_0}\frac{-3}{r^4}\nabla r \tag{6.28}$$

$$= -\frac{\boldsymbol{p}}{4\pi\epsilon_0 r^3} + \frac{3\boldsymbol{p}\cdot\boldsymbol{x}}{4\pi\epsilon_0 r^4}\frac{\boldsymbol{x}}{r} \tag{6.29}$$

or finally with $\boldsymbol{n} = \boldsymbol{x}/r$,

$$\boxed{\boldsymbol{E}^{(1)} = \frac{3(\boldsymbol{n}\cdot\boldsymbol{p})\boldsymbol{n} - \boldsymbol{p}}{4\pi\epsilon_0 r^3}.} \tag{6.30}$$

Note that $\boldsymbol{E}^{(1)} \sim r^{-3}$. You can also directly Taylor expand $\boldsymbol{E}$ to obtain $\boldsymbol{E}^{(0)}, \boldsymbol{E}^{(1)}, \cdots$.

The components of the electric field $\boldsymbol{E}$ can be obtained easily from Equation (6.30). Assuming $\boldsymbol{p} = p\boldsymbol{e}_z$, then, with $\theta = \langle \boldsymbol{n}, \boldsymbol{e}_z \rangle$, $\boldsymbol{e}_\perp \perp \boldsymbol{e}_z$.

$$E_z = \frac{3(\boldsymbol{n}\cdot\boldsymbol{p})\boldsymbol{n} - \boldsymbol{p}}{4\pi\epsilon_0 r^3}\cdot \boldsymbol{e}_z = p\frac{3\cos^2\theta - 1}{4\pi\epsilon_0 r^3}, \tag{6.31}$$

$$E_\perp = \frac{3(\boldsymbol{n}\cdot\boldsymbol{p})\boldsymbol{n} - \boldsymbol{p}}{4\pi\epsilon_0 r^3}\cdot \boldsymbol{e}_\perp = p\frac{3\cos\theta\sin\theta}{4\pi\epsilon_0 r^3}. \tag{6.32}$$

Or if using spherical coordinates,

$$E_r = \boldsymbol{E}\cdot\boldsymbol{n} = \frac{3(\boldsymbol{n}\cdot\boldsymbol{p})(\boldsymbol{n}\cdot\boldsymbol{n}) - \boldsymbol{p}\cdot\boldsymbol{n}}{4\pi\epsilon_0 r^3} = p\frac{2\cos\theta}{4\pi\epsilon_0 r^3} = p\frac{\cos\theta}{2\pi\epsilon_0 r^3}, \tag{6.33}$$

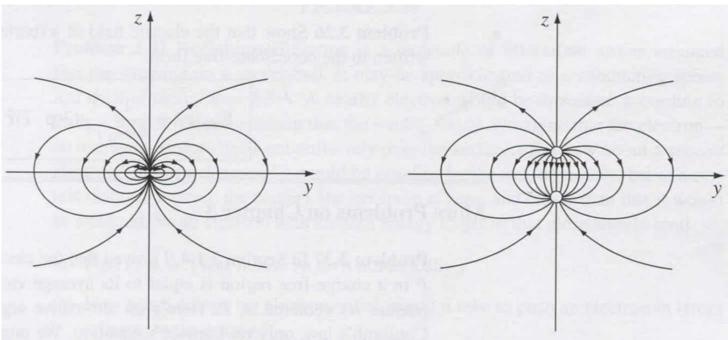
$$E_\theta = \boldsymbol{E}\cdot\boldsymbol{e}_\theta = \frac{3(\boldsymbol{n}\cdot\boldsymbol{p})(\boldsymbol{n}\cdot\boldsymbol{e}_\theta) - \boldsymbol{p}\cdot\boldsymbol{e}_\theta}{4\pi\epsilon_0 r^3} = p\frac{\sin\theta}{4\pi\epsilon_0 r^3}. \tag{6.34}$$

Two useful facts about the dipole moment are worth mentioning. First, if the total charge $\sum_a q = 0$, then $\boldsymbol{p}$ does not depends on the choice of the origin of coordinates. Consider two systems, $\boldsymbol{x}'' = \boldsymbol{x}' + \boldsymbol{a}$, then

$$\boldsymbol{p}' = \sum q\boldsymbol{x}'' = \sum q\boldsymbol{x}' + \sum q\boldsymbol{a} = \sum q\boldsymbol{x}' = \boldsymbol{p}. \tag{6.35}$$

On the other hand, if $Q \equiv \sum_a q \neq 0$, then $\boldsymbol{p}' = \boldsymbol{p} + Q\boldsymbol{a}$. Second, a simple and frequently used dipole moment is for two point charges $|q|$ at $\boldsymbol{x}'_+$ and $-|q|$ at $\boldsymbol{x}'_-$.

$$\boldsymbol{p} = |q|\boldsymbol{x}'_+ - |q|\boldsymbol{x}'_- = |q|\boldsymbol{l}, \quad \text{with } \boldsymbol{l} = \boldsymbol{x}'_+ - \boldsymbol{x}'_-. \tag{6.36}$$



(a) Field of a "pure" dipole (b) Field of a "physical" dipole

Let's now continue to the second order term $\phi^{(2)}$; this term will dominate if both Q and $\boldsymbol{p}$ equal 0. Rewrite Equation (6.23) here,

$$\phi^{(2)} = \frac{1}{8\pi\epsilon_0}\sum_a q x'_i x'_j \frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{r}. \tag{6.37}$$

Define temporarily the *quadrupole moment* of the system

$$\mathsf{D} = \sum_a 3q\boldsymbol{x}'\boldsymbol{x}', \quad \text{or} \quad D_{ij} = \sum_a 3q x'_i x'_j, \tag{6.38}$$

then

$$\phi^{(2)} = \frac{D_{ij}}{24\pi\epsilon_0}\frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{r}. \tag{6.39}$$

is called the *quadrupole* potential. There are 6 components of D,

$$D_{11}, D_{22}, D_{33}, D_{12} = D_{21}, D_{13} = D_{31}, D_{23} = D_{32}. \tag{6.40}$$

But only 5 of them are independent. Let's prove this fact. Since $\boldsymbol{x}$ is located outside the system of charges,

$$\nabla^2 \frac{1}{r} = 0 \quad \text{(Recall it's } -4\pi\delta(r) \text{ in general.)} \tag{6.41}$$

We can rewrite the above equation as

$$\nabla^2 \frac{1}{r} = \frac{\partial^2}{\partial x_i \partial x_i} \frac{1}{r} = \delta_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{r} = 0. \tag{6.42}$$

Using Equation (6.42), the potential $\phi^{(2)}$ can also be written as

$$\phi^{(2)} = \frac{1}{24\pi\epsilon_0} \left(D_{ij} - r'^2 \delta_{ij} \right) \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{r}. \tag{6.43}$$

where $r' = |\boldsymbol{x}'|$. Thus we formally define the *quadrupole moment* as

$$\boxed{D_{ij} = \sum_a q(3x'_i x'_j - r'^2 \delta_{ij}),} \tag{6.44}$$

The trace of the $D_{ij} = D_{11} + D_{22} + D_{33} = 0$, indicating that there are only 5 independent terms in D_{ij}. From now on, we'll use this D as the *quadrupole moment*.

6.4 Electrostatic field energy

In case of a electrostatic field, the field energy is

$$U = \frac{\epsilon_0}{2} \int E^2 \mathrm{d}V, \tag{6.45}$$

the integral is taken over all space. Using $E = -\nabla\phi$ changes equation (6.45) to

$$U = -\frac{\epsilon_0}{2} \int \boldsymbol{E} \cdot \nabla\phi \mathrm{d}V = \frac{\epsilon_0}{2} \int \nabla \cdot (\boldsymbol{E}\phi) \mathrm{d}V + \frac{\epsilon_0}{2} \int \phi \nabla \cdot \boldsymbol{E} \mathrm{d}V. \tag{6.46}$$

Using Gauss's theorem,

$$\frac{\epsilon_0}{2} \int \nabla \cdot (\boldsymbol{E}\phi) \mathrm{d}V = \frac{\epsilon_0}{2} \oint \phi \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = 0, \tag{6.47}$$

because $\phi\boldsymbol{E} \sim R^{-3}$ and $S \sim R^2$, and the integration surface is a sphere with radius R, therefore this integration varies as R^{-1}. As $R \to \infty$, the integration equals 0. The electrostatic field energy is therefore

$$U = \frac{\epsilon_0}{2} \int \phi \nabla \cdot \boldsymbol{E} \mathrm{d}V = \frac{1}{2} \int \rho\phi \mathrm{d}V, \tag{6.48}$$

where we have applied $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$. For a system of point charges, $\rho = \sum q_i \delta(\boldsymbol{r} - \boldsymbol{r}_a)$,

$$U = \frac{1}{2} \int \rho\phi \mathrm{d}V = \frac{1}{2} \sum_a q_a \phi_a, \tag{6.49}$$

where $\phi_a \equiv \phi(\boldsymbol{r}_a)$ is the potential of all charges at $\boldsymbol{r}_a$.

Consider now a single point charge, $U = q\phi/2$. However, $\phi = q/r$, and in this case, $r = 0$ leads to $U = \infty$! If the energy U is infinity, then $m = U/c^2$ must be infinity. This absurd result suggest that the validity of EM theory must be restricted to some limits.

The problem results from that the charge is point-like. If we consider an electron with radius R_0, then $U = e^2/8\pi\epsilon_0 R_0$. Requiring that the self-potential energy is on the same order as mc^2, we can define

$$R_0 \sim \frac{e^2}{4\pi\epsilon_0 mc^2}, \tag{6.50}$$

which is the classical electron radius. The classical electromagnetic theory is problematic for $r < R_0$.

Let's now return to a system of charges,

$$\phi_a = \frac{1}{4\pi\epsilon_0}\sum_b \frac{q_b}{r_{ab}}, \tag{6.51}$$

where $r_{ab} = |\boldsymbol{x}'_a - \boldsymbol{x}'_b|$, the distance between the two charges. This ϕ contains two parts, the infinity self-energy, The energy of interaction of the charges. From what we just discussed in the previous paragraph, we will, from now on, only consider part 2, the energy of interaction.

Since we only consider the energy of interaction,

$$U^* = \frac{1}{2}\sum_a q_a \phi_a^*, \tag{6.52}$$

with

$$\phi_a^* = \frac{1}{4\pi\epsilon_0}\sum_{b\neq a} \frac{q_b}{r_{ab}}, \tag{6.53}$$

the potential produced by all other charges at charge a. The electrostatic field energy of interaction is

$$U^* = \frac{1}{2}\sum_a q_a \sum_{b\neq a}\frac{q_b}{4\pi\epsilon_0 r_{ab}} = \frac{1}{8\pi\epsilon_0}\sum_{a\neq b}\frac{q_a q_b}{r_{ab}}. \tag{6.54}$$

For example, U^* for two charges is

$$U^* = \frac{1}{4\pi\epsilon_0}\left(\underbrace{\frac{1}{2}\frac{q_1 q_2}{r_{12}}}_{i=1,j=2} + \underbrace{\frac{1}{2}\frac{q_2 q_1}{r_{21}}}_{i=2,j=1}\right) = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}}. \tag{6.55}$$

6.5 A system of charges in an external field

We calculate the potential energy of a system of charges in an external field. The potential of the external field is given by $\phi(\boldsymbol{r})$, and

$$U = \sum_a q_a \phi(\boldsymbol{x}'_a) = \sum_a q\phi(\boldsymbol{x}'). \text{ dropping the "}a\text{" subscript} \tag{6.56}$$

Again the origin of the coordinate is located somewhere in the system of charges. The radius vector of charge a is $\boldsymbol{x}'_a$. Assume ϕ varies slowly over the region of charges, then we can expand $\phi(\boldsymbol{x}')$ by

$$\phi(\boldsymbol{x}') = \phi(0) + \boldsymbol{x}' \cdot \nabla\phi + \frac{1}{2}x'_i x'_j \frac{\partial^2\phi}{\partial x_i \partial x_j} + \cdots . \tag{6.57}$$

To see why we required that ϕ varies slowly over the regions of charges, we compare the first and second terms on the RHS.

$$\frac{\boldsymbol{x}' \cdot \nabla\phi}{\phi(0)} \sim \frac{|\boldsymbol{x}'|\phi/L}{\phi} \sim \frac{|\boldsymbol{x}'|}{L}, \tag{6.58}$$

where L is the characteristic scale of ϕ. Therefore, only when $|\boldsymbol{x}'|/L \ll 1$ can we say that the second terms is an order smaller than the first term.

Like the multipole expansion used in deriving fields of a system of charges, we have

$$U = U^{(0)} + U^{(1)} + U^{(2)} + \cdots , \tag{6.59}$$

and

$$U^{(0)} = \sum_a q\phi(0), \tag{6.60}$$

$$U^{(1)} = \sum_a q\boldsymbol{x}' \cdot (\nabla\phi)_0 = \boldsymbol{p} \cdot \nabla\phi, \tag{6.61}$$

$$U^{(2)} = \sum_a q\frac{1}{2}x'_i x'_j \frac{\partial^2\phi}{\partial x_i \partial x_j}. \tag{6.62}$$

Let's again introduce these terms one by one.

Since $U^{(0)} = \sum_a q\phi(0)$, this is just like the monopole term we introduced in previous sections. This is the same as ignoring the internal distributions of charges treat them as a single point charge with $Q = \sum_a q$.

The first order term, or the dipole moment term, $U^{(1)} = \boldsymbol{p} \cdot \nabla\phi$, expressed using $\boldsymbol{E}$, is $U^{(1)} = -\boldsymbol{p} \cdot \boldsymbol{E}(0)$. One important conclusion is the force felt by the charges, which is

$$\boldsymbol{f} = -\nabla(U^{(0)} + U^{(1)} + \cdots) \approx \sum_a q\boldsymbol{E}(0) + \nabla(\boldsymbol{p} \cdot \boldsymbol{E}) \tag{6.63}$$

$$= Q\boldsymbol{E}(0) + (\boldsymbol{p} \cdot \nabla)\boldsymbol{E}. \tag{6.64}$$

If $Q = 0$, then

$$\boldsymbol{f} = (\boldsymbol{p} \cdot \nabla)\boldsymbol{E}. \tag{6.65}$$

In deriving the last equation, we have used [1]

$$\nabla(\boldsymbol{p} \cdot \boldsymbol{E}) = (\boldsymbol{p} \cdot \nabla)\boldsymbol{E} + \boldsymbol{p} \times (\nabla \times \boldsymbol{E}) = (\boldsymbol{p} \cdot \nabla)\boldsymbol{E}, \tag{6.66}$$

since for electrostatic fields, $\nabla \times \boldsymbol{E} = 0$.

The second order term $U^{(2)}$ is

$$U^{(2)} = \sum_a q\frac{1}{2}x'_i x'_j \frac{\partial^2\phi}{\partial x_i \partial x_j}. \tag{6.67}$$

[1] Note that $\nabla \cdot \boldsymbol{p} = 0$ and $\nabla \times \boldsymbol{p} = 0$ since $\boldsymbol{p}$ is defined in particle coordinates $\boldsymbol{x}'$, while $\nabla \equiv \partial/\partial \boldsymbol{x}$.

Noting that for an external field [2]

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x_i \partial x_i} = \delta_{ij}\frac{\partial^2\phi}{\partial x_i \partial x_j} = 0. \tag{6.68}$$

[2] See Gauss's law, $\nabla^2\phi = -\rho/\epsilon$. External field here means that the source charge density $\rho = 0$ at the location of interest ($\boldsymbol{x}'$ here).

Therefore, we rewrite $U^{(2)}$ as

$$U^{(2)} = \frac{1}{6}\sum_a q\left(3x_i'x_j' - r'^2\delta_{ij}\right)\frac{\partial^2\phi}{\partial x_i \partial x_j} = \frac{D_{ij}}{6}\frac{\partial^2\phi}{\partial x_i \partial x_j} \tag{6.69}$$

$$= \frac{1}{6}\mathsf{D} : \nabla\nabla\phi = -\frac{1}{6}\mathsf{D} : \nabla\boldsymbol{E}. \tag{6.70}$$

6.6 Polarization in Dielectrics

We have two kinds of objects: *conductors* and *insulators* (or *dielectrics*). In conductors, there are charges, most likely electrons, that are free to move through the material. On the other hand, *dielectrics* do not have free charges. Instead, all electrons are attached to some atoms or molecules–they can not move freely; the only thing they can do is to move a bit within the atom or molecules. Their cumulative effects, however, are important; these effects can produce new fields inside the dielectrics. In this section, we'll what we learned in previous sections to study the electric fields in matter.

6.6.1 Induced dipoles

Suppose we have an neutral atom in an external field $\boldsymbol{E}$. The atom as a whole does not feel a force, since it's neutral [3]. However, the positive core, the nucleus, and the negative electrons feel the $\boldsymbol{E}$ force. If the $\boldsymbol{E}$ field is very strong, the atom can be pulled apart, or ionized. However, if the $\boldsymbol{E}$ field is not that strong, an equilibrium is soon established. The electrons and the core feel the $\boldsymbol{E}$ force in opposite directions, making the atom *polarized*, creating a tiny dipole $\boldsymbol{p}$, which points in the same direction as $\boldsymbol{E}$. It turns out that $\boldsymbol{p}$ is approximately proportional to the $\boldsymbol{E}$ field, or

[3] the monopole term is 0

$$\boldsymbol{p} = \alpha\boldsymbol{E} \tag{6.71}$$

This constant α is called *atomic polarizability*. Its value depends on the atom in question.

Here, let's adopt a simple model of atoms, consisting of a positively charged core q and a negatively charged electron cloud with radius a and total charge $-q$ surrounding the core. Without an external $\boldsymbol{E}$ field, the nucleus is located at the center of the cloud. If there is an electric field, however, the nucleus will move slightly in the direction of $\boldsymbol{E}$; let's say it's now d from the center of the cloud. Let's approximately assume that the electron cloud is still a sphere. The magnitude of the electric field of the electron cloud at the nucleus is

$$E_e = \frac{qd}{4\pi\epsilon_0 a^3}. \tag{6.72}$$

At equilibrium, then $E_e = E$, or

$$\frac{qd}{4\pi\epsilon_0 a^3} = E, \quad \text{or} \quad qd = 4\pi\epsilon_0 a^3 E. \tag{6.73}$$

Note that qd is just the dipole moment of the newly formed dipole, therefore

$$\boldsymbol{p} = 4\pi\epsilon_0 a^3 \boldsymbol{E}, \text{ and } \alpha = 4\pi\epsilon_0 a^3 \tag{6.74}$$

for simple atoms.

For complicated molecules, things in general get more complicated. Frequently they polarize more readily in some direction than in others. For example, for CO_2 molecules, $\boldsymbol{p} = \alpha_\perp \boldsymbol{E}_\perp + \alpha_\parallel \boldsymbol{E}_\parallel$. In general, the relationship can be expressed using a tensor A, the *polarizability tensor*, or

$$\boldsymbol{p} = \mathsf{A} \cdot \boldsymbol{E}. \tag{6.75}$$

Note that no matter what the cause of the dipole is, when there is an external field $\boldsymbol{E}$, the dipole will feel a torque $\boldsymbol{L} = \boldsymbol{p} \times \boldsymbol{E}$, which will try to align the dipole along $\boldsymbol{E}$.

7
Boundary Value Problems in Electrostatics

7.1 General boundary conditions

From the integral form of Gauss's law, on the boundary layer between region I and region II,

$$\oint \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = \int \rho \mathrm{d}V = Q = \sigma A, \tag{7.1}$$

where σ is the surface charge density, and A is the surface area. Construct a Gaussian pillbox with thickness ε, and the disk area is A, then Equation (7.4.3) gives

$$(D_\perp^{\mathrm{I}} - D_\perp^{\mathrm{II}})A = \sigma A, \tag{7.2}$$

where we've let $\varepsilon \to 0$; therefore,

$$\epsilon_{\mathrm{I}} E_\perp^{\mathrm{I}} - \epsilon_{\mathrm{II}} E_\perp^{\mathrm{II}} = \sigma \text{ or } (\epsilon_{\mathrm{I}} \boldsymbol{E}^{\mathrm{I}} - \epsilon_{\mathrm{II}} \boldsymbol{E}^{\mathrm{II}}) \cdot \boldsymbol{n} = \sigma, \tag{7.3}$$

with $\boldsymbol{n}$ the unit vector pointing from region II to region I.

The tangential components of $\boldsymbol{E}$ are continuous, as can be obtained by

$$\int \nabla \times \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = 0. \tag{7.4}$$

If we construct a thin rectangular loop around the boundary, with the width of the two sides ε and let $\varepsilon \to 0$, then from Equation (7.4)

$$E_\parallel^{\mathrm{I}} - E_\parallel^{\mathrm{II}} = 0. \tag{7.5}$$

You can combine Equation (7.5) and Equation (7.3) to get the boundary condition as

$$\epsilon_{\mathrm{I}} \boldsymbol{E}^{\mathrm{I}} - \epsilon_{\mathrm{II}} \boldsymbol{E}^{\mathrm{II}} = \sigma \boldsymbol{n}. \tag{7.6}$$

You can see that the potential is continuous across any boundary, since

$$\varphi^{\mathrm{I}} - \varphi^{\mathrm{II}} = -\int_a^b \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}, \tag{7.7}$$

as $a \to b$, RHS equals 0; therefore $\varphi^{\mathrm{I}} = \varphi^{\mathrm{II}}$. However the gradient of the potential is not zero, since

$$\boldsymbol{n} \cdot \nabla \varphi = -\frac{\partial \varphi}{\partial n} = \boldsymbol{E} \cdot \boldsymbol{n}, \tag{7.8}$$

or

$$\epsilon_{\mathrm{I}} \frac{\partial \varphi^{\mathrm{I}}}{\partial n} - \epsilon_{\mathrm{II}} \frac{\partial \varphi^{\mathrm{II}}}{\partial n} = -\sigma. \tag{7.9}$$

7.2 Boundary value problems and the first uniqueness theorem

Normally in the region we are interested in, there are no charges. Hence in this section, we will mainly consider the electrostatic problems of the type

$$\nabla^2 \varphi = 0, \tag{7.10}$$

with certain boundary conditions. Now the problem is, which kind of boundary conditions do we need to determine completely the φ inside the region?

Let's start from a 1D problem, and the equation is

$$\frac{\mathrm{d}^2 \varphi}{\mathrm{d}x^2} = 0, \quad \boldsymbol{x} \in [a, b]. \tag{7.11}$$

This equation can be easily solved to give $\varphi = Ax + B$. To determine φ, we need A and B. The following boundary condition can uniquely determine φ:

- if we know $\varphi(a)$ and $\varphi(b)$;
- if we know $\partial\varphi(a)/\partial x$ and $\varphi(b)$;
- if we know $\partial\varphi(b)/\partial x$ and $\varphi(a)$.

On the other hand, if we know $\partial\varphi(b)/\partial x$ and $\partial\varphi(a)/\partial x$, we cannot determine A and B. Or if we just know $\varphi(a)$, we cannot determine both A and B.

In 3D cases, things get more complicated. We'll now prove the following theorem, which is called the *first uniqueness theorem*:

"The solution to $\nabla^2\varphi = 0$ is uniquely determined if φ is specified on the boundary S."

There are different kind of uniqueness theorems in electrostatics. The proof of these unique theorems shares some basic format. Let me show you the format by proving the first uniqueness theorem.

Proof: Suppose there are two solutions to $\nabla^2\varphi = 0$ called φ_1 and φ_2; $\nabla^2\varphi_1 = 0$ and $\nabla^2\varphi_2 = 0$. Now to prove the first uniqueness theorem, we just need to prove that $\varphi_1 = \varphi_2$ if we specify φ on the boundary S.

Define $\varphi_3 \equiv \varphi_1 - \varphi_2$, then φ_3 also satisfies the Laplace equation $\nabla^2\varphi_3 = 0$. Since both $\varphi_1 = \varphi$ and $\varphi_2 = \varphi$ on the boundary S, $\varphi_3 = 0$ on boundary S. In short, the equation for φ_3 is

$$\nabla^2\varphi_3 = 0, \text{ and } \varphi_3 = 0 \text{ on } S. \tag{7.12}$$

You have proved as an exercise that if a function satisfies $\nabla^2 f = 0$, then f cannot have maximum or minimum inside the domain; extreme values of f can only occur on boundary. Since $\varphi_3 = 0$ on the boundary and it cannot have maximum or minimum inside S, then $\varphi_3 = 0$ everywhere. Therefore,

$$\varphi_1 = \varphi_2. \tag{7.13}$$

Proof of the first uniqueness theorem is now completed.

The first uniqueness theorem has an interesting consequence. It's easy to show that if φ satisfies

$$\nabla^2 \varphi = -\frac{\rho}{\epsilon_i},^{1} \tag{7.14}$$

charge density ρ is specified throughout the domain, and φ is given on the boundary S, then the solution of φ to Equation (7.14) is uniquely determined. The proof is almost identical to the proof of the first uniqueness theorem given above.

[1] the footnote i represents the dielectric constant of the i^{th} isotropic area

The uniqueness theorem allows you to solve boundary value problems in electrostatics by guessing. For a problem satisfying the conditions of these uniqueness theorems, if you can guess a solution φ that satisfies the Poisson/Laplace equation, and at the same time satisfies the boundary condition, then the solution you guessed *is* the solution to the equation. We'll see the power of the uniqueness theorem in the method of images latter in the course.

7.2.1 Conductors and the second uniqueness theorems

Sometimes we encounter electrostatic problems where we do not know the potential at the boundary, but we do know the charges on various conductors. For example, we put Q_a on conductor a, and Q_b on conductor b. As soon as we put charges on the conductors, they will redistribute themselves in a way we do not control. Let's say there are some specified charge density ρ in the region between conductors. The *second uniqueness theorem* then says:

"In a region V surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the total charge on each conductor is given. The region as a whole can be bounded by another conductor or else unbounded."

Proof: The proof is similar to the one above, but slightly more complicated. Suppose there are two solutions $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ satisfying Gauss's law in space between conductors,

$$\nabla \cdot \boldsymbol{E}_1 = \frac{\rho}{\epsilon} \text{ and } \nabla \cdot \boldsymbol{E}_2 = \frac{\rho}{\epsilon}, \tag{7.15}$$

and they obey Gauss's law in integral form for a Gaussian surface enclosing each conductor:

$$\oint_{i^{\text{th}} \text{ conductor}} \boldsymbol{E}_1 \cdot \mathrm{d}\boldsymbol{s} = \frac{Q_i}{\epsilon} \text{ and } \oint_{i^{\text{th}} \text{ conductor}} \boldsymbol{E}_2 \cdot \mathrm{d}\boldsymbol{s} = \frac{Q_i}{\epsilon}. \tag{7.16}$$

Likewise, for the outer boundary,

$$\oint_{\text{outer boundary}} \boldsymbol{E}_1 \cdot \mathrm{d}\boldsymbol{s} = \frac{Q_{\text{tot}}}{\epsilon} \text{ and } \oint_{\text{outer boundary}} \boldsymbol{E}_2 \cdot \mathrm{d}\boldsymbol{s} = \frac{Q_{\text{tot}}}{\epsilon}. \tag{7.17}$$

As before, we define $\boldsymbol{E}_3 = \boldsymbol{E}_1 - \boldsymbol{E}_2$. To prove the second uniqueness theorem, all we need to do is to prove $\boldsymbol{E}_3 = 0$.

Apparently $\boldsymbol{E}_3$ satisfies $\nabla\cdot\boldsymbol{E}_3 = 0$ in the region between conductors and

$$\oint \boldsymbol{E}_3 \cdot \mathrm{d}\boldsymbol{s} = 0 \tag{7.18}$$

over each boundary surface.

We know that each conductor is an equipotential, and therefore φ_1 and φ_2 and φ_3 are all constants. Here is the trick we need to use

$$\nabla\cdot(\varphi_3\boldsymbol{E}_3) = \varphi_3\nabla\cdot\boldsymbol{E}_3 + \boldsymbol{E}_3\cdot\nabla\varphi_3 = -E_3^2, \tag{7.19}$$

where we have used $\nabla\cdot\boldsymbol{E}_3 = 0$ and $\nabla\varphi_3 = -\boldsymbol{E}_3$. Integrating over the whole domain, we have

$$\int \nabla\cdot(\varphi_3\boldsymbol{E}_3)\mathrm{d}V = \oint_S \varphi_3\boldsymbol{E}_3\cdot\mathrm{d}\boldsymbol{s} = \varphi_3\oint_S \boldsymbol{E}_3\cdot\mathrm{d}\boldsymbol{s} = 0, \tag{7.20}$$

here we have used that φ_3 is a constant over each surface. From Equation (7.19), however, Equation (7.20) can also be written as

$$\int \nabla\cdot(\varphi_3\boldsymbol{E}_3)\mathrm{d}V = -\int E_3^2\,\mathrm{d}V \leq 0. \tag{7.21}$$

Combining Equations (7.21) and (7.20), we see that

$$\int E_3^2\,\mathrm{d}V = 0, \tag{7.22}$$

or $E_3 = 0$. Therefore $\boldsymbol{E}_1 = \boldsymbol{E}_2$.

7.3 *The method of images*

7.3.1 *The classic image problem*

The method of images doesn't apply to very general situations, but it is a neat method when it works. The classic image problem is this:

A point charge q is hold at a distance d above an infinite grounded conducting plane. What's the potential above the plane? For ease of description, let's establish a coordinate system, and the charge is located at $(0, 0, d)$ on the x-y plane.

Mathematically, we want to solve the Poisson equation in region $z > 0$ with q at $(0, 0, d)$, subjecting to the the following boundary condition,

- $\varphi = 0$ when $z = 0$ (grounded conducting plane).
- $\varphi = 0$ when $z \to \infty$.

Therefore the first uniqueness theorem applies: if we can guess a solution satisfying the boundary condition, then it is *exactly* the solution we want.

Here is the trick to solve this problem: instead of solving the original problem directly, we solve a different problem. The new problem has two charges q and $-q$ at $(0,0,d)$ and $(0,0,-d)$, respectively. Then the total potential at an arbitrary point (x,y,z) is

$$\tilde{\varphi} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right). \quad (7.23)$$

You can easily check that

- $\tilde{\varphi} = 0$ when $z = 0$ (grounded conducting plane).
- $\tilde{\varphi} = 0$ when $z \to \infty$.

From the first uniqueness theorem, $\varphi = \tilde{\varphi}$ is the solution we want to the original problem.

Note that this method is not limited to a single point charge; any stationary charge distribution near a grounded conducting plane can be treated by introducing its mirror image; hence the name *method of images*. You see the crucial role the uniqueness theorem play in the method of images.

7.3.2 *Induced surface charge*

The surface charge induced can be calculated easily. The field inside the conductor is 0; therefore,

$$\sigma = -\epsilon_0\frac{\partial\varphi}{\partial n} = -\epsilon_0\frac{\partial\varphi}{\partial z}. \quad (7.24)$$

At $z = 0$ and using Equation (7.23),

$$\sigma = -\frac{qd}{2\pi}\frac{1}{(x^2+y^2+d^2)^{3/2}}. \quad (7.25)$$

The total induced charge can be obtained by integrating over x and y:

$$Q = \int \sigma \mathrm{d}x\mathrm{d}y = \int \sigma r \mathrm{d}r\mathrm{d}\phi = -q. \quad (7.26)$$

Left as an exercise, you can also calculated the force felt by q due to the induced charge.

7.3.3 *Other image problems*

Suppose a point charge q is situated at a distance a from the center of a grounded conducting sphere of radius R. We can use the method of images to find the potential outside the sphere.

The solution using method of images is to solve a different problem: a point charge $q' = -(R/a)q$ is placed a distance

$$b = R^2/a \quad (7.27)$$

to the right of the center of the sphere. Now these two point charges will produce a potential

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}(\frac{q}{R_q} + \frac{q'}{R'_q}), \quad (7.28)$$

where R_q is the distance to q and R'_q is the distance to q'. You can see that $\varphi(\boldsymbol{r}) = 0$ at all points on the sphere. Therefore $\varphi(\boldsymbol{r})$ is the solution to the original problem.

Again, the method of images is very powerful and simple when it works. But most of the time, finding the "image" is a mission impossible.

7.4 *Separation of variables*

Since you're very familiar with the method of separation of variables. I'll just show a few examples of using the method to solve electrostatic problems.

