1. Model analytics

For the general statisfical model of asset return (see `am_standards`) given by

$$y = f(\psi) + \epsilon. \tag{1}$$

we define a series of analyses that inform financial modeling, interpretability, statistical estimation, hypothesis testing and performance evaluation.

The covariance matrix plays a primary role in the distibution of asset return:

$$\mathrm{Var}(y) = \Sigma. \tag{2}$$

In practice, we analyze the estimated covariance matrix $\hat{\Sigma}$, but here we use Σ for simplicity. An asset model typically assumes some particular structure for Σ that suggests particular analyses to perform. We motivate different types of analyses for the various models in `am_standards`. (Coversions between the various models is often possible).

2. Analytics prototypes

```
Input: model with attributes
-- attribute : type (factor, graphical, etc)
-- attribute : estimation_window (start_date, end_data, freq)
-- attribute : method (pca, mtfa, glasso, etc)

Analytics for type = factor:
-- compute_signal -- arg : factor_id
-- compute_noise
-- compute_size
-- compute_snr
-- compute_skew
-- compute_meltup
-- compute_correlation
-- compute_entropy
```

3. Factor model analytics

Consider the **canonical factor model** for which the covariance has the form

$$\Sigma = \Pi \Psi \Pi^{\top} + \Omega \tag{3}$$

with a diagonal, $q \times q$ postive definite matrix Ψ and a full rank $p \times q$ matrix Π with othogonal columns $\pi_1, \dots, \pi_q$ (i.e., $\langle \pi^k, \pi^\ell \rangle = 0$ for all $k \neq \ell$),[1] and a symmertic positive definite matrix Ω with bounded eigenvalues (in dimension p).

It is important to undertand the evolution, over time, of both components in the decomposition of Σ above. In practice, some estimation procedure (e.g. PCA, see `pca`), will produce estimates of the triplet (Π, Ψ, Ω) which conforms to the standards in `am_standards`. The following analytics help study the structure of (Π, Ψ, Ω)

The systematic risk component of the canonical factor model may be written as

$$\Pi \Psi \Pi^\top = \sigma_1^2 \pi^1 (\pi^1)^\top + \cdots + \sigma_q^2 \pi^q (\pi^q)^\top \tag{4}$$

and we take $\sigma^2 \pi\pi = (\sigma\pi)(\sigma\pi)^\top$ as a representative term letting $\eta = \sigma\pi \in \mathbb{R}^p$ which represents the product of the factor volatility σ times the factor exposures π. In practice, there is no way to estimate the σ and π separately, and we can write

$$\eta\eta^\top = \mathrm{c}^2 \left(\frac{\eta}{\mathrm{c}}\right)\left(\frac{\eta}{\mathrm{c}}\right)^\top \tag{5}$$

for any $\mathrm{c} \neq 0$ so that the variance is now identified as $\sigma^2 = \mathrm{c}^2$ while $\pi = \frac{\eta}{\mathrm{c}}$.

This motivates the analysis of the structure of the vector η to determine how to best model the individual components σ and π among other concerns.

Let $\eta = \sigma\pi \in \mathbb{R}^p$, the product of the factor volatility and exposure. We are interested in the evolution over time of the following quantities.

- **Signal.** Denoted by $\mathrm{m}(\eta)$ and defined by

$$\mathrm{m}(\eta) = \textstyle\sum_{i=1}^p \eta_i / p \geq 0 \tag{6}$$

  where we flip the sign of η to ensure nonnegativity.

- **Noise.** Denoted by $\mathrm{s}(\eta)$ and defined by

$$\mathrm{s}(\eta) = \sqrt{\textstyle\sum_{i=1}^p (\eta_i - \mathrm{m}(\eta))^2 / p}. \tag{7}$$

- **Exposure SNR.** A signal-to-noise ratio defined by

$$\mathrm{SNR}(\eta) = \frac{\mathrm{m}(\eta)}{\mathrm{s}(\eta)}. \tag{8}$$

- **Size.** Denoted by $\mathrm{z}(\eta)$ and defined as a scaled length

$$\mathrm{z}(\eta) = |\eta| / \sqrt{p} = \sqrt{\langle(\eta), \eta\rangle / p} = \sqrt{\mathrm{m}^2(\eta) + \mathrm{s}^2(\eta)}. \tag{9}$$

[1] The inner product could be a weighted one but here we continue with the standard inner product.

- **Skewness.** Denoted by $\mathrm{k}(\eta)$ and defined as

$$\mathrm{k}(\eta) = \frac{\sum_{i=1}^{p}(\eta_i - \mathrm{m}(\eta))^3/p}{\mathrm{s}^3(\eta)} \tag{10}$$

- **Meltup.** Denoted by $\mathrm{u}(\eta)$ and defined as

$$\mathrm{u}(\eta) = \sum_{i=1}^{p}\mathrm{sgn}(\eta_i)/p \tag{11}$$

- **Correlation.** Denoted by $\rho(\eta)$ and defined as

$$\rho(\eta) = \sum_{i\neq j}\eta_i\eta_j/(p^2 - p) \tag{12}$$

- **Entropy.** Denoted by $\mathrm{H}(\eta)$ and defined as

$$\mathrm{H}(\eta) = -\sum_{i=1}^{p}\mathrm{p}_i(\eta)\log\mathrm{p}_i(\eta) \tag{13}$$

  for $\mathrm{p}_i(\eta) = \phi(\eta)$ for some probability density ϕ (e.g. normal).

