CLASE 02

Series de Fourier

- PD1 No hay
- Pc1: 6 de setiembre
- Nuevo horario: Martes 9-12.

- **Convergencia puntual**

$f_n : I \to \mathbb{R}$.

$$f_n \xrightarrow[n\to\infty]{} f \text{ pointwise} \equiv \forall x \in I : (f_n(x)) \xrightarrow[n\to\infty]{} (f(x))$$

- **Convergencia uniforme**

$$f_n \xrightarrow[n\to\infty]{} f \equiv \forall \epsilon > 0 : \exists n_0 \in \mathbb{N} \,/\, n \geq n_0 \Rightarrow$$
$$|f_n(x) - f(x)| < \epsilon, \ \forall x \in I.$$

- $\displaystyle\sum_{n=1}^{N} f_n \xrightarrow[N\to\infty]{} \sum_{n\geq 1} f_n$:

$\forall \epsilon > 0, \exists N_0 \in \mathbb{N} : N \geq N_0 \Rightarrow$

$$\left| \sum_{n=1}^{N} f_n(x) - \sum_{n\geq 1} f_n(x) \right| < \epsilon, \ \forall x \in I$$

$$\left(\left| \sum_{n>N} f_n(x) \right| < \epsilon, \ \forall x \in I \right).$$

- Considere $I = [a, b]$ y f_n continuas en I.

Si $\displaystyle\sum_{n\geq 1} f_n$ converge uniformemente en I, entonces:

- $\sum_{n\geq 1} f_n$ es continua en I.
- $\int_a^b \left(\sum_{n\geq 1} f_n(x)\right)dx = \sum_{n\geq 1} \int_a^b f_n(x)\,dx$.

Lema: Sean $n, m \geq 1$.

- $\int_{-L}^{L} \cos(n\pi x/L)\cos(m\pi x/L)\,dx = \begin{cases} L, & n=m \\ 0, & n\neq m \end{cases}$
- $\int_{-L}^{L} \sin(n\pi x/L)\sin(m\pi x/L)\,dx = \begin{cases} L, & n=m \\ 0, & n\neq m \end{cases}$
- $\int_{-L}^{L} \cos(n\pi x/L)\sin(m\pi x/L)\,dx = 0$.

Ejemplo: Considere f continua y periódica con periodo $2L$, tal que

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \operatorname{sen}\left(\frac{n\pi x}{L}\right)\right).$$

- $\int_{-L}^{L}$: Llegas a $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$
- $m \geq 1$, por $\cos(\pi m x/L)$, e integramos $\int_{-L}^{L}$:

$\Rightarrow$ $$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{\pi m x}{L}\right)dx, \quad m \geq 0$$

- Análogamente: $b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\pi m x/L)\,dx,\ m \geq 1.$

Def: Sea $f: \mathbb{R} \to \mathbb{R}$ de periodo $2L$, f integrable y absolutamente integrable en cualquier intervalo acotado.

Definimos los coeficientes de Fourier como:

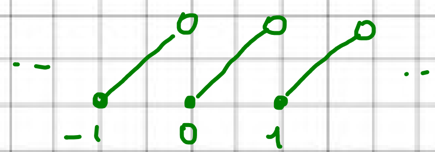
$$a_m = \frac{1}{L} \cdot \int_{-L}^{L} f(x)\cos\left(\frac{\pi m x}{L}\right)dx,\quad m \geq 0;$$

$$b_m = \frac{1}{L} \cdot \int_{-L}^{L} f(x)\sin(\pi m x/L)\,dx,\quad m \geq 1.$$

Ejemplos:

- $\operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$

- $f(x) = x - [x]$



- $g(x) = \begin{cases} 1, & 0 \leq x \leq \pi \\ 0, & -\pi \leq x < 0 \end{cases}$ extendida con periodo 2π.

