CLASE 09

PC2 : • Pregunta 2

$f\ [0,2\pi] \to \mathbb{R}$, $f \in C^2([0,2\pi])$, $f''(x) \geq 0$, $0 \leq x \leq 2\pi$

Prove $\hat{f}(k) + \hat{f}(-k) \geq 0$, $\forall k \in \mathbb{Z}$, $k \neq 0$

Fije $k \in \mathbb{Z}$, $k \neq 0$. $\hat{f}(k) + \hat{f}(-k) \sim \int_0^{2\pi} f(x) \cos(kx)\, dx$

$= -\int_0^{2\pi} f'(x) \left(\sin(kx)/k \right) dx = \ldots$ Finish

• Pregunta 4

$$D_N(\theta) = \sum_{n=-N}^{N} e^{in\theta} .$$

$$\int_{-\pi}^{\pi} |D_N(\theta)|\, d\theta = 2 \int_0^{\pi} \left| \sin((N+1/2)\theta) / \sin(\theta/2) \right| d\theta .$$

Note $f(x) := \begin{cases} x/\sin x & , x \in]0, \pi/2] \\ 1 & , x = 0 \end{cases}$ es

continua en el compacto $[0, \pi/2]$, y, no se anula.

$\Rightarrow \exists\, c > 0 : |x/\sin x| \geq c$, $\forall x \in]0, \pi/2]$.

$$\Rightarrow \int_{-\tau}^{\tau} |D_N(\theta)|\,d\theta \geq 4\int_0^{\pi} \frac{|\sin((N+1/2)\theta)|\,d\theta}{\theta} = 4\int_0^{\pi(N+1/2)} \frac{|\sin w|\,dw}{w}$$

$$\geq 4\sum_{j=0}^{N-1} \int_{j\pi}^{(j+1)\pi} |\sin w|/w\,dw = 4\sum_{j=0}^{N-1}\int_0^{\pi} \underline{\sin\theta} \quad \text{Finish}$$

Ecuación de Dirichlet en el disco y Funciones armónicas

$\Omega = \{ (x,y) \in \mathbb{R}^2 \;\; x^2+y^2 < 1 \}$. Buscamos:

"3"

1) $u \in C^2(\Omega) \cap C(\overline{\Omega})$

2) $\Delta u = 0$ 3) $u|_{\partial\Omega} = f \in C(\partial\Omega)$

Para emplear separación de variables, empleamos coordenadas polares:

$x = r\cos\theta$, $y = r\sin\theta$, $r \in [0,1[$, $\theta \in [0, 2\pi[$.

$u(x,y) = V(r,\theta)$, $f(\cos\theta, \sin\theta) = g(\theta)$.

1) $V \in C^2([0,1[\times \mathbb{R}) \cap C([0,1] \times \mathbb{R})$

2) $r^2 V_{rr} + rV_r + V_{\theta\theta} = 0$, $r \in [0,1[$, $\theta \in \mathbb{R}$

4) $V(r,\theta) = V(r, \theta + 2\pi)$, $r \in [0,1]$, $\theta \in \mathbb{R}$

3) $V(1,\theta) = g(\theta)$

Por separación de variables: $V(r,\theta) = \varphi(r)\cdot\psi(\theta)$

De (2): $r^2\varphi''(r)\psi(\theta) + r\varphi'(r)\psi(\theta) + \varphi(r)\psi''(\theta) = 0$

$$\Rightarrow \frac{r^2\varphi''(r) + r\varphi'(r)}{\varphi(r)} = -\frac{\psi''(\theta)}{\psi(\theta)} = \lambda \quad (cte).$$

- $\varphi \in C^2(]0,1[) \cap C([0,1])$, $r^2\varphi''(r) + r\varphi'(r) - \lambda\varphi(r) = 0$
- $\psi \in C^2(\mathbb{R})$, $\psi(\theta+2\pi) = \psi(\theta)$, $\forall\theta\in\mathbb{R}$, $\psi''(\theta) + \lambda\psi(\theta) = 0$

$$\lambda\int_0^{2\pi}(\psi(\theta))^2 d\theta = -\int_0^{2\pi}\psi(\theta)\psi''(\theta)\,d\theta = -\left(-\int_0^{2\pi}(\psi'(\theta))^2 d\theta\right),$$

por periodicidad e integración por partes.

$$\Rightarrow \lambda\int_0^{2\pi}\psi^2(\theta)\,d\theta = \int_0^{2\pi}(\psi'(\theta))^2 d\theta \quad \Rightarrow \lambda \geq 0.$$

Obs: Si $\lambda = 0 \Rightarrow \psi = cte$

Suponga entonces $\lambda > 0$.

$$\Rightarrow \psi_\lambda(\theta) = A_\lambda\cos(\sqrt{\lambda}\,\theta) + B_\lambda\sin(\sqrt{\lambda}\,\theta). \quad \ldots (\Delta)$$

Tarea: De (Δ) y porque ψ tiene periodo 2π, $\Rightarrow \sqrt{\lambda} = k \in \mathbb{N}$.

