CLASE 03

Ejer: Sean $p \in \mathbb{R}^n_+ \setminus \{0\}$, $\alpha \geq 0$, real. Defina

$$L(x) := \max_{\substack{1 \leq i \leq n \\ p_i \neq 0}} \{p_i x_i\}$$ y el "hiperplano"

$\mathcal{H}(p,\alpha) := \{x \in \mathbb{R}^n_+ : L(x) = \alpha\}$. Muestre un ejemplo de $\mathcal{H}(p,\alpha)$, para $n=2$.

Sol:

Considere $p = (3,6)$, $\alpha = 30$.

$$\mathcal{H}(p,\alpha) = \{x \in \mathbb{R}^2_+ : \max\{3x_1, 6x_2\} = 30\}$$

$$= \{x \in \mathbb{R}^2_+ : (3x_1 = 30 \wedge 3x_1 \geq 6x_2) \vee (6x_2 = 30 \wedge 6x_2 \geq 3x_1)\}$$

$$= \{x \in \mathbb{R}^2_+ : (x_1 = 10 \wedge x_1 \geq 2x_2) \vee (x_2 = 5 \wedge 2x_2 \geq x_1)\}$$

$\mathcal{H}((3,6),30)$:

Ejer: Sea $C \neq \emptyset$. Si C convexo $\Rightarrow ri(C) \neq \emptyset$.

Sol:

Sea $\emptyset \neq C \subseteq \mathbb{R}^n$. Fije $v \in \mathbb{R}^n$.

Pd: $ri(v + C) = v + ri(C)$

$(\subseteq)$ $x \in ri(v+C)$ (Trivial if empty)

$\Rightarrow \exists \epsilon > 0 : B_\epsilon(x) \cap aff(v+C) \subseteq v + C$

$\{x\} \subseteq B_\epsilon(x) \cap (v + aff(C)) \subseteq v + C$

Pd: $A \cap B - v = (A - v) \cap (B - v)$

$(\subseteq)$ $x = ab - v$, $ab \in A \cap B \Rightarrow x \in A - v$, $x \in B - v$

$(\supseteq)$ $x = a - v$, $x = b - v$; $a \in A$, $b \in B$

$x + v = a = b$; $a \in A \cap B$. ✓

$\Rightarrow (B_\epsilon(x) - v) \cap aff(C) \subseteq C$

- $y \in B_\epsilon(x) - v \Rightarrow y = z - v$, $\|z - x\| < \epsilon$

$\Rightarrow \|y + v - x\| < \epsilon \Rightarrow y \in B_\epsilon(x - v)$

- $y \in B_\epsilon(x-v) \Rightarrow \|y - x + v\| < \epsilon$

$y = (y+v) - v \in B_\epsilon(x) - v$.

$\therefore B_\epsilon(x) - v = B_\epsilon(x - v)$

$\Rightarrow B_\epsilon(x-v) \cap aff(C) \subseteq C$

$$x = v + (x - v) \quad , \quad x - v \in ri(C)$$

$(\supseteq)$ $x \in v + ri(C)$

$$x = v + y \; ; \; y \in ri(C)$$

$$\exists \, \epsilon > 0 : B_\epsilon(x - v) \cap aff(C) \subseteq C$$

$$+v : \; B_\epsilon(x) \cap aff(v + C) \subseteq v + C$$

$$\Rightarrow x \in ri(v + C).$$

Por lo tanto, sea $x \in C \neq 0$:

$$ri(\underbrace{-x + C}_{\ni 0}) = ri(C) - x.$$

Así, podemos suponer $0 \in C$.

$\Rightarrow aff(C) = \langle C \rangle$. ... (1)

Si $C = \{0\} \Rightarrow ri(C) = \{0\} \neq \emptyset$.

Suponga entonces $C - \{0\} \neq \emptyset$.

$\Rightarrow$ De (1): $\exists \, d \in \mathbb{Z}^+ ; \; v_1, \dots, v_d \in C$

con $\{v_i\}_1^d$ base de $aff(C)$.

Considere $T \in \mathcal{L}(\mathbb{R}^d, \mathrm{aff}(C))$ definido por
$T(a_1, \dots, a_d) = \sum_{i=1}^{d} a_i v_i$, isomorfismo lineal
de coordenadas.