7.4.1 *Cartesian coordinates*

First, let's solve an electrostatic problem in 3D using Cartesian coordinates.

Example: An infinitely long rectangular metal pipe with sides a and b is grounded, but one end, $x = 0$, is maintained at a specified potential $\varphi_0(y, z)$. Find the potential inside the pipe.

Solution: The equation we need to solve is

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0, \tag{7.29}$$

with the boundary conditions

$$\varphi = 0 \text{ at } y = 0, y = a, z = 0, z = b; \tag{7.30}$$

$$\varphi = 0 \text{ at } x \to \infty; \tag{7.31}$$

$$\varphi = \varphi_0(y, z) \text{ at } x = 0. \tag{7.32}$$

Using separation of variables, we let $\varphi(x, y, z) = X(x)Y(y)Z(z)$. Putting this into Equation (7.29), we have

$$\frac{1}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \frac{1}{Y}\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} + \frac{1}{z}\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} = 0. \tag{7.33}$$

Therefore,

$$\frac{1}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = C_1, \tag{7.34}$$

$$\frac{1}{Y}\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = C_2, \tag{7.35}$$

$$\frac{1}{Z}\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} = C_3, \tag{7.36}$$

with $C_1 + C_2 + C_3 = 0$.

Considering boundary conditions given by Equations (7.30)-(7.32), we let $C_2 = k^2$, $C_3 = l^2$, and then $C_1 = -(k^2 + l^2)$. Putting these into Equations (7.34) - (7.36), we have

$$X = Ae^{\sqrt{k^2+l^2}x} + Be^{-\sqrt{k^2+l^2}x}, \tag{7.37}$$

$$Y = C\sin ky + D\cos ky, \tag{7.38}$$

$$Z = E\sin lz + F\cos lz. \tag{7.39}$$

From Equation (7.31), it's easy to see that $A = 0$. At $y = 0$, $Y = 0$, then we must have $D = 0$. At $y = a$, $Y = 0$, we then have $k = n\pi/a$. Similarly $F = 0$ and $l = m\pi/b$. The final solution of φ is

$$\varphi = \sum_{n,m} C_{n,m} e^{-\pi\sqrt{(n/a)^2+(m/b)^2}x} \sin(n\pi y/a)\sin(m\pi z/b). \tag{7.40}$$

The remaining constant C can be determined from the boundary condition $\varphi(x=0) = \varphi_0(y,z)$. From which we have

$$C_{n,m} = \frac{4}{ab}\int \varphi_0(y,z)\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right)\mathrm{d}y\mathrm{d}z. \tag{7.41}$$

In case $\varphi_0(y,z)$ is a constant φ_0, then

$$C_{n,m} = \begin{cases} 0, & \text{if } n \text{ or } m \text{ is even,} \\ 16\varphi_0/\pi^2 mn, & \text{if } n \text{ and } m \text{ are odd.} \end{cases} \tag{7.42}$$

7.4.2 *Spherical Coordinates*

Sometimes we have problems with round objects. In these cases, using spherical coordinates is a natural choice.

In spherical coordinates (r,θ,ϕ), the Laplace equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2} = 0. \tag{7.43}$$

In this class, we'll only deal with problems with azimuthal symmetry; i.e., φ is independent of ϕ. Therefore Equation (7.43) becomes

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) = 0. \tag{7.44}$$

Using the method of separation of variables, we let $\varphi = R(r)\Theta(\theta)$, then Equation (7.44) becomes

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{1}{\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) = 0. \tag{7.45}$$

The first and the second term then equals a constant of opposite signs. For convenience, in spherical coordinates, we write

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) = l(l+1)R, \tag{7.46}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) = -l(l+1)\sin\theta\,\Theta. \tag{7.47}$$

You should have learned from your mathematical methods for physics class that these two Equations give

$$R(r) = Ar^l + \frac{B}{r^{l+1}}, \tag{7.48}$$

$$\Theta(\theta) = P_l(\cos\theta). \tag{7.49}$$

Here $P_l(\cos\theta)$ are Legendre polynomials in the variable $\cos\theta$. The first few Legendre polynomials are

$$P_0(x) = 1, \tag{7.50}$$
$$P_1(x) = x, \tag{7.51}$$
$$P_2(x) = (3x^2 - 1)/2, \tag{7.52}$$
$$P_3(x) = (5x^3 - 3x)/2, \tag{7.53}$$
$$P_4(x) = (35x^4 - 30x^2 + 3)/8. \tag{7.54}$$

Notice that $P_l(x)$ is the lth-order polynomial in x; it contains only *even* powers if l is *even*, and *odd* powers if l is *odd*. The Legendre polynomials are *orthogonal* functions; i.e.,

$$\begin{aligned}\int_{-1}^{1} P_l(x)P_{l'}(x)\mathrm{d}x &= \int_0^{\pi} P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\mathrm{d}\theta \\ &= \begin{cases} 0, & \text{if } l' \neq l, \\ \dfrac{2}{2l+1}, & \text{if } l' = l. \end{cases}\end{aligned} \tag{7.55}$$

We'll of course need this property of the Legendre polynomials in using the method of separation of variables.

The final general solution of the Laplace equation (7.44) is then

$$\varphi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta). \tag{7.56}$$

Example 1: The potential $\varphi_0(\theta)$ is specified on the surface of a hollow sphere of radius R. Find the potential inside the sphere.

Solution: Because we need a finite φ at $r = 0$, $B_l = 0$; therefore,

$$\varphi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta). \tag{7.57}$$

At $r = R$,

$$\varphi(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \varphi_0(\theta). \tag{7.58}$$

Using Equation (7.55), we can find

$$A_l = \frac{2l+1}{2R^l}\int_0^{\pi}\varphi_0(\theta)P_l(\cos\theta)\sin\theta\mathrm{d}\theta. \tag{7.59}$$

Example 2: An uncharged metal sphere of radius R is placed in an otherwise uniform electric field $\boldsymbol{E} = E_0\boldsymbol{e}_z$. Find the potential in the region outside the sphere.

Let's determine the boundary conditions. First, the conducting sphere is an equipotential, we can set its potential to be a constant or simply 0. Second, since there is an external uniform $\boldsymbol{E}$ field, at large z, $\varphi = -E_0 z + C$. The problem is totally symmetric at $z = 0$ (φ does not change in the $z = 0$ plane), therefore $C = 0$. In summary, the boundary condition for this problem is

$$\varphi = 0, \quad \text{if } r = R, \tag{7.60}$$
$$\varphi = -E_0 r\cos\theta, \quad \text{if } r \gg R. \tag{7.61}$$

Condition (7.60) yields

$$A_l R^l + \frac{B_l}{R^{l+1}} = 0, \tag{7.62}$$

or

$$B_l = -A_l R^{2l+1}. \tag{7.63}$$

Equation (7.56) then becomes

$$\varphi(r,\theta) = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}} \right) P_l(\cos\theta). \tag{7.64}$$

For $r \gg R$, the second term in parentheses is negligible compared to the first term; therefore boundary condition (7.61) gives

$$\varphi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r \cos\theta. \tag{7.65}$$

If you recall that $P_1(x) = x$ or $P_1(\cos\theta) = \cos\theta$, you can immediately reach the conclusion that

$$A_1 = -E_0, \tag{7.66}$$

and all other A_l's are 0. Therefore, the final solution of φ outside the sphere is

$$\varphi(r,\theta) = -E_0 \left(r - \frac{R^3}{r^2} \right) \cos\theta. \tag{7.67}$$

Apparently, the first term is the external field, while the second term is due to the induced charge. The induced charge density can be calculated directly using

$$\sigma(\theta) = -\epsilon_0 \frac{\partial \varphi}{\partial r}\bigg|_R = 3\epsilon_0 E_0 \cos\theta. \tag{7.68}$$

Clearly σ is positive in the "northern" hemisphere, and "negative" in the "southern" hemisphere.

Example 3: A spherified charge density $\sigma_0(\theta)$ is glued over the surface of a spherical shell of radius R. Find the potential field inside and outside the sphere.

Solution: We'll solve the equation in two regions, inside and outside, and match them at the boundary $r = R$ to satisfy the boundary condition.

For $r \leq R$,

$$\varphi = \sum_l A_l r^l P_l(\cos\theta). \tag{7.69}$$

For $r \geq R$,

$$\varphi = \sum_l \frac{B_l}{r^{l+1}} P_l(\cos\theta). \tag{7.70}$$

The potential is continuous at $r = R$, therefore,

$$A_l R^l = \frac{B_l}{R^{l+1}}, \tag{7.71}$$

or

$$B_l = A_l R^{2l+1}. \tag{7.72}$$

Then using Equation (7.9),

$$\epsilon_0 \left.\frac{\partial \varphi_{\text{out}}}{\partial r}\right|_{r=R} - \epsilon_0 \left.\frac{\partial \varphi_{\text{in}}}{\partial r}\right|_{r=R} = -\sigma_0(\theta). \tag{7.73}$$

or

$$\epsilon_0 \sum_l (2l+1) A_l R^{l-1} P_l(\cos\theta) = \sigma_0(\theta). \tag{7.74}$$

With Equation (7.55),

$$A_l = \frac{1}{2\epsilon_0 R^{l-1}} \int_0^{\pi} \sigma_0(\theta) P_l(\cos\theta) \sin\theta \mathrm{d}\theta. \tag{7.75}$$

This finishes our solution of *Example* 3.

7.4.3 *Boundary value problems with linear dielectrics*

For linear dielectrics, the boundary condition Equation gives

$$(\epsilon^{\text{I}} \boldsymbol{E}^{\text{I}} - \epsilon^{\text{II}} \boldsymbol{E}^{\text{II}}) \cdot \boldsymbol{n} = \sigma_f, \tag{7.76}$$

where $\boldsymbol{n}$ is the unit vector pointing from region II to region I, or in terms of potential

$$\epsilon^{\text{I}} \frac{\partial \varphi^{\text{I}}}{\partial n} - \epsilon^{\text{II}} \frac{\partial \varphi^{\text{II}}}{\partial n} = -\sigma_f. \tag{7.77}$$

The potential itself is still continuous,

$$\varphi^{\text{II}} = \varphi^{\text{I}}. \tag{7.78}$$

A useful note sometimes we use is that for a homogeneous isotropic linear dielectric, the bound charge density ρ_b is proportional to the free charge density ρ_f,

$$\rho_b = -\nabla \cdot \boldsymbol{P} = -\nabla \cdot \boldsymbol{D} \left(\frac{\epsilon - \epsilon_0}{\epsilon}\right) = -\left(\frac{\epsilon - \epsilon_0}{\epsilon}\right) \rho_f.^2 \tag{7.79}$$

[2] Notice that $\boldsymbol{P} = \boldsymbol{D} - \epsilon_0 \boldsymbol{E} = (1 - \epsilon_0/\epsilon)\boldsymbol{D}$

Example: A sphere of homogeneous linear dielectric material is placed in an otherwise uniform field $\boldsymbol{E}_0$. Find the electric field inside the sphere.

Solution: Since the material is homogeneous linear dielectric, then $\rho_b = \rho_f = 0$ inside the sphere. Therefore the equation to solve is

$$\nabla^2 \varphi = 0 \tag{7.80}$$

both inside and outside the sphere. The boundary conditions are

$$\varphi_{\text{in}} = \varphi_{\text{out}}, \qquad \varphi \text{ is continuous} \tag{7.81}$$

$$\epsilon \frac{\partial \varphi_{\text{in}}}{\partial r} = \epsilon_0 \frac{\partial \varphi_{\text{out}}}{\partial r}, \qquad \sigma_f = 0 \text{ on the surface,} \tag{7.82}$$

$$\varphi_{\text{out}} = -E_0 r \cos\theta, \qquad \text{for } r \gg R. \tag{7.83}$$

Inside the sphere, the solution is

$$\varphi = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta); \tag{7.84}$$

outside the sphere,

$$\varphi = -E_0 r\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta). \tag{7.85}$$

Boundary condition Equation (7.81) requires that

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = -E_0 R\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta), \tag{7.86}$$

which gives

$$A_1 R = -E_0 R + \frac{B_1}{R^2}, \tag{7.87}$$

$$A_l R^l = \frac{B_l}{R^{l+1}}, \quad \text{for } l \neq 1. \tag{7.88}$$

Condition (7.82) yields

$$\epsilon \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = -\epsilon_0 \boldsymbol{E}\cos\theta - \epsilon_0 \sum_{l=0}^{\infty} \frac{(l+1)B_l}{R^{l+2}} P_l(\cos\theta), \tag{7.89}$$

so

$$\epsilon l A_l R^{l-1} = -\frac{\epsilon_0 (l+1) B_l}{R^{l+2}}, \text{ for } l \neq 1, \tag{7.90}$$

$$\epsilon A_1 = -\epsilon_0 E_0 - \frac{2\epsilon_0 B_1}{R^3}. \tag{7.91}$$

Equations (7.88) and (7.90) give

$$A_l = B_l = 0, \text{ for } l \neq 1, \tag{7.92}$$

while Equation (7.91) and (7.87) can be solved to give

$$A_1 = -\frac{3\epsilon_0}{\epsilon + 2\epsilon_0} E_0, \tag{7.93}$$

$$B_1 = \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} R^3 E_0. \tag{7.94}$$

Therefore

$$\varphi_{\text{in}} = -\frac{3\epsilon_0}{\epsilon + 2\epsilon_0} E_0 r\cos\theta = -\frac{3\epsilon_0}{\epsilon + 2\epsilon_0} E_0 z, \tag{7.95}$$

and the field inside the sphere is

$$\boldsymbol{E} = \frac{3\epsilon_0}{\epsilon + 2\epsilon_0} \boldsymbol{E}_0. \tag{7.96}$$

8
General Magnetostatics

8.1 Static magnetic field

Let's first derive the general or strictly valid equations (like Coulomb's law) for static magnetic fields. The Maxwell equations describing static magnetic fields are

$$\nabla \cdot \boldsymbol{B} = 0, \tag{8.1}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}. \tag{8.2}$$

We have introduced the vector potential $\boldsymbol{A}$, which can be used to present $\boldsymbol{B}$ by

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}. \tag{8.3}$$

Substituting Equation (8.3) into Equation (8.2) gives the equation $\boldsymbol{A}$ satisfies

$$\nabla \times (\nabla \times \boldsymbol{A}) = \mu_0 \boldsymbol{j} \tag{8.4}$$

Note that $\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$, and choose Coulomb gauge

$$\nabla \cdot \boldsymbol{A} = 0, \tag{8.5}$$

then the equation for $\boldsymbol{A}$ is

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{j}. \tag{8.6}$$

This is completely analogous to the Poisson equation

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}. \tag{8.7}$$

Therefore we can directly write the solution of Equation (8.6) for $\boldsymbol{A}$ using the knowledge we have learned from the electrostatic problem. Recall that for a point charge,

$$\phi = \frac{1}{4\pi\epsilon_0} \frac{q}{|\boldsymbol{x} - \boldsymbol{x}'|}, \tag{8.8}$$

and the general solution of the Poisson equation (8.7) is

$$\phi = \frac{1}{4\pi\epsilon_0} \int \frac{\rho}{|\boldsymbol{x} - \boldsymbol{x}'|} \mathrm{d}V', \quad \text{with } \mathrm{d}V' = \mathrm{d}^3 x' \tag{8.9}$$

Here $\boldsymbol{x}$ is for the field point and $\boldsymbol{x}'$ for the charge. Therefore the solution of $\boldsymbol{A}$ from Equation (8.6) is, with $\rho/\epsilon_0 \to \mu_0 \boldsymbol{j}$,

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{j}}{|\boldsymbol{x} - \boldsymbol{x}'|} \mathrm{d}V'. \tag{8.10}$$

The static magnetic field can be obtained from $\boldsymbol{A}$ using $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, or

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{\mu_0}{4\pi} \int \nabla \times \frac{\boldsymbol{j}(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \mathrm{d}V'; \tag{8.11}$$

i.e., we can bring the ∇ operator inside the integral, since the ∇ operator is the differentiation w.r.t. $\boldsymbol{x}$, not $\boldsymbol{x}'$. Performing the differentiation, we have

$$\boldsymbol{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\frac{\boldsymbol{j}}{R} \right) \mathrm{d}V' \tag{8.12}$$

$$= \frac{\mu_0}{4\pi} \int \nabla \frac{1}{R} \times \boldsymbol{j} \, \mathrm{d}V'. \tag{8.13}$$

Or finally, the static magnetic field produced by current $\boldsymbol{j}$ is

$$\boxed{\boldsymbol{B} = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{j} \times \boldsymbol{R}}{R^3} \mathrm{d}V'}. \tag{8.14}$$

This is the law of Biot and Savart. In case of a localized loop current, we replace $\boldsymbol{j}\mathrm{d}V' \to I\mathrm{d}\boldsymbol{l}$, where I is the current. Then

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \oint \frac{I\mathrm{d}\boldsymbol{l}}{R}, \tag{8.15}$$

$$\boldsymbol{B} = \frac{\mu_0}{4\pi} \oint \frac{I\mathrm{d}\boldsymbol{l} \times \boldsymbol{R}}{R^3}. \tag{8.16}$$

8.2 *Magnetic field multiple moments*

In this section, we consider the field at large distances $\boldsymbol{x}$ produced by a localized current. The technique used is similar to the multiple expansion in the electrostatic case. Put the origin of the coordinate inside the current system, then $r \gg r'$,

$$f(\boldsymbol{x} - \boldsymbol{x}') = f(\boldsymbol{x}) - \boldsymbol{x}' \cdot \nabla f(\boldsymbol{x}) + \cdots \tag{8.17}$$

Therefore, with $r \equiv |\boldsymbol{x}|$,

$$\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} \approx \frac{1}{r} - \boldsymbol{x}' \cdot \nabla \frac{1}{r} + \cdots, \quad i = 1, 2, 3. \tag{8.18}$$

Everything works similarly to the electrostatic case. However, because the magnetic potential $\boldsymbol{A}$ is a vector, the general multiple expansion is more complicated. We'll only consider the dipole moment term.

The complete form of $\boldsymbol{A}$ at $\boldsymbol{x}$ is

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{j}(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \mathrm{d}V' \tag{8.19}$$

The 0th and 1st order term in the multiple expansion of $\boldsymbol{A}$ is

$$\boldsymbol{A}^{(0)} = \frac{\mu_0}{4\pi r}\int \boldsymbol{j}\mathrm{d}V' \tag{8.20}$$

$$\boldsymbol{A}^{(1)} = -\frac{\mu_0}{4\pi}\int \boldsymbol{j}(\boldsymbol{x}'\cdot\nabla\frac{1}{r})\mathrm{d}V' \tag{8.21}$$

$$= \frac{\mu_0}{4\pi}\frac{\boldsymbol{x}}{r^3}\cdot\int \boldsymbol{x}'\boldsymbol{j}\mathrm{d}V'. \tag{8.22}$$

We ignore higher order terms, since they are rarely used and also they are complicated.

First we consider $\boldsymbol{A}^{(0)}$, the monopole term. Intuitively, $\boldsymbol{A}^{(0)} = 0$, since there exists no magnetic monopoles. Let's prove this directly. We'll need to use

$$\int (f\boldsymbol{j}\cdot\nabla' g + g\boldsymbol{j}\cdot\nabla' f)\mathrm{d}V' = 0. \tag{8.23}$$

where $\nabla' = \partial/\partial\boldsymbol{x}'$. For simplicity, we call this identity the $f-g$ identity. The proof of this identity will be left as an exercise. Addition: $\boldsymbol{j}$ is local steady current density, and it has $\boldsymbol{j}\cdot\mathrm{d}\boldsymbol{S} = 0$ on the boundary.

Let $f = 1$ and $g = \boldsymbol{x}'\cdot\boldsymbol{e}_i = x_i'$, so that $\nabla' g = \boldsymbol{e}_i$. Using the f-g identity (8.23),

$$A_i^{(0)} = \int \boldsymbol{j}\cdot\boldsymbol{e}_i\mathrm{d}V' = 0, \quad i = 1,2,3. \tag{8.24}$$

All together, we have

$$\boldsymbol{A}^{(0)} = 0. \tag{8.25}$$

Again this is related to the fact that there is no magnetic monopole.

To calculate the 1st order term, we need to calculate

$$\boldsymbol{x}\cdot\int \boldsymbol{x}'\boldsymbol{j}\,\mathrm{d}V'. \tag{8.26}$$

Letting $f = x_i'$ and $g = x_l'$, so that

$$\int (x_i'\boldsymbol{j}\cdot\boldsymbol{e}_l + x_l'\boldsymbol{j}\cdot\boldsymbol{e}_i)\,\mathrm{d}V' = 0, \tag{8.27}$$

or

$$\int (x_i'j_l + x_l'j_i)\,\mathrm{d}V' = 0. \Rightarrow \int x_l'j_i\mathrm{d}V' = -\int x_i'j_l\,\mathrm{d}V' \tag{8.28}$$

The $A^{(1)}$ related term can be rewritten as

$$\boldsymbol{x}\cdot\int \boldsymbol{x}'j_i\,\mathrm{d}V' = x_l\int x_l'j_i\mathrm{d}V' \tag{8.29}$$

$$= x_l\int\left(\frac{x_l'j_i - x_i'j_l}{2}\right)\mathrm{d}V' \tag{8.30}$$

$$= \frac{1}{2}x_l\int \epsilon_{kli}(\boldsymbol{x}'\times\boldsymbol{j})_k\mathrm{d}V' \tag{8.31}$$

$$= -\frac{1}{2}\epsilon_{ilk}x_l\int(\boldsymbol{x}'\times\boldsymbol{j})_k\mathrm{d}V' \tag{8.32}$$

$$= -\frac{1}{2}\left[\boldsymbol{x}\times\int(\boldsymbol{x}'\times\boldsymbol{j})\mathrm{d}V'\right]_i. \tag{8.33}$$

Define the *magnetic moment* to be

$$\boxed{\boldsymbol{m} = \frac{1}{2}\int \boldsymbol{x}' \times \boldsymbol{j}\mathrm{d}V',} \tag{8.34}$$

then

$$\boldsymbol{A}^{(1)} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{x}}{r^3}. \tag{8.35}$$

Since $\boldsymbol{A}^{(0)}$ always vanishes, normally the dipole moment term $\boldsymbol{A}^{(1)}$ dominates. For a small current loop, $\boldsymbol{j}\mathrm{d}V' \to I\mathrm{d}\boldsymbol{l}'$,

$$\boxed{\boldsymbol{m} = \frac{1}{2}\oint \boldsymbol{x}' \times I\mathrm{d}\boldsymbol{l}'.} \tag{8.36}$$

Define $\boldsymbol{S} \equiv \frac{1}{2}\oint \boldsymbol{x}' \times \mathrm{d}\boldsymbol{l}'$. For a plane loop, $\boldsymbol{m} = I\boldsymbol{S}$, with $\boldsymbol{S}$ the area of the loop. The direction of $\boldsymbol{S}$ is determined from the current using the right-hand rule.

The magnetic field $\boldsymbol{B}$ can be calculated directly from $\boldsymbol{A}$.

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} = \frac{\mu_0}{4\pi}\nabla\times\left(\frac{\boldsymbol{m}\times\boldsymbol{x}}{r^3}\right) \tag{8.37}$$

or

$$\boxed{\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{3(\boldsymbol{n}\cdot\boldsymbol{m})\boldsymbol{n} - \boldsymbol{m}}{r^3}, \quad \text{with } \boldsymbol{n} \equiv \frac{\boldsymbol{x}}{r}.} \tag{8.38}$$

This is analogous to the electrostatic dipole field with $\boldsymbol{p}$ replaced by $\boldsymbol{m}$.

8.3 A localized current in an external magnetic field

We consider a localized current distribution in an external $\boldsymbol{B}$. The purpose is to calculate the force and the torque on the current system. The force felt by the current system is

$$\boldsymbol{F} = \int \boldsymbol{j}\times\boldsymbol{B}(\boldsymbol{x}')\mathrm{d}V'. \tag{8.39}$$

To find the force, we first estimate the magnetic field, $\boldsymbol{B} = B_i\boldsymbol{e}_i$,

$$B_i(\boldsymbol{x}') = B_i(0) + \boldsymbol{x}'\cdot\nabla B_i(0) + \cdots, \tag{8.40}$$

The origin of the coordinates is within the current system.

Substituting the equation of $\boldsymbol{B}(\boldsymbol{x}')$ into $\boldsymbol{F}$, then

$$F_i = \int [(\boldsymbol{j}\times\boldsymbol{B}(0))_i + \left(\boldsymbol{j}\times(\boldsymbol{x}'\cdot\nabla)\boldsymbol{B}\right)_i + \cdots]\mathrm{d}V' \tag{8.41}$$

$$= F_i^{(0)} + F_i^{(1)} + \cdots. \tag{8.42}$$

We first calculate $\boldsymbol{F}^{(0)}$,

$$\boldsymbol{F}^{(0)} = \left(\int \boldsymbol{j}\mathrm{d}V'\right)\times\boldsymbol{B}(0) = 0, \tag{8.43}$$

from conclusions obtained in the last section. The force of the dipole term is

$$\boldsymbol{F}^{(1)} = \int \boldsymbol{j} \times (\boldsymbol{x}' \cdot \nabla)\boldsymbol{B}(0)\mathrm{d}V'. \tag{8.44}$$

Its ith component is

$$F_i^{(1)} = \epsilon_{iln} \int j_l \boldsymbol{x}' \cdot \nabla B_n(0)\mathrm{d}V' = \epsilon_{iln} \int j_l x_k' \nabla_k B_n(0)\mathrm{d}V'. \tag{8.45}$$

Recall that when deriving the multiple expansion of $\boldsymbol{A}$, we have used

$$A_l^{(1)} = \frac{\mu_0}{4\pi}\frac{x_k}{r^3} \int x_k' j_l \mathrm{d}V' \Rightarrow A_l^{(1)} = \frac{\mu_0}{4\pi}\frac{(\boldsymbol{m} \times \boldsymbol{x})_l}{r^3} \tag{8.46}$$

We can conclude that

$$\int x_k' j_l x_k \mathrm{d}V' = (\boldsymbol{m} \times \boldsymbol{x})_l \tag{8.47}$$

Replacing $x_k \to \nabla_k B_n$, we have

$$\int x_k' j_l \nabla_k B_n(0)\mathrm{d}V' = (\boldsymbol{m} \times \nabla)_l B_n(0), \tag{8.48}$$

therefore

$$F_i^{(1)} = \epsilon_{iln}(\boldsymbol{m} \times \nabla)_l B_n(0), \tag{8.49}$$

or

$$\boldsymbol{F}^{(1)} = (\boldsymbol{m} \times \nabla) \times \boldsymbol{B} \tag{8.50}$$

Using the middle-outer rule, $\nabla = \nabla_{\boldsymbol{B}}$, and the fact that the ∇ operator is the differentiation w.r.t. $\boldsymbol{x}$, not $\boldsymbol{x}'$, we can easily derive,

$$(\boldsymbol{m} \times \nabla) \times \boldsymbol{B} = \nabla(\boldsymbol{m} \cdot \boldsymbol{B}) - \boldsymbol{m}(\nabla \cdot \boldsymbol{B}),$$

Because $\nabla \cdot \boldsymbol{B} = 0$, we have:

$$\boxed{\boldsymbol{F}^{(1)} = \nabla(\boldsymbol{m} \cdot \boldsymbol{B}).} \tag{8.51}$$

This equation could be put in another form. Using the identity,

$$\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A}) + (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A}$$

and that $\nabla \equiv \partial/\partial \boldsymbol{x}$, we have

$$\nabla(\boldsymbol{m} \cdot \boldsymbol{B}) = \boldsymbol{m} \times (\nabla \times \boldsymbol{B}) + (\boldsymbol{m} \cdot \nabla)\boldsymbol{B}; \tag{8.52}$$

Outside the region of current which generates the external field, $\nabla \times \boldsymbol{B} = 0$; therefore

$$\boxed{\boldsymbol{F}^{(1)} = (\boldsymbol{m} \cdot \nabla)\boldsymbol{B}.} \tag{8.53}$$

This is the force felt by a magnetic dipole in an external $\boldsymbol{B}$. It's analogous to the force felt by the electrostatic dipole moment in an external field with $\boldsymbol{p}$ replaced by $\boldsymbol{m}$. The potential energy of a localized current in external field $\boldsymbol{B}$ is

$$U = -\boldsymbol{m} \cdot \boldsymbol{B}, \tag{8.54}$$

which can be easily see from that the force is

$$\boldsymbol{F} = -\nabla U = \nabla(\boldsymbol{m} \cdot \boldsymbol{B}). \tag{8.55}$$

Compare this with the energy of electrostatic dipole moment, which is $U = \boldsymbol{p} \cdot \boldsymbol{E}$. It can be seen from Equation (8.54) that a magnetic dipole will always try to orient itself parallel to $\boldsymbol{B}$ to achieve the lowest potential energy U.

The torque felt by the localized current is

$$\boldsymbol{K} = \int \boldsymbol{x'} \times \boldsymbol{f} \mathrm{d}V' = \int \boldsymbol{x}' \times (\boldsymbol{j} \times \boldsymbol{B}) \mathrm{d}V'. \tag{8.56}$$

Here the lowest order term, which contributes, is

$$\boldsymbol{K} \approx \boldsymbol{K}^{(0)} = \int \boldsymbol{x}' \times [\boldsymbol{j} \times \boldsymbol{B}(0)] \mathrm{d}V' \tag{8.57}$$

$$= \int [(\boldsymbol{x}' \cdot \boldsymbol{B})\boldsymbol{j} - \boldsymbol{B}(\boldsymbol{x}' \cdot \boldsymbol{j})] \mathrm{d}V'. \tag{8.58}$$

The first term on the right hand side is

$$\int (\boldsymbol{x}' \cdot \boldsymbol{B}) \boldsymbol{j} \mathrm{d}V' = \int [\boldsymbol{B} \cdot \boldsymbol{x}') \boldsymbol{j} \mathrm{d}V' = \boldsymbol{m} \times \boldsymbol{B}(0). \tag{8.59}$$

The second term on the right hand side is 0 from the f-g identity (Equation (8.23)) with $f = g = r'$. Therefore the lowest order torque is

$$\boldsymbol{K} = \boldsymbol{m} \times \boldsymbol{B}(0). \tag{8.60}$$

With $\boldsymbol{K}$ and $\boldsymbol{m}$, we introduce an interesting phenomena. The angular momentum $\boldsymbol{L}$ is related to the torque by

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \boldsymbol{K} = \boldsymbol{m} \times \boldsymbol{B}_0, \tag{8.61}$$

where $\boldsymbol{B}_0 = \boldsymbol{B}(0)$, and $\boldsymbol{L} = \sum \boldsymbol{x}' \times \boldsymbol{p} = \sum \boldsymbol{x}' \times m\boldsymbol{v}$. Here the summation is over all charged particles. Now the magnetic dipole moment is

$$\boldsymbol{m} = \frac{1}{2} \int \boldsymbol{x}' \times \boldsymbol{j} \mathrm{d}V' = \frac{1}{2} \int \boldsymbol{x}' \times \rho \boldsymbol{v} \mathrm{d}V' = \frac{1}{2} \sum \boldsymbol{x}' \times q\boldsymbol{v}. \tag{8.62}$$

We see that, if q/m is the same for all particles,

$$\boldsymbol{m} = \frac{1}{2} \sum (\boldsymbol{x}' \times m\boldsymbol{v}) \frac{q}{m} = \frac{q}{2m} \sum \boldsymbol{x}' \times m\boldsymbol{v} = \frac{q}{2m} \boldsymbol{L}. \tag{8.63}$$

So the angular momentum equation becomes

$$\boxed{\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \frac{q}{2m} \boldsymbol{L} \times \boldsymbol{B}_0 = \boldsymbol{L} \times \boldsymbol{\Omega}_L} \tag{8.64}$$

with $\boldsymbol{\Omega}_L = (q/2m)\boldsymbol{B}_0$ the Larmor precession frequency.

8.4 Magnetization

8.4.1 Diamagnets, Paramagnets, Ferromagnets

All magnetic fields are produced by currents, or electric charges in motion microscopically. Electrons orbits around nuclei and spin about their axes; we can view these tiny current loops as magnetic dipoles. Normally, these magnetic dipoles are randomly oriented; their magnetic fields cancel each other. When a magnetic field is applied, however, a net alignment of dipoles occurs, and the medium becomes magnetically polarized, or *magnetized*.

