Convex Optimization Homework # 1

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1.

(a)

$f((1-\theta)x+\theta x', (1-\theta)y+\theta y') = 2(1-\theta)x+2\theta x' + (1-\theta)y+\theta y' = (1-\theta)(2x+y)+\theta(2x'+y') = (1-\theta)f(x,y)+\theta f(x',y')$

$\Rightarrow$ $2x+y$, $x,y \in \mathbb{R}_+$ is convex

Or simply by using "Any line segment is convex" we can see that the objective function is convex

Since "sum of the convex function is also convex", here just show that $\frac{1}{x}$ is convex

$$\frac{1}{\theta x+(1-\theta)x'} \le \frac{\theta}{x}+\frac{(1-\theta)}{x'} \Rightarrow \frac{1}{\theta(x-x')+x'} \le \frac{\theta(x'-x)+x}{xx'} \Rightarrow xx' \le [\theta(x-x')+x'][\theta(x'-x)+x] = -\theta^2(x-x')^2+\theta x'(x'-x)+\theta x(x-x')+xx'$$

$$= -\theta^2(x-x')^2+\theta(x-x')(x-x')+xx' = -\theta^2(x-x')^2+\theta(x-x')^2+xx'$$

$$= (\theta-\theta^2)(x-x')^2+xx' = \underbrace{\theta(1-\theta)}_{\ge 0}\underbrace{(x-x')^2}_{\ge 0}+xx'$$

So $\frac{1}{x}$ is convex $\Rightarrow$ $\underbrace{\overbrace{\frac{1}{x}+\frac{1}{y}}^{\text{If } f_1 \text{ and } f_2 \text{ are convex, then } f_1+f_2 \text{ also convex}}+(-1)}_{Ax+b \text{ (affine transformation) preserve convexity.}}$ is also convex.

$\Rightarrow$ objective function and constraint function are both convex $\Rightarrow$ convex problem #

(b)

optimal value : 5.828427 699 2547915

optimal point : $x^* = 1.7071422256133406$, $y^* = 2.4141425413321234$

$\Rightarrow$ $2 \cdot x^* + y^* = 5.8284269925479 15$

$\Rightarrow$ $\frac{1}{x^*}+\frac{1}{y^*} = 1.00000000 2329 26394$

(應該是因為 x^*, y^* 表示位數的關係，被四捨五入後，所以才會在小數點後很多位有非0值出現)

2.

(a)

For objective function,

We know that p-norm is convex and a^Tx (affine transformation) is convex

by "if f_1 and f_2 are convex, then f_1+f_2 is also convex" we know that

$\|x\|_p + a^Tx$ is convex

For the constraint,

From previous problem we know $\frac{1}{x}$ is convex, then the combination

$\Sigma_{i=1}^{n} \frac{1}{x_i}$ is also convex.

($\Sigma_{i=1}^{n} \frac{1}{x_i} - 1$ is also convex)

$\Rightarrow$ both objective function and constraint are convex $\Rightarrow$ convex problem #

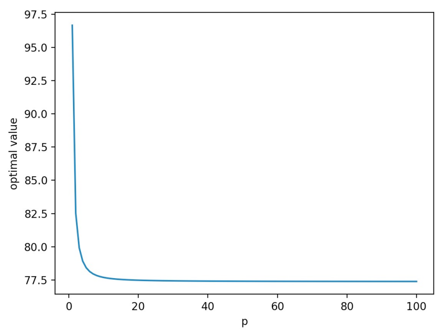
(b)

```
status: optimal
optimal value 79.9188521765671
optimal var [7.11494804 5.79055313 4.93701801 4.35359597 3.92913067]
```

optimal value : 79.9188521765671

optimal point : (7.11494804, 5.79055313, 4.93701801, 4.35359597, 3.92913067)

(c)



3. $\underset{x \in \mathbb{R}^2_+}{\text{maximize}} \quad x_1 x_2$

subject to $x_1 + c x_2 \le 1$

$\|x\| \le \frac{1}{\sqrt{2}}$

(a) For the objective function $f_0(x) = x_1 x_2$ with $\mathrm{dom} f = \mathbb{R}^2_+$

The Hessian of $f_0(x)$ is $\nabla^2 f(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ → NOT positive semidefinite

↘ NOT negative semidefinite

⇒ $f_0(x)$ is neither convex nor concave.

