Timelike Geodesic of a Neutral Particle Around a Massless Reissner-Nordström Spacetime

The computation is performed in Python using TensorFlow, a widely used open-source package for deep learning, and the code is implemented on the Jupyter Notebook platform.
We consider the metric in spherical coordinates

$$ds^2 = -f(r)dt^2 + [f(r)]^{-1}\, dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)$$

with $f(r) = 1 + \frac{Q^2}{r^2}$ and Q the charge of the body. It is regular in these coordinates, except for the singularity in $r = 0$ as described by the Kretschmann scalar $K = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{56Q^4}{r^8}$. Given the symmetry of the spacetime, we consider the motion along the direction of constant polar angle $\theta = \pi/2$, for which

$$ds^2 = -f(r)dt^2 + [f(r)]^{-1}dr^2 + r^2 d\phi^2.$$

Given other symmetries expressed by the Killing fields, the equations of motion are

$$\begin{cases} \ddot{r} = \left(J^2 + Q^2 + \dfrac{2Q^2J^2}{r^2}\right)\dfrac{1}{r^3} \equiv s_1(\tau) \\ \dot{r}(0) = v_0 \\ r(0) = r_0 \end{cases} \qquad \begin{cases} \dot{\phi} = \dfrac{J}{r^2} \equiv s_2(\tau) \\ \phi(0) = \phi_0 \end{cases}$$

with J as the angular momentum per unit rest mass of the particle and the derivatives with respect to the proper time τ of the particle along the curve. We can design the neural network (nn) in this way: an input of one neuron for the time τ, one hidden layer of 16 neurons with a sigmoid activation function and two output neurons for $nn(\tau)[0]$ and $nn(\tau)[1]$. Then, we can construct the solutions as

$$r(\tau) = (nn(\tau)[0] - nn(0)[0] + v_0)\tau + r_0$$

$$\phi(\tau) = nn(\tau)[1]\tau + \phi_0.$$

It is straightforward to verify that each function satisfies the initial conditions. Then, the loss function to be minimized is defined as

$$\mathcal{J} = \sum_\tau (\ddot{r} - s_1(\tau))^2 + \sum_\tau (\dot{\phi} - s_2(\tau))^2.$$

Below, there are the plots of $r(\tau)$ and $(x(\tau), y(\tau))$ with $x(\tau) = r(\tau)\cos(\phi(\tau))$ and $y(\tau) = r(\tau)\sin(\phi(\tau))$. In particular, they are in agreement with the non-existence of stable orbits for each values of Q and J (which have been fixed to unity) of the effective potential. Initially, $r(\tau)$ decreases as $v_0 < 0$ has been chosen.

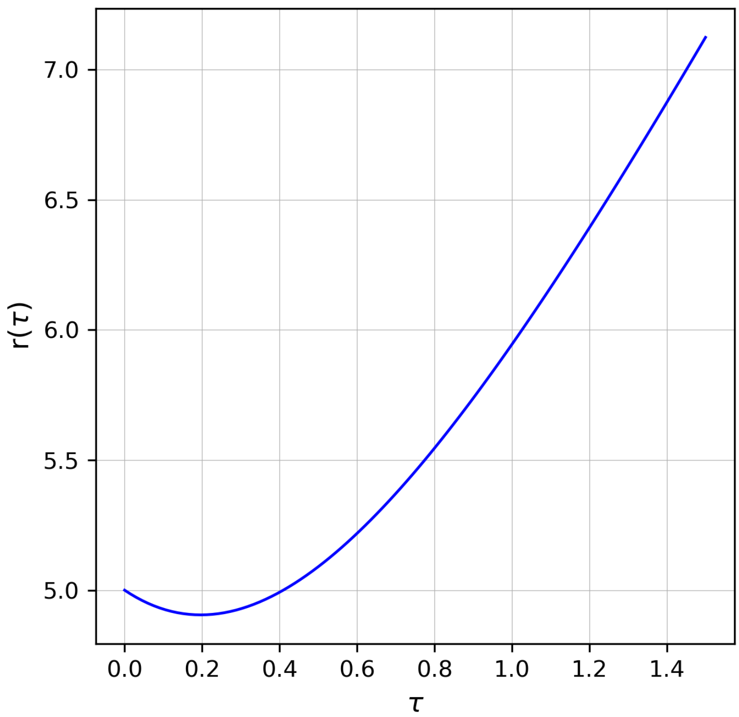

Figure 1: Plot of $r(\tau)$ vs τ.

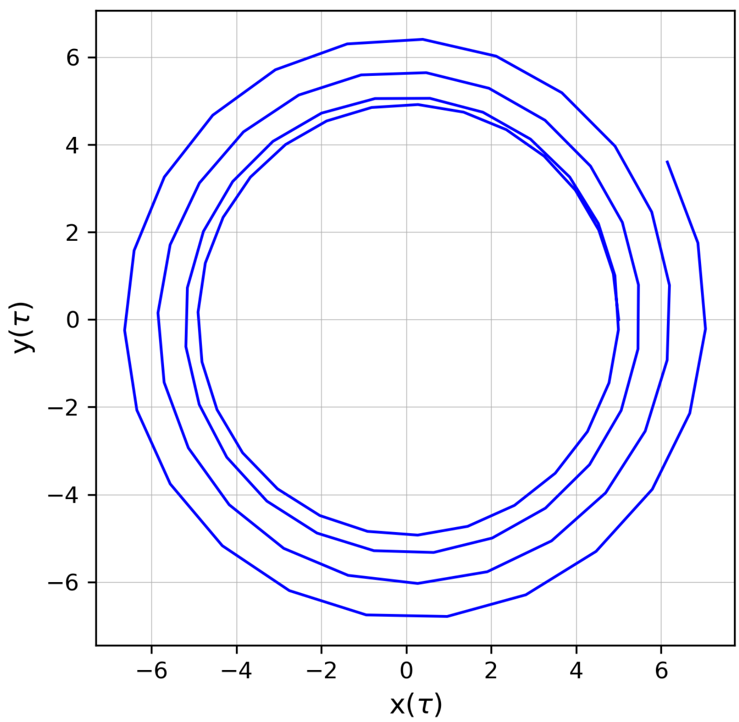

Figure 2: Parametric plot of $(x(\tau), y(\tau))$