De $\dim(\mathbb{R}^d) = d < \infty$: T^{-1} es continuo,
por ser T isomorfismo lineal.

$\Rightarrow$ $T(A)$ es abierto en $\mathrm{aff}(C)$, para todo abierto A de $\mathbb{R}^d$.

Sea $\Lambda := \{ \lambda \in \mathbb{R}^d_{++} : \sum_{i=1}^{d} \lambda_i < 1 \}$, se cumple
$\Lambda = \mathbb{R}^d_{++} \cap B_1(0) = B_1(0) \cap \bigcap_{i=1}^{d} \pi_i^{-1}(]0, \infty[)$,
intersección finita de abiertos en $\mathbb{R}^d$.

$\Rightarrow$ Λ open en $\mathbb{R}^d$ $\Rightarrow$ $T(\Lambda)$ open en $\mathrm{aff}(C)$.

$\therefore$ $\exists\, V \subseteq \mathbb{R}^n$ open : $T(\Lambda) = V \cap \mathrm{aff}(C)$.

De $((2d)^{-1}, \dots, (2d)^{-1}) \in \Lambda$: $\Lambda \neq \emptyset$.

Sea $\lambda \in \Lambda$. $\quad T(\lambda) = \sum_{i=1}^{d} \underbrace{\lambda_i}_{\in [0;1]} \overbrace{v_i}^{\in C} + \underbrace{\left(1 - \sum_{i=1}^{d} \lambda_i\right)}_{\in [0;1]} \overbrace{0}^{\in C}$

$\Rightarrow T(\lambda) \in co(C) = C$.

$\therefore \quad C \supseteq T(\Lambda) = V \cap aff(C)$.

Sea $x \in V \cap aff(C)$.

$\Rightarrow \exists\, \epsilon > 0 : B_\epsilon(x) \subseteq V \,,\; x \in aff(C)$

$\Rightarrow B_\epsilon(x) \cap aff(C) \subseteq V \cap aff(C) = T(\Lambda) \subseteq C$

$\therefore \quad x \in C \,,\; \exists\, \epsilon > 0 : B_\epsilon(x) \cap aff(C) \subseteq C$

$\Rightarrow x \in ri(C) \Rightarrow (\emptyset \neq)\, T(\Lambda) \subseteq ri(C)$

$\therefore \quad ri(C) \neq \emptyset$.

EJER: C convexo $\Rightarrow int(C)$ convexo.

Proof:

Denote $U := int(C)$.

Pd: $\quad tU + (1-t)U \subseteq U \,,\; \forall\, t \;]0;1[$

Fije $t \in \,]0;1[$.

$tU + (1-t)U \subseteq C$, pues $U \subseteq C$ y C es convexo.

$$tU + (1-t)U = \bigcup_{u \in U} (tU + (1-t)u),$$

unión de abiertos; ergo, abierto, pues traslación de open es open, y:

<u>Pd:</u> $\phi \neq A \subseteq \mathbb{R}^n$ open $\Rightarrow \alpha A$ open, $\forall \alpha > 0$.

Fije $\alpha > 0$. Sea $v \in \alpha A$.

$\Rightarrow v = \alpha a$, $a \in A$.

$\frac{v}{\alpha} = a \in A$, open $\Rightarrow \exists\, \epsilon > 0 : B_\epsilon\left(\frac{v}{\alpha}\right) \subseteq A$

Sea $w \in B_{\alpha\epsilon}(v) \Rightarrow \|v - w\| < \alpha\epsilon$

$\|v/\alpha - w/\alpha\| < \epsilon \Rightarrow w/\alpha \in A \Rightarrow w \in \alpha A$

$\Rightarrow B_{\alpha\epsilon}(v) \subseteq \alpha A$, $\alpha\epsilon > 0$. $\therefore \alpha A$ open.

$$\therefore \underbrace{tU + (1-t)U}_{\text{open}} \subseteq C$$

$$\Rightarrow tU + (1-t)U \subseteq \overbrace{\operatorname{int}(C)}^{\text{biggest open en } C} = U$$

$\therefore \operatorname{int}(C)$ convex.