Resume computation HGF

Lucile F.

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Contents

1 SIMPLIFICATION

The inversion problem is to find the posterior $p\big(x^{(k)}, \chi | u^{(1...k)}\big)$. According to the formula for conditional probability, we have :

$$p\big(x^{(k)}, \chi | u^{(1...k)}\big) = p\big(x^{(k)} | \chi, u^{(1...k)}\big) p\big(\chi | u^{(1...k)}\big) \tag{1}$$

We considere that $p\big(\chi | u^{(1...k)}\big) = \delta(\chi - \chi_a)$, where χ_a are the fixed parameter values that characterize a particular agent at a particular time. So it remains to compute $p\big(x^{(k)} | \chi, u^{(1...k)}\big)$.
The problem can be formulated as follow:
Knowing the structure of the environment - ie the law followed by each x_i -, the realisation up to time k and the parameters χ, we want to know $x^{(k)}$

2 EXACT INVERSION

$p(k_{i+1}|k_i) \sim \mathcal{N}(k_i, l)$

According to Bayes' theorem :

$$p\big(x^{(k)} | \chi, u^{(1...k)}\big) = \frac{p\big(\chi, u^{(1...k)} | x^{(k)}\big) p\big(x^{(k)}\big)}{p\big(\chi, u^{(1...k)}\big)} = \frac{p\big(\chi, u^{(1...k)} | x^{(k)}\big) p\big(x^{(k)}\big)}{\int p\big(\chi, u^{(1...k)}, x^{(k)}\big) dx^{(k)}} \tag{2}$$

To compute the red expression we need to perform a marginalization over $x^{(k)}$. But this is intractable.

3 ASSUMPTIONS

1. **Variational assumption.** The posterior distribution is approximated by a **variational distribution**:
$$p\big(x^{(k)}|\chi, u^{(1...k)}\big) \approx q\big(x^{(k)}|\chi, u^{(1...k)}\big)$$

   The distribution q is restricted to belong to a family of distributions of simpler form than p and selected with the intention of making q similar to p. Here we will choose :
$$q\big(x^{(k)}|\chi, u^{(1...k)}\big) = \mathcal{N}\Big(x^{(k)}, \mu^{(k)}, \sigma^{(k)}\Big)$$

   In fact we make minimal assumptions about the form of the approximate posteriors by following the ***maximum entropy principle*** *with the first two moments.*

2. **Mean field approximation.** $q\big(x^{(k)}|\chi, u^{(1...k)}\big)$ is fully factorized over the hidden variables :
$$q\big(x^{(k)}|\chi, u^{(1...k)}\big) = \prod_{i=1}^{n} q_i\big(x_i^{(k)}|\chi, u^{(1...k)}\big) \tag{3}$$

   where $x = \Big(x^{(1)}, \ldots, x^{(k)}\Big)$ and $x^{(j)} = \Big(x_1^{(j)}, \ldots, x_n^{(j)}\Big)$ (There are k steps of times and n levels of hidden variables)

Thanks to theses assumptions, the problem of finding the posterior p, became finding $q_i, i \in \{1, ..., n\}$ the more similar to p. However $q\big(x^{(k)}|\chi, u^{(1...k)}\big) = \mathcal{N}\Big(x^{(k)}, \mu^{(k)}, \sigma^{(k)}\Big)$ and so for each level i, $q_i\big(x_i^{(k)}|\chi, u^{(1...k)}\big) = \mathcal{N}\Big(x_i^{(k)}, \mu_i^{(k)}, \sigma_i^{(k)}\Big)$ -$q_i, i \in \{1, ..., n\}$ is characterize by its mean and variance, it is what we search-.

4 VARIATIONAL INVERSION

- **Kullback–Leibler divergence**

$$D_{KL}(Q||P) := \sum_{Z} Q(Z) \ln \frac{Q(Z)}{P(Z|X)}$$

The Kullback–Leibler divergence (KL-divergence) of P from Q can be see as the choice of dissimilarity function. So we can reformulate our problem as the minimization of $D_{KL}(q||p)$

4.1 Minimize $D_{KL}(q||p)$ is equivalent to maximize the low bound

To do so we will use the **variational Bayesian** where it is a question of determining the posterior distributions $p\big(x^{(k)}, \chi|u^{(1...k)}\big)$ by maximizing the **log-model evidence** $\mathscr{L}$.
The log-model evidence $\mathscr{L}(x^{(k)}, u^{(1...k)})$ is the negative surprise about the data given a model. It is approximated by a lower bound, **the negative free energy**. The variational inversion will give us a expression of q

4.2 Compute the log-evidence $\mathscr{L}$

By the formula of the total probability we can write the posterior as :

$$p\big(\chi|u^{(1...k)}\big) = \int p\big(x^{(k)}, \chi|u^{(1...k)}\big) dx^{(k)} \tag{4}$$

The log-model evidence $\mathscr{L}(x^{(k)}, u^{(1...k)})$ is the negative surprise about data given a model :

$$\begin{aligned} \mathscr{L}(x^{(k)}, u^{(1...k)}) &:= \ln\big(p\big(x^{(k)}, \chi|u^{(1...k)}\big)\big) && \text{by definition of the log- model evidence} \\ &= \ln\Big(\int p\big(x^{(i)}, \chi|u^{(1...k)}\big) dx^{(k)}\Big) && \text{by } (4) \end{aligned} \tag{5}$$

4.3 Compute the negative free energy $\mathscr{F}$

- **Jensen's inequality.** For φ a concave function,

$$\varphi\left(\frac{\sum_{i=1}^{n} a_i x_i}{\sum_{i=1}^{n} a_i}\right) \geq \frac{\sum_{i=1}^{n} a_i \varphi(x_i)}{\sum_{i=1}^{n} a_i} \tag{6}$$

Let's define an auxiliary distribution over the hidden variables :

$$q_{x^{(k)}}(x^{(k)}) := p(x^{(k)}|\chi, u^{(1...k)})$$

We will introduce it into the expression of the log-evidence $\mathscr{L}$ and then minore it.

$$\begin{aligned}
\mathscr{L}(x^{(k)}, u^{(1...k)}) &= \ln\left(\int p(x^{(k)}, \chi|u^{(1...k)}) dx^{(k)}\right) && \text{by (5)} \\
&= \ln\left(\int q_{x^{(k)}}\left(x^{(k)}\right) \frac{p\left(x^{(k)}, \chi|u^{(1...k)}\right)}{q_{x^{(k)}}\left(x^{(k)}\right)} dx^{(k)}\right) && \text{Introduction of } q_{x^{(k)}} \\
&\geq \int q_{x^{(k)}}\left(x^{(k)}\right) \ln\left(\frac{p\left(x^{(k)}, \chi|u^{(1...k)}\right)}{q_{x^{(k)}}\left(x^{(k)}\right)}\right) dx^{(k)} && \text{by (6), Jensen's inequality *} \\
&=: \mathscr{F}\left(q_{x^{(k)}}\left(x^{(k)}\right)\right)
\end{aligned}$$

* We applied the Jensen's inequality with :

- $\varphi := \ln$, a concave function
- $x_i := \frac{p\left(x^{(k)}, \chi|u^{(1...k)}\right)}{q_{x^{(k)}}\left(x^{(k)}\right)}$
- $a_i := q_{x^{(k)}}\left(x^{(k)}\right)$ so $\sum_{i=1}^{n} a_i = \int q_{x^{(k)}}\left(x^{(k)}\right) = 1$

$\mathscr{F}$ is a low bound of $\mathscr{L}$.

