ASSIGNMENT 4

Due 3 p.m. on December 11, 2025

Instructions. Hand-in is in electronic form via email to `sztachera@econ.uni-frankfurt.de`. If you submit multiple files, please combine them in a single *.zip* file. List clearly the names and student numbers of the group members. There should be sufficient documentation for third parties to execute the code. In each exercise, make additional assumptions if necessary and explain them.

Exercise 1 (up to Lecture 5) A Stylized Deterministic 6-Period OLG Model: Steady State.

Households live for $T = 6$ periods. In the first $T_w = 4$ years of life, they choose how many hours $l_t^s \in [0, 1]$ to work in each period. In the last $T_r = T - T_w = 2$ years of life, they are retired and only consume and save: $l_t^s = 0$. The value function depends on age s, which is a state variable.

$$
\begin{aligned}
V_t^s(k_t^s) &= \max_{c_t^s, l_t^s, k_{t+1}^{s+1}} \frac{\left(c_t^s(1-l_t^s)^\gamma\right)^{1-\eta} - 1}{1-\eta} + \beta V_{t+1}^{s+1}(k_{t+1}^{s+1}) \\
\text{s.t. } c_t^s + k_{t+1}^{s+1} &= (1 + r_t)k_t^s + y_{i,t}^s \\
y_{i,t}^s &= \begin{cases} (1-\tau)w_t l_{i,t}^s & s = 1, ..., T_w \\ b_t & s = T_w + 1, ..., T \end{cases}
\end{aligned}
$$

Agents are born with a zero capital endowment $k_t^1 = 0$. Firms operate a Cobb-Douglas technology $Y_t = K_t^\alpha L_t^{1-\alpha}$. Firms' optimal capital and labor demand imply that factors are paid according to their marginal products.

$$
\begin{aligned}
r_t &= \alpha \left(\frac{K_t}{L_t}\right)^{\alpha-1} \\
w_t &= (1-\alpha) \left(\frac{K_t}{L_t}\right)^{\alpha}
\end{aligned}
$$

Government operates a PAYG pension system. The pension replacement rate is proportional to the average after-tax labor income in the economy, where $\bar{l}_t$ denotes average hours worked $\bar{l}_t = \sum_{s=1}^{T_w} \frac{l_t^s}{T_w}$.

$$
\begin{aligned}
\tau_t w_t L_t &= \frac{T - T_w}{T} b_t \\
b_t &= rr(1-\tau_t)w_t \bar{l}_t
\end{aligned}
$$

Capital, labor, and goods markets clear.

$$
\begin{aligned}
\sum_{s=1}^{T} \frac{k_t^s}{T} &= K_t^d \\
\sum_{s=1}^{T_w} \frac{l_t^s}{T} &= L_t^d \\
K_t^\alpha L_t^{1-\alpha} &= \sum_{s=1}^{T} \left(\frac{c_t^s}{T} + \frac{k_{t+1}^{s+1}}{T} - (1-\delta)\frac{k_t^s}{T}\right)
\end{aligned}
$$

Assume the following parameter values: $\beta = 0.9$, $\eta = 2$, $\gamma = 2$, $\alpha = 0.3$, $\delta = 0.4$, and $rr = 0.3$.

a) Define the recursive competitive equilibrium of this economy.

b) Derive the household's optimality conditions for each period of life s.

c) Show in steps how you can use the direct method outlined on Slides 22-23 of Lecture 5 to solve the household problem for given prices r, w, and government policy τ, b. Code up a suitable routine for any number of working life and lifetime periods T_w and T.

d) Compute the equilibrium capital holdings K^*, labor supply L^*, and tax rate τ^*. What are the equilibrium prices r and w? Plot the life-cycle capital holdings k^s, consumption c^s, and labor supply l^s.

e) Compute the model's equilibrium using VFI or EGM for the household problem. Do you recover the same equilibrium capital holdings, labor supply, and prices? Compare the runtime to using the direct method to solve the household problem.

Exercise 2 (up to Lecture 5) A Stylized Deterministic 6-Period OLG Model: Transition Path.

In Exercise 1, you found the steady state of the deterministic 6-Period OLG economy.

The objective of this exercise is to compute the transition path of this economy in response to a once-and-for-all unannounced increase in the retirement age from $T_w + 1 = 5$ to $T_w + 1 = 6$. Assume that the reform is effective on the day of the announcement: Agents of age 5 who would retire under the previous policy, unexpectedly work an additional period.

In the baseline economy, agents work for $T_w = 4$ periods and get pension income for $T_r = T - T_w = 2$ periods. In the new economy, agents work for $T'_w = 5$ periods and get pension income for $T'_r = T - T'_w = 1$ period. All other model parameters, including the replacement rate, stay constant.