$\therefore \lambda = k^2$, $k\in\mathbb{N}$.

$$\psi_k(\theta) = A_k\cos(k\theta) + B_k\sin(k\theta), \quad k\in\mathbb{Z}^+.$$

Note φ_k cubre el caso $\lambda = 0$, si $k = 0$.

$\therefore \varphi_k(\theta) = A_k \cos(k\theta) + B_k \sin(k\theta)$, $k \in \mathbb{Z}_{\geq 0}$.

Ahora, respecto a ϕ:

$$r^2 \phi''(r) + r\phi'(r) - k^2 \phi(r) = 0$$

1) Si $k = 0 \Rightarrow r^2 \phi''(r) + r\phi'(r) = 0$

$$(r\phi'(r))' = 0 \Rightarrow \phi'(r) = A/r \Rightarrow \phi(r) = A\ln(r) + B$$

$\dots (1)$

2) Si $k \in \mathbb{N} \Rightarrow$ Supongamos $\phi(r) = r^\alpha$, $\alpha \in \mathbb{R}$

$$\Rightarrow r^2 \alpha(\alpha-1) r^{\alpha-2} + r\alpha r^{\alpha-1} - k^2 r^\alpha = 0$$

$$\alpha(\alpha-1) + \alpha - k^2 = 0 \Rightarrow \alpha^2 = k^2 \Rightarrow \alpha = \pm k$$

$$\Rightarrow \phi(r) = C_k r^k + D_k r^{-k}. \quad \dots (2)$$

Para tener compatibilidad entre las soluciones $\phi(r)$ de (1) y (2), consideramos $A = \underbrace{0 = D_k}_{\text{evitar singularidad}}$.

$\therefore \phi_k(r) = C_k r^k$, $k \in \mathbb{Z}_{\geq 0}$.

$$\Rightarrow V(r,\theta) = \sum_{k=0}^{\infty} \left(A_k^* \cos(k\theta) + B_k^* \sin(k\theta)\right) r^k$$

$$= \sum_{k\in\mathbb{Z}} C_k e^{ik\theta} r^{|k|} \quad \text{(suponiendo conve. uniforme).}$$

De 3), sería bonito si: $C_k = \frac{1}{2\pi}\int_0^{2\pi} g(t)\, e^{-ikt}\, dt$

$$V(r,\theta) = \sum_{k\in\mathbb{Z}} \left(\frac{1}{2\pi}\int_0^{2\pi} g(t)\, e^{-ikt}\, dt \right) e^{ik\theta}\, r^{|k|}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} g(t) \underbrace{\left(\sum_{k\in\mathbb{Z}} e^{ik(\theta-t)}\, r^{|k|} \right)}_{P_r(\theta-t)} dt$$

$$\Rightarrow V(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} g(t)\, P_r(\theta-t)\, dt = (g * P_r)(\theta),$$

donde $P_r(\theta) = P(r,\theta) := \sum_{k\in\mathbb{Z}} e^{ik\theta}\, r^{|k|}$.

<u>Poisson kernel</u> $P(r,\theta)$, $r<1$.

Por test-Weierstrass, converge absolutamente; y uniformemente en $\mathbb{R}$ $(r\neq 0)$.

$0\le r<1$, $\theta\in\mathbb{R}$.

$$P_r(\theta) = 1 + \sum_{k\geq 1} (e^{i\theta} r)^k + \sum_{k\geq 1} (e^{-i\theta} r)^k$$

$$= 1 + e^{i\theta} r / (1 - e^{i\theta} r) + e^{-i\theta} r / (1 - e^{-i\theta} r).$$

<u>Simplificando:</u> $P(r,\theta) = \dfrac{1-r^2}{1-2r\cos\theta + r^2}$.

Theorem: Defina $g(\theta) = f(\cos\theta, \sin\theta)$ y $V(r,\theta) = (g * P_r)(\theta)$, $0 \le r < 1$, $\theta \in \mathbb{R}$.

Entonces, $u : \bar{\Omega} \to \mathbb{R}$ definida por

$$u(x,y) = \begin{cases} V(r,\theta) : & x = r\cos\theta,\ y = r\sin\theta, \\ & 0 \le r < 1,\ \theta \in \mathbb{R}. \\ f(x,y) : & x^2 + y^2 = 1 \end{cases}$$

es solución de EDP "3".

Proof:

1) $P(\cdot,\cdot) \in C^\infty([0,1[\times \mathbb{R})$

2) $P_{rr} + \frac{1}{r} P_r + \frac{1}{r^2} P_{\theta\theta} = 0$ en $]0,1[\times \mathbb{R}$.

3) $r \in [0,1[$, $\theta \in \mathbb{R}$

$$V(r,\theta) = (g * P_r(\theta)) = \frac{1}{2\pi} \int_0^{2\pi} g(t) P(r, \theta - t)\, dt$$

Por Feynman's integral Trick, 1) implica también $V \in C^\infty([0,1[\times \mathbb{R})$.