4.4 Mean field approximation

In the variational approach we can constrain the posterior distributions to be of a particular tractable form, for example factorized over the hidden variables :

- **Assumption**
  We do the **mean field assumption** : $q_{x^{(k)}}\left(x^{(k)}\right)$ is fully factorized over the hidden variables :

$$q_{x^{(k)}}\left(x^{(k)}\right) = \prod_{i=1}^{n} q_{x_i^{(k)}}\left(x_i^{(k)}\right) \tag{7}$$

  where $x = \left(x^{(1)}, \ldots, x^{(k)}\right)$ and $x^{(j)} = \left(x_1^{(j)}, \ldots, x_n^{(j)}\right)$ (There are k steps of times and n levels of hidden variables)

So the low bound become :

$$\begin{aligned}
\mathscr{F}\Big(q_{x^{(k)}}\big(x^{(k)}\big)\Big) &= \int q_{x^{(k)}}\big(x^{(k)}\big)\ln\Big(\frac{p(x^{(k)},\chi|u^{(1...k)})}{q_{x^{(k)}}\big(x^{(k)}\big)}dx^{(k)}\Big) && \text{by definition}\\
&= \int q_{x^{(k)}}\big(x^{(k)}\big)\ln\big(p(x^{(k)},\chi|u^{(1...k)})\big)dx^{(k)}\\
&\quad - \int q_{x^{(k)}}\big(x^{(k)}\big)\ln\big(q_{x^{(k)}}\big(x^{(k)}\big)\big)dx^{(k)} && \text{by property of the logarithm}\\
&= \int \prod_{i=1}^{n} q_{x_i^{(k)}}\big(x_i^{(k)}\big)\ln\big(p(x^{(k)},\chi|u^{(1...k)})\big)dx^{(k)}\\
&\quad \underbrace{- \int \prod_{i=1}^{n} q_{x_i^{(k)}}\big(x_i^{(k)}\big)\ln\big(\prod_{i=1}^{n} q_{x_i^{(k)}}\big(x_i^{(k)}\big)\big)dx^{(k)}}_{=:*}\\
&&& \text{by (7), mean field assumption}
\end{aligned}$$

We notice that $*$ is the definition of the entropy. By property of the entropy we have :

$$* = -\sum_{j=1}^{n} q_{x_j^{(i)}}\big(x_j^{(i)}\big)\ln\big(q_{x_j^{(i)}}\big(x_j^{(i)}\big)\big) \tag{8}$$

So :

$$\begin{aligned}
\mathscr{F}\Big(q_{x^{(k)}}\big(x^{(k)}\big)\Big) &= \int \prod_{j=1}^{n} q_{x_j^{(k)}}\big(x_j^{(k)}\big)\ln\big(p(x^{(k)},\chi|u^{(1...k)})\big)dx^{(k)}\\
&\quad - \sum_{j=1}^{n} q_{x_j^{(k)}}\big(x_j^{(k)}\big)\ln\big(q_{x_j^{(k)}}\big(x_j^{(k)}\big)\big) && \text{by (8)}
\end{aligned}$$

4.5 Lagrange multipliers to find the form of q

We use Lagrange multipliers : $\{\lambda_i\}$ with $i \in \{1,..,n\}$ on $\mathscr{F}$, with the constraints of normalisation: $\int q_{x_i^{(k)}}\big(x_i^{(k)}\big)dx_i^{(k)} = 1$.
We have the new functional :

$$\tilde{\mathscr{F}}\Big(q_{x^{(k)}}\big(x^{(k)}\big)\Big) = \mathscr{F}\Big(q_{x^{(k)}}\big(x^{(k)}\big)\Big) + \sum_{i=1}^{n}\lambda_i\Big(\int q_{x_i^{(k)}}\big(x_i^{(k)}\big)dx_i^{(k)} - 1\Big)$$

We then take the functional derivative of this expression with respect to each $q_{x_i^{(k)}}\big(x_i^{(k)}\big)$, and equalize to zero. Let's fix $i \in \{1,..,k\}$ we have :

$$\begin{aligned}
\frac{\partial\tilde{\mathscr{F}} q_{x_j^{(i)}}\big(x_j^{(i)}\big)}{\partial q_{x_j^{(i)}}\big(x_j^{(i)}\big)} &= \frac{\partial\mathscr{F} q_{x_j^{(i)}}\big(x_j^{(i)}\big)}{\partial q_{x_j^{(i)}}\big(x_j^{(i)}\big)} + \lambda_{ij}\\
&= \int \prod_{\substack{h=1\\h\neq j}}^{n} q_{x_h^{(i)}}\big(x_h^{(i)}\big)\ln\big(p(x^{(i)},\chi|u^{(1...k)})\big)dx_{\setminus j}^{(i)} - \ln\big(q_{x_j^{(i)}}\big(x_j^{(i)}\big)\big) + \lambda_{ij} = 0
\end{aligned}$$

Where $dx_{\setminus j}^{(i)} := \{dx_h^{(i)}\}_{h\neq j}$.
Isolating $q_{x_j^{(i)}}\big(x_j^{(i)}\big)$:

$$\begin{aligned}
\ln\big(q_{x_j^{(i)}}\big(x_j^{(i)}\big)\big) &= \int \prod_{\substack{h=1\\h\neq j}}^{n} q_{x_h^{(i)}}\big(x_h^{(i)}\big)\ln\big(p(x^{(i)},\chi|u^{(1...k)})\big)dx_{\setminus j}^{(i)} + \lambda_{ij}\\
\implies q_{x_j^{(i)}}\big(x_j^{(i)}\big) &= e^{\int \prod_{\substack{h=1\\h\neq j}}^{n} q_{x_h^{(i)}}\big(x_h^{(i)}\big)\ln\big(p(x^{(i)},\chi|u^{(1...k)})\big)dx_{\setminus j}^{(i)}}\, e^{\lambda_{ij}}
\end{aligned} \tag{9}$$

We inject this expression of $q_{x_j^{(i)}}\left(x_j^{(i)}\right)$ in the constraints of normalisation equation $\int q_{x_j^{(i)}}\left(x_j^{(i)}\right)dx_j^{(i)} = 1$ and isolate λ_{ij} :

$$
\begin{aligned}
&\int q_{x_j^{(i)}}\left(x_j^{(i)}\right)dx_j^{(i)} = 1 \\
\implies &\int e^{\lambda_{ij}} e^{\int \prod_{\substack{h=1\\h\neq j}}^{n} q_{x_h^{(i)}}\left(x_h^{(i)}\right) \ln\left(p(x^{(i)},\chi|u^{(1...k)})\right)dx_{\setminus j}^{(i)}} dx_j^{(i)} = 1 \\
\implies &e^{\lambda_{ij}} \int e^{\int \prod_{\substack{h=1\\h\neq j}}^{n} q_{x_h^{(i)}}\left(x_h^{(i)}\right) \ln\left(p(x^{(i)},\chi|u^{(1...k)})\right)dx_{\setminus j}^{(i)}} dx_j^{(i)} = 1 \\
\implies &e^{\lambda_{ij}} = \left(\int e^{\int \prod_{\substack{h=1\\h\neq j}}^{n} q_{x_h^{(i)}}\left(x_h^{(i)}\right) \ln\left(p(x^{(i)},\chi|u^{(1...k)})\right)dx_{\setminus j}^{(i)}} dx_j^{(i)}\right)^{-1}
\end{aligned}
\tag{10}
$$