Different from electric polarization, where $\boldsymbol{P} \| \boldsymbol{E}$, there are three different kinds of magnetization: a magnetization parallel to $\boldsymbol{B}$ (*paramagnets*), magnetization opposite to $\boldsymbol{B}$ (*diamagnets*), and for a few substances, magnetization depending on the whole magnetic "history" (*ferromagnets*). Ferromagnets are very complicated; I will not discuss it in this class. You can read the corresponding chapters in the textbook if you are interested. In the following sessions, let me explain the microscopic mechanism for *paramagnets* and *diamagnets*.

8.4.2 Paramagnetism

Electrons spin, and each electron constitutes a magnetic dipole. We have learned before that when a $\boldsymbol{B}$ is applied, each magnetic dipole feels a torque; this torque tends to align the dipole parallel to the $\boldsymbol{B}$ to minimize the potential energy. This torque accounts for *paramagnetism*, since the resulting magnetic field from the magnetic dipole is in the same direction as $\boldsymbol{B}$. You might expect paramagnetism to be universal; however, quantum mechanics tends to lock electrons in pairs with opposing spins. The torque on the each pair of electrons is close to 0. As a result, paramagnetism most often occurs in atoms with an odd number of electrons, where the extra unpaired electron is subject to the magnetic torque.

8.4.3 Diamagnetism

Electrons not only spin; they also orbit around nucleus. Assume the radius of the orbit is R; the resulting current is

$$I = -\frac{e}{T} = -\frac{ev}{2\pi R}. \tag{8.65}$$

The current is not steady, but unless we work on a time scale comparable to T, the approximation is not bad. Let z direction be the direction of angular velocity; then the resulting magnetic dipole is

$$\boldsymbol{m} = IS = -\frac{evR}{2}\boldsymbol{e}_z \tag{8.66}$$

When we apply an external magnetic field $\boldsymbol{B}$, the orbit is subject to a torque $\boldsymbol{m} \times \boldsymbol{B}$. But it's a lot harder to tilt the entire orbit than it is the spin, so the orbital contribution to paramagnetism is small.

However, the electron *speeds up* or *slows down*, depending on the direction of $\boldsymbol{B}$. To see this, before applying $\boldsymbol{B}$, we have

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} = m_e\frac{v^2}{R}. \tag{8.67}$$

When we apply an external field $\boldsymbol{B}$, this is an external force $-e(\boldsymbol{v}/c)\times\boldsymbol{B}$, and the electron's new velocity is $\overline{v}$. Assuming $\boldsymbol{B} = B\boldsymbol{e}_z$, the new force equation is

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} + e\overline{v}B = m_e\frac{\overline{v}^2}{R}. \tag{8.68}$$

Letting $\Delta v = \overline{v} - v$ and keeping only first order terms in Δv, Equations (8.68) and (8.67) gives

$$e\overline{v}B = m_e\frac{\overline{v}}{R}2\Delta v, \tag{8.69}$$

or

$$\Delta v = \frac{eRB}{2m_e}. \tag{8.70}$$

When $\boldsymbol{B} = B\boldsymbol{e}_z$ is applied, the electron speeds up. Equation (8.66) indicates that there is a change in $\boldsymbol{m}$,

$$\Delta\boldsymbol{m} = -\frac{eR}{2}\Delta v\boldsymbol{e}_z = -\frac{e^2R^2}{4m_e}\boldsymbol{B}. \tag{8.71}$$

Note that $\Delta\boldsymbol{m}$ is opposite to the direction $\boldsymbol{B}$. Therefore, when an external $\boldsymbol{B}$ is applied, each orbiting electron picks up a little extra dipole moment, and the increments are all anti-parallel to $\boldsymbol{B}$. This is the mechanism responsible for *diamagnetism*. It's universal, but typically much much weaker than *paramagnetism*. Therefore, diamagnetism is mainly observed in atoms with even number of electrons, where paramagnetism is usually absent.

9
Time-varying Electromagnetic Fields

In this chapter, we will focus on time-varying electromagnetic fields (electromagnetic waves) in vacuum and linear media.

9.1 The Wave Equation in Vacuum

Let's start from probably the simplest case: the electromagnetic waves in vacuum. The Maxwell equations in vacuum are

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}, \tag{9.1}$$

$$\nabla\cdot\boldsymbol{E} = 0, \tag{9.2}$$

$$\nabla\times\boldsymbol{B} = \frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t}, \tag{9.3}$$

$$\nabla\cdot\boldsymbol{B} = 0. \tag{9.4}$$

If $\partial/\partial t = 0$, then since $\rho = 0, \boldsymbol{j} = 0$, $\boldsymbol{E} = 0$ and $\boldsymbol{B} = 0$ from Coulomb's law and Biot-Savart's law. To obtain nonzero $\boldsymbol{E}$ and $\boldsymbol{B}$ solutions from above equations, $\partial/\partial t \neq 0$.

Applying $\partial/\partial t$ to the $\nabla\times\boldsymbol{E}$ equation,

$$\nabla\times\frac{\partial\boldsymbol{E}}{\partial t} = \frac{\partial}{\partial t}\nabla\times\boldsymbol{E} = -\frac{\partial^2\boldsymbol{B}}{\partial t^2}. \tag{9.5}$$

Using Ampere's law, we rewrite the equation as

$$\nabla\times\frac{\partial\boldsymbol{E}}{\partial t} = \nabla\times\left(c^2\nabla\times\boldsymbol{B}\right). \tag{9.6}$$

or

$$\nabla\times(\nabla\times\boldsymbol{B}) = -\frac{1}{c^2}\frac{\partial^2\boldsymbol{B}}{\partial t^2}. \tag{9.7}$$

Using the vector identity,

$$\nabla\times(\nabla\times\boldsymbol{B}) = \nabla(\nabla\cdot\boldsymbol{B}) - \nabla^2\boldsymbol{B} = -\nabla^2\boldsymbol{B}, \tag{9.8}$$

so the last equation on the previous slide becomes

$$-\nabla^2\boldsymbol{B} = -\frac{1}{c^2}\frac{\partial^2\boldsymbol{B}}{\partial t^2}, \tag{9.9}$$

or

$$\boxed{\frac{\partial^2\boldsymbol{B}}{\partial t^2} - c^2\nabla^2\boldsymbol{B} = 0.} \tag{9.10}$$

Similarly, one can derive the equation for $\boldsymbol{E}$,

$$\boxed{\frac{\partial^2 \boldsymbol{E}}{\partial t^2} - c^2 \nabla^2 \boldsymbol{E} = 0.} \tag{9.11}$$

Using Maxwell equations in covariant form, one can directly derive the wave equation for ϕ and $\boldsymbol{A}$,

$$\boxed{\begin{aligned} \frac{\partial^2 \phi}{\partial t^2} - c^2 \nabla^2 \phi = 0, \\ \frac{\partial^2 \boldsymbol{A}}{\partial t^2} - c^2 \nabla^2 \boldsymbol{A} = 0. \end{aligned}}$$

These are sometimes called the homogeneous d'Alembert equations, and

$$\boxed{\Box \equiv \frac{\partial^2}{\partial t^2} - c^2 \nabla^2,} \tag{9.12}$$

is the d'Alembert operator.

9.2 *The Electromagnetic Plane Waves*

To understand better the d'Alembert equations for field variables, we now study a prototype equation of $\boldsymbol{E}$ and $\boldsymbol{B}$,

$$\frac{\partial^2 f}{\partial t^2} - c^2 \frac{\partial^2 f}{\partial x^2} = 0, \tag{9.13}$$

this is equivalent to assume $\boldsymbol{E} = \boldsymbol{E}(x)$ or $\boldsymbol{B} = \boldsymbol{B}(x)$. If $\boldsymbol{E}$ or $\boldsymbol{B}$ depends only on one spatial coordinate, the wave is called a *plane wave*.

Let's now solve the plane wave equation in two ways, allowing us to understand the equation from different points of view.

The first method involves writing the plane wave equation as

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) f = 0. \tag{9.14}$$

We can define two new variables ξ and η given by

$$\xi = t - \frac{x}{c}, \tag{9.15}$$

$$\eta = t + \frac{x}{c}. \tag{9.16}$$

These equations gives $t = (\eta + \xi)/2$ and $x = (\eta - \xi)/2$. The definitions of ξ and η leads to

$$\frac{\partial}{\partial \xi} = \frac{1}{2}\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right), \tag{9.17}$$

$$\frac{\partial}{\partial \eta} = \frac{1}{2}\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right). \tag{9.18}$$

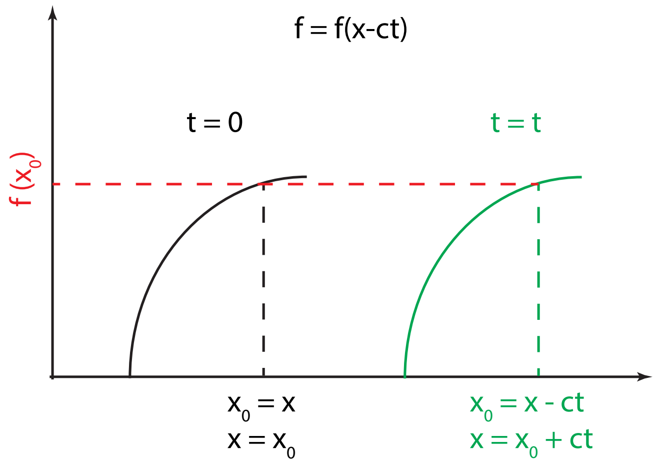
Therefore the equation of f becomes

$$\frac{\partial^2 f}{\partial \xi \partial \eta} = 0 \Rightarrow f = f_1(\xi) + f_2(\eta). \tag{9.19}$$

In t and x,

$$f = f_1(x - ct) + f_2(x + ct). \tag{9.20}$$

It's not difficult to see that $f(x - ct)$ represents a plane wave moving in the positive x-direction, and $f(x + ct)$ represents a plane wave moving in the negative x-direction. The solution $f(x - ct)$ is illustrated in the following figure.



Now consider the electromagnetic plane wave propagating in the positive x-direction. We know now that $\boldsymbol{E}, \boldsymbol{B}, \phi, \boldsymbol{A}$ all are functions of $\xi = x - ct$. Let's see which properties of EM waves we can derive.

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} = -\frac{\partial \phi}{\partial x}\boldsymbol{e}_x - \frac{\partial \boldsymbol{A}}{\partial t}, \tag{9.21}$$

so

$$E_x = -\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t}. \tag{9.22}$$

However, $\xi = x - ct$, $\phi = \phi(\xi)$, $\boldsymbol{A} = \boldsymbol{A}(\xi)$, therefore

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial \xi} \text{ and } \frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial \xi}(-c). \tag{9.23}$$

or

$$\frac{\partial \phi}{\partial x} = -\frac{1}{c}\frac{\partial \phi}{\partial t}, \tag{9.24}$$

Similarly, one can find

$$\frac{\partial A_x}{\partial x} = -\frac{1}{c}\frac{\partial A_x}{\partial t}, \tag{9.25}$$

Therefore, E_x can be written as

$$E_x = -\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} = \frac{1}{c}\frac{\partial \phi}{\partial t} + c\frac{\partial A_x}{\partial x}. \tag{9.26}$$

Choosing Lorenz gauge,

$$\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla \cdot \boldsymbol{A} = 0, \tag{9.27}$$

we immediately arrive at the conclusion that $E_x = 0$. Since $E_x = 0$,

$$E_y = -\frac{\partial A_y}{\partial t}, \text{ and } E_z = -\frac{\partial A_z}{\partial t} \tag{9.28}$$

Now

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{\partial}{\partial x}\boldsymbol{e}_x \times \boldsymbol{A} = -\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{e}_x \times \boldsymbol{A} = \frac{1}{c}\boldsymbol{e}_x \times \left(-\frac{\partial \boldsymbol{A}}{\partial t}\right) \tag{9.29}$$

$$= \frac{1}{c}\boldsymbol{e}_x \times (\boldsymbol{E} + \nabla\phi) = \frac{1}{c}\boldsymbol{e}_x \times \left(\boldsymbol{E} + \frac{\partial \phi}{\partial x}\boldsymbol{e}_x\right) = \frac{1}{c}\boldsymbol{e}_x \times \boldsymbol{E}. \tag{9.30}$$

This result becomes very general if we let $\boldsymbol{e}_x = \hat{\boldsymbol{n}}$, the direction of propagation, then

$$\boxed{\boldsymbol{B} = \hat{\boldsymbol{n}} \times \boldsymbol{E}/c.} \tag{9.31}$$

So for electromagnetic plane waves, we have the following properties:

1. $B = E/c$ in SI units.
2. $\boldsymbol{B} \perp \hat{\boldsymbol{n}}$, $\boldsymbol{E} \perp \hat{\boldsymbol{n}}$, and $\boldsymbol{B} \perp \boldsymbol{E}$.

Both $\boldsymbol{E}$ and $\boldsymbol{B}$ are perpendicular to $\boldsymbol{n}$, the wave is said to be *transverse*. Electromagnetic waves in vacuum are transverse waves.

The energy Poynting flux of the plane wave is

$$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E} \times \boldsymbol{B} \tag{9.32}$$

$$= \frac{1}{\mu_0}\boldsymbol{E} \times (\hat{\boldsymbol{n}} \times \boldsymbol{E}/c) \tag{9.33}$$

$$= \frac{1}{\mu_0 c}[\hat{\boldsymbol{n}}(\boldsymbol{E}\cdot\boldsymbol{E}) - \boldsymbol{E}(\boldsymbol{E}\cdot\hat{\boldsymbol{n}})] \tag{9.34}$$

$$= \frac{1}{\mu_0 c}E^2\hat{\boldsymbol{n}}. \tag{9.35}$$

So the energy flow direction is also $\hat{\boldsymbol{n}}$. The energy density of the EM plane wave is

$$w = \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2 = \epsilon_0 E^2, \tag{9.36}$$

therefore

$$\boldsymbol{S} = cw\hat{\boldsymbol{n}}. \tag{9.37}$$

The velocity

$$\boldsymbol{v} = \frac{\boldsymbol{S}}{w} = c\hat{\boldsymbol{n}}, \tag{9.38}$$

is the group velocity of the EM plane waves.

9.3 The Monochromatic Plane Waves

The wave equation can be solved by other means, e.g., separation of variables or Fourier analysis. [1]

[1] Normally Fourier transform is used, but since you probably haven't learned it, I'll use separation of variables here.

Letting $f(t,x) = g(t)h(x)$, then

$$\frac{\partial^2 f}{\partial t^2} = \frac{\mathrm{d}^2 g}{\mathrm{d}t^2} h(x), \tag{9.39}$$

$$\frac{\partial^2 f}{\partial x^2} = g(t)\frac{\mathrm{d}^2 h}{\mathrm{d}x^2}. \tag{9.40}$$

The wave equation becomes

$$\frac{\mathrm{d}^2 g}{\mathrm{d}t^2} h(x) - c^2 g(t)\frac{\mathrm{d}^2 h}{\mathrm{d}x^2} = 0. \tag{9.41}$$

Re-organizing terms of the equation leads to

$$\frac{1}{g}\frac{\mathrm{d}^2 g}{\mathrm{d}t^2} = c^2 \frac{1}{h}\frac{\mathrm{d}^2 h}{\mathrm{d}x^2}. \tag{9.42}$$

From the method of separation of variables, we let

$$\frac{1}{g_n}\frac{\mathrm{d}^2 g_n}{\mathrm{d}t^2} = -\omega_n^2, \tag{9.43}$$

$$\frac{1}{h_n}\frac{\mathrm{d}^2 h_n}{\mathrm{d}x^2} = -k_n^2 \quad \text{with } k_n = \omega_n/c. \tag{9.44}$$

The solutions of the equations are

$$g_n = g_{n0} e^{-\mathrm{i}\omega_n t}, \tag{9.45}$$

$$h_n = h_{n0} e^{\mathrm{i}k_n x}. \tag{9.46}$$

so

$$f_n = g_n h_n = f_{n0} e^{\mathrm{i}(k_n x - \omega_n t)}, \tag{9.47}$$

$$f = \sum_n f_n = \sum_n f_{n0} e^{\mathrm{i}(k_n x - \omega_n t)}. \tag{9.48}$$

In wave literature,

$$f = f_0 e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}, \tag{9.49}$$

is called a monochromatic (single frequency) wave. Here ω is the angular frequency [2], $\boldsymbol{k}$ is the wave vector, and $\varphi = \boldsymbol{k}\cdot\boldsymbol{r} - \omega t$ is called the wave phase. You can see that $\partial\varphi/\partial t = -\omega$ and $\nabla\varphi = \boldsymbol{k}$. For electromagnetic waves in vacuum, the phase velocity $v_p \equiv \omega/k = c$ equals the wave group velocity (v_g) [3]. The solution f_n is therefore a monochromatic plane wave with frequency ω_n and wave number k_n. From Equation (9.48), general plane waves can be considered to be consisted of a series of monochromatic waves.

[2] In theoretical physics, this is simply called frequency.

[3] Note that this is a special case and in general is not true.

Using complex variables to represent real physical variables related to waves is very convenient and has a lot of advantages. For example, for electromagnetic plane waves in vacuum,

$$f = f_0 e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}, \tag{9.50}$$

we can immediately have

$$\frac{\partial f}{\partial t} = -\mathrm{i}\omega f; \quad \nabla f = \mathrm{i}\boldsymbol{k} f. \tag{9.51}$$

If we use real functions, like cos and sin, e.g., $f = f_0 \cos(kx - \omega t)$, then

$$\frac{\partial f}{\partial t} = \omega f_0 \sin(kx - \omega t); \quad \frac{\partial f}{\partial x} = -k f_0 \sin(kx - \omega t). \tag{9.52}$$

This is too tedious and unnecessarily complicated.

If we use a complex function f to represent a physical quantity, then the real physical variable is just $\mathrm{Re}(f)$. But under what condition can we do this? *The equation that f satisfies must be a linear equation.* Suppose $\mathcal{L}$ is a linear operator, then

$$\mathcal{L}\{f\} = \mathcal{L}\{\mathrm{Re}(f) + \mathrm{i}\,\mathrm{Im}(f)\} = \mathcal{L}\{\mathrm{Re}(f)\} + \mathrm{i}\,\mathcal{L}\{\mathrm{Im}(f)\} = 0, \tag{9.53}$$

meaning $\mathcal{L}\{\mathrm{Re}(f)\} = 0$ and $\mathcal{L}\{\mathrm{Im}(f)\} = 0$. Maxwell equations are linear equations, so we can use complex numbers to represent $\boldsymbol{E}$ and $\boldsymbol{B}$.

E.g., for a monochromatic wave variable f (scalar) and $\boldsymbol{A}$ (vector),

$$\nabla \times \boldsymbol{A} = \mathrm{i}\boldsymbol{k} \times \boldsymbol{A}, \quad \frac{\partial \boldsymbol{A}}{\partial t} = -\mathrm{i}\omega \boldsymbol{A}, \quad \nabla f = \mathrm{i}\boldsymbol{k} f \quad \text{and} \quad \frac{\partial f}{\partial t} = -\mathrm{i}\omega f. \tag{9.54}$$

Therefore, for an electromagnetic monochromatic plane wave,

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} = -\mathrm{i}\boldsymbol{k}\phi + \mathrm{i}\omega\boldsymbol{A}, \tag{9.55}$$

$$\boldsymbol{B} = \mathrm{i}\boldsymbol{k} \times \boldsymbol{A} = \boldsymbol{k} \times \boldsymbol{E}/\omega. \tag{9.56}$$

From this we see that $\boldsymbol{E} \perp \boldsymbol{B}$, and $\boldsymbol{B} \perp \boldsymbol{k}$. The physical "$\boldsymbol{E}$" and "$\boldsymbol{B}$" are just Re($\boldsymbol{E}$) and Re($\boldsymbol{B}$).

What's the energy density of the field if we use complex $\boldsymbol{E}$? From now, let's denote complex $\boldsymbol{E}$ by $\hat{\boldsymbol{E}}$ so that

$$\boldsymbol{E} = \mathrm{Re}\hat{\boldsymbol{E}}. \tag{9.57}$$

The energy density of the electric field is $w = \epsilon_0 \boldsymbol{E} \cdot \boldsymbol{E}/2$. But is the equation

$$w = \mathrm{Re}\left\{\frac{\epsilon_0}{2}\hat{\boldsymbol{E}} \cdot \hat{\boldsymbol{E}}\right\}, \tag{9.58}$$

correct? We need to look at this carefully. Suppose $\hat{\boldsymbol{E}} = \boldsymbol{E}_0 e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}$, and for simplicity, $\boldsymbol{E}_0$ is real.[4]

[4] In general, it is a complex constant.

$$\hat{\boldsymbol{E}} \cdot \hat{\boldsymbol{E}} = \boldsymbol{E}_0^2 e^{2\mathrm{i}\varphi} \quad \text{with } \varphi \equiv \boldsymbol{k} \cdot \boldsymbol{x} - \omega t, \tag{9.59}$$

then

$$Re\left\{\frac{\epsilon_0}{2}\hat{\boldsymbol{E}} \cdot \hat{\boldsymbol{E}}\right\} = \frac{\epsilon_0}{2} E_0^2 \cos 2\varphi. \tag{9.60}$$

However, the energy density of the field is, using real variables $\boldsymbol{E} = \boldsymbol{E}_0 \cos\varphi$,

$$w = \frac{\epsilon_0}{2}\boldsymbol{E}_0^2 \cos^2\varphi \neq Re\left\{\frac{\epsilon_0}{2}\hat{\boldsymbol{E}} \cdot \hat{\boldsymbol{E}}\right\}. \tag{9.61}$$

The reason for this is obvious: w is not a linear function of $\boldsymbol{E}$. How to properly represent the energy density using complex variables will be left as an exercise.

9.4 *Wave Polarization*

Let's now introduce wave polarization. Consider a plane monochromatic EM wave with $\boldsymbol{k} = k\boldsymbol{e}_x$, then

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{\mathrm{i}(kx-\omega t)}, \tag{9.62}$$

Since $\boldsymbol{k} = k\boldsymbol{e}_x$, $\boldsymbol{E} = \boldsymbol{E}_y + \boldsymbol{E}_z$, or

$$\boldsymbol{E} = (\boldsymbol{E}_{0y} + \boldsymbol{E}_{0z})e^{\mathrm{i}(kx-\omega t)} = (\hat{E}_{0y}\boldsymbol{e}_y + \hat{E}_{0z}\boldsymbol{e}_z)e^{\mathrm{i}(kx-\omega t)}. \tag{9.63}$$

Note here $\hat{E}_{0y}$ and $\hat{E}_{0z}$ are in general complex constants.

Let's start from the simplest case, the linear polarization. If $\hat{E}_{0y}$ and $\hat{E}_{0z}$ have the same phase α; i.e.,

$$\hat{E}_{0y} = E_{0y}e^{\mathrm{i}\alpha} \quad \text{and} \quad \hat{E}_{0z} = E_{0z}e^{\mathrm{i}\alpha}. \tag{9.64}$$

Then

$$\boldsymbol{E} = (E_{0y}\boldsymbol{e}_y + E_{0z}\boldsymbol{e}_z)e^{\mathrm{i}(kx-\omega t+\alpha)}, \tag{9.65}$$

You can see here that the electric field vector oscillates along a line with a fixed direction $\hat{\boldsymbol{n}}$; the angle between $\hat{\boldsymbol{n}}$ and $\boldsymbol{e}_y$ is

$$\langle \hat{\boldsymbol{n}}, \boldsymbol{e}_y \rangle = \tan^{-1}\left(\frac{E_{0z}}{E_{0y}}\right), \tag{9.66}$$

This wave is called *linearly polarized* or *plane polarized*.

In general $\hat{E}_{0y}$ and $\hat{E}_{0z}$ have different phases, and the wave is said to be *elliptically polarized*.

Let's start from the simplest case of circular polarization, where $E_{0y} = E_{0z} = E_0$, i.e.,

$$\hat{E}_{0y} = E_0 e^{\mathrm{i}\alpha} \quad \text{and} \quad \hat{E}_{0z} = E_0 e^{\mathrm{i}(\alpha\pm\pi/2)}, \tag{9.67}$$

then

$$\boldsymbol{E} = E_0(\boldsymbol{e}_y \pm \mathrm{i}\boldsymbol{e}_z)e^{\mathrm{i}(kx-\omega t+\alpha)}. \tag{9.68}$$

The actual electric field is Re($\boldsymbol{E}$), i.e.,

$$E_y = E_0\cos(kx - \omega t + \alpha), \tag{9.69}$$
$$E_z = \mp E_0\sin(kx - \omega t + \alpha). \tag{9.70}$$

At a fixed point $z = z_0$, the electric field sweeps around in a circle at a frequency ω with constant magnitude E_0. This wave is called circularly polarized. Note that $\hat{E}_{0z}/\hat{E}_{0y} = \mathrm{i}$ or $-\mathrm{i}$, the directions of rotation (DOR) are different. When the observer is facing at the oncoming wave,

1. $\hat{E}_{0z}/\hat{E}_{0y} = \mathrm{i}$, DOR is counter-clockwise.
2. $\hat{E}_{0z}/\hat{E}_{0y} = -\mathrm{i}$, DOR is clockwise.

$$\boldsymbol{E} = E_0(\boldsymbol{e}_y + \mathrm{i}\boldsymbol{e}_z)e^{\mathrm{i}(kx-\omega t+\alpha)}$$

Circular polarization may be referred to as right-handed or left-handed, but there is a lot of confusion about this definition. There are at least three different ways I'm aware of to define these terms.

Way I: From the point of view of the source

Pointing one's left or right thumb **away** from the source, in the **same** direction as $\boldsymbol{k}$, matching the curling of figures to the temporal rotation of the field.

1. $\hat{E}_{0z}/\hat{E}_{0y} = \mathrm{i}$, right-handed polarization.
2. $\hat{E}_{0z}/\hat{E}_{0y} = -\mathrm{i}$, left-handed polarization.

Typical communities using Way I are like engineering, quantum physics, and radio astronomy.

Way II: From the point of view of the receiver

Pointing one's left or right thumb **toward** the source, **against** the direction of $\boldsymbol{k}$, matching the curling of figures to the temporal rotation of the field.

1. $\hat{E}_{0z}/\hat{E}_{0y} = \mathrm{i}$, left-handed polarization.
2. $\hat{E}_{0z}/\hat{E}_{0y} = -\mathrm{i}$, right-handed polarization.

Typical community using Way II is optics.

Way III: From the direction of the local magnetic field

Pointing one's left or right thumb **along** local $\boldsymbol{B}_0$, **regardless** of the direction of $\boldsymbol{k}$, matching the curling of figures to the temporal rotation of the field.

1. $\hat{E}_{0z}/\hat{E}_{0y} = \mathrm{i}$, left or right polarization.
2. $\hat{E}_{0z}/\hat{E}_{0y} = -\mathrm{i}$, left or right polarization.

Typical community using Way III is plasma physics. The reason for this is that R-mode waves rotate in the same sense as an electron, L-mode waves rotate in the same sense as an ion.

Circular polarization is a special case of the *elliptical polarization*. Now let's take Landau's approach and study elliptical polarization.

Recall that

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{\mathrm{i}(kx-\omega t)}, \tag{9.71}$$

Suppose $\boldsymbol{E}_0 = \boldsymbol{b}e^{\mathrm{i}\alpha}$, where $\boldsymbol{b}$ is a complex vector,

$$\boldsymbol{b} = \boldsymbol{b}_1 + \mathrm{i}\boldsymbol{b}_2, \tag{9.72}$$

where $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are real vectors. Here we choose $\boldsymbol{b}$ so that $\boldsymbol{b}\cdot\boldsymbol{b} = \boldsymbol{b}_1^2 + \boldsymbol{b}_2^2 + 2\mathrm{i}\boldsymbol{b}_1\cdot\boldsymbol{b}_2$ is real[5], then $\boldsymbol{b}_1\cdot\boldsymbol{b}_2$ must be 0, or $\boldsymbol{b}_1 \perp \boldsymbol{b}_2$. For simplicity of discussion, we choose $\boldsymbol{b}_1 = b_1\boldsymbol{e}_1$ with $b_1 > 0$.

[5] It's customary to use $\boldsymbol{b}^2 \equiv \boldsymbol{b}\cdot\boldsymbol{b}$.

The other direction $\boldsymbol{e}_2$ is defined so that $\boldsymbol{e}_x \times \boldsymbol{e}_1 = \boldsymbol{e}_2$, then

$$\boldsymbol{b}_2 = \pm b_2\boldsymbol{e}_2, \quad b_2 > 0. \tag{9.73}$$

Let's now calculate the e_1 and e_2 components of E.

$$\boldsymbol{E} = \boldsymbol{b}e^{\mathrm{i}(kx-\omega t+\alpha)} = (b_1\boldsymbol{e}_1 \pm \mathrm{i}b_2\boldsymbol{e}_2)e^{\mathrm{i}(kx-\omega t+\alpha)}. \tag{9.74}$$

$$\hat{\boldsymbol{E}}_1 = b_1 e^{\mathrm{i}(kx-\omega t+\alpha)}\boldsymbol{e}_1, \tag{9.75}$$

$$\hat{\boldsymbol{E}}_2 = \pm b_2 e^{\mathrm{i}(kx-\omega t+\alpha+\pi/2)}\boldsymbol{e}_2. \tag{9.76}$$

Therefore, in $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ directions,

$$E_1 = b_1\cos(kx-\omega t+\alpha), \tag{9.77}$$

$$E_2 = \mp b_2\sin(kx-\omega t+\alpha). \tag{9.78}$$

You can then easily verify that E_1 and E_2 satisfy

$$\frac{E_1^2}{b_1^2} + \frac{E_2^2}{b_2^2} = 1. \tag{9.79}$$

At any fixed point in space, the $\boldsymbol{E}$ vector rotates in a plane perpendicular to the direction of propagation. The endpoint of the $\boldsymbol{E}$ vector describes the ellipse given by the equation; the wave is *elliptically polarized*. If $b_1 = b_2$, then the wave becomes *circularly polarized*. If $b_1 = 0$ or $b_2 = 0$ (but not both!), then $E_1 = 0$ or $E_2 = 0$, respectively. The wave is *linearly polarized*.

9.5 Propagation of Electromagnetic Waves in Linear Media

From now on, we consider electromagnetic waves in matter. For linear media without free current and free charges, we know that the Maxwell equations are

$$\nabla\cdot\boldsymbol{E} = 0, \tag{9.80}$$

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}, \tag{9.81}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{9.82}$$

$$\nabla\times\boldsymbol{B} = \mu\epsilon\frac{\partial\boldsymbol{E}}{\partial t}. \tag{9.83}$$

The boundary conditions are the same as before,

$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}, \tag{9.84}$$

$$\boldsymbol{E}_1^{\parallel} = \boldsymbol{E}_2^{\parallel}, \tag{9.85}$$

$$B_1^{\perp} = B_2^{\perp}, \tag{9.86}$$

$$\frac{\boldsymbol{B}_1^{\parallel}}{\mu_1} = \frac{\boldsymbol{B}_2^{\parallel}}{\mu_2}. \tag{9.87}$$

We have assumed here that μ and ϵ are independent of wave frequency. Now following what we have down before, the components of $\boldsymbol{E}$ and $\boldsymbol{B}$ satisfy the wave equation

$$\frac{\partial^2 u}{\partial t^2} - v^2\nabla^2 u = 0, \tag{9.88}$$

where $v = 1/\sqrt{\mu\epsilon}$ is the phase velocity of light in this substance. The plane-wave solution of Equation (9.88) has been obtained before, in general

$$u(x,t) = f(x - vt) + g(x + vt). \tag{9.89}$$

Or in terms of monochromatic plane wave solution,

$$u = u_0 e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}, \tag{9.90}$$

with

$$k = \frac{\omega}{v} = \omega\sqrt{\mu\epsilon}, \tag{9.91}$$

the wave number. Note that we can define the index of refraction by [6]

$$n = \frac{c}{v} = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} = \sqrt{\epsilon_r\mu_r}. \tag{9.92}$$

[6] Again, please note that n is the refraction index while the vector $\hat{\boldsymbol{n}}$ denotes unit vector in the propagation direction

For most substances, $\mu_r \approx 1$, and therefore $n \approx \sqrt{\epsilon_r}$. Equation (9.91) shows that the phase velocity of EM waves in linear media is scaled by a factor $1/n$.

