

CLASE 03

Ejer. Sean $p \in \mathbb{R}_+^n \setminus \{0\}$, $\alpha > 0$, real. Defina

$$L(x) := \max_{\substack{1 \leq i \leq n \\ p_i \neq 0}} \{ p_i x_i \} \quad y \text{ el "hiperplano"}$$

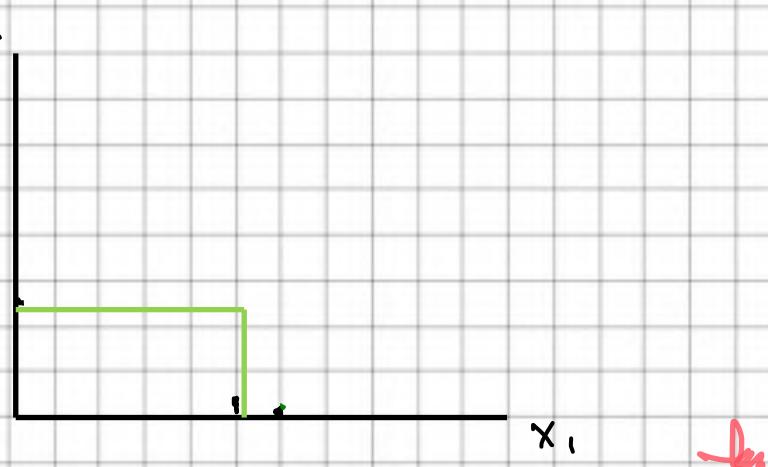
$H(p, \alpha) = \{ x \in \mathbb{R}_+^n : L(x) = \alpha \}$. Muestre un ejemplo de $H(p, \alpha)$, para $n=2$

Sol:

Considera $p = (3, 6)$, $\alpha = 30$.

$$\begin{aligned} H(p, \alpha) &= \{ x \in \mathbb{R}_+^2 : \max \{ 3x_1, 6x_2 \} = 30 \} \\ &= \{ x \in \mathbb{R}_+^2 : (3x_1 = 30 \wedge 3x_1 \geq 6x_2) \\ &\quad \vee \\ &\quad (6x_2 = 30 \wedge 6x_2 \geq 3x_1) \} \\ &= \{ x \in \mathbb{R}_+^2 : (x_1 = 10 \wedge x_1 \geq 2x_2) \vee (x_2 = 5 \wedge 2x_2 \geq x_1) \} \end{aligned}$$

$H((3, 6), 30)$:



Ejer. Sea $C \neq \emptyset$. Si C convexo $\Rightarrow r_1(C) \neq \emptyset$.

Sol:

Sea $\phi \neq C \subseteq \mathbb{R}^n$. Fixe $v \in \mathbb{R}^n$.

Pd: $v \in v + C = v + \text{ri}(C)$

(\subseteq) $x \in \text{ri}(v + C)$ (Trivial if empty)

$\Rightarrow \exists \epsilon > 0 : B_\epsilon(x) \cap \text{aff}(v + C) \subseteq v + C$

$\{x\} \subseteq B_\epsilon(x) \cap (v + \text{aff}(C)) \subseteq v + C$

Pd: $A \cap B - v = (A - v) \cap (B - v)$

(\subseteq) $x = ab - v, ab \in A \cap B \Rightarrow x \in A - v, x \in B - v$

(\supseteq) $x = a - v, x = b - v, a \in A, b \in B$

$x + v = a = b, a \in A \cap B.$ ✓

$\Rightarrow (B \in(x) - v) \cap \text{aff}(C) \subseteq C$

• $y \in B \in(x) - v \Rightarrow y = z - v, \|z - x\| < \epsilon$

$\Rightarrow \|y + v - x\| < \epsilon \Rightarrow y \in B_\epsilon(x - v)$

• $y \in B_\epsilon(x - v) \Rightarrow \|y - x + v\| < \epsilon$

$y = (y + v) - v \in B_\epsilon(x) - v.$

$\therefore B_\epsilon(x) - v = B_\epsilon(x - v)$

$\Rightarrow B_\epsilon(x - v) \cap \text{aff}(C) \subseteq C$

$$x = v + (x-v), \quad x-v \in r_1(C)$$

$$(2) \quad x \in v + r_1(C)$$

$$x = v + y, \quad y \in r_1(C)$$

$$\exists \epsilon > 0 \quad B_{\epsilon}(x-v) \cap \text{aff}(C) \subseteq C$$

$$+v \therefore B_{\epsilon}(x) \cap \text{aff}(v+C) \subseteq v+C$$
$$\Rightarrow x \in r_1(v+C).$$

Por lo tanto, sea $x \in C \neq \emptyset$:

$$r_1(\underbrace{-x+C}_{\exists 0}) = r_1(C) - x.$$

Así, podemos suponer $0 \in C$.

$$\Rightarrow \text{aff}(C) = \langle C \rangle. \quad \dots (1)$$

$$\text{Si } C = \{0\} \Rightarrow r_1(C) = \{0\} \neq \emptyset$$

Suponga entonces $C - \{0\} \neq \emptyset$.

$$\Rightarrow \text{De (1): } \exists d \in \mathbb{Z}^+; \quad v_1, \dots, v_d \in C$$

con $\{v_i\}_1^d$ base de $\text{aff}(C)$.