Now we substitute this expression into the one of $q_{x_j^{(i)}}\left(x_j^{(i)}\right)$ (9)

$$
\begin{aligned}
q_{x_j^{(i)}}\left(x_j^{(i)}\right) &= e^{\int \prod_{\substack{h=1\\h\neq j}}^{n} q_{x_h^{(i)}}\left(x_h^{(i)}\right) \ln\left(p(x^{(i)},\chi|u^{(1...k)})\right)dx_{\setminus j}^{(i)}} e^{\lambda_{ij}} && \text{by (9)} \\
&= e^{\underbrace{\int \prod_{\substack{h=1\\h\neq j}}^{n} q_{x_h^{(i)}}\left(x_h^{(i)}\right) \ln\left(p(x^{(i)},\chi|u^{(1...k)})\right)dx_{\setminus j}^{(i)}}_{=:I\left(x_j^{(i)}\right)}} \left(\underbrace{\int e^{\int \prod_{\substack{h=1\\h\neq j}}^{n} q_{x_h^{(i)}}\left(x_h^{(i)}\right) \ln\left(p(x^{(i)},\chi|u^{(1...k)})\right)dx_{\setminus j}^{(i)}} dx_j^{(i)}}_{=:Z_j^{(i)}}\right)^{-1} && \text{by (10)}
\end{aligned}
$$

So we have :

$$
q_{x_j^{(i)}}\left(x_j^{(i)}\right) = \frac{1}{Z_j^{(i)}} e^{I\left(x_j^{(i)}\right)} \tag{11}
$$

With:

$$
Z_j^{(i)} = \int e^{\int \prod_{\substack{h=1\\h\neq j}}^{n} q_{x_h^{(i)}}\left(x_h^{(i)}\right) \ln\left(p(x^{(i)},\chi|u^{(1...k)})\right)dx_{\setminus j}^{(i)}} dx_j^{(i)}
$$

and

$$
I\left(x_j^{(i)}\right) = \int \prod_{\substack{h=1\\h\neq j}}^{n} q_{x_h^{(i)}}\left(x_h^{(i)}\right) \ln\left(p(x^{(i)},\chi|u^{(1...k)})\right)dx_{\setminus j}^{(i)} \tag{12}
$$

5 QUADRATIC APPROXIMATION TO THE VARIATIONAL ENERGY I

5.1 Compute $p(x^{(k)}, \chi | u^{(1...k)})$

1. Factorization
   Let fix i, we definie : $x_{\backslash i} = \{x_j\}_{j \neq i}$.
   Then, we have :

$$
\begin{aligned}
p\Big(x^{(k)}, \chi | u^{(1...k)}\Big) &= p\Big(x_i^{(k)}, x_{\backslash i}^{(k)}, \chi | u^{(1...k)}\Big) \\
&= \int_{x_1^{(k-1)}} \cdots \int_{x_n^{(k-1)}} \prod_{j=1}^{n} p\Big(x_j^{(k)} | x_j^{(k-1)}, x_{\backslash j}^{(k)}, \chi, u^{(1...k)}\Big) \\
&\quad\ p\Big(x_j^{(k-1)} | u^{(1...k-1)}\Big) dx_1^{(k-1)} \ldots dx_n^{(k-1)} \\
&= \prod_{j=1}^{n} \int_{x_j^{(k-1)}} p\Big(x_j^{(k)} | x_j^{(k-1)}, x_{\backslash j}^{(k)}, \chi, u^{(1...k)}\Big) p\Big(x_j^{(k-1)} | u^{(1...k-1)}\Big) dx_j^{(k-1)}
\end{aligned}
$$

   where :

   - the first equality by the notation definied
   - the second equality by the formula of the total probability (cf appendix)
   - in the third equality we factorize the integral since each term i depends only on $x_i^{(k-1)}$.

2. Compute $\int p\Big(x_j^{(k)} | x_j^{(k-1)}, x_{\backslash j}^{(k)}, \chi\Big) p\Big(x_j^{(k-1)} | u^{(1...k-1)}\Big) dx_j^{(k-1)}$ for $j \in \{1...n\}$ fixed

   - **By definition of the model HFG**, the next inference at level j, time k, knowing the previous inferences and all the other level at time k folllow a normal walk centred in the inference at the previous time, same level, and with a variance of $t^{(k)} e^{\kappa_j x_{j+1}^{(k)} + \omega_j}$, (see appendice A Mathys 2014) for the justification)

$$
p\Big(x_j^{(k)} | x_j^{(k-1)}, x_{\backslash j}^{(k)}, \chi\Big) = \mathscr{N}\left(x_j^{(k)}, x_j^{(k-1)}, t^{(k)} e^{\kappa_j x_{j+1}^{(k)} + \omega_j}\right)
$$

     Where for $j = n$, the variance is considered as constante : $t^{(k)} e^{\kappa_n x_{n+1}^{(k)} + \omega_n} = \theta$

   - **By assumption of the form of the posterior.** We did the assumption that the posterior at each level j and time $k-1$ follows a normal distribution centred on $\mu_j^{(k-1)}$, and with a variance of $\sigma_j^{(k-1)}$.

$$
p\Big(x_j^{(k-1)} | u^{(1...k-1)}\Big) = \mathscr{N}\left(x_j^{(k-1)}, \mu_j^{(k-1)}, \sigma_j^{(k-1)}\right)
$$

   So, we obtain :

$$
\begin{aligned}
&\int_{x_j^{(k-1)}} p\Big(x_j^{(k)} | x_j^{(k-1)}, x_{\backslash j}^{(k)}, \chi\Big) p\Big(x_j^{(k-1)} | u^{(1...k-1)}\Big) dx_j^{(k-1)} \\
&= \int_{x_j^{(k-1)}} \mathscr{N}\left(x_j^{(k)}, x_j^{(k-1)}, t^{(k)} e^{\kappa_j x_{j+1}^{(k)} + \omega_j}\right) \mathscr{N}\left(x_j^{(k-1)}, \mu_j^{(k-1)}, \sigma_j^{(k-1)}\right) dx_j^{(k-1)} \\
&= \mathscr{N}\left(x_j^{(k)}, \mu_j^{(k-1)}, \sigma_j^{(k-1)} + t^{(k)} e^{\kappa_j x_{j+1}^{(k)} + \omega_j}\right)
\end{aligned}
$$

   Where the ultimate equality come from the following property, proved in appendix :

$$
\int_{\mu} \mathscr{N}\left(z | \mu, \sigma_0^2\right) \mathscr{N}\left(\mu | \mu_0, \sigma_1^2\right) d\mu = \mathscr{N}\left(z | \mu_0, \sigma_0^2 + \sigma_1^2\right) \tag{13}
$$

3. Compute $p\left(x^{(k)}, \chi | u^{(1...k)}\right)$
   So we can deduce :

$$p\left(x^{(k)}, \chi | u^{(1...k)}\right) = \prod_{j=1}^{n} \mathcal{N}\left(x_j^{(k)}, \mu_j^{(k-1)}, \sigma_j^{(k-1)} + t^{(k)} e^{\kappa_j x_{j+1}^{(k)} + \omega_j}\right) \tag{14}$$

5.2 Compute I

We will now use the expression of $p\left(x^{(k)}, \chi | u^{(1...k)}\right)$ to compute I.