$f_0(x)$ is NOT quasiconvex. (since its sublevel sets are not convex)

It is quasiconcave since its superlevel sets $\{(x_1, x_2) \in \mathbb{R}^2_+ \mid x_1 x_2 \ge \alpha\}$ are convex.

For the constraints,

$x_1 + c x_2$ → linear hence convex

$\|x\|$ → norm → convex.

(b) $f(x) = x_1 x_2$

$\Rightarrow f(x) = x_1 x_2 \ge t \iff 1 \ge t / x_1 x_2 \Rightarrow \frac{1}{x_1 x_2} \le \frac{1}{t}$ or $\frac{t}{x_1 x_2} - 1 \le 0$

$\Rightarrow \phi_t = \frac{1}{x_1 x_2} - \frac{1}{t}$ is convex for all t

$\phi_t = \frac{t}{x_1 x_2} - 1$ is convex for all t.

(c)

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(c) ----------------------------------------
Using qcp solver...
/Users/yangyating/anaconda3/envs/cvxpy/lib/python3.9/site-packages/cvxpy/problem
s/problem.py:1245: UserWarning: Solution may be inaccurate. Try another solver,
adjusting the solver settings, or solve with verbose=True for more information.
  warnings.warn(
optimal value p* 0.1250004768371582
optimal point x*  [0.50000489 0.24999755]
Bisection method...
/Users/yangyating/anaconda3/envs/cvxpy/lib/python3.9/site-packages/cvxpy/problem
s/problem.py:1245: UserWarning: Solution may be inaccurate. Try another solver,
adjusting the solver settings, or solve with verbose=True for more information.
  warnings.warn(
--------------------------------------------------
value of t:  0.1250001754611731
optimal value p*  0.12500001998892027
optimal point x*  [0.49999978 0.25000015]
```

Using qcp = True

p^*: 0.1250004768371582

$x_1 = 0.50000489$, $x_2 = 0.24999755$

Using bisection method

p^*: 0.1250001754611731

$x_1 = 0.49999978$, $x_2 = 0.25000015$

(d)

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(d) ----------------------------------------
When c =  0.5
value of t:  0.2500002305954695
optimal value p*  0.25000013519607256
optimal point x*  [0.50000185 0.49999842]
When c =  1
value of t:  0.25000018365681176
optimal value p*  0.2500000248058436
optimal point x*  [0.50000003 0.50000002]
When c =  2
value of t:  0.1250001754611731
optimal value p*  0.12500001998892027
optimal point x*  [0.49999978 0.25000015]
When c =  3
value of t:  0.08333358950912956
optimal value p*  0.08333334744806407
optimal point x*  [0.49999972 0.16666679]
When c =  4
value of t:  0.06250033564865592
optimal value p*  0.06250002777212062
optimal point x*  [0.4999376  0.12501566]
```

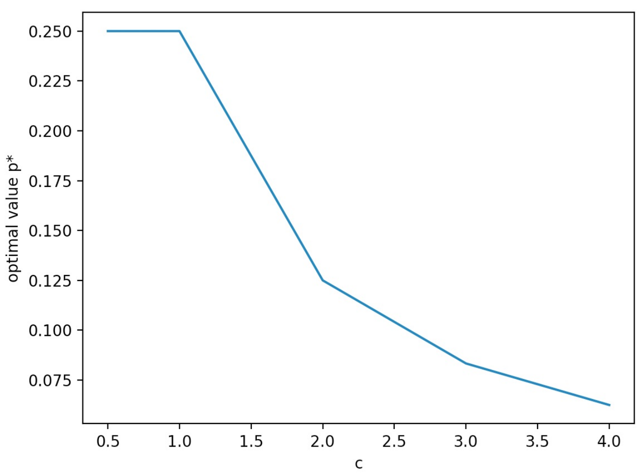

(e)

C	0.5	1.0	2.0	3.0	4.0
$x_1 + C x_2 \le 1$	inactive	active	active	active	active
$\lVert x \rVert \le \frac{1}{\sqrt{2}}$	active	active	inactive	inactive	inactive