Consideremos $T \in \mathcal{L}(\mathbb{R}^d, \text{aff}(c))$ definido por

$$T(a_1, \dots, a_d) = \sum_{i=1}^d a_i v_i, \text{ isomorfismo lineal}$$

de coordenadas.

De $\dim(\mathbb{R}^d) = d < \infty$: T^{-1} es continuo,

por ser T isomorfismo lineal.

$\Rightarrow T(A)$ es abierto en $\text{aff}(c)$, para todo abierto A de \mathbb{R}^d .

Sea $\Lambda := \{ \lambda \in \mathbb{R}_{++}^d : \sum_{i=1}^d \lambda_i < 1 \}$, se cumple

$$\Lambda = \mathbb{R}_{++}^d \cap B_1(0) = B_1(0) \cap \bigcap_{i=1}^d \bar{\pi}_i([0, \infty[),$$

intersección finita de abiertos en \mathbb{R}^d .

$\Rightarrow \Lambda$ open en $\mathbb{R}^d \Rightarrow T(\Lambda)$ open en $\text{aff}(c)$.

$\therefore \exists V \subseteq \mathbb{R}^n$ open : $T(\Lambda) = V \cap \text{aff}(c)$.

De $((2d)^{-1}, \dots, (2d)^{-1}) \in \Lambda \therefore \Lambda \neq \emptyset$.

$$\text{Sea } \lambda \in \Lambda. \quad T(\lambda) = \sum_{i=1}^d \lambda_i \underbrace{v_i}_{\in C} + \underbrace{(1 - \sum_{i=1}^d \lambda_i)}_{\in [0,1]} \underbrace{0}_{\in C}$$

$$\Rightarrow T(\lambda) \in \text{co}(C) = C.$$

$$\therefore C \supseteq T(\Lambda) = V \cap \text{aff}(C).$$

$$\text{Sea } x \in V \cap \text{aff}(C)$$

$$\Rightarrow \exists \epsilon > 0 : B_\epsilon(x) \subseteq V, x \in \text{aff}(C)$$

$$\Rightarrow B_\epsilon(x) \cap \text{aff}(C) \subseteq V \cap \text{aff}(C) = T(\Lambda) \subseteq C$$

$$\therefore x \in C, \exists \epsilon > 0 : B_\epsilon(x) \cap \text{aff}(C) \subseteq C$$

$$\Rightarrow x \in r_i(C) \Rightarrow (\phi+) T(\Lambda) \subseteq r_i(C)$$

$$\therefore r_i(C) \neq \emptyset. \quad \text{Im}$$

EJER: C convexo $\Rightarrow \text{int}(C)$ convexo

Proof.

Denote $U := \text{int}(C)$.

$$\underline{\text{Pd}}: tU + (1-t)U \subseteq U, \forall t \in [0,1]$$

Fije $t \in [0,1]$.

$tU + (1-t)V \subseteq C$, pues $U \subseteq C$ y C es convexo.

$$tU + (1-t)V = \bigcup_{u \in U} (tU + (1-t)u),$$

unión de abiertos, pero, abierto,
pues Traslación de open es open, y:

Pd: $\emptyset \neq A \subseteq \mathbb{R}^n$ open $\Rightarrow \alpha A$ open, $\forall \alpha > 0$

Fije $\alpha > 0$. Sea $v \in \alpha A$.

$\Rightarrow v = \alpha a, a \in A$.

$\frac{v}{\alpha} = a \in A$, open $\Rightarrow \exists \epsilon > 0 : B_\epsilon(\frac{v}{\alpha}) \subseteq A$

Sea $w \in B_{\alpha\epsilon}(v) \Rightarrow \|v - w\| < \alpha\epsilon$

$\|v/\alpha - w/\alpha\| < \epsilon \Rightarrow w/\alpha \in A \Rightarrow w \in \alpha A$

$\Rightarrow B_{\alpha\epsilon}(v) \subseteq \alpha A, \epsilon > 0$, $\therefore \alpha A$ open

$\therefore \underbrace{tU + (1-t)V}_{\text{open}} \subseteq C$

biggest open en C

$\Rightarrow tU + (1-t)V \subseteq \overline{\text{int}(C)} = U$

$\therefore \text{int}(C)$ convex *lma*