$$\begin{aligned}
I(x_i^{(k)}) &= \int \cdots \int q\left(x_{\setminus i}^{(k)}\right) \ln\left(p\left(x^{(k)}, \chi | u^{(1...k)}\right)\right) dx_{\setminus i}^{(k)} && \text{by (12) : definition of } I \\
&= \int \cdots \int q\left(x_{\setminus i}^{(k)}\right) \ln\left(\prod_{j=1}^{n} \mathcal{N}\left(x_j^{(k)}, \mu_j^{(k-1)}, \sigma_j^{(k-1)} + t^{(k)} e^{\kappa_j x_{j+1}^{(k)} + \omega_j}\right)\right) dx_{\setminus i}^{(k)} && \text{by (14)} \\
&= \sum_{j=1}^{n} \int \cdots \int q\left(x_{\setminus i}^{(k)}\right) \ln\left(\mathcal{N}\left(x_j^{(k)}, \mu_j^{(k-1)}, \sigma_j^{(k-1)} + t^{(k)} e^{\kappa_j x_{j+1}^{(k)} + \omega_j}\right)\right) dx_{\setminus i}^{(k)} && \text{by property of the logarithm} \\
&= \sum_{\substack{j=1 \\ j \neq i-1 \\ j \neq i}}^{n} \underbrace{\int \cdots \int q\left(x_{\setminus i}^{(k)}\right) \ln\left(\mathcal{N}\left(x_j^{(k)}, \mu_j^{(k-1)}, \sigma_j^{(k-1)} + t^{(k)} e^{\kappa_j x_{j+1}^{(k)} + \omega_j}\right)\right) dx_{\setminus i}^{(k)}}_{\text{don't depend on } x_i^{(k)} \text{ so it is a constante}} && \longleftarrow \text{others terms} \\
&\quad + \underbrace{\int \cdots \int q\left(x_{\setminus i}^{(k)}\right) \ln\left(\mathcal{N}\left(x_{i-1}^{(k)}, \mu_{i-1}^{(k-1)}, \sigma_{i-1}^{(k-1)} + t^{(k)} e^{\kappa_{i-1} x_i^{(k)} + \omega_{i-1}}\right)\right) dx_{\setminus i}^{(k)}}_{=:\star} && \longleftarrow \text{term } j = i-1 \\
&\quad + \underbrace{\int \cdots \int q\left(x_{\setminus i}^{(k)}\right) \ln\left(\mathcal{N}\left(x_i^{(k)}, \mu_i^{(k-1)}, \sigma_i^{(k-1)} + t^{(k)} e^{\kappa_i x_{i+1}^{(k)} + \omega_i}\right)\right) dx_{\setminus i}^{(k)}}_{=:\star\star} && \longleftarrow \text{term } j = i
\end{aligned}$$

5.2.1 Recall

1. Definition of $q\left(x_{\setminus i}^{(k)}\right)$

$$q\left(x_{\setminus i}^{(k)}\right) = \prod_{\substack{j=1 \\ j \neq i}}^{n} q\left(x_j^{(k)}\right) \tag{15}$$

2. Property of distribution of probability

$$\int q\left(x_i^{(k)}\right) dx_i^{(k)} = 1 \tag{16}$$

3. Definition of the normal distribution, (where here σ is the variance)

$$\mathcal{N}(x, \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma}} \tag{17}$$

4. Definition of the mean

$$\int q\left(x_i^{(k)}\right) x_i^{(k)} dx_i^{(k)} = \mu_i^{(k)} \tag{18}$$

5. Definition of the variance

$$\int q\left(x_i^{(k)}\right) \left(x_i^{(k)} - \mu_i^{(k)}\right)^2 dx_i^{(k)} = \sigma_i^{(k)} \tag{19}$$

5.2.2 compute $\star$

Using $x_{\setminus i} = \{x_j\}_{j\neq i}$, and the definition of $q\left(x_{\setminus i}^{(k)}\right)$ (15) :

$$\star = \int_{x_n^{(k)}} \cdots \int_{x_1^{(k)}} q\left(x_{\setminus i}^{(k)}\right) \ln\left(\mathscr{N}\left(x_{i-1}^{(k)}, \mu_{i-1}^{(k-1)}, \sigma_{i-1}^{(k-1)} + t^{(k)} e^{\kappa_{i-1} x_i^{(k)} + \omega_{i-1}}\right)\right) dx_1^{(k)} \dots dx_{i-1}^{(k)} dx_{i+1}^{(k)} \dots dx_n^{(k)}$$

$$= \int_{x_n^{(k)}} \cdots \int_{x_1^{(k)}} \prod_{\substack{j=1\\ j\neq i}}^{n} q\left(x_j^{(k)}\right) \ln\left(\mathscr{N}\left(x_{i-1}^{(k)}, \mu_{i-1}^{(k-1)}, \sigma_{i-1}^{(k-1)} + t^{(k)} e^{\kappa_{i-1} x_i^{(k)} + \omega_{i-1}}\right)\right) dx_1^{(k)} \dots dx_{i-1}^{(k)} dx_{i+1}^{(k)} \dots dx_n^{(k)}$$

Where by the definition of the normal distribution (17) :

$$\mathscr{N}\left(x_{i-1}^{(k)}, \mu_{i-1}^{(k-1)}, \sigma_{i-1}^{(k-1)} + t^{(k)} e^{\kappa_{i-1} x_i^{(k)} + \omega_{i-1}}\right) = \frac{1}{\sqrt{2\pi\left(\sigma_{i-1}^{(k-1)} + t^{(k)} e^{\kappa_{i-1} x_i^{(k)} + \omega_{i-1}}\right)}} e^{-\frac{1}{2}\frac{\left(x_{i-1}^{(k)} - \mu_{i-1}^{(k-1)}\right)^2}{\sigma_{i-1}^{(k-1)} + t^{(k)} e^{\kappa_{i-1} x_i^{(k)} + \omega_{i-1}}}} \tag{20}$$

Since (20) depend only on $x_{i-1}^{(k)}$ and $x_i^{(k)}$ we can put this term outside of the integrals on $dx_j^{(k)}$,with $j \neq i-1$:

$$\begin{aligned}
\star &= \int_{x_n^{(k)}} \cdots \int_{x_2^{(k)}} \prod_{\substack{j=2\\ j\neq i}}^{n} q\left(x_j^{(k)}\right) \ln\left(\mathscr{N}\left(x_{i-1}^{(k)}, \mu_{i-1}^{(k-1)}, \sigma_{i-1}^{(k-1)} + t^{(k)} e^{\kappa_{i-1} x_i^{(k)} + \omega_{i-1}}\right)\right) \\
&\qquad \underbrace{\int_{x_1^{(k)}} q\left(x_1^{(k)}\right) dx_1^{(k)}}_{=1 \text{ by } (16)} \dots dx_{i-1}^{(k)} dx_{i+1}^{(k)} \dots dx_n^{(k)} \\
&= \dots \\
&= \int q\left(x_{i-1}^{(k)}\right) \ln\left(\mathscr{N}\left(x_{i-1}^{(k)}, \mu_{i-1}^{(k-1)}, \sigma_{i-1}^{(k-1)} + t^{(k)} e^{\kappa_{i-1} x_i^{(k)} + \omega_{i-1}}\right)\right) dx_{i-1}^{(k)}
\end{aligned}$$