With a monochromatic plane wave solution like Equation (9.90), we can see from the Maxwell equations that

$$\boldsymbol{k}\cdot\boldsymbol{E} = 0 \text{ and } \boldsymbol{k}\cdot\boldsymbol{B} = 0, \tag{9.93}$$

therefore, the wave is transverse. Furthermore, Faraday's law gives

$$\mathrm{i}\boldsymbol{k}\times\boldsymbol{E} = -(-\mathrm{i}\omega)\boldsymbol{B} \tag{9.94}$$

or

$$\boldsymbol{n}\times\frac{\boldsymbol{E}}{c} = \boldsymbol{B}, \tag{9.95}$$

where $\boldsymbol{n} = n\widehat{\boldsymbol{k}} = \boldsymbol{k}c/w$. Equation (9.95) shows that $B = nE/c$ for a transverse wave in media.

In above discussions, we assumed that k and ω are real. In more general cases, to represent wave growth or damping, k or ω can be complex. Let's say for now that $\boldsymbol{k} = \boldsymbol{k}_r + \mathrm{i}\boldsymbol{k}_i$ is complex and ω is real. Then the plane wave solution is

$$u = u_0 e^{\mathrm{i}(\boldsymbol{k}_r\cdot\boldsymbol{x} - \omega t)} e^{-\boldsymbol{k}_i\cdot\boldsymbol{x}}. \tag{9.96}$$

This means that the wave damps ($k_i > 0$) or grows ($k_i < 0$) spatially, we'll use this in studying wave propagation in conducting medium. On the other hand, if k is real and $\omega = \omega_r + \mathrm{i}\omega_i$, then

$$u = u_0 e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega_r t)} e^{\omega_i t}. \tag{9.97}$$

means that the wave grows ($\omega_i > 0$) or damps ($\omega_i < 0$) temporally.

9.6 Reflection and refraction of electromagnetic waves

9.6.1 General information

In this section, we will study reflection and refraction (折射) of light, mainly using Equations (9.84)-(9.87). We will explain, using electrodynamics, the following two basic properties you are familiar with:

- The incident, reflected, and refracted wave vectors form a plane.
- Angle of reflection equals angle of incidence.
- Snell's law; i.e.,

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{n'}{n}, \tag{9.98}$$

  where θ_i and θ_r are the angles of incidence and refraction, respectively, while n' and n are the corresponding indices of refraction.

We will also study the following things that you may not know

- Intensities of reflected and refracted radiation.
- Phase changes and polarization.

The coordinate system is like this. For $z > 0$, we have a linear media with μ' and ϵ'; for $z < 0$, μ and ϵ. The indices of refraction are then $n' = \sqrt{\mu'\epsilon'/\mu_0\epsilon_0}$ for $z > 0$ and $n = \sqrt{\mu\epsilon/\mu_0\epsilon_0}$ for $z < 0$. A plane wave with wave vector $\boldsymbol{k}$ and frequency ω is incident from $z < 0$, medium μ and ϵ. The refracted and reflected wave have wave vectors $\boldsymbol{k}'$ and $\boldsymbol{k}''$, respectively. Define θ_i the incident angle, $\theta_{r'}$ the angle of reflection, θ_r the angle of refraction.

There are three waves.

- INCIDENT

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}, \tag{9.99}$$

$$\boldsymbol{B} = \boldsymbol{n}\times\frac{\boldsymbol{E}}{c} = \sqrt{\mu\epsilon}\frac{\boldsymbol{k}}{k}\times\boldsymbol{E}. \tag{9.100}$$

- REFRACTED

$$\boldsymbol{E}' = \boldsymbol{E}_0' e^{\mathrm{i}(\boldsymbol{k}'\cdot\boldsymbol{x}-\omega t)}, \tag{9.101}$$

$$\boldsymbol{B}' = \boldsymbol{n}'\times\frac{\boldsymbol{E}'}{c} = \sqrt{\mu'\epsilon'}\frac{\boldsymbol{k}'}{k'}\times\boldsymbol{E}'. \tag{9.102}$$

- REFLECTED

$$\boldsymbol{E}'' = \boldsymbol{E}_0'' e^{\mathrm{i}(\boldsymbol{k}''\cdot\boldsymbol{x}-\omega t)}, \tag{9.103}$$

$$\boldsymbol{B}'' = \boldsymbol{n}''\times\frac{\boldsymbol{E}''}{c} = \sqrt{\mu\epsilon}\frac{\boldsymbol{k}''}{k}\times\boldsymbol{E}''. \tag{9.104}$$

Note that here we have used $n'' = n = \sqrt{\mu\epsilon/\mu_0\epsilon_0}$ and $k'' = k = \omega\sqrt{\mu\epsilon/\mu_0\epsilon_0}/c = \omega\sqrt{\mu\epsilon}$. Of course, $k' = \omega\sqrt{\mu'\epsilon'/\mu_0\epsilon_0}/c = \omega\sqrt{\mu'\epsilon'}$.

The field in regions below $z = 0$ is $\boldsymbol{E} + \boldsymbol{E}''$ and $\boldsymbol{B} + \boldsymbol{B}''$, and above $z > 0$, $\boldsymbol{E}'$ and $\boldsymbol{B}'$. Boundary conditions (9.84)-(9.87) will be something like, at $z = 0$,

$$(\cdots)e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} + (\cdots)e^{\mathrm{i}(\boldsymbol{k}''\cdot\boldsymbol{x}-\omega t)} = (\cdots)e^{\mathrm{i}(\boldsymbol{k}'\cdot\boldsymbol{x}-\omega t)}. \tag{9.105}$$

This condition must be satisfied at all points, at all times, when $z = 0$. Therefore we must have that all the phase factors are the same, or

$$\boldsymbol{k}\cdot\boldsymbol{x} = \boldsymbol{k}''\cdot\boldsymbol{x} = \boldsymbol{k}'\cdot\boldsymbol{x}, \text{ at } z = 0, \tag{9.106}$$

or

$$k_x x + k_y y = k''_x x + k''_y y = k'_x x + k'_y y. \tag{9.107}$$

This equation will be satisfied for all points (x and y) only if

$$k_x = k''_x = k'_x, \tag{9.108}$$

and

$$k_y = k''_y = k'_y. \tag{9.109}$$

Let's orient our axes so that $\boldsymbol{k}$ is in the xz plane; i.e., $k_y = 0$. Then according to equation (9.109), $k''_y = k'_y = 0$; i.e., $\boldsymbol{k}''$ and $\boldsymbol{k}'$ are both in the xz plane. So we have the first conclusion:

> The incident, reflected, and refracted wave vectors form a plane (called the plane of incidence), which also includes the normal to the surface (here $\boldsymbol{e}_z$).

Second, Equation (9.108) gives

$$k \sin\theta_i = k'' \sin\theta_{r'} = k' \sin\theta_r. \tag{9.110}$$

Because $k = k''$, we immediate have that $\theta_i = \theta_{r'}$, or

> The angle of incidence equals the angle of reflection.

Regarding the angle of refraction θ_r, we have

$$\boxed{\frac{\sin\theta_i}{\sin\theta_r} = \frac{k'}{k} = \frac{n'}{n}.} \tag{9.111}$$

This is just *the law of refraction – Snell's law*.

To obtain intensity and phase relations, we need to use boundary conditions. Also for simplicity, we assume that the incident wave is linearly polarized. A more general elliptically polarized wave can be decomposed into two linearly polarized waves. We consider two cases, one is the polarization vector $\boldsymbol{E}$ is within the plane of incidence (xz plane). The second one with the polarization vector $\boldsymbol{E}$ normal to the plane of incidence.

We first consider $\boldsymbol{E}$ normal to plane of incidence. Let all $\boldsymbol{E}$ in y direction. The electric field of the incident wave is $\boldsymbol{E}_0 = E_0\boldsymbol{e}_y$, etc.

All $\boldsymbol{B}$ fields are then defined by $\boldsymbol{B} = \boldsymbol{n} \times \boldsymbol{E}/c$. Applying boundary conditions given by Equations (9.85) and (9.87), we have

$$E_0 + E_0'' - E_0' = 0, \tag{9.112}$$

$$\sqrt{\frac{\epsilon}{\mu}}(E_0 - E_0'')\cos\theta_i - \sqrt{\frac{\epsilon'}{\mu'}}E_0'\cos\theta_r = 0. \tag{9.113}$$

The other two boundary conditions yield nothing new. The relative amplitudes of the reflected and refracted waves are

$$\frac{E_0'}{E_0} = \frac{2n\cos\theta_i}{n\cos\theta_i + (\mu/\mu')\sqrt{n'^2 - n^2\sin^2\theta_i}}, \tag{9.114}$$

$$\frac{E_0''}{E_0} = \frac{n\cos\theta_i - (\mu/\mu')\sqrt{n'^2 - n^2\sin^2\theta_i}}{n\cos\theta_i + (\mu/\mu')\sqrt{n'^2 - n^2\sin^2\theta_i}}, \tag{9.115}$$

where we have used Snell's law. This is for $\boldsymbol{E}$ perpendicular to the plane of incidence.

Now if the $\boldsymbol{E}$ field is parallel to the plane of incidence. Let's now make all magnetic field components in $-\boldsymbol{e}_y$ direction. Boundary conditions (9.85) and (9.87) give

$$\cos\theta_i(E_0 - E_0'') - \cos\theta_r E_0' = 0, \tag{9.116}$$

$$\sqrt{\frac{\epsilon}{\mu}}(E_0 + E_0'') - \sqrt{\frac{\epsilon'}{\mu'}}E_0' = 0. \tag{9.117}$$

These can be combined to give

$$\frac{E_0'}{E_0} = \frac{2nn'\cos\theta_i}{(\mu/\mu')n'^2\cos\theta_i + n\sqrt{n'^2 - n^2\sin^2\theta_i}}, \tag{9.118}$$

$$\frac{E_0''}{E_0} = \frac{(\mu/\mu')n'^2\cos\theta_i - n\sqrt{n'^2 - n^2\sin^2\theta_i}}{(\mu/\mu')n'^2\cos\theta_i + n\sqrt{n'^2 - n^2\sin^2\theta_i}}. \tag{9.119}$$

For normal incidence $\theta_i = 0$, both cases give the same results

$$\frac{E_0'}{E_0} = \frac{2}{\sqrt{\mu\epsilon'/\mu'\epsilon} + 1}, \tag{9.120}$$

$$\frac{E_0''}{E_0} = \frac{\sqrt{\mu\epsilon'/\mu'\epsilon} - 1}{\sqrt{\mu\epsilon'/\mu'\epsilon} + 1}. \tag{9.121}$$

If further we assume $\mu' = \mu$, then

$$\frac{E_0'}{E_0} = \frac{2n}{n' + n}, \tag{9.122}$$

$$\frac{E_0''}{E_0} = \frac{n' - n}{n' + n}. \tag{9.123}$$

9.6.2 *Polarization by reflection*

There is an interesting thing about reflection. If $\boldsymbol{E}$ is in the plane of incidence, then from equation (9.119), you can see that there is a special angle of incidence, θ_B, at which the reflection total vanishes; i.e., $E_0'' = 0$. Now let's assume $\mu' = \mu$ for simplicity, this angle is given by

$$n'^2 \cos\theta_B - n\sqrt{n'^2 - n^2 \sin^2\theta_B} = 0, \tag{9.124}$$

which can be solved to give

$$\boxed{\theta_B = \tan^{-1}\left(\frac{n'}{n}\right).} \tag{9.125}$$

This angle is called *Brewster's angle*. Waves with $\boldsymbol{E}$ perpendicular to the plane of incidence do not show such behavior. Now consider a light with mixed polarization is incident at the Brewster angle, then the reflected light is completely polarized with $\boldsymbol{E}$ perpendicular to the plane of incidence. At other angles, the reflected light is partially polarized. This is the physics behind the circular polarizer widely used in photography or sunglasses to remove reflections or glares from water surface.

9.6.3 *Total internal reflection*

You are probably very familiar with *total internal reflection* from other cases. If $n > n'$, then there is angle of incidence θ_0, at which $\theta_r = \pi/2$. This follows directly from Snell's law; i.e.,

$$n \sin\theta_0 = n' \sin\left(\frac{\pi}{2}\right) = n' \Rightarrow \theta_0 = \sin^{-1}\left(\frac{n'}{n}\right). \tag{9.126}$$

At θ_0, the refracted wave has a $\boldsymbol{k}'$ parallel to the interface; there can be no energy flow across the surface. Hence when $\theta_i = \theta_0$, there must be total reflection.

But what about $\theta_i > \theta_0$? To solve this, we note that for $\theta_i > \theta_0$,

$$\sin\theta_r = \frac{n}{n'} \sin\theta_i > \frac{n}{n'} \sin\theta_0 = 1. \tag{9.127}$$

Therefore θ_r must be a complex angle, and its cosine is purely imaginary, given by

$$\cos\theta_r = \sqrt{1 - \sin^2\theta_r} = \mathrm{i}\sqrt{\left(\frac{\sin\theta_i}{\sin\theta_0}\right)^2 - 1}. \tag{9.128}$$

Putting $\cos\theta_r$ into $k_z' = \boldsymbol{k}' \cdot \boldsymbol{e}_z = k' \cos\theta_r$, we immediately get

$$\boldsymbol{k}' = k' \sin\theta_r \boldsymbol{e}_x + \mathrm{i}k'\sqrt{\left(\frac{\sin\theta_i}{\sin\theta_0}\right)^2 - 1}\boldsymbol{e}_z = k_x' \boldsymbol{e}_x + \mathrm{i}k_{zi}' \boldsymbol{e}_z. \tag{9.129}$$

That is, k_z' is now purely imaginary, and it equals $\mathrm{i}k_{zi}'$. Therefore, for refracted wave,

$$e^{\mathrm{i}\boldsymbol{k}' \cdot \boldsymbol{x}} = e^{-k_{zi}' z} e^{\mathrm{i}k_x' x}. \tag{9.130}$$

This equation shows that, for $\theta_i > \theta_0$, the refracted wave will be quickly damped/attenuated, typically within a few wavelengths.

You can also verify that there is no energy flow through the surface. The time-averaged Poynting vector along $\boldsymbol{e}_z$ is

$$\langle \boldsymbol{S} \rangle \cdot \boldsymbol{e}_z \propto \mathrm{Re}\left[\boldsymbol{e}_z \cdot (\boldsymbol{E}_0' \times \boldsymbol{H_0}'^{*})\right], \tag{9.131}$$

where $\boldsymbol{H}_0' = \boldsymbol{n}' \times \boldsymbol{E}_0'/\mu' c$. Average equation (9.131) gives

$$\langle \boldsymbol{S} \rangle \cdot \boldsymbol{e}_z \propto \mathrm{Re}\left[(\boldsymbol{e}_z \cdot \boldsymbol{n}')|\boldsymbol{E}_0'|^2\right], \tag{9.132}$$

But $\boldsymbol{e}_z \cdot \boldsymbol{n}' \propto \boldsymbol{e}_z \cdot \boldsymbol{k}'$ is purely imaginary, therefore $\langle \boldsymbol{S} \rangle \cdot \boldsymbol{e}_z = 0$.

9.7 *The frequency dependence of permittivity*

In the previous section, we assumed that ϵ and μ were independent of frequency, therefore normally $n \approx \sqrt{\epsilon_r}$ is also independent of frequency. In reality, however, all materials show some kind of *dispersion*: the dependence of refractive index on frequency. The medium is correspondingly called *dispersive*. In this section, we develop a simple model to describe the dependence of ϵ, therefore n, on frequency ω.

The equation of motion of a bounded electron (dielectric) is given by

$$\ddot{\boldsymbol{x}} + \gamma \dot{\boldsymbol{x}} + \omega_0^2 \boldsymbol{x} = -\frac{e}{m}\boldsymbol{E}(\boldsymbol{x}, t) = -\frac{e}{m}\boldsymbol{E}_0 e^{-\mathrm{i}\omega t}. \tag{9.133}$$

Here ω_0 is the natural frequency of the electron, and $\gamma \ll (\omega_0, \omega)$. We know this equation has a solution

$$\boldsymbol{x} = -\frac{e}{m}\frac{1}{\omega_0^2 - \omega^2 - \mathrm{i}\omega\gamma}\boldsymbol{E}. \tag{9.134}$$

The homogeneous solution is discarded due to attenuation. The corresponding dipole moment is

$$\boldsymbol{p} = -e\boldsymbol{x} = \frac{e^2}{m}\frac{1}{\omega_0^2 - \omega^2 - \mathrm{i}\omega\gamma}\boldsymbol{E}. \tag{9.135}$$

Now suppose that there are N molecules per unit volume and Z electrons per molecule. Instead of a single natural frequency for all electrons, there are f_j electrons per molecule with natural frequency ω_j and damping coefficient γ_j, then polarization vector $\boldsymbol{P}$ is

$$\boldsymbol{P} = \frac{Ne^2}{m}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - \mathrm{i}\omega\gamma_j}\boldsymbol{E} = \chi_e \epsilon_0 \boldsymbol{E}. \tag{9.136}$$

Where χ_e is electric susceptibility. Therefore the relative permittivity ϵ is

$$\epsilon_r = 1 + \chi_e = 1 + \frac{Ne^2}{m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - \mathrm{i}\omega\gamma_j}. \tag{9.137}$$

Now that ϵ_r is complex, correspondingly $n = \sqrt{\epsilon_r}$ and $k = n\omega/c$ are both complex. If $k = k_r + \mathrm{i}k_i$, then the wave is damped/attenuated if $k_i > 0$. In our case, because of energy conservation, we expect $k_i > 0$ because of a finite γ.

Normally, since the intensity of the wave is proportional to $Re(\boldsymbol{E})^2$, it's customary to define

$$k = \beta + \mathrm{i}\frac{\alpha}{2}, \tag{9.138}$$

so that the intensity of the wave falls off as $e^{-\alpha x}$. For gases, the second term on the RHS of equation (9.137) is small. From $\sqrt{1+x} \approx 1 + x/2$, approximately we have

$$n \approx \sqrt{\epsilon_r} \approx 1 + \frac{Ne^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - \mathrm{i}\omega\gamma_j} \tag{9.139}$$

Now we separate n into a real part and an imaginary part, $n = n_r + \mathrm{i}n_i$, or

$$n_r = 1 + \frac{Ne^2}{2m\epsilon_0}\sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + (\omega\gamma_j)^2}, \tag{9.140}$$

$$n_i = \frac{Ne^2\omega}{2m\epsilon_0}\sum_j \frac{f_j\gamma_j}{(\omega_j^2 - \omega^2)^2 + (\omega\gamma_j)^2} \tag{9.141}$$

Here n_r is the normal index of refraction, and n_i characterize the damping/growth of the wave. Putting $k = n\omega/c$ and using Equation (9.141), we have

$$\alpha = \frac{Ne^2\omega^2}{m\epsilon_0 c}\sum_j \frac{f_j\gamma_j}{(\omega_j^2 - \omega^2)^2 + (\omega\gamma_j)^2} \tag{9.142}$$

Therefore the index of refraction is given by Equation (9.141) and the damping coefficient is given by (9.142).

You can see by plotting Equation (9.141) that normally the index of refraction increases with frequency. However, in the immediate neighborhood of a resonance, the index of refraction drops sharply with increasing frequency; this behavior is called *anomalous dispersion*[7]. At the same time from (9.142), you see that α becomes very large near resonances, a large amount of energy is dissipated by the damping mechanism.

[7] The name "anomalous dispersion" is kind of misleading, because it is a very common phenomenon in many substances. This nomination is due to historical reasons. People firstly found "normal dispersion", the phenomenon that dispersion rate increases as wave frequency increases. Later people also found dispersion rate can also decrease as wave frequency increases, so it is named "anomalous dispersion".

If we stay away from resonances, the index of refraction can be approximately written as

$$n_r = 1 + \frac{Ne^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2}. \tag{9.143}$$

For most substances, natural frequencies ω_j are scattered all over the spectrum. For transparent materials, the nearest significant resonances typically lie in the ultraviolet, so that $\omega < \omega_j$. In this case,

$$\frac{1}{\omega_j^2 - \omega^2} \approx \frac{1}{\omega_j^2}\left(1 + \frac{\omega^2}{\omega_j^2}\right), \tag{9.144}$$

therefore,

$$n_r = 1 + \left(\frac{Ne^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2}\right) + \omega^2\left(\frac{Ne^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^4}\right). \tag{9.145}$$

In terms of the wavelength in vacuum ($\lambda = 2\pi c/\omega$):

$$n = 1 + \boldsymbol{A}(1 + \frac{\boldsymbol{B}}{\lambda^2}). \tag{9.146}$$

This is known as Cauchy's formula; the constant A is called the coefficient of refraction, and B is called the coefficient of dispersion. Cauchy's equation applies reasonably well to most gases, in the optical region.

Before we end this section, let's quickly discuss three limiting cases for ϵ.

Case 1): The low frequency limit $\omega \to 0$. If all $\omega_j \neq 0$ (like insulators), then you'll recover molecular polarizability if $\omega = 0$. Now we consider a different case. Suppose that some electrons are free (conductors), not bound to any molecules, then the natural frequencies of these electrons are 0. Use $j = 0$ to denote these electrons. If we separate the contribution of free electrons,

$$\epsilon_r(\omega) = 1 + \frac{Ne^2}{m\epsilon_0}\sum_{j\neq 0}\frac{f_j}{\omega_j^2 - \omega^2 - \mathrm{i}\omega\gamma_j} + \frac{Ne^2}{m\epsilon_0}\frac{f_0}{\omega_0^2 - \omega^2 - \mathrm{i}\omega\gamma_0} \tag{9.147}$$

$$\approx \epsilon_e + \mathrm{i}\frac{Ne^2 f_0}{m\epsilon_0\omega(\gamma_0 - \mathrm{i}\omega)}, \tag{9.148}$$

where ϵ_e are due to all bounded electrons ($j \neq 0$). Putting this ϵ_r into Faraday's law, we have, for conductors,

$$\nabla\times\boldsymbol{H} = \frac{\partial\boldsymbol{D}}{\partial t} = -\mathrm{i}\omega\epsilon_r(\omega)\epsilon_0\boldsymbol{E}. \tag{9.149}$$

On the other hand, for conductors, we can view ϵ_e as the normal dielectric constant due to bounded electrons. For free electrons, the current density can be expressed using Ohm's law $\boldsymbol{j_f} = \sigma\boldsymbol{E}$, where σ is the electric conductivity. Putting this into Ampere's law, we get

$$\nabla\times\boldsymbol{H} = \boldsymbol{j_f} + \frac{\partial\boldsymbol{D}}{\partial t} = -\mathrm{i}\omega\left(\epsilon_e\epsilon_0 + \mathrm{i}\frac{\sigma}{\omega}\right)\boldsymbol{E}. \tag{9.150}$$

Combining these two equations, we have

$$\sigma = \frac{Ne^2 f_0}{m(\gamma_0 - \mathrm{i}\omega)}. \tag{9.151}$$

This is the model of Drude for the electric conductivity, with $f_0 N$ being the number of free electrons per unit volume.

Case 2):The other limit is the high frequency limit. If $\omega \gg \omega_j$ for all j, then

$$\epsilon_r = 1 + \frac{Ne^2}{m\epsilon_0}\sum_j\frac{f_j}{\omega_j^2 - \omega^2 - \mathrm{i}\omega\gamma_j} \approx 1 - \frac{NZe^2}{m\epsilon_0}\frac{1}{\omega^2} \equiv 1 - \frac{\omega_p^2}{\omega^2}. \tag{9.152}$$

Here

$$\omega_p^2 = \frac{NZe^2}{m\epsilon_0}, \tag{9.153}$$

is the very famous *plasma frequency* of the medium. The corresponding dispersion relation is, using $n^2 = \epsilon_r$,

$$\omega^2 = \omega_p^2 + c^2k^2. \tag{9.154}$$

This is valid only for $\omega \gg \omega_p$ in our cases. In actual plasmas, if $\omega < \omega_p$, the wave number $k = \sqrt{\omega^2 - \omega_p^2}/c$ is imaginary, and the refracted wave is attenuated. You can then use this to measure the plasma density. For $\omega < \omega_p$, the wave is reflected.

9.8 Electromagnetic Waves in Conductors

As we have discussed in the previous section, the dielectric constant ϵ is normally complex. The imaginary part of ϵ can be neglected for insulators for many purposes. However, for conductors, the imaginary part of ϵ becomes important. It is customary to separate the complex dielectric constant of the conducting medium formally into two parts: a real dielectric constant ϵ and a real conductivity σ, as we did in the last section, case 1).

Now with these in mind, we write the Maxwell equations for conductors, assuming μ and ϵ.

$$\nabla \cdot \boldsymbol{E} = \frac{\rho_f}{\epsilon}, \tag{9.155}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{9.156}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{9.157}$$

$$\nabla \times \boldsymbol{B} = \mu\epsilon \frac{\partial \boldsymbol{E}}{\partial t} + \mu\sigma\boldsymbol{E}. \tag{9.158}$$

Here we have used $\boldsymbol{j}_f = \sigma \boldsymbol{E}$.

Now consider the continuity equation and Gauss' law (Equation (9.155)), we have

$$\frac{\partial \rho_f}{\partial t} = -\nabla \cdot \boldsymbol{j}_f = -\sigma\left(\nabla \cdot \boldsymbol{E}\right) = -\frac{\sigma}{\epsilon}\rho_f. \tag{9.159}$$

This equation can be solved to give

$$\rho_f = \rho_{f0} e^{-t/\tau}, \tag{9.160}$$

where $\tau = \epsilon/\sigma$. This equation clearly shows that any free charge will disappear within a time scale of τ. For a "perfect" conductor, $\sigma = \infty$, and $\tau = 0$. In reality, we can compare τ with the relevant time scale to measure how good a conductor is. The relevant time scale for a oscillatory system with frequency ω is $1/\omega$. For "good" conductors, $\tau \ll 1/\omega$. For "poor" conductors, $\tau \gg 1/\omega$. These two equations can also be written like this:

$$\begin{cases} \dfrac{\sigma}{\epsilon\omega} & \gg 1, \quad for''good''conductors \\ \dfrac{\sigma}{\epsilon\omega} & \ll 1, \quad for''poor''conductors \end{cases} \tag{9.161}$$

Now let's consider good conductors, where τ is typically very small, on the order of 10^{-14} or smaller. Then we are not interested in the transient behavior of ρ explained above. We wait for any accumulated free charge to disappear. Then $\rho_f = 0$, and the Maxwell equations are

$$\nabla \cdot \boldsymbol{E} = 0, \tag{9.162}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{9.163}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{9.164}$$

$$\nabla \times \boldsymbol{B} = \mu\epsilon\frac{\partial \boldsymbol{E}}{\partial t} + \mu\sigma\boldsymbol{E}. \tag{9.165}$$

These equations are just like the Maxwell equations we discussed in the previous section, except the extra term in Ampere' law.

As before, we can obtain wave equations for $\boldsymbol{E}$ and $\boldsymbol{B}$. Both $\boldsymbol{E}$ and $\boldsymbol{B}$ satisfy

$$\nabla^2 \boldsymbol{E} = \mu\epsilon\frac{\partial^2 \boldsymbol{E}}{\partial t^2} + \mu\sigma\frac{\partial \boldsymbol{E}}{\partial t}, \tag{9.166}$$

$$\nabla^2 \boldsymbol{B} = \mu\epsilon\frac{\partial^2 \boldsymbol{B}}{\partial t^2} + \mu\sigma\frac{\partial \boldsymbol{B}}{\partial t}. \tag{9.167}$$

Assuming $\boldsymbol{E} = \boldsymbol{E}_0 e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}$ and similarly for $\boldsymbol{B}$, we have a complex k, which is given by

$$k^2 = \mu\epsilon\omega^2 + \mathrm{i}\mu\sigma\omega. \tag{9.168}$$

Let $k = \beta + \mathrm{i}\alpha/2$, then we get

$$\beta = \sqrt{\frac{\mu\epsilon}{2}}\omega\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2}, \tag{9.169}$$

$$\frac{\alpha}{2} = \sqrt{\frac{\mu\epsilon}{2}}\omega\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2}. \tag{9.170}$$

For a poor conductor, $\sigma/\epsilon\omega \ll 1$, then

$$k \approx \sqrt{\mu\epsilon}\omega + \mathrm{i}\frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}. \tag{9.171}$$

In this case, $\mathrm{Re}(k) \gg \mathrm{Im}(k)$. For a good conductor, however, $\sigma/\epsilon\omega \gg 1$, $\beta \approx \alpha/2$, and

$$k = (1+\mathrm{i})\sqrt{\mu\sigma\omega/2} \tag{9.172}$$

The imaginary part of k, as explained before, means the attenuation of the wave. The distance it takes to reduce the amplitude by a factor of $1/e \approx 1/3$ is called the skin depth δ, given by

$$\delta = \frac{2}{\alpha} \approx \sqrt{\frac{2}{\mu\sigma\omega}}. \tag{9.173}$$

This equation is valid only for good conductors. For a conductor like copper, $\delta \approx 0.85\,\mathrm{cm}$ for 60 Hz frequency wave, and $0.71 \times 10^{-3}\,\mathrm{cm}$ for a 100 MHz wave.

The whole calculation above for k can be obtained simply by using a complex reflective index $n = \sqrt{\mu\hat{\epsilon}/\mu_0\epsilon_0}$ with $\hat{\epsilon} = \epsilon + \mathrm{i}\sigma/\omega$(see formula (1.158)), and $k = n\omega/c$.

We can obtain the relation between $\boldsymbol{B}$ and $\boldsymbol{E}$ using Faraday's' law. For a good conductor,

$$\boldsymbol{B} = \boldsymbol{n} \times \frac{\boldsymbol{E}}{c} = \frac{c\boldsymbol{k}}{\omega} \times \frac{\boldsymbol{E}}{c} = \frac{1}{\omega}\sqrt{\mu\sigma\omega/2}(1+\mathrm{i})\boldsymbol{e}_k \times \boldsymbol{E}, \tag{9.174}$$

which is

$$\boldsymbol{B} = \sqrt{\frac{\mu\sigma}{\omega}}\, e^{\mathrm{i}\pi/4}\boldsymbol{e}_k \times \boldsymbol{E}. \tag{9.175}$$

Therefore, $\boldsymbol{B}$ lags $\boldsymbol{E}$ in phase by $\pi/4$ for a good conductor. Note that the energy density in a linear media is

$$w = \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{B}\cdot\boldsymbol{H}) = \frac{1}{2}(\epsilon E^2 + \frac{1}{\mu}B^2). \tag{9.176}$$

Hence in a good conductor, the ratio of magnetic energy density to electric energy density is

$$\frac{w_B}{w_E} = \frac{B^2/\mu}{\epsilon E^2} = \frac{\sigma}{\epsilon\omega} \gg 1. \tag{9.177}$$

Therefore most energy is in the magnetic field.

9.9 *Reflection at a conducting surface*

We only deal with normal incidence from vacuum to a good conductor here for simplicity. The relevant boundary conditions are

$$\boldsymbol{E}_1^{\|} = \boldsymbol{E}_2^{\|}, \tag{9.178}$$

$$\boldsymbol{H}_1^{\|} - \boldsymbol{H}_1^{\|} = \boldsymbol{K}_f \times \boldsymbol{n}. \tag{9.179}$$

Now for a good conductor with finite σ, $\boldsymbol{j}_f = \sigma\boldsymbol{E}$. This indicates you cannot get a δ-function like free surface current ($\sigma \to 0$), unless you have $\boldsymbol{E}$ infinite [8]. The boundary conditions for good conductors are

$$\boldsymbol{E}_1^{\|} = \boldsymbol{E}_2^{\|}, \tag{9.180}$$

$$\boldsymbol{H}_1^{\|} = \boldsymbol{H}_1^{\|}. \tag{9.181}$$

Using notations from the previous chapter and sessions, we have

$$E_0 + E_0'' = E_0', \tag{9.182}$$

$$H - H'' = H'. \tag{9.183}$$

Assuming $\mu_r \approx \mu_r' \approx 1$ and using

$$H' = \sqrt{\frac{\sigma}{2\mu_0\omega}}(1+\mathrm{i})E_0', \tag{9.184}$$

$$H = E/\mu_0 c, \tag{9.185}$$

$$H'' = E''/\mu_0 c. \tag{9.186}$$

[8] In case of a perfect conductor, currents are confined to the surface and there is true surface current.

we write Equation (9.183) as

$$E_0 - E_0'' = c\sqrt{\frac{\mu_0\sigma}{2\omega}}(1+\mathrm{i})E_0'. \tag{9.187}$$

Solving Equation (9.187) and (9.182) together, we get

$$\frac{E_0''}{E_0} = -\frac{1+\mathrm{i}-\sqrt{2\omega/\mu_0\sigma}/c}{1+\mathrm{i}+\sqrt{2\omega/\mu_0\sigma}/c}. \tag{9.188}$$

The corresponding reflection coefficient for normal incidence is

$$R = \left|\frac{E_0''}{E_0}\right|^2 = \frac{\left(1-\sqrt{2\omega/\mu_0\sigma}/c\right)^2+1}{\left(1+\sqrt{2\omega/\mu_0\sigma}/c\right)^2+1} \approx 1-\frac{2}{c}\sqrt{\frac{2\omega}{\mu_0\sigma}} = 1-2\sqrt{\frac{2\omega\epsilon_0}{\sigma}}. \tag{9.189}$$

For perfect conductors, $\sigma \to \infty$, and $R \to 1$. This is why excellent conductors make good mirrors[9].

