Problema 4

Ecuación del calor: • $u_t = 4u_{xx}$ en $(0,1)\times(0,\infty)$

- $u(0,t) = 0 = u(1,t)$, $t \geq 0$
- $u(x,0) = f(x)$, $x \in [0,1]$, donde

$$f(x) = \begin{cases} x, & \text{si } 0 \leq x \leq 1/2 \\ 1-x, & \text{si } 1/2 < x \leq 1 \end{cases}.$$

Se busca solución $u \in C^2((0,1)\times(0,\infty)) \cap C([0,1]\times[0,\infty))$.

Solución:

Como f es afín por partes, es claro:

$$f'(x) = \begin{cases} 1, & 0 \leq x < 1/2 \\ -1, & 1/2 < x \leq 1 \end{cases}. \text{ Así, } f' \in SC[0,1].$$

Además, $f \in C([0,1])$, pues $f \in SC([0,1])$ y $f(1/2^-) = 1/2 = f(1/2^+) = 1 - 1/2$.

Asimismo, note $f(0) = 0 = f(1) = 1-1$.

Así, de $f(0)=f(1)=0$, $f\in C[0,1]$, f' existe salvo en un punto de su dominio, y que $f'\in SC[0,1]$, concluimos (vía un teorema de clase) que la ecuación del calor planteada tiene solución $u(x,t)$ en $C^2((0,1)\times(0,\infty))\cap C([0,1]\times[0,\infty))$:

$$u(x,t)=\sum_{n\geq 1} b_n \sin(n\pi x)\exp(-4n^2\pi^2 t),$$

donde $b_n = 2\int_0^1 f(x)\sin(n\pi x)dx$, $n\geq 1$.

Hallemos b_n:

Fije $n\in\mathbb{Z}^+$.

$$\int_0^1 f(x)\sin(n\pi x)dx = \int_0^{1/2} x\sin(n\pi x)dx + \int_{1/2}^1 (1-x)\sin(n\pi x)dx.$$

- $\int x\sin(n\pi x)dx$: x, $\sin(n\pi x)$ son continuas en $\mathbb{R}$.

$u=x$, $dv=\sin(n\pi x)dx$

$\Rightarrow du=dx$, $v=\dfrac{-1}{n\pi}\cos(n\pi x)$

$$\Rightarrow \int_0^{1/2} x\, \text{Sin}(n\pi x)\,dx = \frac{-1}{n\pi}\left(x\, \text{Cos}(n\pi x)\right)\Big|_0^{1/2} + \frac{1}{n\pi}\int_0^{1/2} \text{Cos}(n\pi x)\,dx$$

$$= -\frac{1}{2n\pi}\text{Cos}(n\pi/2) + \frac{1}{n\pi}\left(\frac{1}{n\pi}\text{Sin}(n\pi x)\right)\Big|_0^{1/2}$$

$$= -\frac{1}{2n\pi}\text{Cos}(n\pi/2) + (n\pi)^{-2}\,\text{Sin}(n\pi/2) \quad \ldots (1)$$

- $$\int_{1/2}^{1}(1-x)\,\text{Sin}(n\pi x)\,dx = \int_{1/2}^{1}\text{Sin}(n\pi x)\,dx - \int_{1/2}^{1} x\,\text{Sin}(n\pi x)\,dx$$

$$= -(n\pi)^{-1}\,\text{Cos}(n\pi x)\Big|_{1/2}^{1} - \Big[-(n\pi)^{-1}\left(x\,\text{Cos}(n\pi x)\right)\Big|_{1/2}^{1} + (n\pi)^{-2}\left(\text{Sin}(n\pi x)\right)\Big|_{1/2}^{1}\Big]$$

$$= -(n\pi)^{-1}\left((-1)^n - \text{Cos}(n\pi/2)\right) - \Big[-(n\pi)^{-1}\left((-1)^n - \text{Cos}(n\pi/2)/2\right) + (n\pi)^{-2}\left(0 - \text{Sin}(n\pi/2)\right)\Big]$$

$$= (n\pi)^{-1}\,\text{Cos}(n\pi/2)/2 + (n\pi)^{-2}\,\text{Sin}(n\pi/2) \quad \ldots (2)$$

<u>De (1) y (2):</u> $b_n/2 = 2(n\pi)^{-2}\,\text{Sin}(n\pi/2)$

$$\therefore\ b_n = \frac{4}{n^2\pi^2}\cdot \text{Sin}\left(\frac{n\pi}{2}\right).$$

$$\Rightarrow \quad \frac{b_n}{(4/(n^2\pi^2))} = \begin{cases} 0 \ ; & \text{si } n \text{ es par} \\ (-1)^K \ ; & n = 2K+1 \ , \ K \in \mathbb{Z} \end{cases}$$

$$\therefore \ u(x,t) = \frac{4}{\pi^2} \cdot \sum_{K=0}^{\infty} \frac{(-1)^K}{(2K+1)^2} \cdot \sin((2K+1)\pi x)\exp(\cdot)$$

$$(-4\pi^2(2K+1)^2 t)$$

Problema 5

Considere el problema (1) en $(0,\pi)^2$:

- $\Delta u_1 = 0$ en $(0,\pi)^2$
- $u_1(x,0) = 0 = u_1(x,\pi)$, $x \in [0,\pi]$
- $u_1(0,y) = 0$, $y \in [0,\pi]$
- $u_1(\pi,y) = \sin(2y)$, $y \in [0,\pi]$.

Asimismo, considere el problema (2) en $(0,\pi)^2$:

- $\Delta u_2 = 0$, en $(0,\pi)^2$
- $u_2(x,0) = 0 = u_2(x,\pi)$, $x \in [0,\pi]$
- $u_2(0,y) = \sin(2y)$, $y \in [0,\pi]$
- $u_2(\pi,y) = 0$, $y \in [0,\pi]$

De clase, sabemos la solución única de (1).

Aplicaremos separación de variables para resolver (2).

Primero, note $u := u_1 + u_2$ es solución del problema planteado en esta pregunta.

- $\Delta u = \Delta u_1 + \Delta u_2 = 0$ en $(0,\pi)^2$
- $u(x,0) = 0 + 0 = 0 = u(x,\pi)$, $x \in [0,\pi]$
- $u(0,y) = 0 + \sin(2y) = \sin(2y) + 0 = u(\pi,y)$, con $y \in [0,\pi]$.

$\therefore$ Basta resolver el problema (2).

Suponga $u_2(x,t) = \varphi(x)\cdot\psi(t)$ en $[0,\pi]^2$.

$u_{2_x} = \psi\varphi_x$, $u_{2_{xx}} = \psi\varphi_{xx}$.

$u_{2_{tt}} = \varphi\psi_{tt}$

$$\Rightarrow \psi\varphi_{xx} = -\varphi\psi_{tt} \Rightarrow \frac{\varphi_{xx}}{\varphi} = -\frac{\psi_{tt}}{\psi} = \lambda \text{ (cte)}$$

$\Rightarrow$ • $\varphi_{xx} = \lambda\varphi$ • $\psi_{tt} = -\lambda\psi$

$$u_2(x, 0) = \varphi\,\psi(0) = 0 = \varphi\,\psi(\pi)$$

$\Rightarrow \psi(0) = \psi(\pi)$, para evitar $\varphi \equiv 0$.