Where in the dots step we use also (16), that lead to vanish all integrals execpt one. We will now use (16), (18)(the definition of the mean) and (19) (the definition of the variance) to compute this integral :

$$\begin{aligned}
\star &= \int q\left(x_{i-1}^{(k)}\right) \ln\left(\mathscr{N}\left(x_{i-1}^{(k)}, \mu_{i-1}^{(k-1)}, \sigma_{i-1}^{(k-1)} + t^{(k)} e^{\kappa_{i-1} x_i^{(k)} + \omega_{i-1}}\right)\right) dx_{i-1}^{(k)} \\
&= \int q\left(x_{i-1}^{(k)}\right) \ln\left(\frac{1}{\sqrt{2\pi\left(\sigma_{i-1}^{(k-1)} + t^{(k)} e^{\kappa_{i-1} x_i^{(k)} + \omega_{i-1}}\right)}} e^{-\frac{1}{2}\frac{\left(x_{i-1}^{(k)} - \mu_{i-1}^{(k-1)}\right)^2}{\sigma_{i-1}^{(k-1)} + t^{(k)} e^{\kappa_{i-1} x_i^{(k)} + \omega_{i-1}}}}\right) dx_{i-1}^{(k)} \\
&= \int q\left(x_{i-1}^{(k)}\right)\left(-\frac{1}{2}\ln\left(2\pi\left(\sigma_{i-1}^{(k-1)} + t^{(k)} e^{\kappa_{i-1} x_i^{(k)} + \omega_{i-1}}\right)\right) - \frac{1}{2}\frac{(x_{i-1}^{(k)} - \mu_{i-1}^{(k-1)})^2}{\sigma_{i-1}^{(k-1)} + t^{(k)} e^{\kappa_{i-1} x_i^{(k)} + \omega_{i-1}}}\right) dx_{i-1}^{(k)} \\
&= -\ln\pi - \frac{1}{2}\ln\left(\sigma_{i-1}^{(k-1)} + t^{(k)} e^{\kappa_{i-1} x_i^{(k)} + \omega_{i-1}}\right) \underbrace{\int q\left(x_{i-1}^{(k)}\right) dx_{i-1}^{(k)}}_{=1 \text{ by } (16)} \\
&\qquad - \frac{1}{2}\frac{1}{\sigma_{i-1}^{(k-1)} + t^{(k)} e^{\kappa_{i-1} x_i^{(k)} + \omega_{i-1}}} \underbrace{\int q\left(x_{i-1}^{(k)}\right) (x_{i-1}^{(k)} - \mu_{i-1}^{(k-1)})^2 dx_{i-1}^{(k)}}_{=:\triangleleft}
\end{aligned}$$

Where:

$$\begin{aligned}
\triangleleft &= \int q\left(x_{i-1}^{(k)}\right)\left({x_{i-1}^{(k)}}^2 - 2x_{i-1}^{(k)}\mu_{i-1}^{(k-1)} + \mu_{i-1}^{(k-1)2}\right)dx_{i-1}^{(k)} \\
&= \mu_{i-1}^{(k-1)2}\underbrace{\int q\left(x_{i-1}^{(k)}\right)dx_{i-1}^{(k)}}_{=1 \text{ by } (16)} - 2\mu_{i-1}^{(k-1)}\underbrace{\int q\left(x_{i-1}^{(k)}\right)x_{i-1}^{(k)}dx_{i-1}^{(k)}}_{=\mu_{i-1}^{(k)} \text{ by } (18)} + \int q\left(x_{i-1}^{(k)}\right){x_{i-1}^{(k)}}^2 dx_{i-1}^{(k)} \\
&= \mu_{i-1}^{(k-1)2} - 2\mu_{i-1}^{(k-1)}\mu_{i-1}^{(k)} + \int q\left(x_{i-1}^{(k)}\right)\left({x_{i-1}^{(k)}}^2 - 2x_{i-1}^{(k)}\mu_{i-1}^{(k)} + {\mu_{i-1}^{(k)}}^2 + 2x_{i-1}^{(k)}\mu_{i-1}^{(k)} - {\mu_{i-1}^{(k)}}^2\right)dx_{i-1}^{(k)} \\
&= \mu_{i-1}^{(k-1)2} - 2\mu_{i-1}^{(k-1)}\mu_{i-1}^{(k)} + \int q\left(x_{i-1}^{(k)}\right)\underbrace{\left({x_{i-1}^{(k)}}^2 - 2x_{i-1}^{(k)}\mu_{i-1}^{(k)} + {\mu_{i-1}^{(k)}}^2\right)}_{\left(x_{i-1}^{(k)} - \mu_{i-1}^{(k)}\right)^2} dx_{i-1}^{(k)} \\
&\qquad + \underbrace{\int q\left(x_{i-1}^{(k)}\right)\left(2x_{i-1}^{(k)}\mu_{i-1}^{(k)} - {\mu_{i-1}^{(k)}}^2\right)dx_{i-1}^{(k)}}_{=2{\mu_{i-1}^{(k)}}^2 - {\mu_{i-1}^{(k)}}^2 = {\mu_{i-1}^{(k)}}^2 \text{ by } (16)} \\
&= \mu_{i-1}^{(k-1)2} - 2\mu_{i-1}^{(k-1)}\mu_{i-1}^{(k)} + \underbrace{\int q\left(x_{i-1}^{(k)}\right)(x_{i-1}^{(k)} - \mu_{i-1}^{(k)})^2 dx_{i-1}^{(k)}}_{=\sigma_{i-1}^{(k)} \text{ by } (19)} + {\mu_{i-1}^{(k)}}^2 \\
&= \sigma_{i-1}^{(k)} + \left(\mu_{i-1}^{(k)} - \mu_{i-1}^{(k-1)}\right)^2
\end{aligned}$$

So :

$$\begin{aligned}
\star &= -\ln\pi - \frac{1}{2}\ln\left(\sigma_{i-1}^{(k-1)} + t^{(k)}e^{\kappa_{i-1}x_i^{(k)}+\omega_{i-1}}\right) - \frac{1}{2}\frac{\triangleleft}{\sigma_{i-1}^{(k-1)} + t^{(k)}e^{\kappa_{i-1}x_i^{(k)}+\omega_{i-1}}} \\
&= -\ln\pi - \frac{1}{2}\ln\left(\sigma_{i-1}^{(k-1)} + t^{(k)}e^{\kappa_{i-1}x_i^{(k)}+\omega_{i-1}}\right) - \frac{1}{2}\frac{\sigma_{i-1}^{(k-1)} + \left(\mu_{i-1}^{(k)} - \mu_{i-1}^{(k-1)}\right)^2}{\sigma_{i-1}^{(k)} + t^{(k)}e^{\kappa_{i-1}x_i^{(k)}+\omega_{i-1}}}
\end{aligned}$$

5.2.3 compute $\star\star$

We will now compute the integrales of the term $j = i$ in the expression of $I(x_i^{(k)})$. Using $x_{\setminus i} = \{x_j\}_{j\neq i}$ and (15) in a similar way as above, we have :

$$\begin{aligned}
\star\star &= \int_{x_n^{(k)}}\cdots\int_{x_1^{(k)}} q\left(x_{\setminus i}^{(k)}\right)\ln\left(\mathcal{N}\left(x_i^{(k)}, \mu_i^{(k-1)}, \sigma_i^{(k-1)} + t^{(k)}e^{\kappa_i x_{i+1}^{(k)}+\omega_i}\right)\right)dx_1^{(k)}\dots dx_{i-1}^{(k)}dx_{i+1}^{(k)}\dots dx_n^{(k)} \\
&= \int_{x_n^{(k)}}\cdots\int_{x_1^{(k)}} \prod_{\substack{j=1\\j\neq i}}^{n} q\left(x_j^{(k)}\right)\ln\left(\mathcal{N}\left(x_i^{(k)}, \mu_i^{(k-1)}, \sigma_i^{(k-1)} + t^{(k)}e^{\kappa_i x_{i+1}^{(k)}+\omega_i}\right)\right)dx_1^{(k)}\dots dx_{i-1}^{(k)}dx_{i+1}^{(k)}\dots dx_n^{(k)}
\end{aligned}$$