[9] Check the back of your mirrors at home, see what that is (most likely some kind of silver).

9.10 Wave guides and resonant cavities

So far, we have only considered plane waves in infinite medium. Now we consider electromagnetic waves confined to hollow metallic pipe. If the pipe has end surfaces, it is called a *cavity*; otherwise, a *wave guide*. We assume that the boundaries are made of perfect conductors. We'll only deal with pipes those have rectangular cross sections, and we assume that the cross sectional size and shape are constant. One of the main purposes of this section is to demonstrate that boundaries can cause the change of wave properties. One main change is that confined waves, in general, are not transverse.

We assume that the pipe is filled with a uniform non-dissipative medium with dielectric constant ϵ and permeability μ. Then the Maxwell equations inside the medium are, assuming $e^{-\mathrm{i}\omega t}$ dependence,

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} = \mathrm{i}\omega\boldsymbol{B}, \tag{9.190}$$

$$\nabla\cdot\boldsymbol{E} = 0, \tag{9.191}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{9.192}$$

$$\nabla\times\boldsymbol{B} = \mu\epsilon\frac{\partial\boldsymbol{E}}{\partial t} = -\mathrm{i}\mu\epsilon\omega\boldsymbol{E}. \tag{9.193}$$

These equations are the same as the equations in the previous chapter, and the equations for $\boldsymbol{E}$ and $\boldsymbol{B}$ are

$$\left(\nabla^2+\mu\epsilon\omega^2\right)\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{B}\end{pmatrix} = 0, \tag{9.194}$$

subject to boundary conditions

$$\boldsymbol{E}^{\parallel} = 0, \tag{9.195}$$

$$B^{\perp} = 0. \tag{9.196}$$

Note that we do not assume the $e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ form for the spatial dependence, as it depends on the type of boundary conditions.

9.10.1 *Wave guides*

Suppose the direction along pipe (in case of a wave guide) is z. In this case, waves can propagate along z direction; therefore, we assume

$$\boldsymbol{E}(x,y,z,t) = \boldsymbol{E}(x,y)e^{\mathrm{i}(kz-\omega t)}, \tag{9.197}$$

$$\boldsymbol{B}(x,y,z,t) = \boldsymbol{B}(x,y)e^{\mathrm{i}(kz-\omega t)}. \tag{9.198}$$

The field equation (9.194) reduces to two dimensions,

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\mu\epsilon\omega^2 - k^2\right)\right]\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{B}\end{pmatrix} = 0. \tag{9.199}$$

You can use separation of variables to solve directly this equation. But before that let's write out the components of Maxwell equations – more specifically, the Faraday's law and the Ampere's law. Then we'll demonstrate that the E_x and E_y are completely determined if we specify E_z and B_z.

The Faraday's law and Ampere's law give

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = \mathrm{i}\omega B_z, \tag{9.200}$$

$$\frac{\partial E_z}{\partial y} - \mathrm{i}kE_y = \mathrm{i}\omega B_x, \tag{9.201}$$

$$\mathrm{i}kE_x - \frac{\partial E_z}{\partial x} = \mathrm{i}\omega B_y, \tag{9.202}$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\mathrm{i}\mu\epsilon\omega E_z, \tag{9.203}$$

$$\frac{\partial B_z}{\partial y} - \mathrm{i}kB_y = -\mathrm{i}\mu\epsilon\omega E_x, \tag{9.204}$$

$$\mathrm{i}kB_x - \frac{\partial B_z}{\partial x} = -\mathrm{i}\mu\epsilon\omega E_y. \tag{9.205}$$

Equations (9.201), (9.202), (9.204) and (9.205) can be combined to give

$$E_x = \frac{\mathrm{i}}{\mu\epsilon\omega^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right), \tag{9.206}$$

$$E_y = \frac{\mathrm{i}}{\mu\epsilon\omega^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right), \tag{9.207}$$

$$B_x = \frac{\mathrm{i}}{\mu\epsilon\omega^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \mu\epsilon\omega\frac{\partial E_z}{\partial y}\right), \tag{9.208}$$

$$B_y = \frac{\mathrm{i}}{\mu\epsilon\omega^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \mu\epsilon\omega\frac{\partial E_z}{\partial x}\right). \tag{9.209}$$

It's clear from these equations that once we specify E_z and B_z, the x,y components of $\boldsymbol{E}$ and $\boldsymbol{B}$ are completely determined. The equations for E_z and B_z are simply obtained from Equation (9.199),

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\mu\epsilon\omega^2 - k^2\right)\right]\begin{pmatrix}E_z\\ B_z\end{pmatrix} = 0. \tag{9.210}$$

Depending on E_z and B_z, there are three types of waves in a wave guide. If $E_z = 0$, we call these *TE* ("transverse electric") waves;

if $B_z = 0$, we call them *TM* ("transverse magnetic") waves. If both $E_z = 0$ and $B_z = 0$, we call them *TEM* waves.

Note that TEM waves cannot occur in a hollow wave guide. If $E_z = 0$ and $B_z = 0$, you cannot use Equations (9.206)-(9.209); they are indeterminate. So we go back to Maxwell equations. From Gauss's law,

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0. \tag{9.211}$$

If $B_z = 0$, Faraday's law gives

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0. \tag{9.212}$$

The vector $\boldsymbol{E} = E_x \boldsymbol{e}_x + E_y \boldsymbol{e}_y$ has zero divergence and zero curl. Therefore it can be written as a scalar potential that satisfies Laplace's equation. But the boundary condition on $\boldsymbol{E}$ requires $\boldsymbol{E}^{\parallel} = 0$, or the boundary surface should be an equipotential. Because Laplace's equation doesn't allow local extreme values, extreme values only occur on boundaries. Therefore, the scalar potential must be a constant inside the domain, and the electric field $\boldsymbol{E} = 0$. *Note that this argument only applies to a completely hollow pipe. If we have a separate conductor down the middle (e.g., a coaxial transmission line), the potential at its surface need not be the same as on the outer wall, and hence a nontrivial solution of potential is possible, allowing* $\boldsymbol{E} \neq 0$.

9.10.2 *TE waves in a rectangular wave guide*

Suppose we have a rectangular wave guide with height a and width b, and we want to know the propagation of TE waves. From discussions in previous section, we only need to solve

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\mu\epsilon\omega^2 - k^2\right)\right] B_z = 0. \tag{9.213}$$

Because $B^{\perp} = 0$, therefore from Equations (9.208) and (9.209), we have boundary conditions

$$\frac{\partial B_z}{\partial x} = 0, \text{ at } x = 0, a, \tag{9.214}$$

$$\frac{\partial B_z}{\partial y} = 0, \text{ at } y = 0, b. \tag{9.215}$$

The equation (9.213) subject to boundary conditions (9.214) forms a typical eigenvalue problem.

Using separation of variables, $B_z(x, y) = X(x)Y(y)$, then

$$\frac{1}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \frac{1}{Y}\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} + \mu\epsilon\omega^2 - k^2 = 0. \tag{9.216}$$

Therefore, we have

$$\frac{1}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = -k_x^2, \tag{9.217}$$

$$\frac{1}{Y}\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = -k_y^2, \tag{9.218}$$

$$k_x^2 + k_y^2 + k^2 = \mu\epsilon\omega^2. \tag{9.219}$$

The general solutions of X and Y are

$$X(x) = A\sin(k_x x) + B\cos(k_x x), \tag{9.220}$$
$$Y(y) = C\sin(k_y y) + D\cos(k_y y). \tag{9.221}$$

Consider boundary conditions

$$X(x) = B\cos(k_x x), \text{ with } k_x = m\pi/a, \tag{9.222}$$
$$Y(y) = D\cos(k_y y), \text{ with } k_y = n\pi/b. \tag{9.223}$$

Here $m = 0, 1, 2, 3, \cdots$, and $n = 0, 1, 2, 3, \cdots$. Therefore,

$$B_z = B_0\cos(m\pi x/a)\cos(n\pi y/b). \tag{9.224}$$

This solution is called the $\mathrm{TE_{mn}}$ mode. Conventionally $m \geq n$. Note that m and n cannot be both 0. The wave number of $\mathrm{TE_{mn}}$ mode is

$$k = \sqrt{\mu\epsilon\omega^2 - \pi^2[(m/a)^2 + (n/b)^2]}. \tag{9.225}$$

If

$$\omega < \frac{1}{\sqrt{\mu\epsilon}}\pi\sqrt{(m/a)^2 + (n/b)^2} \equiv \omega_{mn}, \tag{9.226}$$

the wave number becomes imaginary, and the wave cannot propagate far. Therefore ω_{mn} is called the *cutoff frequency* for $\mathrm{TE_{mn}}$ mode. The *lowest* cutoff frequency for a given wave guide is clearly

$$\boxed{\omega_{10} = \frac{\pi}{\sqrt{\mu\epsilon}a}.} \tag{9.227}$$

Frequencies below this will not propagate at all in the given wave guide.

The wave number k can be written using the cutoff frequency as

$$k = \sqrt{\mu\epsilon}\sqrt{\omega^2 - \omega_{mn}^2}. \tag{9.228}$$

The wave phase velocity is

$$v_{ph} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}\sqrt{1 - (\omega_{mn}/\omega)^2}}. \tag{9.229}$$

The group velocity is

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{1}{\mathrm{d}k/\mathrm{d}\omega} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{1 - (\omega_{mn}/\omega)^2}. \tag{9.230}$$

If $\mu = \epsilon = 1$, then $v_{ph} > c$ and $v_g < c$.

10
Fields of Moving Charges

Starting from this chapter, we discuss how moving charges generate fields.

10.1 Potentials of general sources

To consider fields generated by general sources, we start from the four-dimensional Maxwell equations and find the potentials,

$$\frac{\partial F^{\alpha\beta}}{\partial x^{\beta}} = \mu_0 j^{\alpha}. \tag{10.1}$$

In terms of $A^{\alpha} = (\phi/c, \boldsymbol{A})$, these equations are

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu_0 \boldsymbol{j}, \tag{10.2}$$

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}. \tag{10.3}$$

Operator

$$\Box \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{10.4}$$

is the d'Alembert operator.

If the fields are static, then

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{j}, \tag{10.5}$$

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}. \tag{10.6}$$

These are static field equations. On the other hand, if $\rho = 0$ and $\boldsymbol{j} = 0$, then

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0, \tag{10.7}$$

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0. \tag{10.8}$$

These are homogeneous wave equations, or wave equations in vacuum. We have dealt with both cases in previous chapters.

The general solution of an inhomogeneous equation is the sum of a special solution (or called a particular solution) and a general solution of the corresponding homogeneous equation. Let's take the ϕ equation for example. The solution is given by

$$\phi = \phi_1 + \phi_0, \tag{10.9}$$

where ϕ_1 is a special solution, and ϕ_0 is the general solution of the homogeneous equation (10.8).

To find the special/particular solution, we use the principle of superposition. We start from considering the potential produced by a point charge $\mathrm{d}q(t)$ at $\boldsymbol{x}'$. The charge density of this charge is $\rho = \mathrm{d}q(t)\delta(\boldsymbol{R})$, with $\boldsymbol{R} = \boldsymbol{x} - \boldsymbol{x}'$. The corresponding ϕ-equation (10.3) can be written as

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\mathrm{d}q(t)}{\epsilon_0}\delta(\boldsymbol{R}). \tag{10.10}$$

Everywhere with $\boldsymbol{R} \neq 0$, Equation (10.10) is

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0. \tag{10.11}$$

The method here is to first solve Equation (10.11) for ϕ at $\boldsymbol{R} \neq 0$, then we match the this solution to ϕ near $\boldsymbol{R} = 0$.

Clearly for a point charge $\mathrm{d}q(t)$, ϕ has central symmetry, i.e., $\phi = \phi(R)$. Using spherical coordinates and considering $\phi = \phi(R)$, we write Equation (10.11) as

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\phi}{\partial R}\right) - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0. \tag{10.12}$$

Letting $\phi(R,t) = \chi(R,t)/R$, Equation (10.12) becomes

$$\frac{\partial^2\chi}{\partial R^2} - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2} = 0. \tag{10.13}$$

This is the equation of plane waves, whose general solution is

$$\chi(R,t) = f(t - R/c) + g(t + R/c). \tag{10.14}$$

Since we only want a particular solution, we can choose either f or g for χ. Note that f represents waves propagating in $+\boldsymbol{R}$ direction, and g represents waves propagating in $-\boldsymbol{R}$ direction. Most of the time, we're interested in fields emitted/radiated, so we choose $\chi = f(t - R/c)$, and

$$\phi = \frac{\chi(t - R/c)}{R}. \tag{10.15}$$

The form of χ is still not known; we determine χ by match ϕ near $\boldsymbol{R} = 0$.

Near the origin, there are various ways to determine χ. We'll introduce Landau's method. As $R \to 0$, $\phi \to \infty$, therefore spatial variation dominates the temporal variation. Neglecting the time derivative, the ϕ equation near $\boldsymbol{R} = 0$ is

$$\nabla^2\phi = -\frac{\mathrm{d}q(t)}{\epsilon_0}\delta(\boldsymbol{R}), \quad \text{near } \boldsymbol{R} = 0. \tag{10.16}$$

Note that this is the electrostatic potential equation for a point charge $\mathrm{d}q$. From Coulomb's law, we immediately have

$$\phi = \frac{\mathrm{d}q(t)}{4\pi\epsilon_0 R} \quad \text{near } \boldsymbol{R} = 0. \tag{10.17}$$

Matching ϕ in Equation (10.15) to this ϕ near $\boldsymbol{R} = 0$ yields

$$\phi = \frac{\mathrm{d}q(t - R/c)}{4\pi\epsilon_0 R} \quad \text{for all } \boldsymbol{R}. \tag{10.18}$$

For an arbitrary distribution of charges, the ϕ-equation solution is just

$$\phi(\boldsymbol{x}, t) = \int \frac{1}{4\pi\epsilon_0 R}\, \rho\left(\boldsymbol{x}', t - \frac{R}{c}\right) \mathrm{d}V' + \phi_0. \tag{10.19}$$

Here $\boldsymbol{x}$ is the field point vector, and $\mathrm{d}V' = \mathrm{d}x'\mathrm{d}y'\mathrm{d}z'$. Similarly, the solution of $\boldsymbol{A}$ is

$$\boldsymbol{A}(\boldsymbol{x}, t) = \int \frac{\mu_0}{4\pi\boldsymbol{R}}\, \boldsymbol{j}\left(\boldsymbol{x}', t - \frac{R}{c}\right) \mathrm{d}V' + \boldsymbol{A}_0. \tag{10.20}$$

Let's now discuss the physical implications of these solutions. The particular solutions of ϕ and $\boldsymbol{A}$ we have found,

$$\phi_1(\boldsymbol{x}, t) = \int \frac{1}{4\pi\epsilon_0 R}\, \rho\left(\boldsymbol{x}', t - \frac{R}{c}\right) \mathrm{d}V', \tag{10.21}$$

$$\boldsymbol{A}_1(\boldsymbol{x}, t) = \int \frac{\mu_0}{4\pi R}\, \boldsymbol{j}\left(\boldsymbol{x}', t - \frac{R}{c}\right) \mathrm{d}V', \tag{10.22}$$

are called "retarded potentials" because of the factor $t - R/c$. From now on, I will only discuss ϕ for simplicity.

Define $t' \equiv t - R/c$, $\phi(t)$ is determined by $\rho(t')$; here t' is called the "retarded time". In words, ϕ at current time t is determined by ρ at a previous time t' [1]. This is because it takes electromagnetic signals a finite amount of time to propagate from $\boldsymbol{x}'$ to $\boldsymbol{x}$; this time is R/c. We've learned this from the special theory of relativity: the maximum propagation speed of a signal is c. Now consider two points in the source $\boldsymbol{x}_1'$ and $\boldsymbol{x}_2'$, whose corresponding distances to $\boldsymbol{x}$ are R_1 and R_2. The potential $\phi(\boldsymbol{x}, t)$ is determined by $\rho(\boldsymbol{x}_1', t - R_1/c)$ and $\rho(\boldsymbol{x}_2', t - R_2/c)$: different points at different "previous" times.

[1] An example of the retarded potentials might help you better understand the concept: the sun light you see "now" is produced by moving charges at the sun at about 8 minutes ago; since it takes light roughly 8 minutes to propagate from the sun to Earth.

The general solutions of homogeneous equations ϕ_0 and $\boldsymbol{A}_0$ are to be determined by boundary/initial conditions. An important use of ϕ_0 and $\boldsymbol{A}_0$ is in this problem: First, an external field is incident on the system from outside. Then the system reacts and emits new radiation. In this case, ϕ_0 and $\boldsymbol{A}_0$ are used to match the external field. And the retarded potentials represent the new radiation. From now on, we'll consider only retarded potentials.

In static case, ρ and $\boldsymbol{j}$ are independent of time, and the retarded potentials become

$$\phi(\boldsymbol{x}, t) = \int \frac{1}{4\pi\epsilon_0 R}\, \rho\left(\boldsymbol{x}'\right) \mathrm{d}V', \tag{10.23}$$

$$\boldsymbol{A}(\boldsymbol{x}, t) = \int \frac{\mu_0}{4\pi R}\, \boldsymbol{j}\left(\boldsymbol{x}'\right) \mathrm{d}V'. \tag{10.24}$$

These are just the Coulomb's law and the Biot-Savart law.

Compare these equations with the retarded potentials, all we did was putting in factor $t - R/c$ on ρ and $\boldsymbol{j}$ in place of t. These "tricks", however, do not apply in general to $\boldsymbol{E}$ and $\boldsymbol{B}$ fields.

To derive the general form of $\boldsymbol{E}$ and $\boldsymbol{B}$ from retarded potentials, recall that

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}, \tag{10.25}$$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}. \tag{10.26}$$

The spatial derivative is with respect to $\boldsymbol{x}$, the field point. Retarded potentials have the form,

$$\phi(\boldsymbol{x}, t) = \int \frac{1}{4\pi\epsilon_0 R}\,\rho\left(\boldsymbol{x}', t - \frac{R}{c}\right) \mathrm{d}V', \tag{10.27}$$

$$\boldsymbol{A}(\boldsymbol{x}, t) = \int \frac{\mu_0}{4\pi R}\,\boldsymbol{j}\left(\boldsymbol{x}', t - \frac{R}{c}\right) \mathrm{d}V'. \tag{10.28}$$

Note that ϕ and $\boldsymbol{A}$ depend on $\boldsymbol{x}$ in two places:

1. explicitly through $R = |\boldsymbol{x} - \boldsymbol{x}'|$.
2. implicitly through $t' = t - R/c$.

Left as an exercise, you can prove that

$$\begin{aligned}\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \Bigg[\left(\frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{1}{|\mathbf{r} - \mathbf{r}'|^2 c} \frac{\partial\rho(\mathbf{r}', t')}{\partial t} \right) (\mathbf{r} - \mathbf{r}') \\ - \frac{1}{|\mathbf{r} - \mathbf{r}'| c^2} \frac{\partial \mathbf{j}(\mathbf{r}', t')}{\partial t} \Bigg] \mathrm{d}^3\mathbf{r}\end{aligned} \tag{10.29}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \left[\frac{\mathbf{j}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{1}{|\mathbf{r} - \mathbf{r}'|^2 c} \frac{\partial \mathbf{j}(\mathbf{r}', t')}{\partial t} \right] \times (\mathbf{r} - \mathbf{r}') \mathrm{d}^3\mathbf{r} \tag{10.30}$$

These are called Jefimenko's equations. But as it's typically easier to calculate fields from retarded potentials than using above equations, they are of limited utility.

10.2 The Lienard-Wiechert potentials

We consider the potential of an arbitrarily moving point charge q. Suppose the trajectory is given by $\boldsymbol{x}'(t)$ and velocity $\boldsymbol{v}(t)$. From retarded potentials, $\phi(\boldsymbol{x}, t)$ and $\boldsymbol{A}(\boldsymbol{x}, t)$ are determined by $\boldsymbol{x}'(t')$ and $\boldsymbol{v}(t')$.

One way to obtain potentials of an arbitrarily moving charge is to use results from the previous section and consider a point charge as the source. We, however, choose to use Lorentz transformation to find $\boldsymbol{A}$ and ϕ just as we did for a uniformly moving charge. The main difference is that K' now moves with $\boldsymbol{v}(t')$ and this velocity itself is not constant. Therefore, K' is an instantaneous inertial reference frame.

Let's discuss a little more about the retarded time t'. For a point charge, the retarded time t' is determined by

$$t' = t - R/c \quad \text{or} \quad |\boldsymbol{x} - \boldsymbol{x}'(t')| = c(t' - t). \tag{10.31}$$

We can find out t' by solving this equation, but how do you know there is only one root? Let's now prove that.

Suppose there are two roots t'_1 and t'_2, then

$$R_1 = c(t - t'_1) \quad \text{and} \quad R_2 = c(t - t'_2), \tag{10.32}$$

from which $R_1 - R_2 = c(t'_2 - t'_1)$. Consider $\boldsymbol{x}, \boldsymbol{x}'(t'_1), \boldsymbol{x}'(t'_2)$ as the three sides of a triangle, clearly

$$R_1 - R_2 = |\boldsymbol{x} - \boldsymbol{x}'(t'_1)| - |\boldsymbol{x} - \boldsymbol{x}'(t'_2)| \leq |\boldsymbol{x}'(t'_2) - \boldsymbol{x}'(t'_1)|, \tag{10.33}$$

hence $|\boldsymbol{x}'(t'_2) - \boldsymbol{x}'(t'_1)| \geq c(t'_2 - t'_1)$, or

$$\frac{|\boldsymbol{x}'(t'_2) - \boldsymbol{x}'(t'_1)|}{t'_2 - t'_1} \geq c. \tag{10.34}$$

This is impossible from Einstein's principle of relativity: no charged particles can travel at or faster than c. Therefore for a given field point $\boldsymbol{x}$ and t, there is only one t'.

The potentials at $\boldsymbol{x}$ and t are determined by $\boldsymbol{x}'(t')$ and $\boldsymbol{v}(t')$. We consider an instantaneous inertial reference frame $\widetilde{K}$ moving with $\boldsymbol{v}(t')$ relative to K, the laboratory frame.

First, we calculate potentials in $\widetilde{K}$, where the particle is at rest. Therefore $\widetilde{\phi}$ and $\widetilde{\boldsymbol{A}}$ are

$$\widetilde{\phi} = \frac{q}{4\pi\epsilon_0 \widetilde{R}}, \quad \text{and} \quad \widetilde{\boldsymbol{A}} = 0. \tag{10.35}$$

Here $\widetilde{R} = c(\widetilde{t} - \widetilde{t'})$ is the distance between $\boldsymbol{x}$ and $\boldsymbol{x}'$ in $\widetilde{K}$.

Now we can transform from $\widetilde{K}$ to K to find ϕ and $\boldsymbol{A}$. Since $\widetilde{K}$ moves with $\boldsymbol{v}(t')$ relative to K,

$$\phi = \frac{\widetilde{\phi}}{\sqrt{1 - v(t')^2/c^2}} = \frac{1}{\sqrt{1 - v(t')^2/c^2}} \frac{q}{4\pi\epsilon_0 \widetilde{R}}, \tag{10.36}$$

$$\boldsymbol{A} = \frac{(\boldsymbol{v}(t')/c^2)\widetilde{\phi}}{\sqrt{1 - v(t')^2/c^2}} = \frac{\boldsymbol{v}(t')}{c^2}\phi. \tag{10.37}$$

We now have the same problem we once faced in Section 4.7.3: ϕ and $\boldsymbol{A}$ are expressed using $\widetilde{R}$ instead of R, the spatial coordinates in K. Hence we need to express $\widetilde{R}$ in terms of R.

Using the Lorentz transform to express $\widetilde{t}$ in terms of t and $\boldsymbol{x}$ gives

$$\widetilde{R} = c(\widetilde{t} - \widetilde{t'}) = \frac{c}{\sqrt{1 - v(t')^2/c^2}} \left[t - t' - \frac{\boldsymbol{v}(t')}{c^2} \cdot (\boldsymbol{x} - \boldsymbol{x}') \right], \tag{10.38}$$

$$= \frac{1}{\sqrt{1 - v^2/c^2}} \left[c(t - t') - \frac{\boldsymbol{v}}{c} \cdot (\boldsymbol{x} - \boldsymbol{x}') \right], \tag{10.39}$$

$$= \gamma(R - \boldsymbol{\beta} \cdot \boldsymbol{R}). \tag{10.40}$$

Important: Note here all variables on the right hand side (RHS) $(\boldsymbol{v}, \boldsymbol{\beta}, \boldsymbol{R}, \gamma)$ are evaluated at t'; i.e., $\gamma = 1/\sqrt{1 - v(t')^2/c^2}, \boldsymbol{\beta} = \boldsymbol{v}(t')/c, \boldsymbol{R} = \boldsymbol{x} - \boldsymbol{x}'(t')$, and $R = c(t - t')$.

Substituting Equation (10.40) into Equations (10.36) and (10.37) results in the retarded potentials for a point charge

$$\boxed{\begin{aligned} \phi(\boldsymbol{x},t) &= \gamma\frac{q}{4\pi\epsilon_0(R-\boldsymbol{\beta}\cdot\boldsymbol{R})}\frac{1}{\gamma} = \frac{q}{4\pi\epsilon_0(R-\boldsymbol{\beta}\cdot\boldsymbol{R})}\,, \qquad (10.41)\\ \boldsymbol{A}(\boldsymbol{x},t) &= \frac{q\boldsymbol{\beta}}{4\pi\epsilon_0(R-\boldsymbol{\beta}\cdot\boldsymbol{R})c}\,. \qquad (10.42)\end{aligned}}$$

Again the RHS variables are all evaluated at time t'; i.e.,

$$\boldsymbol{\beta} = \boldsymbol{v}/c = \boldsymbol{v}(t')/c, \quad \text{and} \quad \boldsymbol{R} = \boldsymbol{x} - \boldsymbol{x}'(t'). \tag{10.43}$$

The retarded potentials for a moving point charge, Equations (10.41) and (10.42), are known as *the Liénard-Wiechert potentials*.

From the potentials, the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields are given by

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial\boldsymbol{A}}{\partial t}, \tag{10.44}$$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}. \tag{10.45}$$

where $\nabla = \partial/\partial\boldsymbol{x}$. These differentiations seem to be simple, but they are not because of explicit or implicit dependences on $t(t')$ and $\boldsymbol{x}(\boldsymbol{x}')$. Obtaining $\boldsymbol{E}$ and $\boldsymbol{B}$ from ϕ and $\boldsymbol{A}$ is a tedious process; we'll first analyze how ϕ and $\boldsymbol{A}$ depend on t and $\boldsymbol{x}$.

Potentials ϕ and $\boldsymbol{A}$ are functions of $R, \boldsymbol{R}$ and $\boldsymbol{v}$, and

$$\boldsymbol{R} = \boldsymbol{x} - \boldsymbol{x}'(t') \tag{10.46}$$

$$\boldsymbol{v} = \boldsymbol{v}(t') \tag{10.47}$$

Here the retarded time t' is determined by

$$|\boldsymbol{x} - \boldsymbol{x}'(t')| = c(t - t') = R, \tag{10.48}$$

therefore $t' = t'(\boldsymbol{x}, t)$. From above analysis, ϕ and $\boldsymbol{A}$

1. depend on t through t' (Equation (10.48)).

2. depend on $\boldsymbol{x}$ explicitly through $\boldsymbol{R}$ and implicitly through t' (Equation (10.46)).

Therefore we may write $\phi = \phi(\boldsymbol{x}, t')$ and $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{x}, t')$. To find $\boldsymbol{E}$ and $\boldsymbol{B}$ from ϕ and $\boldsymbol{A}$, we must evaluate $\partial/\partial t$ and $\nabla \equiv \partial/\partial\boldsymbol{x}$.