En Ω ($r \in]0,1[$):

$$\Delta u = V_{rr} + \frac{1}{r} V_r + \frac{1}{r^2} V_{\theta\theta} = \frac{1}{2\pi} \int_0^{2\pi} g(t) \Big(P_{rr}(r, \theta - t) +$$

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$\frac{1}{r} P_r(r, \theta - t) + \frac{1}{r^2} P_{\theta\theta}(r, \theta - t) \Big) dt = 0$

$\Rightarrow \Delta u = 0$ en Ω.

- $V(r,\theta) \longrightarrow g(\theta)$ uniformemente en $\mathbb{R}$, cuando $r \to 1^-$.

Dado $\epsilon > 0$, $\exists \delta > 0 : 0 < 1 - r < \delta \Rightarrow \forall \theta \in \mathbb{R}$:

$$|V(r,\theta) - g(\theta)| < \epsilon.$$

$$V(r,\theta) - g(\theta) = (g * P_r)(\theta) = \frac{1}{2\pi} \int_0^{2\pi} P_r(t)(g(\theta - t) - g(\theta)) dt$$

(por periodo 2π)

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t)(g(\theta - t) - g(\theta)) dt \quad \ldots (*)$$

Recuerda g es continua y periódica en $[0, 2\pi]$, así, su extensión en $\mathbb{R}$ es uniformemente continua en $\mathbb{R}$.

$\exists \eta_\epsilon > 0 : |z - y| < \eta_\epsilon,\ z, y \in \mathbb{R} \Rightarrow |g(z) - g(y)| < \epsilon$

Así: $|\theta - t - \theta| < \eta_\epsilon \Rightarrow |g(\theta - t) - g(\theta)| < \epsilon$

$|t| < \eta_\epsilon \Rightarrow |g(\theta - t) - g(\theta)| < \epsilon$.

Asimismo: $\exists B > 0 : |g(\theta)| \leq B,\ \forall \theta \in \mathbb{R}$.

En $(*)$:

$$|V(r,\theta) - g(\theta)| \leq \frac{1}{2\pi} \int_{|t|<\eta_\epsilon} |P_r(t)| \, |g(\theta - t) - g(\theta)| \, dt$$

$$+ \frac{1}{2\pi} \int_{\eta_\epsilon \leq |t| \leq \pi} |P_r(t)| \, |g(\theta - t) - g(\theta)| \, dt$$

$$\leq \frac{\epsilon}{2\pi} \int_{|t|<\eta_\epsilon} P_r(t) dt + \frac{2B}{2\pi} \int_{\eta_\epsilon \leq |t| \leq \pi} |P_r(t)| \, dt$$

$$< \epsilon + \frac{2B}{\pi} \int_{\eta_\epsilon \leq |t| \leq \pi} P_r(t) dt \leq \epsilon + \frac{2B}{\pi} P_r(\eta_\epsilon) \cdot (\pi - \eta_\epsilon)$$

$< \epsilon + 2B P_r(\eta_\epsilon)$. Basta notar $P_r(\eta_\epsilon)$ Tiende a cero, cuando $r \to 1$.

Funciones armónicas

$\Omega \subseteq \mathbb{R}^2$, abierto.

$u: \Omega \to \mathbb{R}$ es armónica si $u \in C^2(\Omega)$ y $\Delta u = 0$.

Recordemos el concepto de función holomorfa:

$f \in \mathcal{O}(\Omega)$ (holomorfa en Ω) si existe

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0), \ \forall z_0 \in \Omega.$$

Considere $f \in \mathcal{O}(\Omega)$, $f(z) = f(x+iy) = f(x,y)$,
$z_0 = x_0 + iy_0$.

$$f'(z_0) = \lim_{h\to 0} \frac{f(x_0+iy_0+h) - f(x_0+iy_0)}{h}$$

$$= \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} = f_x(z_0)$$

$$f'(z_0) = \lim_{h\to 0} \frac{f(x_0+iy_0+ih) - f(x_0+iy_0)}{ih}$$

$$= \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{ih} = \frac{1}{i} f_y(z_0)$$

$$\Rightarrow f_x(z_0) = -i f_y(z_0).$$

$f = u + iv$, $u = \operatorname{Re} f$, $v = \operatorname{Im} f$

$f_x = u_x + iv_x$, $f_y = u_y + iv_y$

$f_x = -f_y$ implica $u_x + iv_x = -i(u_y + iv_y)$

$\Rightarrow u_x = v_y$, $u_y = -v_x$ (ecuaciones de Cauchy-Riemann)

Notación: Definimos operadores:

- $$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

- $$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

En análisis complejo, se demuestra:

$$f \in \mathcal{O}(\Omega) \iff f \in C^1(\Omega) \wedge \underbrace{\frac{\partial f}{\partial \bar{z}} = 0}_{\text{ecu. Cauchy-Rie}}.$$