Again the argument of the integral depend only on $x_i^{(k)}$ and $x_{i+1}^{(k)}$ we can put this term outside of the integrals on $dx_j^{(k)}$,with $j \neq i+1$, and obtain:

$$\begin{aligned}
\star\star &= \int_{x_n^{(k)}}\cdots\int_{x_2^{(k)}} \prod_{\substack{j=2\\j\neq i}}^{n} q\left(x_j^{(k)}\right)\ln\left(\mathcal{N}\left(x_i^{(k)}, \mu_i^{(k-1)}, \sigma_i^{(k-1)} + t^{(k)}e^{\kappa_i x_{i+1}^{(k)}+\omega_i}\right)\right)\underbrace{\int_{x_1^{(k)}} q\left(x_1^{(k)}\right)dx_1^{(k)}}_{=1 \text{ by } (16)}\dots dx_{i+1}^{(k)}dx_{i+1}^{(k)}\dots dx_n^{(k)} \\
&= \dots \\
&= \int q\left(x_{i+1}^{(k)}\right)\ln\left(\mathcal{N}\left(x_i^{(k)}, \mu_i^{(k-1)}, \sigma_i^{(k-1)} + t^{(k)}e^{\kappa_i x_{i+1}^{(k)}+\omega_i}\right)\right)dx_{i+1}^{(k)}
\end{aligned}$$

Applying the definition of the normal distribution, we separe the terms depending on $x_i^{(k)}$ and those not :

$$\star\star = \int q\left(x_{i+1}^{(k)}\right) \ln\left(\frac{1}{\sqrt{2\pi\left(\sigma_i^{(k-1)} + t^{(k)} e^{\kappa_i x_{i+1}^{(k)} + \omega_i}\right)}} e^{-\frac{1}{2}\frac{\left(x_{i+1}^{(k)} - \mu_i^{(k-1)}\right)^2}{\sigma_i^{(k-1)} + t^{(k)} e^{\kappa_i x_i^{(k)} + \omega_i}}}\right) dx_{i+1}^{(k)}$$

$$= \int q\left(x_{i+1}^{(k)}\right)\left(-\frac{1}{2}\ln\left(2\pi\left(\sigma_i^{(k-1)} + t^{(k)} e^{\kappa_i x_{i+1}^{(k)} + \omega_i}\right)\right) - \frac{1}{2}\frac{(x_{i+1}^{(k)} - \mu_i^{(k-1)})^2}{\sigma_i^{(k-1)} + t^{(k)} e^{\kappa_i x_i^{(k)} + \omega_i}}\right) dx_{i+1}^{(k)} \tag{21}$$

$$= \underbrace{\int q\left(x_{i+1}^{(k)}\right)\left(-\frac{1}{2}\ln\left(2\pi\left(\sigma_i^{(k-1)} + t^{(k)} e^{\kappa_i x_{i+1}^{(k)} + \omega_i}\right)\right) dx_{i+1}^{(k)}\right.}_{=\text{constante because do not depend on } x_i^{(k)}}$$

$$- \frac{1}{2}(x_{i+1}^{(k)} - \mu_i^{(k-1)})^2 \underbrace{\int q\left(x_{i+1}^{(k)}\right)\left(\sigma_i^{(k-1)} + t^{(k)} e^{\kappa_i x_i^{(k)} + \omega_i}\right)^{-1} dx_{i+1}^{(k)}}_{=:\odot}$$

We develop $\left(\sigma_i^{(k-1)} + t^{(k)} e^{\kappa_i x_i^{(k)} + \omega_i}\right)^{-1}$ with a Taylor serie around $\mu_{i+1}^{(k-1)}$:

Taylor development of f around a : $f(x) \approx f(a) + f'(a)(x - a)$

Here :

$$\begin{cases} x & = x_{i+1}^{(k)} \\ a & = \mu_{i+1}^{(k-1)} \\ f\left(x_{i+1}^{(k)}\right) & = \left(\sigma_i^{(k-1)} + t^{(k)} e^{\kappa_i x_i^{(k)} + \omega_i}\right)^{-1} \\ f'\left(\mu_{i+1}^{(k-1)}\right) & = -\frac{\kappa_i t^{(k)} e^{\kappa_i \mu_{i+1}^{(k-1)} + \omega_i}}{\left(\sigma_i^{(k-1)} + t^{(k)} e^{\kappa_i \mu_{i+1}^{(k-1)} + \omega_i}\right)^2} \end{cases}$$

So:

$$\begin{aligned} &\left(\sigma_i^{(k-1)} + t^{(k)} e^{\kappa_i x_i^{(k)} + \omega_i}\right)^{-1} \approx \\ &\left(\sigma_i^{(k-1)} + t^{(k)} e^{\kappa_i \mu_i^{(k-1)} + \omega_i}\right)^{-1} - \frac{\kappa_i t^{(k)} e^{\kappa_i \mu_{i+1}^{(k-1)} + \omega_i}}{\left(\sigma_i^{(k-1)} + t^{(k)} e^{\kappa_i \mu_{i+1}^{(k-1)} + \omega_i}\right)^2}\left(x_{i+1}^{(k)} - \mu_{i+1}^{(k-1)}\right) \end{aligned} \tag{22}$$

And then, we can simplify the integral $\odot$:

$$\odot = \int q\left(x_{i+1}^{(k)}\right)\left(\sigma_i^{(k-1)} + t^{(k)} e^{\kappa_i x_i^{(k)} + \omega_i}\right)^{-1} dx_{i+1}^{(k)}$$

$$= \int q\left(x_{i+1}^{(k)}\right)\left(\left(\sigma_i^{(k-1)} + t^{(k)} e^{\kappa_i \mu_{i+1}^{(k-1)} + \omega_i}\right)^{-1} - \frac{\kappa_i t^{(k)} e^{\kappa_i \mu_{i+1}^{(k-1)} + \omega_i}}{\left(\sigma_i^{(k-1)} + t^{(k)} e^{\kappa_i \mu_{i+1}^{(k-1)} + \omega_i}\right)^2}\left(x_{i+1}^{(k)} - \mu_{i+1}^{(k-1)}\right)\right) dx_{i+1}^{(k)} \text{ by } (22)$$

$$= \left(\sigma_i^{(k-1)} + t^{(k)} e^{\kappa_i \mu_{i+1}^{(k-1)} + \omega_i}\right)^{-1} \underbrace{\int_{x_{i+1}^{(k)}} q\left(x_{i+1}^{(k)}\right) dx_{i+1}^{(k})}_{=1 \text{ by } (16)}$$

$$- \frac{\kappa_i t^{(k)} e^{\kappa_i \mu_i^{(k-1)} + \omega_i}}{\left(\sigma_i^{(k-1)} + t^{(k)} e^{\kappa_i \mu_i^{(k-1)} + \omega_i}\right)^2}\left(\underbrace{\int q\left(x_{i+1}^{(k)}\right) x_{i+1}^{(k)} dx_{i+1}^{(k)}}_{=\mu_{i+1}^{(k-1)} \text{ by } (18)} - \mu_{i+1}^{(k-1)} \underbrace{\int q\left(x_{i+1}^{(k)}\right) dx_{i+1}^{(k)}}_{=1 \text{ by } (16)}\right)$$

So:

$$\odot = \left(\sigma_i^{(k-1)} + t^{(k)} e^{\kappa_i \mu_{i+1}^{(k-1)} + \omega_i}\right)^{-1} \tag{23}$$

We can now compute $\star\star$:

$$\begin{aligned} \star\star &= cst - \frac{1}{2}(x_{i+1}^{(k)} - \mu_i^{(k-1)})^2 \int q\left(x_{i+1}^{(k)}\right)\left(\sigma_i^{(k-1)} + t^{(k)} e^{\kappa_i x_i^{(k)} + \omega_i}\right)^{-1} dx_{i+1}^{(k)} && \text{by } (21) \\ &= cst - \frac{1}{2}(x_{i+1}^{(k)} - \mu_i^{(k-1)})^2 \left(\sigma_i^{(k-1)} + t^{(k)} e^{\kappa_i \mu_{i+1}^{(k-1)} + \omega_i}\right)^{-1} dx_{i+1}^{(k)} && \text{by } (23) \end{aligned}$$