Let's first calculate $\partial/\partial t$, since the dependence on t is only through t'. Therefore

$$\left.\frac{\partial}{\partial t}\right|_{\boldsymbol{x}} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'}. \tag{10.49}$$

Since variables like $\boldsymbol{v}, R$ are all functions evaluated at t', we do not need to do anything about $\partial/\partial t'$. However, we still need to calculate $\partial t'/\partial t$. Using $R = c(t - t')$,

$$\frac{\partial R}{\partial t} = c\left(1 - \frac{\partial t'}{\partial t}\right) = \frac{\partial R}{\partial t'}\frac{\partial t'}{\partial t}. \tag{10.50}$$

The purpose is to solve for $\partial t'/\partial t$ from this equation and we still need $\partial R/\partial t'$. Note $R = |\boldsymbol{x} - \boldsymbol{x}'|$, or $R^2 = \boldsymbol{R}\cdot\boldsymbol{R} = \boldsymbol{x}^2 + \boldsymbol{x}'(t')^2 - 2\boldsymbol{x}\cdot\boldsymbol{x}'(t')$,

$$\begin{aligned}\frac{\partial R^2}{\partial t'} &= 2\boldsymbol{x}'\cdot\frac{\partial \boldsymbol{x}'}{\partial t'} - 2\boldsymbol{x}\cdot\frac{\partial \boldsymbol{x}'}{\partial t'} = 2[\boldsymbol{x}'(t') - \boldsymbol{x}]\cdot\boldsymbol{v}(t') && (10.51)\\ &= -2\boldsymbol{R}\cdot\boldsymbol{v}. && (10.52)\end{aligned}$$

Therefore

$$\frac{\partial R}{\partial t'} = -\frac{\boldsymbol{R}\cdot\boldsymbol{v}}{R}. \tag{10.53}$$

Substituting Equation (10.53) into Equation (10.50) gives

$$c\left(1 - \frac{\partial t'}{\partial t}\right) = -\frac{\boldsymbol{R}\cdot\boldsymbol{v}}{R}\frac{\partial t'}{\partial t}, \tag{10.54}$$

from which we find

$$\frac{\partial t'}{\partial t} = \frac{c}{c - \boldsymbol{R}\cdot\boldsymbol{v}/R} = \frac{1}{1 - \boldsymbol{R}\cdot\boldsymbol{v}/Rc} \equiv \frac{R}{S(\boldsymbol{x}, t')}, \tag{10.55}$$

where

$$\boxed{S(\boldsymbol{x}, t') = R - \boldsymbol{\beta}\cdot\boldsymbol{R}} \tag{10.56}$$

Finally, substituting Equation (10.55) into Equation (10.49) gives

$$\frac{\partial}{\partial t} = \frac{1}{1 - \boldsymbol{R}\cdot\boldsymbol{v}/Rc}\frac{\partial}{\partial t'} = \frac{R}{S(\boldsymbol{x}, t')}\frac{\partial}{\partial t'}. \tag{10.57}$$

Potentials depend on $\boldsymbol{x}$ explicitly through $\boldsymbol{R} = \boldsymbol{x} - \boldsymbol{x}'$ and implicitly through t', hence

$$\nabla f(\boldsymbol{x}, t') = \nabla f(\boldsymbol{x}, t')\big|_{t'} + \frac{\partial f}{\partial t'}\nabla t'. \tag{10.58}$$

To calculate ∇, we need $\nabla t'$, which is the partial derivative of t' with respect to $\boldsymbol{x}$ while fixing t. We again start from $R = c(t - t')$,

$$\nabla R = -c\nabla t' \Rightarrow \nabla t' = -\frac{1}{c}\nabla R. \tag{10.59}$$

But note that $R = R(\boldsymbol{x}, t')$,

$$\nabla R(\boldsymbol{x}, t') = \nabla R(\boldsymbol{x}, t')\big|_{t'} + \frac{\partial R}{\partial t'}\nabla t'. \tag{10.60}$$

Therefore

$$\nabla t' = -\frac{1}{c}\left(\nabla R|_{t'} + \frac{\partial R}{\partial t'}\nabla t'\right). \tag{10.61}$$

Fortunately, we've just shown that

$$\frac{\partial R}{\partial t'} = -\frac{\boldsymbol{R}\cdot\boldsymbol{v}}{R}. \tag{10.62}$$

Now we only need to calculate $\nabla R|_{t'}$. Again using $R^2 = \boldsymbol{R}\cdot\boldsymbol{R}$ and $\boldsymbol{R} = \boldsymbol{x} - \boldsymbol{x}'(t')$,

$$2R\,\nabla R|_{t'} = 2\,\boldsymbol{R}\cdot\nabla\boldsymbol{R}|_{t'} = 2\boldsymbol{R}\cdot\nabla\boldsymbol{x} = 2\boldsymbol{R}\cdot\mathsf{I} = 2\boldsymbol{R}, \tag{10.63}$$

or

$$\nabla R|_{t'} = \boldsymbol{R}/R. \tag{10.64}$$

Substituting Equation (10.64) for $\nabla R|_{t'}$ and Equation (10.62) for $\partial R/\partial t'$ into Equation (10.61) gives

$$\nabla t' = -\frac{1}{c}\left(\frac{\boldsymbol{R}}{R} - \frac{\boldsymbol{R}\cdot\boldsymbol{v}}{R}\nabla t'\right), \tag{10.65}$$

from which we can find

$$\nabla t' = -\frac{1}{c}\frac{\boldsymbol{R}}{R}\frac{1}{1-\boldsymbol{R}\cdot\boldsymbol{v}/Rc} = -\frac{\boldsymbol{R}}{cS} \tag{10.66}$$

With these preparations, we're now ready to calculate fields. Using S, ϕ and $\boldsymbol{A}$ are

$$\phi = \frac{q}{4\pi\epsilon_0 S} \quad \text{and} \quad \boldsymbol{A} = \frac{\boldsymbol{v}}{c^2}\phi. \tag{10.67}$$

Therefore

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} = \frac{q}{4\pi\epsilon_0 S^2}\nabla S - \frac{1}{c^2}\frac{\partial}{\partial t}(\boldsymbol{v}\phi) \tag{10.68}$$

$$= \frac{q}{4\pi\epsilon_0 S^2}\nabla S - \frac{1}{c^2}\left[\frac{\partial \boldsymbol{v}}{\partial t}\phi + \boldsymbol{v}\frac{\partial\phi}{\partial t}\right] \tag{10.69}$$

$$= \frac{q}{4\pi\epsilon_0 S^2}\nabla S - \frac{1}{c^2}\left[\boldsymbol{a}\frac{\partial t'}{\partial t}\phi + \boldsymbol{v}\left(-\frac{q}{4\pi\epsilon_0 S^2}\frac{\partial S}{\partial t}\right)\right], \tag{10.70}$$

where $\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t'$ is the acceleration of the particle at t'.

The structure of $\boldsymbol{E}$ tells us we need ∇S and $\partial S/\partial t$. First, let's calculate ∇S.

$$\nabla S = \nabla(R - \boldsymbol{\beta}\cdot\boldsymbol{R}) = \nabla R - \nabla(\boldsymbol{v}\cdot\boldsymbol{R})/c, \tag{10.71}$$

$$\nabla R = -c\nabla t' = -c\left(-\frac{\boldsymbol{R}}{cS}\right) = \frac{\boldsymbol{R}}{S}, \tag{10.72}$$

$$\nabla(\boldsymbol{v}\cdot\boldsymbol{R}) = \nabla\boldsymbol{v}\cdot\boldsymbol{R} + \nabla\boldsymbol{R}\cdot\boldsymbol{v}, \tag{10.73}$$

$$\nabla\boldsymbol{v} = \nabla t'\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t'} = -\frac{\boldsymbol{R}}{cS}\boldsymbol{a}, \tag{10.74}$$

$$\nabla\boldsymbol{R} = \nabla[\boldsymbol{x} - \boldsymbol{x}'(t')] = \mathsf{I} - \nabla t'\boldsymbol{v} = \mathsf{I} + \frac{\boldsymbol{R}}{cS}\boldsymbol{v}, \tag{10.75}$$

$$\nabla(\boldsymbol{v}\cdot\boldsymbol{R}) = -\frac{\boldsymbol{R}}{cS}(\boldsymbol{a}\cdot\boldsymbol{R}) + \boldsymbol{v} + \frac{\boldsymbol{R}}{cS}v^2. \tag{10.76}$$

In summary,

$$\nabla S = \frac{\boldsymbol{R}}{S} + \frac{\boldsymbol{R}}{c^2 S}(\boldsymbol{a}\cdot\boldsymbol{R}) - \boldsymbol{\beta} - \beta^2\frac{\boldsymbol{R}}{S} \tag{10.77}$$

$$= \frac{\boldsymbol{R}}{S}\left(1 - \beta^2 + \frac{\boldsymbol{a}\cdot\boldsymbol{R}}{c^2}\right) - \boldsymbol{\beta} \tag{10.78}$$

Now let's calculate $\partial S/\partial t$,

$$\frac{\partial S}{\partial t} = \frac{\partial R}{\partial t} - \frac{1}{c}\frac{\partial \boldsymbol{v}\cdot\boldsymbol{R}}{\partial t} \tag{10.79}$$

$$= \frac{\partial R}{\partial t'}\frac{\partial t'}{\partial t} - \frac{1}{c}\left(\frac{\partial \boldsymbol{v}}{\partial t}\cdot\boldsymbol{R} + \frac{\partial \boldsymbol{R}}{\partial t}\cdot\boldsymbol{v}\right) \tag{10.80}$$

$$= -\frac{\boldsymbol{R}\cdot\boldsymbol{v}}{R}\frac{R}{S} - \frac{1}{c}\left(\frac{R}{S}\frac{\partial \boldsymbol{v}(t')}{\partial t'}\cdot\boldsymbol{R} - \frac{\partial \boldsymbol{x}'(t')}{\partial t}\cdot\boldsymbol{v}\right) \tag{10.81}$$

$$= -\frac{\boldsymbol{R}\cdot\boldsymbol{v}}{S} - \frac{1}{c}\left(\frac{R}{S}\boldsymbol{a}\cdot\boldsymbol{R} - \frac{R}{S}\frac{\partial \boldsymbol{x}'(t')}{\partial t'}\cdot\boldsymbol{v}\right) \tag{10.82}$$

$$= -\frac{\boldsymbol{R}\cdot\boldsymbol{v}}{S} - \frac{1}{c}\left(\frac{R}{S}\boldsymbol{a}\cdot\boldsymbol{R} - \frac{R}{S}\boldsymbol{v}\cdot\boldsymbol{v}\right). \tag{10.83}$$

Or in summary,

$$\frac{\partial S}{\partial t} = -\frac{\boldsymbol{R}\cdot\boldsymbol{v}}{S} - \frac{R}{S}\frac{\boldsymbol{a}\cdot\boldsymbol{R}}{c} + \frac{v^2}{c}\frac{R}{S}. \tag{10.84}$$

Using ∇S and $\partial S/\partial t$, we can easily find

$$\nabla\phi = -\frac{q}{4\pi\epsilon_0 S^2}\nabla S = -\frac{q}{4\pi\epsilon_0 S^2}\left[\frac{\boldsymbol{R}}{S}\left(1-\beta^2+\frac{\boldsymbol{a}\cdot\boldsymbol{R}}{c^2}\right)-\boldsymbol{\beta}\right],$$

$$\frac{\partial \boldsymbol{A}}{\partial t} = \frac{1}{c^2}\left[\boldsymbol{a}\frac{qR}{4\pi\epsilon_0 S^2} - \frac{q\boldsymbol{v}}{4\pi\epsilon_0 S^2}\left(-\frac{\boldsymbol{R}\cdot\boldsymbol{v}}{S} - \frac{R}{S}\frac{\boldsymbol{a}\cdot\boldsymbol{R}}{c} + \frac{v^2}{c}\frac{R}{S}\right)\right].$$

Combining these two,

$$\begin{aligned}\boldsymbol{E} = \frac{q}{4\pi\epsilon_0 S^2}&\left\{\frac{\boldsymbol{R}}{S}\left(1-\beta^2+\frac{\boldsymbol{a}\cdot\boldsymbol{R}}{c^2}\right)-\boldsymbol{\beta}\right.\\ &\left.+\frac{\boldsymbol{v}}{c^2}\left(-\frac{\boldsymbol{R}\cdot\boldsymbol{v}}{S} - \frac{R}{S}\frac{\boldsymbol{a}\cdot\boldsymbol{R}}{c} + \frac{v^2}{c}\frac{R}{S}\right) - \frac{R}{c^2}\boldsymbol{a}\right\}\end{aligned} \tag{10.85}$$

We divide terms inside $\{\}$ in Equation (10.85) into two parts. Part ① are those not related to acceleration $\boldsymbol{a}$,

$$① = \frac{\boldsymbol{R}}{S}\left(1-\beta^2\right) - \boldsymbol{\beta} + \frac{\boldsymbol{v}}{c^2}\left(-\frac{\boldsymbol{R}\cdot\boldsymbol{v}}{S} + \frac{v^2}{c}\frac{R}{S}\right) \tag{10.86}$$

$$= \frac{1}{S}\left[\boldsymbol{R}\left(1-\beta^2\right) - \boldsymbol{\beta}S - \boldsymbol{\beta}(\boldsymbol{R}\cdot\boldsymbol{\beta}) + \beta^2 R\boldsymbol{\beta}\right] \tag{10.87}$$

$$= \frac{1}{S}\left[\boldsymbol{R}\left(1-\beta^2\right) - \boldsymbol{\beta}(R - \boldsymbol{\beta}\cdot\boldsymbol{R}) - \boldsymbol{\beta}(\boldsymbol{R}\cdot\boldsymbol{\beta}) + \beta^2 R\boldsymbol{\beta}\right] \tag{10.88}$$

$$= \frac{1}{S}\left[\boldsymbol{R}\left(1-\beta^2\right) - \boldsymbol{\beta}R(1-\beta^2)\right] \tag{10.89}$$

$$= \frac{1}{S}(1-\beta^2)(\boldsymbol{R} - R\boldsymbol{\beta}). \tag{10.90}$$

Part ② are those terms related to acceleration $\boldsymbol{a}$,

$$② = \frac{\boldsymbol{R}}{S}\frac{\boldsymbol{a}\cdot\boldsymbol{R}}{c^2} - \frac{\boldsymbol{v}}{c^2}\frac{R}{S}\frac{\boldsymbol{a}\cdot\boldsymbol{R}}{c} - \frac{R}{c^2}\boldsymbol{a} \tag{10.91}$$

$$= \frac{1}{Sc^2}\left[\boldsymbol{R}(\boldsymbol{a}\cdot\boldsymbol{R}) - (\boldsymbol{a}\cdot\boldsymbol{R})R\boldsymbol{\beta} - RS\boldsymbol{a}\right] \tag{10.92}$$

$$= \frac{1}{Sc^2}\left\{(\boldsymbol{a}\cdot\boldsymbol{R})(\boldsymbol{R}-R\boldsymbol{\beta}) - \left[R^2 - (\boldsymbol{\beta}\cdot\boldsymbol{R})R\right]\boldsymbol{a}\right\} \tag{10.93}$$

$$= \frac{1}{Sc^2}\left\{(\boldsymbol{a}\cdot\boldsymbol{R})(\boldsymbol{R}-R\boldsymbol{\beta}) - \left[\boldsymbol{R}\cdot(\boldsymbol{R}-R\boldsymbol{\beta})\right]\boldsymbol{a}\right\} \tag{10.94}$$

$$= \frac{1}{Sc}\boldsymbol{R}\times\left[(\boldsymbol{R}-R\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]. \tag{10.95}$$

Combining parts ① and ② leads to

$$\boldsymbol{E} = \frac{q}{4\pi\epsilon_0 S^3}\left\{(1-\beta^2)(\boldsymbol{R}-R\boldsymbol{\beta}) + \frac{1}{c}\boldsymbol{R}\times\left[(\boldsymbol{R}-R\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right\} \tag{10.96}$$

$$= \frac{q}{4\pi\epsilon_0\gamma^2}\frac{\boldsymbol{R}-R\boldsymbol{\beta}}{S^3} + \frac{q}{4\pi\epsilon_0 c}\frac{\boldsymbol{R}\times\left[(\boldsymbol{R}-R\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{S^3}. \tag{10.97}$$

The first part (called part ①) on the RHS does not depend on $\boldsymbol{a}$, and it is called the "*velocity field*". The part (called part ②) dependent on $\boldsymbol{a}$ is called the "*radiation field*.".

Note that $S \sim R$, so ① $\sim R^{-2}$ and ② $\sim R^{-1}$. Therefore the total energy flow through a sphere with radius R per unit time is

$$P^{①} \sim E^2R^2 \sim R^{-2}, \tag{10.98}$$

hence as $R \to \infty$, $P^{①} \to 0$. The velocity field exists even if $\boldsymbol{a} = 0$, it equals the Coulomb field if $\boldsymbol{v} = 0$ (sometimes called the *generalized Coulomb field*), or the Lorentz transformation of the Coulomb field for a partial moving uniformly with the velocity $\boldsymbol{v}$. The velocity field moves with the charge and does not transfer energy to infinity.

For the second part, note that ② $\sim R^{-1}$, so at large distances

$$\boldsymbol{E} \approx ②, \tag{10.99}$$

i.e., the second part of $\boldsymbol{E}$ dominates at large distances. Also consider the energy flow through a sphere with radius $\boldsymbol{R}$,

$$P^{②} \sim E^2R^2 \sim \text{const}, \tag{10.100}$$

i.e., it can radiate energy to large distances. This part is responsible for the electromagnetic radiation. We will focus on radiation in the next chapter.

The magnetic field $\boldsymbol{B}$ is $\boldsymbol{B} = \nabla\times\boldsymbol{A}$,

$$\begin{aligned}
\boldsymbol{B} &= \nabla\times\boldsymbol{A} = \nabla\times(\phi\boldsymbol{\beta}/c) = \phi\nabla\times\boldsymbol{\beta}/c - \boldsymbol{\beta}\times\nabla\phi/c \\
&= \phi\nabla t'\times\dot{\boldsymbol{\beta}}/c - \boldsymbol{\beta}\times\nabla\phi/c \\
&= \frac{q}{4\pi\epsilon_0 cS}\left(-\frac{\boldsymbol{R}}{cS}\right)\times\dot{\boldsymbol{\beta}} - \boldsymbol{\beta}\times\left(-\frac{q}{4\pi\epsilon_0 cS^2}\right)\left[\frac{\boldsymbol{R}}{S}\left(1-\beta^2+\frac{\dot{\boldsymbol{\beta}}\cdot\boldsymbol{R}}{c}\right)-\boldsymbol{\beta}\right] \\
&= \frac{q}{4\pi\epsilon_0 cS^3}\boldsymbol{\beta}\times\boldsymbol{R}(1-\beta^2) + \frac{q}{4\pi\epsilon_0 cS^2}\boldsymbol{\beta}\times\frac{\boldsymbol{R}}{S}\frac{\dot{\boldsymbol{\beta}}\cdot\boldsymbol{R}}{c} - \frac{q}{4\pi\epsilon_0 c^2S^2}\boldsymbol{R}\times\dot{\boldsymbol{\beta}} \\
&= \boldsymbol{e}_R\times\left\{-\frac{q}{4\pi\epsilon_0 cS^3}\frac{R\boldsymbol{\beta}}{\gamma^2} - \frac{q}{4\pi\epsilon_0 c^2S^3}R\boldsymbol{\beta}(\dot{\boldsymbol{\beta}}\cdot\boldsymbol{R}) - \frac{q}{4\pi\epsilon_0 c^2S^3}\dot{\boldsymbol{\beta}}RS\right\} \\
&= \boldsymbol{e}_R\times\left\{-\frac{q}{4\pi\epsilon_0 cS^3}\frac{R\boldsymbol{\beta}}{\gamma^2} - \frac{q}{4\pi\epsilon_0 c^2S^3}R\left[\boldsymbol{\beta}(\dot{\boldsymbol{\beta}}\cdot\boldsymbol{R}) - \dot{\boldsymbol{\beta}}(\boldsymbol{\beta}\cdot\boldsymbol{R}) + \dot{\boldsymbol{\beta}}R\right]\right\} \\
&= \boldsymbol{e}_R\times\left\{-\frac{q}{4\pi\epsilon_0 cS^3}\frac{R\boldsymbol{\beta}}{\gamma^2} + \frac{q}{4\pi\epsilon_0 c^2S^3}\left[-R\boldsymbol{\beta}(\dot{\boldsymbol{\beta}}\cdot\boldsymbol{R}) - \dot{\boldsymbol{\beta}}(\boldsymbol{R}-R\boldsymbol{\beta})\cdot\boldsymbol{R}\right]\right\}
\end{aligned}$$

The terms in $\{\}$ look similar to $\boldsymbol{E}$, except a few terms are missing. However, noticing that the missed terms are all proportional to $\boldsymbol{R}$, we can add in those terms to make them $\boldsymbol{E}$; i.e.,

$$\begin{aligned}
\boldsymbol{B} = \boldsymbol{e}_R\times\Bigg\{&-\frac{q}{4\pi\epsilon_0 cS^3}\frac{R\boldsymbol{\beta}}{\gamma^2} - \frac{q}{4\pi\epsilon_0 cS^3}R\boldsymbol{\beta}\left(\frac{\dot{\boldsymbol{\beta}}\cdot\boldsymbol{R}}{c}\right) \\
&-\frac{q}{4\pi\epsilon_0 c^2S^3}\dot{\boldsymbol{\beta}}(R^2 - R\boldsymbol{\beta}\cdot\boldsymbol{R})\Bigg\}
\end{aligned}$$

Since $\boldsymbol{e}_R \times \boldsymbol{R} = 0$, we can insert two terms: $\boldsymbol{R}/\gamma^2$ and $\boldsymbol{R}(\dot{\boldsymbol{\beta}} \cdot \boldsymbol{R})$

$$= \boldsymbol{e}_R \times \left\{ \frac{q}{4\pi\epsilon_0 c S^3} \frac{1}{\gamma^2} (\boldsymbol{R} - R\boldsymbol{\beta}) + \frac{q}{4\pi\epsilon_0 c^2 S^3} \left[\boldsymbol{R}(\dot{\boldsymbol{\beta}} \cdot \boldsymbol{R}) - R\boldsymbol{\beta} \left(\dot{\boldsymbol{\beta}} \cdot \boldsymbol{R}\right) - \dot{\boldsymbol{\beta}}(\boldsymbol{R} - R\boldsymbol{\beta}) \cdot \boldsymbol{R} \right] \right\}$$
$$= \boldsymbol{e}_R \times \left\{ \frac{q}{4\pi\epsilon_0 c S^3} \frac{1}{\gamma^2} (\boldsymbol{R} - R\boldsymbol{\beta}) + \frac{q}{4\pi\epsilon_0 c^2 S^3} \boldsymbol{R} \times \left[(\boldsymbol{R} - R\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right\}.$$

Or in summary,

$$\boldsymbol{B} = \boldsymbol{e}_R \times \boldsymbol{E}/c, \quad \text{where } \boldsymbol{e}_R \equiv \boldsymbol{R}/R. \tag{10.101}$$

11
Radiation of Electromagnetic Waves

The word "radiation" here means the transportation of energy in the form of electromagnetic waves out to infinity. Therefore in this chapter we are mainly focused on calculating the power and energy radiated by moving charges.

11.1 Radiation of a Point Charge

We start from the simple case about the radiation of a point charge and calculating the corresponding power radiated. We will calculate this using the radiation electric and magnetic fields. From the previous chapter, the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields are

$$\boldsymbol{E} = \frac{q}{4\pi\epsilon_0\gamma^2}\frac{\boldsymbol{R} - R\boldsymbol{\beta}}{S^3} + \frac{q}{4\pi\epsilon_0 c}\frac{\boldsymbol{R}\times\left[(\boldsymbol{R} - R\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{S^3}, \tag{11.1}$$

$$\boldsymbol{B} = \boldsymbol{e}_R \times \boldsymbol{E}/c. \tag{11.2}$$

We have already learned that the first term on the RHS of Equation (11.1) is the "velocity" field, and it varies as R^{-2}, therefore it does not radiate. The second term varies as R^{-1}, and therefore can transport energy to infinity. Therefore, we now only consider the radiation part of the field, using $S = R - \boldsymbol{\beta}\cdot\boldsymbol{R}$,

$$\boldsymbol{E} = \frac{q}{4\pi\epsilon_0 c}\frac{\boldsymbol{R}\times\left[(\boldsymbol{R} - R\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{(R - \boldsymbol{\beta}\cdot\boldsymbol{R})^3} \tag{11.3}$$

$$= \frac{q}{4\pi\epsilon_0 c}\frac{\boldsymbol{e}_R\times\left[(\boldsymbol{e}_R - \boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{(1 - \boldsymbol{\beta}\cdot\boldsymbol{e}_R)^3 R}, \tag{11.4}$$

$$\boldsymbol{B} = \boldsymbol{e}_R \times \boldsymbol{E}/c. \tag{11.5}$$

The Poynting flux is, noting that $\boldsymbol{E}\cdot\boldsymbol{e}_R = 0$ [1],

$$\boldsymbol{\mathcal{S}}(\boldsymbol{r},t) = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B} = \frac{1}{\mu_0 c}E^2\boldsymbol{e}_R, \tag{11.6}$$

$$= \frac{1}{\mu_0 c}\left(\frac{q}{4\pi\epsilon_0 c}\right)^2\frac{\left|\boldsymbol{e}_R\times\left[(\boldsymbol{e}_R - \boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right|^2}{(1 - \boldsymbol{\beta}\cdot\boldsymbol{e}_R)^6 R^2}\boldsymbol{e}_R \tag{11.7}$$

The Poynting vector means the amount of energy passing through unit area in the direction $\boldsymbol{\mathcal{S}}$ in unit time. Therefore, if we want to calculate the energy radiated from $t_1(t_1')$ to $t_2(t_2')$ is

$$\Delta E = \int_{t_1}^{t_2}\boldsymbol{\mathcal{S}}\cdot\boldsymbol{e}_R \mathrm{d}t. \tag{11.8}$$

[1] Please note: $S = R - \boldsymbol{\beta}\cdot\boldsymbol{R}$ and $\boldsymbol{\mathcal{S}}$ is the Poynting flux.

Note here that terms on the RHS of Equation (11.6) are explicit functions of t' instead of t, therefore it is convenient to make a transformation from t to t', and write ΔE as

$$\Delta E = \int_{t_1}^{t_2} \boldsymbol{\mathcal{S}} \cdot \boldsymbol{e}_R \frac{\partial t}{\partial t'} \mathrm{d}t', \tag{11.9}$$

here the meaning of the integrand

$$\mathrm{d}E(t') = \boldsymbol{\mathcal{S}} \cdot \boldsymbol{e}_R \frac{\partial t}{\partial t'} \mathrm{d}t', \tag{11.10}$$

is the amount of energy radiated by the particle in $\mathrm{d}t'$ through unit area in direction of $\boldsymbol{e}_R$. The total energy radiated by the particle in $\mathrm{d}t'$ is then,

$$\Delta \mathcal{E} = \boldsymbol{\mathcal{S}} \cdot \boldsymbol{e}_R \frac{\partial t}{\partial t'} R^2 \mathrm{d}\Omega \Delta t', \tag{11.11}$$

here $\mathrm{d}\Omega = \sin\theta \mathrm{d}\theta \mathrm{d}\phi$ is the solid angle. The radiation power is

$$\mathrm{d}P \equiv \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t'} = \boldsymbol{\mathcal{S}} \cdot \boldsymbol{e}_R R^2 \frac{\partial t}{\partial t'} \mathrm{d}\Omega, \tag{11.12}$$

and the power radiated by the particle per solid angle $\mathrm{d}\Omega$ is

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \boldsymbol{\mathcal{S}} \cdot \boldsymbol{e}_R R^2 \frac{\partial t}{\partial t'}. \tag{11.13}$$

Now using

$$\frac{\partial t}{\partial t'} = 1 - \boldsymbol{e}_R \cdot \boldsymbol{\beta}, \tag{11.14}$$

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \boldsymbol{\mathcal{S}} \cdot \boldsymbol{e}_R R^2 (1 - \boldsymbol{e}_{\boldsymbol{R}} \cdot \boldsymbol{\beta}). \tag{11.15}$$

Substituting Equation (11.7) into Equation (11.15) leads to

$$\boxed{\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{q^2 \mu_0 c}{16\pi^2} \frac{\left|\boldsymbol{e}_R \times \left[(\boldsymbol{e}_R - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right]\right|^2}{(1 - \boldsymbol{\beta} \cdot \boldsymbol{e}_R)^5}.} \tag{11.16}$$

This is the general equation we'll use to discuss various special cases.

11.1.1 *The radiation of a slow particle*

First, let's see the radiation from a non-relativistic particle, $\beta \ll 1$, then $1 - \boldsymbol{\beta} \cdot \boldsymbol{e}_R \approx 1$, and Equation (11.16) becomes

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{q^2 \mu_0 c}{16\pi^2} \left|\boldsymbol{e}_R \times (\boldsymbol{e}_R \times \dot{\boldsymbol{\beta}})\right|^2 \tag{11.17}$$

$$= \frac{q^2 \mu_0 c}{16\pi^2} \left|(\boldsymbol{e}_R \cdot \dot{\boldsymbol{\beta}})\boldsymbol{e}_R - \dot{\boldsymbol{\beta}}\right|^2 \tag{11.18}$$

$$= \frac{q^2 \mu_0 c}{16\pi^2} \left|\dot{\boldsymbol{\beta}}_\perp\right|^2 \tag{11.19}$$

$$= \frac{q^2 \mu_0}{16\pi^2 c} \dot{v}^2 \sin^2 \Theta, \tag{11.20}$$

here $\Theta \equiv \langle \dot{\boldsymbol{v}}, \boldsymbol{e}_R \rangle = \langle \boldsymbol{a}, \boldsymbol{e}_R \rangle$. If we further note that the dipole moment [2] for a single point charge is $\boldsymbol{p} = \sum q\boldsymbol{r}' = q\boldsymbol{r}'$, then $\dot{\boldsymbol{p}} = q\boldsymbol{v}$ and $\ddot{\boldsymbol{p}} = q\dot{\boldsymbol{v}} = q\boldsymbol{a}$. Therefore Equation (11.20) can be written using $\boldsymbol{p}$ as

[2] Note that in this chapter, the scalar P is power, $\mathfrak{p}$ is momentum and $\boldsymbol{p}$ is the dipole moment.

$$\boxed{\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{\mu_0 |\ddot{\boldsymbol{p}}|^2}{16\pi^2 c} \sin^2 \Theta.} \tag{11.21}$$

The total power radiated by the slow charge is

$$\boxed{P = \int \mathrm{d}P = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\Theta \frac{\mu_0 |\ddot{\boldsymbol{p}}|^2}{16\pi^2 c} \sin^3 \Theta = \frac{\mu_0 |\ddot{\boldsymbol{p}}|^2}{6\pi c} = \frac{\mu_0 q^2 a^2}{6\pi c}.} \tag{11.22}$$

This is the famous *Larmor formula*. Note from the formula that the power radiated by a point charge is proportional to the square of its acceleration.

11.1.2 *The radiation of a fast particle*

For a fast particle where $\beta \sim 1$, we cannot ignore β or $\boldsymbol{\beta}$ in Equation (11.16) and things in general are complicated. Here we first discuss two special cases with $\boldsymbol{\beta} \| \dot{\boldsymbol{\beta}}$ and $\boldsymbol{\beta} \perp \dot{\boldsymbol{\beta}}$, and leave the general case as your exercise.

Let's first calculate the radiation power for $\boldsymbol{\beta} \| \dot{\boldsymbol{\beta}}$. If $\boldsymbol{\beta} \| \dot{\boldsymbol{\beta}}$, then Equation (11.16) becomes

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{q^2 \mu_0 c}{16\pi^2} \frac{\left| \boldsymbol{e}_R \times \left[\boldsymbol{e}_R \times \dot{\boldsymbol{\beta}} \right] \right|^2}{(1 - \boldsymbol{\beta} \cdot \boldsymbol{e}_R)^5}. \tag{11.23}$$

Define $\theta = \langle \boldsymbol{\beta}, \boldsymbol{e}_R \rangle$. For $\boldsymbol{\beta} \| \dot{\boldsymbol{\beta}}$, $\langle \dot{\boldsymbol{\beta}}, \boldsymbol{e}_R \rangle = \theta$.

$$\boxed{\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{q^2 \mu_0 c}{16\pi^2} \frac{\dot{\beta}^2 \sin^2 \theta}{(1 - \beta \cos\theta)^5},} \tag{11.24}$$

The angular distribution of radiation is interesting. If $\theta = 0$, then $\mathrm{d}P/\mathrm{d}\Omega = 0$. However, for ultra-relativistic particles, $\beta \to 1$, then

$$1 - \beta \cos\theta \sim 1 - \cos\theta \to 0, \quad \text{as } \theta \to 0. \tag{11.25}$$

Therefore $\mathrm{d}P/\mathrm{d}\Omega$ becomes very large near $\theta \sim 0$ but not at $\theta = 0$. Note that the angular distribution of the power radiated is independent of the sign of $\boldsymbol{a}$ (whether the particle is accelerating or de-accelerating). When a high speed electron is de-accelerating, for example, when it hits a target, it radiates [3]. This kind of radiation is called bremsstrahlung, *braking radiation*, 轫致辐射，刹车辐射或制动辐射.

[3] One example is the X-ray radiation caused by relativistic electrons from space hitting atmosphere. People actually can use the radiated X-ray to deduce the energy spectrum of the precipitated electrons. Google "NASA BARREL mission" for more information.

The total power radiated for $\boldsymbol{\beta} \| \dot{\boldsymbol{\beta}}$ is

$$P_{\|} = \int \frac{q^2 \mu_0 c}{16\pi^2} \frac{\dot{\beta}^2 \sin^2\theta}{(1-\beta\cos\theta)^5} \mathrm{d}\Omega \tag{11.26}$$

$$= \frac{q^2 \mu_0 c \dot{\beta}^2}{16\pi^2} \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \mathrm{d}\theta \tag{11.27}$$

$$= \frac{q^2 \mu_0 c \dot{\beta}^2}{16\pi^2} 2\pi \frac{4}{3(1-\beta^2)^3} \tag{11.28}$$

$$= \frac{q^2 \mu_0 c \dot{\beta}^2}{6\pi(1-\beta^2)^3} \tag{11.29}$$

$$= \frac{q^2 \mu_0}{6\pi c} \gamma^6 \dot{v}^2 = \frac{q^2 \mu_0}{6\pi c} \gamma^6 a^2. \tag{11.30}$$

Sometimes it's convenient to express the radiation power in terms of the force felt by the particle. For $\boldsymbol{v} \| \dot{\boldsymbol{v}}$, we have from the time rate change of momentum $\mathfrak{p}$,

$$\boldsymbol{F} = \frac{\mathrm{d}\mathfrak{p}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m \boldsymbol{v}) = \dot{\gamma} m \boldsymbol{v} + \gamma m \dot{\boldsymbol{v}}. \tag{11.31}$$

Note that $\beta^2 = (\gamma^2 - 1)/\gamma^2$, and for $\boldsymbol{v} \| \dot{\boldsymbol{v}}$,

$$\dot{\gamma} m \boldsymbol{v} = \gamma^3 \beta^2 m \dot{\boldsymbol{v}} = (\gamma^3 - \gamma) m \dot{\boldsymbol{v}}. \tag{11.32}$$

Substituting Equation (11.32) into Equation (11.31) yields

$$\boldsymbol{F} = \frac{\mathrm{d}\mathfrak{p}}{\mathrm{d}t} = \gamma^3 m \dot{\boldsymbol{v}}. \tag{11.33}$$

Therefore in terms of force $\boldsymbol{F}$, the total radiation power for $\boldsymbol{\beta} \| \dot{\boldsymbol{\beta}}$ is

$$\boxed{P_{\|} = \frac{q^2 \mu_0}{6\pi c m^2} \left(\frac{\mathrm{d}\mathfrak{p}}{\mathrm{d}t}\right)^2.} \tag{11.34}$$

Therefore the radiation power is independent of γ for a given force.