Calculemos los coeficientes de Fourier de g:

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \int_{0}^{\pi}\cos(mx)\,dx \,/\, \pi$$

$\Rightarrow a_0 = \pi / \pi = 1.$

$m \neq 0 \Rightarrow a_m = \sin(mx)/(m\pi) \Big|_0^{\pi} = 0$

$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx) dx = \int_0^{\pi} \operatorname{sen}(mx) dx / \pi$

$= - \frac{\cos(mx)}{m\pi} \Big|_0^{\pi} \Rightarrow b_m = \frac{1-(-1)^m}{\pi m}.$

Def: Denotamos por

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{\pi n x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right).$$

$$\therefore g(x) \sim \frac{1}{2} + \sum_{m=1}^{\infty} \left(\frac{1-(-1)^m}{\pi m} \right) \sin(mx).$$

Def: $f : \mathbb{R} \to \mathbb{R}$. f es seccionalmente continua si para todo intervalo acotado $[a,b]$ si existen $a \leq x_1 < x_2 < \cdots < x_k \leq b$ tal que:

- f continua en $]x_i, x_{i+1}[$.
- Existen $\lim_{x \to x_i^+} f(x)$ y $\lim_{x \to x_i^-} f(x)$.

Def: f es seccionalmente diferenciable si f es seccionalmente continua y f' es seccionalmente continua.

Teorema de Fourier: Sea f seccionalmente diferenciable de periodo $2L$, entonces:

$$\frac{f(x^+)+f(x^-)}{2} = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{\pi m x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right), \forall x.$$

Donde $f(x^+) := \lim_{t \to x^+} f(t)$.

Ejemplo:

Para la computación previa de $g(x)$, g es seccionalmente diferenciable.

$$\Rightarrow g(\pi/2) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{\pi(2k-1)} \sin(\pi(2k-1)/2)$$

$$= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{\pi(2k-1)} (-1)^{k+1}.$$

De $g(\pi/2) = 1$: $\quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

Ejemplo: $f(x) = \begin{cases} L-x, & 0 \leq x \leq L \\ L+x, & -L \leq x \leq 0 \end{cases}$ de periodo $2L$.

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{\pi m x}{L}\right) dx = \frac{1}{L}\int_{-L}^{L} (L-|x|) \cos\left(\frac{\pi m x}{L}\right) dx$$

$$a_m = \frac{2}{L}\int_0^L (L-x) \cos\left(\frac{\pi m x}{L}\right) dx \; ; \; b_m = 0.$$

$$\int_0^L (L-x) \cos(\pi m x/L)\, dx = \frac{L}{\pi m}\left[(L-x) \sin\left(\frac{\pi m x}{L}\right)\Big|_0^L\right.$$

$$-\int_0^L (-1)\sin(\pi m x/L)\,dx \Big] = \frac{L}{\pi m}\int_0^L \sin\left(\frac{\pi m x}{L}\right)dx$$

$$= -\left(\frac{L}{\pi m}\right)^2 \cdot \cos\left(\frac{\pi m x}{L}\right)\Big|_0^L = \left(\frac{L}{\pi m}\right)^2\left(1-(-1)^m\right)$$

$\therefore\ a_m = \dfrac{2L}{\pi^2 m^2}\left(1-(-1)^m\right),\ m>0,\quad a_0 = L.$

Note f es seccional diferenciable.

$$\Rightarrow f(x) = \frac{L}{2} + \sum_{m=1}^{\infty} \frac{2L}{\pi^2 m^2}\left(1-(-1)^m\right)\cos\left(\frac{\pi m x}{L}\right)$$

$$= \frac{L}{2} + \sum_{k=1}^{\infty} \frac{2L}{\pi^2(2k-1)^2}\cos\left(\frac{\pi(2k-1)x}{L}\right)$$

Para $x=0$:

$$L = \frac{L}{2} + \sum_{k\geq 1} \frac{2L}{\pi^2(2k-1)^2}$$

$$\therefore\ \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$