6 UPDATE EQUATIONS

6.1 Assumption on the law of q

We assume a fixed form q We make minimal assumptions about the form of the approximate posteriors by following the **maximum entropy principle**:
Given knowledge of, or assumptions about, constraints on a distribution, the least arbitrary choice of distribution is the one that maximizes entropy (Jaynes, 1957). "To keep the description of the posteriors simple and bio- logically plausible, we take them to be characterized only by their first two moments; i.e., by their mean and variance. "

We make the assumption that the posterior q follows a normal law : $q_{x_i^{(k)}} \sim \mathcal{N}(\mu_i^{(k)}, \sigma_i^{(k)})$ i.e. :

$$q_{x_i^{(k)}}\left(x_i^{(k)}\right) = \frac{1}{\sqrt{2\pi\sigma_i^{(k)}}} e^{-\frac{(x_i^{(k)} - \mu_i^{(k)})^2}{2\sigma_i^{(k)}}}$$

We will now find an expression (by iteration : update equations) of the parameters : $\mu_i^{(k)}$ and $\sigma_i^{(k)}$). To do so we will need to compute I' and I''.

6.2 Notations

Let first definie some notations :

$$\begin{aligned}
&A := \sigma_{i-1}^{(k)} + \left(\mu_{i-1}^{(k)} - \mu_{i-1}^{(k-1)}\right)^2 \\
&v_i^{(k)} := \begin{cases} t^{(k)} e^{\kappa_i \mu_{i+1}^{(k-1)} + \omega_i}, & i = 1, ..., n-1 \\ t^{(k)} \vartheta, & i = n \end{cases} \\
&\hat{\mu}_i^{(k)} := \mu_i^{(k-1)} \\
&\hat{\pi}_i^{(k)} := \frac{1}{\sigma_i^{(k-1)} + v_i^{(k)}} \\
&\delta_i^{(k)} := \left(\sigma_i^{(k)} + \left(\mu_i^{(k)} - \hat{\mu}_i^{(k)}\right)^2\right)\hat{\pi}_i^{(k)} - 1
\end{aligned}$$

With this notation and by the previous point, we have that :

$$I(x_i^{(k)}) = -\frac{1}{2}\left(\ln\left(\sigma_{i-1}^{(k-1)} + t^{(k)} e^{\kappa_{i-1} x_i^{(k)} + \omega_{i-1}}\right) + \frac{A}{\sigma_{i-1}^{(k-1)} + t^{(k)} e^{\kappa_{i-1} x_i^{(k)} + \omega_{i-1}}} + \hat{\pi}_i^{(k)}\left(x_i^{(k)} - \mu_i^{(k-1)}\right)^2\right)$$

6.3 Computation of I'

Performing one derivation :

$$-2I'\left(x_i^{(k)}\right) = \frac{\kappa_{i-1} t^{(k)} e^{\kappa_{i-1} x_i^{(k)} + \omega_{i-1}}}{\sigma_{i-1}^{(k-1)} + t^{(k)} e^{\kappa_{i-1} x_i^{(k)} + \omega_{i-1}}} - \frac{A\kappa_{i-1} t^{(k)} e^{\kappa_{i-1} x_i^{(k)} + \omega_{i-1}}}{\left(\sigma_{i-1}^{(k)} + t^{(k)} e^{\kappa_{i-1} x_i^{(k)} + \omega_{i-1}}\right)^2} + 2\hat{\pi}_i^{(k)}\left(x_i^{(k)} - \mu_i^{(k-1)}\right)$$

Evaluating I' in $\mu_i^{(k-1)}$ and using the notation definie earlier, we obtain :

$$I'\left(\mu_i^{(k-1)}\right) = \frac{1}{2}\kappa_{i-1} v_{i-1}^{(k)} \hat{\pi}_{i-1}^{(k)} \delta_{i-1}^{(k)} \tag{24}$$

6.4 Computation of I''

Performing the second derivation :

$$-2I''(x_i^{(k)}) =$$
$$\frac{\kappa_{i-1}^2 t^{(k)} e^{\kappa_{i-1}x_i^{(k)}+\omega_{i-1}}\left(\sigma_{i-1}^{(k-1)} + t^{(k)} e^{\kappa_{i-1}x_i^{(k)}+\omega_{i-1}}\right) - \left(\kappa_{i-1} t^{(k)} e^{\kappa_{i-1}x_i^{(k)}+\omega_{i-1}}\right)^2}{\left(\sigma_{i-1}^{(k-1)} + t^{(k)} e^{\kappa_{i-1}x_i^{(k)}+\omega_{i-1}}\right)^2}$$
$$- A\frac{\kappa_{i-1}^2 t^{(k)} e^{\kappa_{i-1}x_i^{(k)}+\omega_{i-1}}\left(\sigma_{i-1}^{(k-1)} + t^{(k)} e^{\kappa_{i-1}x_i^{(k)}+\omega_{i-1}}\right)^2 - \left(\kappa_{i-1} t^{(k)} e^{\kappa_{i-1}x_i^{(k)}+\omega_{i-1}}\right)^2 2\left(\sigma_{i-1}^{(k-1)} + t^{(k)} e^{\kappa_{i-1}x_i^{(k)}+\omega_{i-1}}\right)}{\left(\sigma_{i-1}^{(k-1)} + t^{(k)} e^{\kappa_{i-1}x_i^{(k)}+\omega_{i-1}}\right)^4}$$
$$+ 2\hat{\pi}_i^{(k)}$$

Evaluating I'' in $\mu_i^{(k-1)}$ and using the notation definie earlier, we obtain :

$$\begin{aligned}
-2I''(\mu_i^{(k-1)}) &= \kappa_{i-1}^2 v_{i-1}^{(k)}\hat{\pi}_{i-1}^{(k)} - \left(\kappa_{i-1} v_{i-1}^{(k)}\hat{\pi}_{i-1}^{(k)}\right)^2 - A\left(\kappa_{i-1}^2 v_{i-1}^{(k)}\hat{\pi}_{i-1}^{(k)\,2} - 2.\kappa_{i-1}^2 v_{i-1}^{(k)^2}\hat{\pi}_{i-1}^{(k)\,3}\right) + 2\hat{\pi}_i^{(k)} \\
&= \left(\kappa_{i-1} v_{i-1}^{(k)}\hat{\pi}_{i-1}^{(k)}\right)^2\left(\frac{1}{v_{i-1}^{(k)}\hat{\pi}_{i-1}^{(k)}} - 1 - A\left(\frac{1}{v_{i-1}^{(k)}} - \frac{2}{\hat{\pi}_{i-1}^{(k)}}\right)\right) + 2\hat{\pi}_i^{(k)} \\
&= \left(\kappa_{i-1} v_{i-1}^{(k)}\hat{\pi}_{i-1}^{(k)}\right)^2\left(\frac{1}{v_{i-1}^{(k)}\hat{\pi}_{i-1}^{(k)}} - 1 - \Bigg(\underbrace{\frac{A}{\hat{\pi}_{i-1}^{(k)}} - 1}_{\delta_{i-1}^{(k)}} + 1\Bigg)\left(\frac{1}{v_{i-1}^{(k)}\hat{\pi}_{i-1}^{(k)}} - 2\right)\right) + 2\hat{\pi}_i^{(k)} \\
&= \left(\kappa_{i-1} v_{i-1}^{(k)}\hat{\pi}_{i-1}^{(k)}\right)^2\left(\frac{1}{v_{i-1}^{(k)}\hat{\pi}_{i-1}^{(k)}} - 1 - \left(\delta_{i-1}^{(k)}\Big(\frac{1}{v_{i-1}^{(k)}\hat{\pi}_{i-1}^{(k)}} - 2\Big) + \Big(\frac{1}{v_{i-1}^{(k)}\hat{\pi}_{i-1}^{(k)}} - 2\Big)\right)\right) + 2\hat{\pi}_i^{(k)} \\
&= \left(\kappa_{i-1} v_{i-1}^{(k)}\hat{\pi}_{i-1}^{(k)}\right)^2\left(1 + \delta_{i-1}^{(k)}\left(2 + \frac{1}{v_{i-1}^{(k)}\hat{\pi}_{i-1}^{(k)}}\right)\right) + 2\hat{\pi}_i^{(k)}
\end{aligned}$$