The second special case we are going to consider is $\boldsymbol{\beta} \perp \dot{\boldsymbol{\beta}}$; this case is slightly more complicated than the previous case. We first establish a coordinate system where $\boldsymbol{v} = v\boldsymbol{e}_z$, and $\dot{\boldsymbol{v}} = \dot{v}\boldsymbol{e}_x$, then

$$\dot{\boldsymbol{\beta}} \cdot \boldsymbol{e}_R = \dot{\beta} \sin\theta \cos\phi, \tag{11.35}$$

$$\boldsymbol{\beta} \cdot \boldsymbol{e}_R = \beta \cos\theta. \tag{11.36}$$

Here (R, θ, ϕ) forms a spherical coordinate system. Using these two equations,

$$\begin{aligned} &\boldsymbol{e}_R \times [(\boldsymbol{e}_R - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}] \\ &= (\boldsymbol{e}_R \cdot \dot{\boldsymbol{\beta}})(\boldsymbol{e}_R - \boldsymbol{\beta}) - [(\boldsymbol{e}_R - \boldsymbol{\beta}) \cdot \boldsymbol{e}_{\boldsymbol{R}}] \dot{\boldsymbol{\beta}} && (11.37) \\ &= \dot{\beta} \sin\theta \cos\phi (\boldsymbol{e}_R - \boldsymbol{\beta}) - (1 - \beta\cos\theta)\dot{\boldsymbol{\beta}}. && (11.38) \end{aligned}$$

Therefore,

$$\begin{aligned} &|\boldsymbol{e}_R \times [(\boldsymbol{e}_R - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2 \\ &= (\dot{\beta} \sin\theta\cos\phi)^2 (\boldsymbol{e}_R - \boldsymbol{\beta}) \cdot (\boldsymbol{e}_R - \boldsymbol{\beta}) + (1 - \beta\cos\theta)^2 \dot{\boldsymbol{\beta}} \cdot \dot{\boldsymbol{\beta}} \\ &\quad - 2\dot{\beta} \sin\theta\cos\phi (1 - \beta\cos\theta)(\boldsymbol{e}_R - \boldsymbol{\beta}) \cdot \dot{\boldsymbol{\beta}} && (11.39) \\ &= \dot{\beta}^2 \left[(1 - \beta\cos\theta)^2 + (\sin\theta\cos\phi)^2(\beta^2 - 1)\right]. && (11.40) \end{aligned}$$

Using Equation (11.40) in Equation (11.16) gives

$$\boxed{\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{q^2\mu_0 c}{16\pi^2}\frac{\dot{\beta}^2[(1-\beta\cos\theta)^2 + (\sin\theta\cos\phi)^2(\beta^2-1)]}{(1-\beta\cos\theta)^5}.} \tag{11.41}$$

Integrating (11.41) gives the total radiation power

$$P_\perp = \frac{q^2\mu_0 c\dot{\beta}^2}{16\pi^2}2\pi\frac{4}{3(1-\beta^2)^2} = \frac{q^2\mu_0}{6\pi c}\dot{v}^2\gamma^4. \tag{11.42}$$

Since the force $\mathrm{d}\mathfrak{p}/\mathrm{d}t$ for $\boldsymbol{v} \perp \dot{\boldsymbol{v}}$ is, noting that γ is constant,

$$\frac{\mathrm{d}\mathfrak{p}}{\mathrm{d}t} = \gamma m\dot{\boldsymbol{v}} \Rightarrow \left(\frac{\mathrm{d}\mathfrak{p}}{\mathrm{d}t}\right)^2 = \gamma^2 m^2 \dot{v}^2, \tag{11.43}$$

the radiation power in terms of the force felt by the particle is

$$\boxed{P_\perp = \frac{q^2\mu_0}{6\pi c}\frac{\gamma^2}{m^2}\left(\frac{\mathrm{d}\mathfrak{p}}{\mathrm{d}t}\right)^2.} \tag{11.44}$$

Therefore for a given force, the radiation increases as γ^2 or energy squared.

The above results for $P_\parallel$ and $P_\perp$ are important for the design of accelerators. There are at least two types of accelerators: linear accelerators ($\boldsymbol{v} \parallel \dot{\boldsymbol{v}}$) and cyclic particle accelerators ($\boldsymbol{v} \perp \dot{\boldsymbol{v}}$). You can see that for a given force, $P_\perp = P_\parallel\gamma^2$. Therefore as energy goes up, the radiation power loss is much much worse in cyclic particle accelerators. So in principle, it is more effective using linear accelerators to accelerate particles to really really high energy. On the other hand, the most important application of "$\boldsymbol{v} \perp \dot{\boldsymbol{v}}$" case is circular motion, and the radiation is called *synchrotron radiation*. Instead of accelerating particles, people get radiation (or called "light") from this kind of accelerators.

Now we are ready to calculate the power radiated by a charge in arbitrary motion. Noting that $\boldsymbol{E}$ and $\boldsymbol{B}$ fields satisfy linear superposition, therefore $\boldsymbol{E} = \boldsymbol{E}(\dot{\boldsymbol{\beta}}_\parallel) + \boldsymbol{E}(\dot{\boldsymbol{\beta}}_\perp)$, and $\parallel$ and $\perp$ are with respect to $\boldsymbol{v}$. The electric field is

$$\boldsymbol{E} \propto \boldsymbol{e}_R \times [(\boldsymbol{e}_R - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}] \tag{11.45}$$

$$= \boldsymbol{e}_R \times [(\boldsymbol{e}_R - \boldsymbol{\beta}) \times (\dot{\boldsymbol{\beta}}_\parallel + \dot{\boldsymbol{\beta}}_\perp)] \tag{11.46}$$

$$= \boldsymbol{e}_R \times [(\boldsymbol{e}_R - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}_\parallel + (\boldsymbol{e}_R - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}_\perp)]. \tag{11.47}$$

Hence if we write $\boldsymbol{E} = \boldsymbol{E}_\parallel + \boldsymbol{E}_\perp$, then

$$E^2 = E_\parallel^2 + E_\perp^2 + 2\boldsymbol{E}_\parallel \cdot \boldsymbol{E}_\perp. \tag{11.48}$$

Therefore Equation (11.16) becomes

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \left(\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right)_\parallel + \left(\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right)_\perp + \left(\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right)_{\text{cross}}. \tag{11.49}$$

From Equation (11.47),

$$\left(\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right)_{\text{cross}} \propto 2\{\boldsymbol{e}_R \times [(\boldsymbol{e}_R - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}_\parallel]\} \cdot \{\boldsymbol{e}_R \times [(\boldsymbol{e}_R - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}_\perp]\}. \tag{11.50}$$

The total power is given by

$$P = \int \left[\left(\frac{dP}{d\Omega} \right)_{\parallel} + \left(\frac{dP}{d\Omega} \right)_{\perp} + \left(\frac{dP}{d\Omega} \right)_{\text{cross}} \right] d\Omega \tag{11.51}$$

$$= P_{\parallel} + P_{\perp} + \int \left(\frac{dP}{d\Omega} \right)_{\text{cross}} d\Omega. \tag{11.52}$$

Left as an exercise, you can prove that

$$\int \left(\frac{dP}{d\Omega} \right)_{\text{cross}} d\Omega = 0. \tag{11.53}$$

Therefore using Equations (11.30) and (11.42),

$$\boxed{P = P_{\parallel} + P_{\perp} = \frac{q^2 \mu_0}{6\pi c} \gamma^6 a_{\parallel}^2 + \frac{q^2 \mu_0}{6\pi c} \gamma^4 a_{\perp}^2 = \frac{q^2 \mu_0}{6\pi c} (\gamma^6 a_{\parallel}^2 + \gamma^4 a_{\perp}^2).} \tag{11.54}$$

This is total power radiated by a relativistic particle in arbitrary motion. It is called *Lienard formula*. Lienard's generalization of the Larmor formula given by Equation (11.22). you can see that Equation (11.54) reduces to (11.22) if $\gamma \to 1$.

11.2 *Radiation reaction*

In this section, we consider the effects of radiation on the motion of a single particle. This is normally called radiation reaction, or radiation damping.

Since radiation carries energy away to infinity, an accelerated charged particle loses energy in this process. For a given force, a charged particle accelerates less than a neutral particle. Equivalently speaking, the radiation exerts a force, called $\boldsymbol{F}_{\text{rad}}$, back on the charge. The purpose of this section is to calculate this force, and for simplicity, we consider only non-relativistic particles. You'll see that, like "self-energy", this "self-force" (the force a charged particle exerts on itself) is not well understood, at least in the framework of classical electromagnetic theory.

From the conservation of energy, you might want to write that the energy lost by radiation equals the amount of work done by $\boldsymbol{F}_{\text{rad}}$, or

$$\boldsymbol{F}_{\text{rad}} \cdot \boldsymbol{v} = -P = -\frac{\mu_0 q^2 a^2}{6\pi c}, \tag{11.55}$$

where the radiation power is given by the Larmor formula. But this is actually not correct. When we calculated the radiation power, we excluded the energy from the velocity field, because we were only interested in the energy that could be transported to infinity. Well, the velocity field do carry energy, they just don't transport it to infinity. As the particle accelerates or de-accelerates, the energy exchanges between the particle and the velocity field, and at the same time, is radiated to infinity. Equation (11.55) only accounts for the radiation field, but not the velocity field. If we want to know the force exerted by the fields on the charge, we need to know the total power lost by the charge at any given moment [4].

[4] The term "radiation reaction" or "radiation damping" are kind of misleading; we are actually talking about the force exerted by the total field (radiation field + velocity field) on the particle, not just the radiation field.

Therefore the total energy lost by the particle should equal the total energy radiated and the total energy pumped into the velocity field. However, if we consider only intervals where the initial state and the final state of the system are identical, e.g., a period system, the energy stored in the velocity field is the same. The only net loss of energy is in the form of radiation. Thus Equation (11.55) is incorrect at any given moment, but is valid if we perform an average over the interval

$$\int_{t_1}^{t_2} \boldsymbol{F}_{\text{rad}} \cdot \boldsymbol{v}\mathrm{d}t = -\frac{\mu_0 q^2}{6\pi c}\int_{t_1}^{t_2} a^2 \mathrm{d}t. \tag{11.56}$$

We want to emphasize again that the state of the system is identical at t_1 and t_2. Integrating the RHS of the equation by parts, we have

$$\int_{t_1}^{t_2} a^2 \mathrm{d}t = \int_{t_1}^{t_2} \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\cdot\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\mathrm{d}t = \boldsymbol{v}\cdot\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{\mathrm{d}^2\boldsymbol{v}}{\mathrm{d}t^2}\cdot\boldsymbol{v}\mathrm{d}t. \tag{11.57}$$

Because the state of system is identical at t_1 and t_2,

$$\int_{t_1}^{t_2} a^2 \mathrm{d}t = -\int_{t_1}^{t_2}\frac{\mathrm{d}^2\boldsymbol{v}}{\mathrm{d}t^2}\cdot\boldsymbol{v}\mathrm{d}t = -\int_{t_1}^{t_2}\dot{\boldsymbol{a}}\cdot\boldsymbol{v}\mathrm{d}t. \tag{11.58}$$

Putting this equation into Equation (11.57) gives

$$\int_{t_1}^{t_2}\left(\boldsymbol{F}_{\text{rad}} - \frac{\mu_0 q^2}{6\pi c}\dot{\boldsymbol{a}}\right)\cdot\boldsymbol{v}\mathrm{d}t = 0. \tag{11.59}$$

Apparently Equation (11.59) will be satisfied if

$$\boldsymbol{F}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c}\dot{\boldsymbol{a}}. \tag{11.60}$$

This is called the *Abraham-Lorentz formula* for the radiation reaction force. However, you should note that you can add any term perpendicular to $\boldsymbol{v}$ and Equation (11.59) will still be satisfied. Equation (11.60) gives the time-average force of the field exerted on the charged particle. It is by far the simplest form of the radiation force can take. We'll use this formula to characterize the radiation reaction force exerted on a charged particle.

However, we need to point out that the Abraham-Lorentz formula has disturbing implications. To see this, let's first write down the equation of motion of the charged particle if we consider the radiation reaction:

$$ma = \frac{\mu_0 q^2}{6\pi c}\dot{\boldsymbol{a}} + F_0, \tag{11.61}$$

here F_0 denotes any other forces, and for simplicity, we shall consider a 1D problem. Equation (11.61) can also be written as

$$a = \tau\dot{a} + F_0/m, \tag{11.62}$$

where

$$\tau = \frac{\mu_0 q^2}{6\pi m c} = \frac{2r_q}{3c}, \tag{11.63}$$

with $r_q = q^2/4\pi\epsilon_0 mc^2$ the classical charged particle radius. For an electron, $\tau \sim 6 \times 10^{-24}$ s. Note that within radiation reaction consider, the acceleration a is a continuous function of time, just like velocity and coordinates. To see that, we integrate Equation (11.62) from $t-\epsilon$ to $t+\epsilon$,

$$\int_{t-\epsilon}^{t+\epsilon} a\mathrm{d}t = \int_{t-\epsilon}^{t+\epsilon} \tau\dot{a}\mathrm{d}t + \int_{t-\epsilon}^{t+\epsilon} F_0/m\mathrm{d}t, \tag{11.64}$$

this is the same as

$$v(t+\epsilon) - v(t-\epsilon) = \tau[a(t+\epsilon) - a(t-\epsilon)] + \int_{t-\epsilon}^{t+\epsilon} F_0/m\mathrm{d}t. \tag{11.65}$$

Note that v is a continuous function of t, then if we let $\epsilon \to 0$,

$$0 = \tau[a(t+\epsilon) - a(t-\epsilon)] + 0, \text{ as } \epsilon \to 0. \tag{11.66}$$

Therefore, $a(t+\epsilon) - a(t-\epsilon) \to 0$ as $\epsilon \to 0$, or a is a continuous function of t.

Now we consider a particle moving from a region with some kind of field so that the particle feels a constant force F_0 to a region with no field at $t = 0$. Let's denote the region with field region I, and the region without field region II. Since the force is constant, the general solution of Equation (11.62) in Region I is

$$a^{\mathrm{I}} = a_0^{\mathrm{I}} e^{t/\tau} + F_0/m, \tag{11.67}$$

and in Region II is

$$a^{\mathrm{II}} = a_0^{\mathrm{II}} e^{t/\tau}. \tag{11.68}$$

Equating a^{II} to a^{I} at $t = 0$, we have

$$a_0^{\mathrm{I}} + F_0/m = a_0^{\mathrm{II}}. \tag{11.69}$$

Since we choose $t = 0$ quite arbitrarily, there is no reason to believe $a_0^{\mathrm{II}} = 0$. As long as $a_0^{\mathrm{II}} \neq 0$, the solution of a in region II tells us that the particle accelerates/de-accelerates indefinitely even if there is no external force! This problem also arises if the radiation damping force dominates, since then the equation of motion is reduced to

$$a = \tau\dot{a}, \tag{11.70}$$

which again gives $a = a_0 e^{t/\tau}$. From what we have discussed above, we shall assume that the radiation damping force is much much smaller than any external force from now on.

11.3 *The field of a system of charges at large distances*

We choose the origin of the coordinates O in the interior of the system of charges, like before in the chapters about electrostatics and magnetostatics. Charge position $\boldsymbol{r}'$ and field position $\boldsymbol{r}$, and

$$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'. \tag{11.71}$$

We normally consider radiation in the form of waves. Correspondingly there are typically three spatial scales, $|\boldsymbol{r}|$, $|\boldsymbol{r}'|$, and λ, which is the typical radiation wave length. To simplify analysis, we make some assumptions. First, we assume that $|\boldsymbol{r}'| \ll (|\boldsymbol{r}|, \lambda)$; i.e., the system is small in size compared to the typical wave length and the distance of the field point to the origin. Then by comparing λ and $|\boldsymbol{r}|$, we can define three zones of radiation.

1. The near zone with $|\boldsymbol{r}| \ll \lambda$.

2. The intermediate zone: $|\boldsymbol{r}| \sim \lambda$.

3. The far/radiation zone: $|\boldsymbol{r}| \gg \lambda$.

In this class, we only consider electromagnetic field in the radiation zone. Further more, since we are considering a system of charges now, we need to use the general expression of potentials, the retarded potentials, to calculate $\boldsymbol{E}$ and $\boldsymbol{B}$.

Now here is one extra note about the condition $|\boldsymbol{r}'| \ll \lambda$. Assume the the characteristic wave period corresponding to λ is T, or $\lambda \sim Tc$, where c is the speed of light in vacuum. Now consider the motion of a particle in the system within T, the distance this particle travels should be about $|\boldsymbol{r}'|$, the scale size of the system. Therefore the particle should move with a characteristic speed $v \sim |\boldsymbol{r}'|/T$. The condition $|\boldsymbol{r}'| \ll \lambda$ then implies

$$|\boldsymbol{r}'| \ll \lambda \sim Tc \tag{11.72}$$

or

$$\frac{|\boldsymbol{r}'|}{T} \sim v \ll c; \tag{11.73}$$

i.e., the condition $|\boldsymbol{r}'| \ll \lambda$ implies that the particles are non-relativistic.

Now let's start our analysis of the radiation field of a system of charges. We can use Fourier transform to analyze the field in far zone, and therefore we consider only the component with frequency ω; i.e., assume the source ρ and $\boldsymbol{j}$ are $\rho(\boldsymbol{r}')e^{-\mathrm{i}\omega t}$ and $\boldsymbol{j} = \boldsymbol{j}(\boldsymbol{r}')e^{-\mathrm{i}\omega t}$, then the retarded potentials give

$$\phi(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho_0(\boldsymbol{r}')e^{-\mathrm{i}\omega(t-R/c)}}{R} \mathrm{d}V', \tag{11.74}$$

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{j}_0(\boldsymbol{r}')e^{-\mathrm{i}\omega(t-R/c)}}{R} \mathrm{d}V'. \tag{11.75}$$

Using $k = \omega/c$ for electromagnetic waves in vacuum, we rewrite above equations as

$$\phi(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho_0(\boldsymbol{r}')e^{\mathrm{i}(kR-\omega t)}}{R} \mathrm{d}V', \tag{11.76}$$

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{j}_0(\boldsymbol{r}')e^{\mathrm{i}(kR-\omega t)}}{R} \mathrm{d}V'. \tag{11.77}$$

Now that since the time dependence in potentials is in the form of $e^{-\mathrm{i}\omega t}$, then

$$\frac{\partial}{\partial t} = -\mathrm{i}\omega, \tag{11.78}$$

therefore outside the source region,

$$\nabla\times\boldsymbol{B} = \frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t} = -\mathrm{i}\frac{\omega}{c^2}\boldsymbol{E} = -\mathrm{i}k\boldsymbol{E}/c, \tag{11.79}$$

or

$$\boldsymbol{E} = \mathrm{i}\frac{c}{k}\nabla\times\boldsymbol{B}, \text{ and } \boldsymbol{B} = \nabla\times\boldsymbol{A}. \tag{11.80}$$

Therefore for radiation of a system of charges, we calculate $\boldsymbol{B}$ first from the vector potential $\boldsymbol{A}$, and then use Equation (11.80) to calculate $\boldsymbol{E}$, the electric field[5]. Note that since the time dependence can be easily separated from the $\boldsymbol{r}$ dependence, we write

[5] Compare this with the radiation field of a point charge, where we calculate $\boldsymbol{E}$ first, and then use $\boldsymbol{B} = \boldsymbol{e}_R \times \boldsymbol{E}$ to obtain $\boldsymbol{B}$.

$$\boldsymbol{A}(\boldsymbol{r},t) = \boldsymbol{A}(\boldsymbol{r})e^{-\mathrm{i}\omega t}, \tag{11.81}$$

where

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int\frac{\boldsymbol{j}_0(\boldsymbol{r}')e^{\mathrm{i}kR}}{R}\mathrm{d}V'. \tag{11.82}$$

From now on we'll only analyze $\boldsymbol{A}(\boldsymbol{r})$, since the time dependence is fixed and easy to calculate.

Now let's go back to Equation (11.77) and simplify the expression of $\boldsymbol{A}(\boldsymbol{r})$ using the fact that we are considering fields in the far zone; i.e., $|\boldsymbol{r}| \gg \lambda \gg |\boldsymbol{r}'|$. Now we need to be careful here: there are two small parameters

$$\epsilon \equiv |\boldsymbol{r}'|/|\boldsymbol{r}| \ll 1, \tag{11.83}$$

and

$$\eta \equiv |\boldsymbol{r}'|/\lambda \sim v/c \ll 1. \tag{11.84}$$

If we compare η and ϵ, we find that

$$\eta/\epsilon = |\boldsymbol{r}|/\lambda \gg 1. \tag{11.85}$$

In our Taylor expansion-type analysis below, we'll throw away high order terms in ϵ, but not η.

First, let's Taylor expand R,

$$R = \sqrt{(\boldsymbol{r}-\boldsymbol{r}')^2} = (\boldsymbol{r}^2 + \boldsymbol{r}'^2 - 2\boldsymbol{r}\cdot\boldsymbol{r}')^{1/2} \approx r - \boldsymbol{r}'\cdot\boldsymbol{n} + \cdots, \tag{11.86}$$

where $r = |\boldsymbol{r}|$, $\boldsymbol{n} \equiv \boldsymbol{r}/r$. Substituting Equation (11.86) into Equation (11.82), we find

$$\boldsymbol{A}(\boldsymbol{r}) \approx \frac{\mu_0}{4\pi r}\int\boldsymbol{j}_0(\boldsymbol{r}')e^{\mathrm{i}kr-\mathrm{i}k\boldsymbol{r}'\cdot\boldsymbol{n}}\mathrm{d}V' = \frac{\mu_0 e^{\mathrm{i}kr}}{4\pi r}\int\boldsymbol{j}_0(\boldsymbol{r}')e^{-\mathrm{i}k\boldsymbol{r}'\cdot\boldsymbol{n}}\mathrm{d}V'. \tag{11.87}$$

There is a reason why we ignored $\boldsymbol{r}' \cdot \boldsymbol{n}$ in the denominator while keeping the term $\mathrm{i}k\boldsymbol{r}' \cdot \boldsymbol{n}$ in the phase. First,

$$\frac{1}{R} = \frac{1}{\boldsymbol{r} - \boldsymbol{r}' \cdot \boldsymbol{n}} \approx \frac{1}{\boldsymbol{r}} \left(1 + \frac{\boldsymbol{r}' \cdot \boldsymbol{n}}{\boldsymbol{r}}\right) \sim \frac{1}{\boldsymbol{r}} \left(1 + r'/r\right) = \frac{1}{\boldsymbol{r}} \left(1 + \epsilon\right), \tag{11.88}$$

On the other hand,

$$e^{-\mathrm{i}k\boldsymbol{r}' \cdot \boldsymbol{n}} \approx 1 - \mathrm{i}k\boldsymbol{r}' \cdot \boldsymbol{n} \sim 1 - \mathrm{i}r'/\lambda = 1 - \mathrm{i}\eta \tag{11.89}$$

Since $\eta/\epsilon \gg 1$ and we are keeping lowest order terms of ϵ, but not η, we cannot ignore $e^{-\mathrm{i}k\boldsymbol{r}' \cdot \boldsymbol{n}}$ in general.

You can also try to understand this term directly using the retarded potentials, in which we have $\rho(\boldsymbol{r}', t')$ or $\boldsymbol{j}(\boldsymbol{r}', t')$. Using Equation (11.86),

$$\rho(\boldsymbol{r}', t') \approx \rho(\boldsymbol{r}', t - r/c + \boldsymbol{r}' \cdot \boldsymbol{n}/c) \tag{11.90}$$

$$\approx \rho(\boldsymbol{r}', t - r/c) + \left.\frac{\partial \rho}{\partial t'}\right|_{t - r/c} \frac{\boldsymbol{r}' \cdot \boldsymbol{n}}{c}. \tag{11.91}$$

Therefore, whether or not you can ignore the second term depends on how much ρ changes during the time interval $\boldsymbol{r}' \cdot \boldsymbol{n}/c$. This is totally different from the factor $1/R$.

The remaining analysis of this chapter makes use of the small parameter $\eta = |\boldsymbol{r}'|/\lambda$, and it's a similar process similar to the multipole expansion of the electrostatic and magnetostatic fields.

11.4 *The dipole radiation*

To lower order in $\eta = |\boldsymbol{r}'|/\lambda$, Equation (11.87) becomes

$$\boldsymbol{A}(\boldsymbol{r}) \approx \frac{\mu_0 e^{\mathrm{i}kr}}{4\pi r} \int \boldsymbol{j}_0(\boldsymbol{r}') \mathrm{d}V'. \tag{11.92}$$

Because $\boldsymbol{j_0}(\boldsymbol{r}') = \rho_0(\boldsymbol{r}')\boldsymbol{v}$, the integration becomes

$$\int \boldsymbol{j}_0(\boldsymbol{r}') \mathrm{d}V' = \int \rho_0(\boldsymbol{r}') \boldsymbol{v} \mathrm{d}V' = \sum_a q_a \boldsymbol{v}_a = \dot{\boldsymbol{p}}. \tag{11.93}$$

Here we have used

$$\rho(\boldsymbol{r}') = \sum q_a \delta(\boldsymbol{r}' - \boldsymbol{r}'_a), \tag{11.94}$$

and the definition of the dipole moment,

$$\boldsymbol{p} = \sum q_a \boldsymbol{r}'_a \Rightarrow \dot{\boldsymbol{p}} = \sum q_a \dot{\boldsymbol{r}}'_a = \sum q_a \boldsymbol{v}_a. \tag{11.95}$$

Therefore, Equation (11.92) becomes

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 e^{\mathrm{i}kr}}{4\pi r} \dot{\boldsymbol{p}}. \tag{11.96}$$

This is the vector potential of the dipole radiation in far zone.

From Equation (11.96), we can easily obtain the magnetic field $\boldsymbol{B}$ by

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{\mu_0 e^{\mathrm{i}kr}}{4\pi} \left(\frac{\mathrm{i}k}{r} - \frac{1}{r^2} \right) \nabla r \times \dot{\boldsymbol{p}} \tag{11.97}$$

$$= \frac{\mathrm{i}k\mu_0 e^{\mathrm{i}kr}}{4\pi r} \left(1 - \frac{1}{\mathrm{i}kr} \right) \boldsymbol{n} \times \dot{\boldsymbol{p}}. \tag{11.98}$$

Using that $1/kr \sim \lambda/r \sim \epsilon/\eta \ll 1$,

$$\boldsymbol{B} = \frac{\mathrm{i}k\mu_0 e^{\mathrm{i}kr}}{4\pi r} \boldsymbol{n} \times \dot{\boldsymbol{p}} = \frac{\mathrm{i}\omega\mu_0 e^{\mathrm{i}kr}}{4\pi c r} \boldsymbol{n} \times \dot{\boldsymbol{p}}. \tag{11.99}$$

Since the time dependence is in the form of $e^{-\mathrm{i}\omega t}$, $\mathrm{i}\omega\dot{\boldsymbol{p}} = -\ddot{\boldsymbol{p}}$, and

$$\boldsymbol{B} = \frac{\mu_0 e^{\mathrm{i}kr}}{4\pi c r} \ddot{\boldsymbol{p}} \times \boldsymbol{n}. \tag{11.100}$$

From the above analysis, we see that when performing ∇ operation and when we only need to keep the lowest order terms in λ/r, we just need to replace ∇ by $\mathrm{i}k\boldsymbol{n}$.

The electric field can be obtained from Equation (11.80),

$$\boldsymbol{E} = \mathrm{i}\frac{c}{k}\nabla \times \boldsymbol{B} = \mathrm{i}\frac{c}{k}\mathrm{i}k\boldsymbol{n} \times \boldsymbol{B} = c\boldsymbol{B} \times \boldsymbol{n} = \frac{\mu_0 e^{\mathrm{i}kr}}{4\pi r}(\ddot{\boldsymbol{p}} \times \boldsymbol{n}) \times \boldsymbol{n}. \tag{11.101}$$

If we define $\theta \equiv \langle \boldsymbol{p}, \boldsymbol{n} \rangle$, noting that $\ddot{\boldsymbol{p}} = -\omega^2 \boldsymbol{p}$, therefore θ is also the angle between $\boldsymbol{p}$ and $\boldsymbol{n}$. The $\boldsymbol{E}$ (Equation (11.101)) and $\boldsymbol{B}$ (Equation (11.100)) fields are

$$\boxed{\begin{aligned} \boldsymbol{B}(\boldsymbol{r}) &= \frac{\mu_0 e^{\mathrm{i}kr}}{4\pi c r} \ddot{p} \sin\theta \boldsymbol{e}_\phi, && (11.102) \\ \boldsymbol{E}(\boldsymbol{r}) &= \frac{\mu_0 e^{\mathrm{i}kr}}{4\pi r} \ddot{p} \sin\theta \boldsymbol{e}_\theta. && (11.103) \end{aligned}}$$

Here (r, θ, ϕ) form the normal spherical coordinates. These are the electric and magnetic fields from dipole radiation.

To calculate the power radiated by the dipole radiation, we start from the calculation of Poynting flux.

$$\boldsymbol{S} = \frac{1}{\mu_0}\mathrm{Re}(\boldsymbol{E}(\boldsymbol{r}, t)) \times \mathrm{Re}(\boldsymbol{B}(\boldsymbol{r}, t)) = \frac{1}{\mu_0}\left(\frac{\boldsymbol{E} + \boldsymbol{E}^*}{2} \right) \times \left(\frac{\boldsymbol{B} + \boldsymbol{B}^*}{2} \right) \tag{11.104}$$

$$= \frac{1}{4\mu_0}\left(\boldsymbol{E} \times \boldsymbol{B} + \boldsymbol{E} \times \boldsymbol{B}^* + \boldsymbol{E}^* \times \boldsymbol{B} + \boldsymbol{E}^* \times \boldsymbol{B}^* \right). \tag{11.105}$$

From equations (11.102) and (11.103),

$$\boldsymbol{E} \times \boldsymbol{B} = \boldsymbol{E}(\boldsymbol{r}) \times \boldsymbol{B}(\boldsymbol{r}) e^{-2\mathrm{i}\omega t}, \tag{11.106}$$

$$\boldsymbol{E} \times \boldsymbol{B}^* = \boldsymbol{E}(\boldsymbol{r}) \times \boldsymbol{B}(\boldsymbol{r}). \tag{11.107}$$

The Poynting flux therefore contains rapidly varying terms proportional to $e^{-2\mathrm{i}\omega t}$. In this case, it's customary to calculate the time averaged Poynting flux

$$\langle \boldsymbol{S} \rangle = \frac{1}{T}\int_t^{t+T} \boldsymbol{S}(\tau) \mathrm{d}\tau. \tag{11.108}$$

Note that

$$\langle e^{\mathrm{i}m\omega t}\rangle = \frac{1}{T}\int_t^{t+T} e^{\mathrm{i}m\omega\tau}\mathrm{d}\tau = 0 \text{ for } m \neq 0. \tag{11.109}$$

Therefore the time-averaged Poynting flux is

$$\langle \boldsymbol{S}\rangle = \frac{1}{4\mu_0}\left(\boldsymbol{E}\times\boldsymbol{B}^* + \boldsymbol{E}^*\times\boldsymbol{B}\right) = \frac{1}{2\mu_0}\mathrm{Re}(\boldsymbol{E}\times\boldsymbol{B}^*). \tag{11.110}$$

For dipole radiation, $\boldsymbol{E} = c\boldsymbol{B}\times\boldsymbol{n}$, then

$$\boldsymbol{E}\times\boldsymbol{B}^* = (c\boldsymbol{B}\times\boldsymbol{n})\times\boldsymbol{B}^* = c\boldsymbol{n}(\boldsymbol{B}\cdot\boldsymbol{B}^*) = c|\boldsymbol{B}|^2\boldsymbol{n}, \tag{11.111}$$

since $\boldsymbol{B}\perp\boldsymbol{n}$ from Equation (11.100). Substituting Equation (11.111) into Equation (11.110) gives

$$\langle \boldsymbol{S}\rangle = \frac{c}{2\mu_0}|\boldsymbol{B}|^2\boldsymbol{n}. \tag{11.112}$$

The angular distribution of the radiation power is, using Equation (11.100),

$$\left\langle \frac{\mathrm{d}P}{\mathrm{d}\Omega}\right\rangle = \langle\boldsymbol{S}\rangle\cdot\boldsymbol{n}r^2 = \frac{\mu_0}{32\pi^2 c}|\ddot{\boldsymbol{p}}\times\boldsymbol{n}|^2, \tag{11.113}$$

or

$$\boxed{\left\langle \frac{\mathrm{d}P}{\mathrm{d}\Omega}\right\rangle = \frac{\mu_0}{32\pi^2 c}|\ddot{\boldsymbol{p}}|^2\sin^2\theta = \frac{\mu_0\omega^4}{32\pi^2 c}|\boldsymbol{p}|^2\sin^2\theta,} \tag{11.114}$$

where we have used $|\ddot{\boldsymbol{p}}| = \omega^2|\boldsymbol{p}|$. The total power radiated is then

$$\begin{aligned}
\langle P\rangle &= \int\left\langle \frac{\mathrm{d}P}{\mathrm{d}\Omega}\right\rangle \mathrm{d}\Omega = \int\left\langle \frac{\mathrm{d}P}{\mathrm{d}\Omega}\right\rangle \sin\theta\mathrm{d}\theta\mathrm{d}\varphi & (11.115)\\
&= \frac{\mu_0}{32\pi^2 c}|\ddot{\boldsymbol{p}}|^2\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\sin^3\theta\mathrm{d}\theta & (11.116)\\
&= \frac{\mu_0|\ddot{\boldsymbol{p}}|^2}{12\pi c} = \frac{\mu_0\omega^4|\boldsymbol{p}_0|^2}{12\pi c}, & (11.117)
\end{aligned}$$

where we have used $\boldsymbol{p} = \boldsymbol{p}_0 e^{-\mathrm{i}\omega t}$.