To build your own algorithm computing the transition path, you can follow the solution steps described in Lecture 5, Slides 29-31/35.

a) Compute the initial and terminal steady states. How do policy functions differ between them, and why?

b) Plot the time paths of aggregate consumption, labor supply, capital, and prices along the transition.

c) Plot policy functions for different cohorts in the economy along the transition path. Focus on the early transition periods. Interpret your results.

d) Compute the welfare change for all transition cohorts and the terminal steady state cohort relative to the initial steady state cohort: $\Delta\mathcal{W} = \frac{V_c^1 - V_{ini}^1}{\left|V_{ini}^1\right|}$ (cross-cohort difference in at-birth value functions). Who gains and who loses from this reform?

Exercise 3 (up to Lecture 6) **A $T = 70$ OLG economy with idiosyncratic risk.**

Households live for $T = 70$ periods. In the first $T_w = 45$ years of life, they choose how many hours $l_t^s \in [0, 1]$ to work in each period. In the last $T_r = T - T_w = 25$ years of life, they are retired and only consume and save: $l_t^s = 0$.
Household labor income is subject to idiosyncratic income risk.

$$y_{i,t}^s = \theta_{i,t} w_t l_{i,t}^s$$

The idiosyncratic income component θ follows a Gaussian AR1 process in logs.

$$\log \theta_{i,t+1}^{s+1} = (1 - \rho)\mu + \rho \log \theta_{i,t}^s + \xi^s$$

Assume the following parameter values for the income process: $\rho = 0.96$, $\sigma_\xi^2 = 0.045$, and $\mu = -\frac{\sigma_\xi^2}{2(1-\rho^2)}$.
Households face a strict borrowing limit $\underline{a} = 0$. Furthermore, households cannot supply negative hours worked to the market. This constraint is binding in some states and needs to be imposed when solving the model numerically.
The household problem takes the following recursive form.

$$\begin{aligned}
V_t(s, \theta^s, a_t^s) &= \max_{c_t^s, l_t^s} \frac{\left((c_t^s)^\gamma (1 - l_t^s)^{1-\gamma}\right)^{1-\eta} - 1}{1 - \eta} + \beta \mathbb{E}_t V_{t+1}(s + 1, \theta^{s+1}, a_{t+1}^{s+1}) \\
\text{s.t. } c_{i,t}^s + a_{i,t+1}^{s+1} &= y_{i,t}^s + (1 + r_t) a_{i,t}^s \\
y_{i,t}^s &= \begin{cases} (1 - \tau)\theta_{i,t} w_t l_{i,t}^s & s = 1, ..., T_w \\ b_t & s = T_w + 1, ..., T \end{cases} \\
a_{i,t+1} &\geq 0 \\
l_{i,t} &\geq 0
\end{aligned}$$

Households are born without assets and leave no bequests at T: $a_{i,t}^1 = a_{i,t}^{T+1} = 0$.
Firms produce using Cobb-Douglas technology $Y_t = K_t^\alpha L_t^{1-\alpha}$.
The aggregate consumption, labor supply, and capital satisfy the following equations.

$$\begin{aligned}
L_t &= \sum_{s=1}^{T_w} \int_{\theta \times \mathcal{A}} \theta_{i,t} l_{i,t}^s dF(s, a, \theta) \\
C_t &= \sum_{s=1}^{T} \int_{\theta \times \mathcal{A}} c_{i,t}^s dF(s, a, \theta) \\
K_t &= \sum_{s=1}^{T} \int_{\theta \times \mathcal{A}} a_{i,t}^s dF(s, a, \theta)
\end{aligned}$$

The government runs a balanced budget.

$$\tau w_t L_t = \frac{T - T_w}{T} b_t$$

The steady-state replacement rate equals $rr = 0.35$.

$$b_t = rr_t (1 - \tau_t) w_t \bar{l}_t$$

Assume the following parameter values: $\alpha = 0.35$, $\delta = 0.083$, $\beta = 1.011$, $\eta = 2$, $\gamma = 0.33$.

a) Define the stationary equilibrium of this economy.

b) Derive the first-order conditions of the household problem.

c) Compute the equilibrium real interest and wage rates. Plot the average life-cycle labor supply and wealth profiles and the steady-state wealth distribution. Are they realistic? How would you extend the model to improve its fit to the empirical labor supply and wealth profiles?

Hints for the solution:

- To solve the household problem given prices, you can build on your code for the deterministic OLG model (but you cannot use the direct method due to the presence of idiosyncratic risk).
- Pay attention to differences in the Euler equation between $t > T_w$, $t = T_w - 1$, and $t < T_w - 1$.
- Use the Markov transition matrix for productivity and the optimal capital policy to form a transition matrix from current to future states.
- The number of rows of the transition matrix should equal the total number of states today: $T \times n_\theta \times n_a$.
- A good initial guess for the stationary distribution should have equal masses across age and have all newborn agents hold no capital.
- If the distribution does not converge, there is likely a problem with the range of your asset grid.
- If the distribution does not sum up to one, there is likely a mistake in the transition matrix.