Where :

- In the second equality we factorize by $\left(\kappa_{i-1} v_{i-1}^{(k)}\hat{\pi}_{i-1}^{(k)}\right)^2$.
- In the third equality we make appear the expression of $\delta_{i-1}^{(k)}$.
- In the fourth equality we develop the term depend on $\delta_{i-1}^{(k)}$ and thoses which not.
- In the last equality we simplified the $\frac{1}{v_{i-1}^{(k)}\hat{\pi}_{i-1}^{(k)}}$ terms.

Then we use the fact that :

$$2 - \frac{1}{v_{i-1}^{(k)}\hat{\pi}_{i-1}^{(k)}} = 1 - \frac{1}{v_{i-1}^{(k)}\pi_{i-1}^{(k-1)}}$$

And we obtain :

$$-I''(\mu_i^{(k-1)}) = \frac{1}{2}\left(\kappa_{i-1} v_{i-1}^{(k)}\hat{\pi}_{i-1}^{(k)}\right)^2\left(1 + \delta_{i-1}^{(k)}\left(1 - \frac{1}{v_{i-1}^{(k)}\pi_{i-1}^{(k-1)}}\right)\right) + \hat{\pi}_i^{(k)} \tag{25}$$

6.5 Update equation of $\sigma_i^{(k)}$

By previous points we have :

$$\begin{cases} q_{x_i^{(k)}}\left(x_i^{(k)}\right) = \frac{1}{Z_i^{(k)}} e^{I\left(x_i^{(k)}\right)} & \text{by (11), the mean field approximation} \\ q_{x_i^{(k)}}\left(x_i^{(k)}\right) = \frac{1}{\sqrt{2\pi\sigma_i^{(k)}}} e^{-\frac{(x_i^{(k)}-\mu_i^{(k)})^2}{2\sigma_i^{(k)}}} & \text{by the assumption on the gaussian form of } q \end{cases}$$

So :

$$q_{x_i^{(k)}}\left(x_i^{(k)}\right) = \frac{1}{\sqrt{2\pi\sigma_i^{(k)}}} e^{-\frac{(x_i^{(k)}-\mu_i^{(k)})^2}{2\sigma_i^{(k)}}} = \frac{1}{Z_i^{(k)}} e^{I\left(x_i^{(k)}\right)} \tag{26}$$

By taking the logarithm on both sides we have :

$$-2\ln\left(2\pi\sigma_i^{(k)}\right) - \frac{\left(x_i^{(k)}-\mu_i^{(k)}\right)^2}{2\sigma_i^{(k)}} = -\ln\left(Z^{(k)}\right) + I\left(x_i^{(k)}\right)$$

Differentiating twice with respect to $x_i^{(k)}$ gives:

$$-\frac{1}{\sigma_i^{(k)}} = I''\left(x_i^{(k)}\right) \tag{27}$$

Since I is a quadratic function, I'' is constant. So in order to find it we may evaluate it in a known point as $\mu_i^{(k-1)}$. Thanks to the previous computation of $I"\left(\mu_i^{(k-1)}\right)$ (25), we obtain the update equations of $\sigma_i^{(k)}$:

$$\pi_i^{(k)} = -I"(\mu_i^{(k-1)}) = \frac{1}{2}\left(\kappa_{i-1}v_{i-1}^{(k)}\hat{\pi}_{i-1}^{(k)}\right)^2\left(1+\delta_{i-1}^{(k)}\left(1-\frac{1}{v_{i-1}^{(k)}\pi_{i-1}^{(k-1)}}\right)\right) + \hat{\pi}_i^{(k)} \tag{28}$$

Where $\pi_i^{(k)} = \frac{1}{\sigma_i^{(k)}}$

6.6 Update equations of $\mu_i^{(k)}$

1. $\mu_i^{(k)}$ is the argument of the maximum of $I\left(x_i^{(k)}\right)$
   In $q_{x_i^{(k)}}\left(x_i^{(k)}\right) = \frac{1}{\sqrt{2\pi\sigma_i^{(k)}}} e^{-\frac{(x_i^{(k)}-\mu_i^{(k)})^2}{2\sigma_i^{(k)}}}$, when $x_i^{(k)} = \mu_i^{(k)}$, the function reach its maximum.
   And as $q_{x_i^{(k)}}\left(x_i^{(k)}\right) = \frac{1}{\sqrt{2\pi\sigma_i^{(k)}}} e^{-\frac{(x_i^{(k)}-\mu_i^{(k)})^2}{2\sigma_i^{(k)}}} = \frac{1}{Z_i^{(k)}} e^{I\left(x_i^{(k)}\right)}$ (26), $\mu_i^{(k)}$ is the argument of the maximum of $\frac{1}{Z_i^{(k)}} e^{I\left(x_i^{(k)}\right)}$. Moreover $Z_i^{(k)}$ is a constante, so $\mu_i^{(k)}$ is the argument of the maximum of $I\left(x_i^{(k)}\right)$.

2. Applying Newton method to find $\arg\max I$
   Starting at any point $x_i^{(k)}$ the exact argmax of a quadratic function can be found in one step by Newton's method:

$$\mu_i^{(k)} = \arg\max I\left(x_i^{(k)}\right) = x_i^{(k)} - \frac{I'\left(x_i^{(k)}\right)}{I''\left(x_i^{(k)}\right)}$$

   We choose : $x_i^{(k)} = \mu_i^{(k)}$ as our expansion point, and use the expression of $\sigma_i^{(k)}$ we found previously.

$$\begin{aligned}\mu_i^{(k)} &= \mu_i^{(k-1)} - \frac{I'\left(\mu_i^{(k-1)}\right)}{I''\left(\mu_i^{(k-1)}\right)} \\ &= \mu_i^{(k-1)} + \frac{1}{\pi_i^{(k)}} I'\left(\mu_i^{(k-1)}\right) \qquad (27)\end{aligned}$$

   Thanks to the previous computation of $I'\left(\mu_i^{(k-1)}\right)$ (24), we obtain the update equations of $\mu_i^{(k)}$.

$$\mu_i^{(k)} = \mu_i^{(k-1)} + \frac{1}{2}\kappa_{i-1}v_{i-1}^{(k)}\frac{\hat{\pi}_{i-1}^{(k)}}{\pi_{i-1}^{(k)}}\delta_{i-1}^{(k)}$$

   Why do we choose this expansion point ?
   Because when we performed the Taylor expansion of I we have agreement of the exact I and the approximation up to the second derivative.

7 PERSPECTIVE

1. code
2. decision model of mathys 2014
3. Interpretation of the update equation
4. computation with less assumption

8 REFERENCE

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