See the dependence on ω^4? This means that the radiated power increases tremendously if we increase frequency. Also this is what's responsible for the "blue" sky, also know as Rayleigh scattering. As electromagnetic waves from the sun, the sunlight, pass through the atmosphere, the atoms in the atmosphere feel the electromagnetic field and get stimulated. If you treat these oscillating atoms as simple harmonics, they oscillate at the "forced" frequency. The tiny dipoles formed by these atoms will then re-radiate electromagnetic waves, more intense in higher frequencies (blue) than in the lower frequencies (red). This is what accounts for the blue sky. At sunset/sunrise, those blue has been removed by the exact same scattering, and you see what's left, beautiful red light. We'll cover more about the scattering of electromagnetic waves later.

As an application of the dipole radiation, let's consider the radiation of a linear antenna whose length $d \ll \lambda$. Choosing the origin of a coordinate system to be the center of the antenna. Let the current I in the antenna be

$$I = I\boldsymbol{e}_z = I(z)e^{-\mathrm{i}\omega t}\boldsymbol{e}_z = I_0(1 - \frac{2|z|}{d})e^{-\mathrm{i}\omega t}\boldsymbol{e}_z. \tag{11.118}$$

From the continuity equation,

$$\nabla\cdot\boldsymbol{j} - \mathrm{i}\omega\rho = 0 \tag{11.119}$$

or $\rho = \nabla\cdot\boldsymbol{j}/\mathrm{i}\omega$, so the linear charge density[6] is

$$\rho' = \rho\Delta A = \frac{\nabla\cdot\boldsymbol{j}\Delta A}{\mathrm{i}\omega} = \frac{\nabla\cdot(I\boldsymbol{e}_z)}{\mathrm{i}\omega} = \frac{1}{\mathrm{i}\omega}\frac{\mathrm{d}I}{\mathrm{d}z} = \mathrm{i}\frac{2I_0}{\omega d}\mathrm{sgn}z. \tag{11.120}$$

[6] Normally people use λ to denote linear charge density; however, we've used λ for wavelength.

The electric dipole moment is

$$\boldsymbol{p} = \sum_a q_a\boldsymbol{r}'_a = \sum q(z)z\boldsymbol{e}_z = \int_{-d/2}^{d/2}\rho' z\mathrm{d}z\boldsymbol{e}_z. \tag{11.121}$$

Substituting Equation (11.120) into (11.121) gives

$$\boldsymbol{p} = 2\int_0^{d/2}\mathrm{i}\frac{2I_0}{\omega d}z\mathrm{d}z\boldsymbol{e}_z = \mathrm{i}\frac{I_0 d}{2\omega}\boldsymbol{e}_z. \tag{11.122}$$

Substituting Equation (11.122) into Equations (11.114) and (11.117) gives

$$\left\langle\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right\rangle = \frac{\mu_0\omega^4}{32\pi^2 c}\left(\frac{I_0 d}{2\omega}\right)^2\sin^2\theta = \frac{\mu_0 c I_0^2}{128\pi^2}(kd)^2\sin^2\theta, \tag{11.123}$$

and

$$\langle P\rangle = \frac{\mu_0 c}{48\pi}I_0^2(kd)^2. \tag{11.124}$$

Another thing is the total radiation power is proportional to kd or d/λ, so to increase radiation power for a given λ, you can increase d. However, we derived the dipole radiation under the assumption that $d/\lambda \ll 1$, or $kd \ll 1$, therefore you cannot increase d without limits but still use Equation (11.124) to calculate the power.

11.5 *Magnetic dipole and electric quadrupole radiation*

Normally the electric dipole radiation (sometimes just called "dipole radiation") dominates. However, in case the electric dipole radiation is 0 or if we want to go to $\mathcal{O}(\eta)$, we need to expand $e^{-\mathrm{i}k\boldsymbol{r}'\cdot\boldsymbol{n}}$ and keep terms of $\mathcal{O}(\eta)$.

The vector potential, when Taylor expanded, is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 e^{\mathrm{i}kr}}{4\pi r}\int\boldsymbol{j}(\boldsymbol{r}')e^{-\mathrm{i}k\boldsymbol{r}'\cdot\boldsymbol{n}}\mathrm{d}V' \tag{11.125}$$

$$\approx \frac{\mu_0 e^{\mathrm{i}kr}}{4\pi r}\int\boldsymbol{j}(\boldsymbol{r}')(1 - \mathrm{i}k\boldsymbol{r}'\cdot\boldsymbol{n})\mathrm{d}V'. \tag{11.126}$$

We have consider the lowest order term, the electric dipole radiation. The first order term in r'/λ is

$$\boldsymbol{A}(\boldsymbol{r}) = -\frac{\mathrm{i}k\mu_0 e^{\mathrm{i}kr}}{4\pi r}\int\boldsymbol{j}\boldsymbol{r}'\cdot\boldsymbol{n}\mathrm{d}V' = -\frac{\mathrm{i}k\mu_0 e^{\mathrm{i}kr}}{4\pi r}\int\boldsymbol{n}\cdot\boldsymbol{r}'\boldsymbol{j}\mathrm{d}V'. \tag{11.127}$$

Any tensor can be written as the summation of a symmetric part and an asymmetric part. So we do this for $\boldsymbol{r}'\boldsymbol{j}$.

$$\boldsymbol{r}'\boldsymbol{j} = \frac{1}{2}(\boldsymbol{r}'\boldsymbol{j} + \boldsymbol{j}\boldsymbol{r}') + \frac{1}{2}(\boldsymbol{r}'\boldsymbol{j} - \boldsymbol{j}\boldsymbol{r}'). \tag{11.128}$$

After dotting Equation (11.128) with $\boldsymbol{n}/c$ from left we have

$$\boldsymbol{n}\cdot\boldsymbol{r}'\boldsymbol{j} = \frac{1}{2}[(\boldsymbol{n}\cdot\boldsymbol{r}')\boldsymbol{j} + (\boldsymbol{n}\cdot\boldsymbol{j})\boldsymbol{r}'] + \frac{1}{2}[(\boldsymbol{n}\cdot\boldsymbol{r}')\boldsymbol{j} - (\boldsymbol{n}\cdot\boldsymbol{j})\boldsymbol{r}'] \tag{11.129}$$

$$= \frac{1}{2}[(\boldsymbol{n}\cdot\boldsymbol{r}')\boldsymbol{j} + (\boldsymbol{n}\cdot\boldsymbol{j})\boldsymbol{r}'] + \frac{1}{2}(\boldsymbol{r}'\times\boldsymbol{j})\times\boldsymbol{n}. \tag{11.130}$$

Substitute Equation (11.128) into Equation (11.127). Let's first consider the anti-symmetric part. The integration for the anti-symmetric part is

$$\frac{1}{2}\int(\boldsymbol{r}'\times\boldsymbol{j})\times\boldsymbol{n}\mathrm{d}V' = -\boldsymbol{n}\times\frac{1}{2}\int(\boldsymbol{r}'\times\boldsymbol{j})\mathrm{d}V' = -\boldsymbol{n}\times\boldsymbol{m}, \tag{11.131}$$

where

$$\boldsymbol{m} = \frac{1}{2}\int(\boldsymbol{r}'\times\boldsymbol{j})\mathrm{d}V' \tag{11.132}$$

is just the magnetic dipole moment we have learned before.

The integration of the symmetric part becomes

$$\frac{1}{2}\int\left[(\boldsymbol{n}\cdot\boldsymbol{r}')\boldsymbol{j} + (\boldsymbol{n}\cdot\boldsymbol{j})\boldsymbol{r}'\right]\mathrm{d}V', \tag{11.133}$$

Note that $\boldsymbol{j} = \sum q_a\delta(\boldsymbol{r}' - \boldsymbol{r}_a')\boldsymbol{v}_a$, the integration becomes

$$\frac{1}{2}\int[(\boldsymbol{n}\cdot\boldsymbol{r}')\boldsymbol{j} + (\boldsymbol{n}\cdot\boldsymbol{j})\boldsymbol{r}']\mathrm{d}V' \tag{11.134}$$

$$= \frac{1}{2}\sum q_a[(\boldsymbol{n}\cdot\boldsymbol{r}_a')\boldsymbol{v}_a + (\boldsymbol{n}\cdot\boldsymbol{v}_a)\boldsymbol{r}_a']. \tag{11.135}$$

Using $\boldsymbol{v}_a = \mathrm{d}\boldsymbol{r}_a'/\mathrm{d}t$, we have

$$(\boldsymbol{n}\cdot\boldsymbol{r}_a')\boldsymbol{v}_a + (\boldsymbol{n}\cdot\boldsymbol{v}_a)\boldsymbol{r}_a' \tag{11.136}$$

$$= (\boldsymbol{n}\cdot\boldsymbol{r}_a')\frac{\mathrm{d}\boldsymbol{r}_a'}{\mathrm{d}t} + (\boldsymbol{n}\cdot\frac{\mathrm{d}\boldsymbol{r}_a'}{\mathrm{d}t})\boldsymbol{r}_a' \tag{11.137}$$

$$= \boldsymbol{n}\cdot\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}_a'\boldsymbol{r}_a'). \tag{11.138}$$

Substituting Equation (11.138) into Equation (11.135) leads to

$$\frac{1}{2}\int\left[(\boldsymbol{n}\cdot\boldsymbol{r}')\boldsymbol{j} + (\boldsymbol{n}\cdot\boldsymbol{j})\boldsymbol{r}'\right]\mathrm{d}V' = \frac{1}{2}\boldsymbol{n}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\sum q\boldsymbol{r}'\boldsymbol{r}'. \tag{11.139}$$

Because $\boldsymbol{B} = \nabla\times\boldsymbol{A} = \mathrm{i}k\boldsymbol{n}\times\boldsymbol{A}$, we can add to $\boldsymbol{A}$ an arbitrary vector in the direction of $\boldsymbol{n}$, so we add to Equation (11.138)

$$-\frac{1}{6}\boldsymbol{n}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum qr'^2\mathsf{I}\right) \tag{11.140}$$

and the resulting summation is

$$\frac{\boldsymbol{n}}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\sum q\boldsymbol{r}'\boldsymbol{r}' - \frac{\boldsymbol{n}}{6}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum qr'^2\mathsf{I}\right) \tag{11.141}$$

$$= \frac{\boldsymbol{n}}{6}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\sum q\left(3\boldsymbol{r}'\boldsymbol{r}' - r'^2\mathsf{I}\right) \tag{11.142}$$

$$= \boldsymbol{n}\cdot\frac{1}{6}\dot{\mathsf{D}} \equiv \frac{1}{6}\dot{\boldsymbol{D}}, \tag{11.143}$$

where the tensor D is the electric quadrupole we have learned before, and the vector $\boldsymbol{D} = \boldsymbol{n} \cdot \mathsf{D}$ is a vector we defined by $\boldsymbol{D} = \boldsymbol{n} \cdot \mathsf{D}$.

Combining the symmetric and the anti-symmetric part we have the total vector potential

$$\boldsymbol{A} = -\frac{\mathrm{i}k\mu_0 e^{\mathrm{i}kr}}{4\pi r}\left(\frac{1}{6}\dot{\boldsymbol{D}} - \boldsymbol{n} \times \boldsymbol{m}\right). \tag{11.144}$$

Therefore the magnetic dipole radiation and the electric quadrupole radiation are of same order. The radiation magnetic field can be obtained from $\boldsymbol{A}$ as

$$\begin{aligned} \boldsymbol{B} = \nabla \times \boldsymbol{A} = \mathrm{i}k\boldsymbol{n} \times \frac{-\mathrm{i}k\mu_0 e^{\mathrm{i}kr}}{4\pi r}\left(\frac{1}{6}\dot{\boldsymbol{D}} - \boldsymbol{n} \times \boldsymbol{m}\right) \quad (11.145) \\ = \frac{k^2\mu_0 e^{\mathrm{i}kr}\boldsymbol{n}}{4\pi r} \times \left(\frac{1}{6}\dot{\boldsymbol{D}} - \boldsymbol{n} \times \boldsymbol{m}\right). \quad (11.146) \end{aligned}$$

We can separate the magnetic dipole part and the electric quadrupole part. The magnetic dipole part is

$$\begin{aligned} \boldsymbol{B_m} &= \frac{k^2\mu_0 e^{\mathrm{i}kr}\boldsymbol{n}}{4\pi r} \times (-\boldsymbol{n} \times \boldsymbol{m}) \\ &= (\boldsymbol{n} \times \boldsymbol{m}) \times \boldsymbol{n}\frac{k^2\mu_0 e^{\mathrm{i}kr}}{4\pi r} \\ &= (\ddot{\boldsymbol{m}} \times \boldsymbol{n}) \times \boldsymbol{n}\frac{\mu_0 e^{\mathrm{i}kr}}{4\pi c^2 r}. \end{aligned} \tag{11.147}$$

The electric quadrupole part is

$$\boldsymbol{B_D} = \boldsymbol{n} \times \dot{\boldsymbol{D}}\frac{k^2\mu_0 e^{\mathrm{i}kr}}{24\pi r} = -\mathrm{i}\,\boldsymbol{n} \times \boldsymbol{D}\frac{k^3 c\mu_0 e^{\mathrm{i}kr}}{24\pi r} = \dddot{\boldsymbol{D}} \times \boldsymbol{n}\frac{\mu_0 e^{\mathrm{i}kr}}{24\pi c^2 r}. \tag{11.148}$$

The electric field can be obtained from $\boldsymbol{E} = c\boldsymbol{B} \times \boldsymbol{n}$, and correspondingly

$$\begin{aligned} \boldsymbol{E_m} = c\boldsymbol{B_m} \times \boldsymbol{n} = (\boldsymbol{n} \times \ddot{\boldsymbol{m}})\frac{\mu_0 e^{\mathrm{i}kr}}{4\pi cr}, \quad (11.149) \\ \boldsymbol{E_D} = c\boldsymbol{B_D} \times \boldsymbol{n} = (\dddot{\boldsymbol{D}} \times \boldsymbol{n}) \times \boldsymbol{n}\frac{\mu_0 e^{\mathrm{i}kr}}{24\pi cr}. \quad (11.150) \end{aligned}$$

Using Equations (11.147) and (11.148), we can obtain the angular distribution of power

$$\left\langle \frac{\mathrm{d}P}{\mathrm{d}\Omega} \right\rangle_{\boldsymbol{m}} = \left\langle \frac{c}{\mu_0}|\boldsymbol{B_m}|^2 r^2 \right\rangle = \left\langle \frac{c}{\mu_0}\left|\frac{\mu_0(\ddot{\boldsymbol{m}} \times \boldsymbol{n}) \times \boldsymbol{n}}{4\pi c^2}\right|^2 \right\rangle = \frac{\mu_0|\ddot{\boldsymbol{m}}|^2}{32\pi^2 c^3}\sin^2\theta. \tag{11.151}$$

$$\left\langle \frac{\mathrm{d}P}{\mathrm{d}\Omega} \right\rangle_{\boldsymbol{D}} = \left\langle \frac{c}{\mu_0}|\boldsymbol{B_D}|^2 r^2 \right\rangle = \left\langle \frac{c}{\mu_0}\left|\frac{\mu_0\dddot{\boldsymbol{D}} \times \boldsymbol{n}}{24\pi c^2}\right|^2 \right\rangle = \frac{\mu_0}{1152\pi^2 c^3}|\dddot{\boldsymbol{D}} \times \boldsymbol{n}|^2. \tag{11.152}$$

The total power of magnetic dipole radiation can be easily obtained

$$\langle P \rangle_{\boldsymbol{m}} = \int \left\langle \frac{\mathrm{d}P}{\mathrm{d}\Omega} \right\rangle \mathrm{d}\Omega = \iint \frac{\mu_0|\ddot{\boldsymbol{m}}|^2}{16\pi^2 c^3}\sin^3\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = \frac{\mu_0|\ddot{\boldsymbol{m}}|^2}{12\pi c^3}. \tag{11.153}$$

This is in exact the same form as the electric dipole radiation power shown in Equation (11.117). Left as an exercise, you need to figure out why the electric dipole radiation is one order lower than the magnetic dipole radiation, even though they have the same form.

Because $\boldsymbol{D} = \mathsf{D} \cdot \boldsymbol{n}$, the angular distribution of power radiated by the electric quadrupole is in general very complicated. Here we'll use a simple example to demonstrate its calculation and the corresponding total radiation power. Consider three charges $(q, q, -2q)$ located at $\boldsymbol{r}'(q) = l\boldsymbol{e}_z$, $\boldsymbol{r}'(q) = -l\boldsymbol{e}_z$, and $\boldsymbol{r}'(-2q) = 0\boldsymbol{e}_z$, then

$$\mathsf{D} = 6ql^2\boldsymbol{e}_z\boldsymbol{e}_z, \tag{11.154}$$

so $\boldsymbol{D} = \mathsf{D} \cdot \boldsymbol{n} = 6ql^2\boldsymbol{e}_z \cos\theta$, then

$$|\boldsymbol{D} \times \boldsymbol{n}| = |6ql^2 \cos\theta \boldsymbol{e}_z \times \boldsymbol{n}| = 6ql^2 \cos\theta \sin\theta, \tag{11.155}$$

and

$$|\dddot{\boldsymbol{D}} \times \boldsymbol{n}|^2 = 36q^2l^4\omega^6 \cos^2\theta \sin^2\theta. \tag{11.156}$$

The total power radiated is then

$$\langle P \rangle_{\boldsymbol{D}} = \frac{\mu_0}{1152\pi^2c^3} \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} 36q^2l^4\omega^6 \cos^2\theta \sin^3\theta \mathrm{d}\theta = \frac{\mu_0 q^2 l^4 \omega^6}{60\pi c^3}.$$

11.6 Scattering by free charges

Now we have considered the radiation by charges. Let's now consider a closely related physical process: scattering of electromagnetic waves by charges. When electromagnetic waves incident on charges, charges feel the Lorentz force and get accelerated. The acceleration leads to radiation of waves in all directions. This is the physical mechanism responsible for scattering of light. And in the most general cases, it is called the scattering of the original wave.

It is customary to characterize the scattering by the ratio of energy emitted by the scattering system in a given direction per unit time [7], denoted by $\langle \mathrm{d}P \rangle$, to the energy flux density of the incident radiation, denoted by $\langle S \rangle$. This ratio has dimension of area, and is called the *effective scattering cross-section*, or simply *cross-section*[8]. Here $\langle \cdots \rangle$ denotes time average. Therefore

$$\mathrm{d}\sigma = \frac{\langle \mathrm{d}P \rangle}{\langle S \rangle}. \tag{11.157}$$

You can integrate $\mathrm{d}\sigma$ to get σ, which is called the *total scattering cross-section*.

In this section, we consider the scattering by a free charge at rest. We only consider non-relativistic case here; i.e., the charge velocity $v \ll c$. Then the ratio of the magnetic force to the electric force is

$$\frac{|\boldsymbol{v} \times \boldsymbol{B}|}{|\boldsymbol{E}|} \sim \frac{vB}{E} \sim \frac{v}{c} \ll 1. \tag{11.158}$$

[7] This is just the power emitted by the scattering system in a given direction.

[8] You can see "cross-section" in lots of places, e.g., collision.

Therefore we neglect the magnetic force and only consider the electric force. Let the incident wave be a linearly polarized monochromatic wave

$$\boldsymbol{E} = \boldsymbol{E}_0 \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t + \alpha), \tag{11.159}$$

Then the equation of motion of the charge is

$$\ddot{\boldsymbol{r}} = \frac{q}{m}\boldsymbol{E} = \frac{q}{m}\boldsymbol{E}_0 \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t + \alpha). \tag{11.160}$$

Now again because we are considering non-relativistic particles, the amplitude of the motion $l \sim vT \ll cT = \lambda$, where T is the wave period and λ is the wavelength. Therefore we can ignore the variation of $\boldsymbol{r}$ in the wave phase factor $\boldsymbol{k} \cdot \boldsymbol{r}$ and let $\boldsymbol{k} \cdot \boldsymbol{r} = \boldsymbol{k} \cdot \boldsymbol{r}_0$. For simplicity, we can set $\boldsymbol{r}_0 = 0$. Equation (11.160) becomes

$$\ddot{\boldsymbol{r}} = \frac{q}{m}\boldsymbol{E}_0 \cos(-\omega t + \alpha). \tag{11.161}$$

In complex notation,

$$\ddot{\boldsymbol{r}} = \frac{q}{m}\widehat{\boldsymbol{E}}_0 e^{-\mathrm{i}\omega t}, \text{ with } \widehat{\boldsymbol{E}}_0 = \boldsymbol{E}_0 e^{\mathrm{i}\alpha}. \tag{11.162}$$

Because the particle moves with a non-zero acceleration, it radiates and the lowest order radiation is given by the dipole radiation. The dipole moment $\boldsymbol{p} = q\boldsymbol{r}$ and

$$\ddot{\boldsymbol{p}} = q\ddot{\boldsymbol{r}} = \frac{q^2}{m}\widehat{\boldsymbol{E}}_0 e^{-\mathrm{i}\omega t}. \tag{11.163}$$

So the averaged radiated field can be calculated using Equation (11.114), which is

$$\left\langle \frac{\mathrm{d}P}{\mathrm{d}\Omega} \right\rangle = \frac{\mu_0}{32\pi^2 c}|\ddot{\boldsymbol{p}}|^2 \sin^2\theta = \frac{\mu_0}{32\pi^2 c}\frac{q^4}{m^2}E_0^2 \sin^2\theta, \tag{11.164}$$

or

$$\langle \mathrm{d}P \rangle = \frac{\mu_0}{32\pi^2 c}\frac{q^4}{m^2}E_0^2 \sin^2\theta \mathrm{d}\Omega. \tag{11.165}$$

The energy flux of the incident wave is

$$S = \frac{1}{\mu_0}|\boldsymbol{E} \times \boldsymbol{B}| = \frac{1}{\mu_0 c}E^2, \tag{11.166}$$

so

$$\langle S \rangle = \frac{1}{\mu_0 c}\frac{E_0^2}{2} = \frac{1}{2\mu_0 c}E_0^2. \tag{11.167}$$

Substituting Equation (11.167) and Equation (11.165) into Equation (11.157) gives the scattering cross-section

$$\mathrm{d}\sigma = \frac{\mu_0^2 q^4}{16\pi^2 m^2}\sin^2\theta \mathrm{d}\Omega = \left(\frac{q^2}{4\pi\epsilon_0 mc^2}\right)^2 \sin^2\theta \mathrm{d}\Omega. \tag{11.168}$$

We see that $\mathrm{d}\sigma$ is independent of the frequency of the incident wave[9].

[9] Compare this with Rayleigh scattering, see below.

The total scattering cross-section is simply $\sigma = \int d\sigma$, or

$$\sigma = \left(\frac{q^2}{4\pi\epsilon_0 mc^2}\right)^2 \int_0^\pi \sin^3\theta d\theta \int_0^{2\pi} d\varphi = \frac{8\pi}{3}\left(\frac{q^2}{4\pi\epsilon_0 mc^2}\right)^2. \tag{11.169}$$

If we use the classical electron radius $r_e = q^2/4\pi\epsilon_0 mc^2$, then

$$\boxed{\sigma = \frac{8\pi}{3} r_e^2.} \tag{11.170}$$

The scattering by a free non-relativistic electron is called Thomson scattering and the cross-section (11.169) or (11.170) is called the *Thomson formula*.

Equation (11.169) is also valid even if the incident wave is unpolarized. In this case, we need to average over the incident wave to find the total cross-section. Let's establish a coordinate system, where $\boldsymbol{E}$ is in x-z plane, and $\phi = \langle \boldsymbol{E}, \boldsymbol{e}_z \rangle$, Note that

$$\cos\theta = \boldsymbol{n}\cdot\boldsymbol{e}, \text{ with } \boldsymbol{e} \equiv \boldsymbol{E}/E, \tag{11.171}$$

or

$$\cos\theta = [(\boldsymbol{n}\cdot\boldsymbol{e}_y)\boldsymbol{e}_y + (\boldsymbol{n}\cdot\boldsymbol{e}_z)\boldsymbol{e}_z]\cdot\boldsymbol{e} = \sin\alpha\cos\phi. \tag{11.172}$$

Therefore

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \sin^2\alpha\cos^2\phi. \tag{11.173}$$

We need to average over ϕ since the incident wave is unpolarized,

$$\langle \sin^2\theta \rangle_\phi = \frac{1}{2\pi}\int_0^{2\pi} (1 - \sin^2\alpha\cos^2\phi) d\phi = 1 - \frac{\sin^2\alpha}{2}, \tag{11.174}$$

and

$$\frac{d\sigma}{d\Omega} = \left(\frac{q^2}{4\pi\epsilon_0 mc^2}\right)^2 \frac{1}{2}(1+\cos^2\alpha), \tag{11.175}$$

and the total cross section is

$$\sigma = \frac{1}{2}\left(\frac{q^2}{4\pi\epsilon_0 mc^2}\right)^2 \int (1+\cos^2\alpha) d\Omega = \frac{8\pi}{3}\left(\frac{q^2}{4\pi\epsilon_0 mc^2}\right)^2. \tag{11.176}$$

11.7 *Scattering by bounded electrons*

Let's now calculate the scattering of an incident electromagnetic wave by a bounded electron, e.g., an electron in an atom. We model the electron motion using a harmonic oscillator with natural frequency ω_0. The corresponding equations of motion of the electron is

$$\ddot{\boldsymbol{x}} + \omega_0^2 \boldsymbol{x} = (\boldsymbol{F}_{\text{damping}} + \boldsymbol{F}_{\text{drive}})/m, \tag{11.177}$$

Here $\boldsymbol{F}_{\text{damping}}$ is any damping force, and we consider radiation damping only, therefore

$$\boldsymbol{F}_{\text{damping}} = m\tau\dot{\boldsymbol{a}} = m\tau\dddot{\boldsymbol{x}} = -m\tau\omega^2\dot{\boldsymbol{x}} = -m\gamma\dot{\boldsymbol{x}}, \tag{11.178}$$

where $\gamma = \tau\omega^2$. On the other hand, $\boldsymbol{F}_{\text{drive}}$ is the force from the incident wave, and it's

$$\boldsymbol{F}_{\text{drive}} = q\boldsymbol{E}e^{-\mathrm{i}\omega t}. \tag{11.179}$$

Putting $\boldsymbol{F}_{\text{damping}}$ and $\boldsymbol{F}_{\text{drive}}$ into Equation (11.177) results in the needed equation

$$\ddot{\boldsymbol{x}} + \omega_0^2\boldsymbol{x} + \gamma\dot{\boldsymbol{x}} = \frac{q}{m}\boldsymbol{E}e^{-\mathrm{i}\omega t}. \tag{11.180}$$

Note here we assume that $\gamma \ll (\omega_0, \omega)$.

From what we have learned about the forced oscillation of a harmonic oscillator, $\boldsymbol{x}$ in Equation (11.180) has a solution of the form $\boldsymbol{x}_0 e^{-\mathrm{i}\omega t}$, and the solution of $\boldsymbol{x}$ is given by

$$\boldsymbol{x} = \frac{(q/m)\boldsymbol{E}_0 e^{-\mathrm{i}\omega t}}{\omega_0^2 - \omega^2 - \mathrm{i}\gamma\omega} = \frac{(q/m)\boldsymbol{E}_0 e^{-\mathrm{i}(\omega t - \delta)}}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}, \tag{11.181}$$

where $-\delta$ is the phase angle of $\omega_0^2 - \omega^2 - \mathrm{i}\gamma\omega$, and

$$\tan\delta = \frac{\gamma\omega}{\omega_0^2 - \omega^2}. \tag{11.182}$$

From Equation (11.181), the second order time derivative of the dipole moment $\boldsymbol{p}$ is

$$\ddot{\boldsymbol{p}} = q\ddot{\boldsymbol{x}} = -\frac{(q^2/m)\omega^2\boldsymbol{E}_0 e^{-\mathrm{i}(\omega t - \delta)}}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}. \tag{11.183}$$

The remaining calculation of the scattering cross-section is almost identical to that in the previous section. The angular distribution of the radiated power from the dipole radiation is

$$\left\langle \frac{\mathrm{d}P}{\mathrm{d}\Omega} \right\rangle = \frac{\mu_0}{32\pi^2 c}|\ddot{\boldsymbol{p}}|^2 \sin^2\theta \tag{11.184}$$

$$= \frac{\mu_0}{32\pi^2 c}\left(\frac{q^2}{m}\right)^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} E_0^2 \sin^2\theta. \tag{11.185}$$

The total radiated power is simply $\int \langle \frac{\mathrm{d}P}{\mathrm{d}\Omega} \rangle \mathrm{d}\Omega$, or

$$\langle P \rangle = \frac{\mu_0 q^4}{12\pi m^2 c} \frac{\omega^4 E_0^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}. \tag{11.186}$$

The total cross section is simply

$$\sigma = \frac{\langle P \rangle}{\langle S \rangle} = \frac{\langle P \rangle}{(1/2\mu_0 c)E_0^2} \tag{11.187}$$

$$= \frac{\mu_0^2 q^4}{6\pi m^2} \frac{\omega^4}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}, \tag{11.188}$$

where we have used Equation (11.167) for $\langle S \rangle$. Using the classical electron radius $r_e = q^2/4\pi\epsilon_0 mc^2$, the cross section is

$$\sigma = \frac{8\pi}{3} r_e^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}. \tag{11.189}$$

Let's now discuss several limiting cases of Equation (11.189). If $\omega \ll \omega_0$, then

$$\boxed{\sigma \approx \frac{8\pi}{3} r_e^2 \frac{\omega^4}{\omega_0^4}.} \tag{11.190}$$

This is the Rayleigh scattering we've mentioned before. If $\omega \gg \omega_0$,

$$\boxed{\sigma \approx \frac{8\pi}{3} r_e^2,} \tag{11.191}$$

which is Thomson scattering cross section, the electron move (ω_0) so slowly that it is essentially a free electron to the incident wave. If, however, $\omega \sim \omega_0$, then

$$\boxed{\sigma = \frac{8\pi}{3} r_e^2 \left(\frac{\omega}{\gamma}\right)^2,} \tag{11.192}$$

which is much much larger than the Thomson scattering cross section because resonance occurs.