CLASE 02

Series de Fourier

- PD1: No hay
- Pc1: 6 de setiembre
- **Nuevo horario:** Martes 9-12.

- Convergencia puntual

$f_n : I \to \mathbb{R}$.

$f_n \xrightarrow[n\to\infty]{} f$ pointwise $\equiv \forall x \in I : (f_n(x)) \xrightarrow[n\to\infty]{} (f(x))$

- Convergencia uniforme

$f_n \xrightarrow[n\to\infty]{} f \equiv \forall \epsilon > 0 : \exists n_0 \in \mathbb{N} \,/\, n \geq n_0 \Rightarrow$
$|f_n(x) - f(x)| < \epsilon, \forall x \in I$.

- $\sum_{n=1}^{N} f_n \xrightarrow[N\to\infty]{} \sum_{n\geq 1} f_n$:

$\forall \epsilon > 0, \exists N_0 \in \mathbb{N} : N \geq N_0 \Rightarrow$

$\left| \sum_{n=1}^{N} f_n(x) - \sum_{n\geq 1} f_n(x) \right| < \epsilon, \forall x \in I$

$\left(\left| \sum_{n > N} f_n(x) \right| < \epsilon, \forall x \in I \right)$.

- Considere $I = [a; b)$ y f_n continuas en I.

Si $\sum_{n\geq 1} f_n$ converge uniformemente en I, entonces:

- $\sum_{n\geq 1} f_n$ es continua en I.
- $\int_a^b \left(\sum_{n\geq 1} f_n(x)\right)dx = \sum_{n\geq 1} \int_a^b f_n(x)\,dx$.

Lema: Sean $n, m \geq 1$.

- $\int_{-L}^{L} \cos(n\pi x/L)\cos(m\pi x/L)\,dx = \begin{cases} L, & n=m \\ 0, & n\neq m \end{cases}$

- $\int_{-L}^{L} \sin(n\pi x/L)\sin(m\pi x/L)\,dx = \begin{cases} L, & n=m \\ 0, & n\neq m \end{cases}$

- $\int_{-L}^{L} \cos(n\pi x/L)\sin(m\pi x/L)\,dx = 0$.

Ejemplo: Considere f continua y periódica con periodo $2L$, tal que

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \operatorname{sen}\left(\frac{n\pi x}{L}\right)\right).$$

- $\int_{-L}^{L}$: Llegas a $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$

- $m \geq 1$, por $\cos(\pi m x/L)$, e integramos $\int_{-L}^{L}$:

$\Rightarrow$ $a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{\pi m x}{L}\right)dx$, $m \geq 0$.

- Análogamente: $b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin(\pi m x / L) dx, \ m \geq 1.$

<u>Def</u>: Sea $f: \mathbb{R} \to \mathbb{R}$ de periodo $2L$, f integrable y absolutamente integrable en cualquier intervalo acotado.

Definimos los coeficientes de Fourier como:

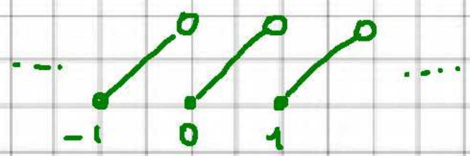
$$a_m = \frac{1}{L} \cdot \int_{-L}^{L} f(x) \cos\left(\frac{\pi m x}{L}\right) dx, \quad m \geq 0 \ ;$$

$$b_m = \frac{1}{L} \cdot \int_{-L}^{L} f(x) \sin(\pi m x / L) \, dx, \quad m \geq 1 .$$

<u>Ejemplos</u>:

- $\operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$

- $f(x) = x - [x]$



- $g(x) = \begin{cases} 1, & 0 \leq x \leq \pi \\ 0, & -\pi \leq x < 0 \end{cases}$ extendida con periodo 2π.

Calculemos los coeficientes de Fourier de g:

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(mx) dx = \int_{0}^{\pi} \cos(mx) dx \ . / \ \pi$$

$\Rightarrow a_0 = \pi/\pi = 1.$

$m \neq 0 \Rightarrow a_m = \sin(mx)/(m\pi) \Big|_0^\pi = 0$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)dx = \int_0^\pi \sin(mx)dx/\pi$$

$$= -\frac{\cos(mx)}{m\pi}\Big|_0^\pi \Rightarrow b_m = \frac{1-(-1)^m}{\pi m}.$$

Def: Denotamos por

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{\pi n x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right).$$

$$\therefore\ g(x) \sim \frac{1}{2} + \sum_{m=1}^{\infty}\left(\frac{1-(-1)^m}{\pi m}\right)\sin(mx).$$

Def: $f: \mathbb{R} \to \mathbb{R}$. f es seccionalmente continua si para todo intervalo acotado $[a;b]$ si existen $a \leq x_1 < x_2 < \dots < x_K \leq b$ tal que:

- f continua en $]x_i; x_{i+1}[$.
- Existen $\lim_{x \to x_i^+} f(x)$ y $\lim_{x \to x_i^-} f(x)$.

Def: f es seccionalmente diferenciable si f es seccionalmente continua y f' es seccionalmente continua.

Teorema de Fourier: Sea f seccionalmente diferenciable de periodo $2L$, entonces:

$$\frac{f(x^+)+f(x^-)}{2} = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{\pi m x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right), \forall x.$$

Donde $f(x^+) := \lim_{t \to x^+} f(t)$.

Ejemplo:

Para la computación previa de $g(x)$, g es seccionalmente diferenciable.

$$\Rightarrow g(\pi/2) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{\pi(2k-1)} \sin(\pi(2k-1)/2)$$

$$= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{\pi(2k-1)} (-1)^{k+1}.$$

De $g(\pi/2) = 1$: $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

Ejemplo: $f(x) = \begin{cases} L-x ; & 0 \le x \le L \\ L+x ; & -L \le x \le 0 \end{cases}$ de periodo $2L$.

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{\pi m x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L}(L-|x|)\cos\left(\frac{\pi m x}{L}\right)dx$$

$$a_m = \frac{2}{L}\int_0^L (L-x)\cos\left(\frac{\pi m x}{L}\right)dx \;;\; b_m = 0.$$

$$\int_0^L (L-x)\cos(\pi m x/L)\,dx = \frac{L}{\pi m}\left[(L-x)\sin\left(\frac{\pi m x}{L}\right)\Big|_0^L\right.$$

$$-\int_0^L (-1)\sin(\pi m x/L)\,dx \Big] = \frac{L}{\pi m}\int_0^L \sin\left(\frac{\pi m x}{L}\right)dx$$

$$= -\left(\frac{L}{\pi m}\right)^2 \cdot \cos\left(\frac{\pi m x}{L}\right)\Big|_0^L = \left(\frac{L}{\pi m}\right)^2\left(1-(-1)^m\right)$$

$$\therefore\ a_m = \frac{2L}{\pi^2 m^2}\left(1-(-1)^m\right),\ m>0;\quad a_0 = L.$$

Note f es seccional. diferenciable.

$$\Rightarrow f(x) = \frac{L}{2} + \sum_{m=1}^{\infty}\frac{2L}{\pi^2 m^2}\left(1-(-1)^m\right)\cos\left(\frac{\pi m x}{L}\right)$$

$$= \frac{L}{2} + \sum_{K=1}^{\infty}\frac{2L}{\pi^2(2K-1)^2}\cos\left(\frac{\pi(2K-1)x}{L}\right)$$

Para $x=0$:

$$L = \frac{L}{2} + \sum_{K\geq 1}\frac{2L}{\pi^2(2K-1)^2}$$

$$\therefore\ \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$