In particular, the size, signal and noise can all serve as a proxy for the factor variance, i.e., (one exception is when the signal $\mathrm{m}(\eta)$ is close or equal to zero)

$$\begin{aligned}\eta\eta^\top &= \mathrm{z}^2(\eta)\Big(\frac{\eta}{\mathrm{z}(\eta)}\frac{\eta^\top}{\mathrm{z}(\eta)}\Big) = \mathrm{s}^2(\eta)\Big(\frac{\eta}{\mathrm{s}(\eta)}\frac{\eta^\top}{\mathrm{s}(\eta)}\Big) = \mathrm{m}^2(\eta)\Big(\frac{\eta}{\mathrm{m}(\eta)}\frac{\eta^\top}{\mathrm{m}(\eta)}\Big)\\ &= \sigma^2\pi\pi^\top.\end{aligned} \tag{14}$$

identifying the σ^2 with different quantities.

3.1. SSIM estimation. An SSIM estimate takes the form

$$\hat{\Sigma} = v^2 hh^\top + \mathrm{D} \quad \text{where} \quad \textstyle\sum_i h_i \geq 0, \quad |h| = 1. \tag{15}$$

There is no way in general to estimate the "sizes" of $v \in (0, \infty)$ and $h \in \mathbb{R}^p$ so a normalization such as $|h| = 1$ has to be selected. Alternatively we can take any,

$$\hat{\Sigma} = \Big(\frac{v^2}{Z^2}\Big)\eta\eta^\top + \mathrm{D} \quad \text{where} \quad \textstyle\sum_i \eta_i \geq 0, \quad |\eta| = Z. \tag{16}$$

A useful example would set Z equal to the average entry of h (hence, η has unit mean).

Given a $p \times n$ return matrix Y (n observations of $y \in \mathbb{R}^p$) we take the sample covariance and correlation matrices

$$\mathrm{S} = \mathrm{Y}\mathrm{Y}^\top/n \qquad \mathrm{C} = \mathrm{V}^{-1/2}\mathrm{S}\mathrm{V}^{-1/2} \qquad \mathrm{V} = \mathrm{diag}(\mathrm{S}). \tag{17}$$

Then we have the following models.

- The PCA covariance model takes h to be the eigenvector of S with largest eigenvalue $\mathscr{s}_1^2$ and sets $v^2 = \mathscr{s}_1^2$. Further, $\mathrm{D} = \mathrm{diag}(\mathrm{S} - v^2hh^\top)$.
- The PCA correlation model takes h to be the eigenvector of C with largest eigenvalue $\mathscr{c}_1^2$ but sets $v^2 = \mathscr{s}_1^2$ (any identity for $\mathscr{s}$ and $\mathscr{c}$?) and D as above.
- James-Stein PCA variation
- MTFA
- Graphical model.

4. Beta financial indicators

These can use the raw beta β or the market-cap beta $\beta^{\mathrm{M}} = \mathrm{V}^{-1/2}\beta$. The latter is more robust and theoretically consistent. It has

$$\beta_i^{\mathrm{M}} \propto \sqrt{\frac{\beta_i^2}{\sigma_{\mathrm{M}}^2\beta_i^2 + \delta_i^2}} \tag{18}$$

in a strict single index model (SSIM).

- **Mean beta.**
$$\mathrm{m}(\beta) = \sum_{i=1}^{p} \beta_i/p$$
- **Beta dispersion.**
$$\mathrm{d}^2(\beta) = \mathrm{s}^2(\beta)/\mathrm{m}^2(\beta)$$
where $\mathrm{s}^2(\beta) = \sum_{i=1}^p (\beta_i^2 - \mu(\beta))^2/p$.

– **Beta skewness.**

$$\mathrm{k}(\beta) = \mathrm{m}_3(\beta)/\mathrm{s}^2(\beta)$$

where $\mathrm{m}_3(\beta) = \sum_{i=1}^{p}(\beta_i - \mu(\beta))^3/p$.

– **Melt up.** (number positive minus number negative beta).

$$\mathrm{M}(\beta) = \sum_{i=1}^{p} \mathrm{sign}(\beta_i)/p$$

– **Average pairwise correlation.** For a covariance matrix $\Sigma = (\Sigma_{ij})$,

$$\rho(\Sigma) = \sum_{i \neq j} \frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}\Sigma_{jj}}}/(p^2 - p)$$

We can also define the average pairwise correlation of Beta (or any factor β) as

$$\rho(\beta) = \sum_{i \neq j} \beta_i \beta_j/(p^2 - p)$$

which is rougly $\mu^2(\beta)$ for p large. Note that in a SSIM,

$$\rho(\beta^{\mathrm{M}}) = \rho(\Sigma).$$

– **Entropy.** Let ϕ be some probability density (e.g.. normal). Let $\mathrm{p}_i(\beta) = \phi(\beta_i)$.

$$\mathrm{H}(\beta) = -\sum_{i=1}^{p} \mathrm{p}_i(\beta)\log(\mathrm{p}_i(\beta))$$

5. Nonbeta financial indicators

Portfolio variances.

– **Market volatility.** This is simply σ_{M} in model (**??**). In the estimated model this may be taken as v in (15), or for the version of (16) that assumes a unit-mean beta,

$$v\mathrm{m}(h). \tag{19}$$

– **Portfolio volatility.** For a portfolio w computed based on covariance Σ,

$$\sqrt{w^{\top}\Sigma w}$$

is the volatility of w. Similarly for an estimated $\hat{w}$ based on $\hat{\Sigma}$ (i.e., $\sqrt{\hat{w}^\top \Sigma \hat{w}}$) . Then,

$$\sqrt{\hat{w}^\top \Sigma \hat{w}}$$

is the realized portfolio variance. (When Σ is not available an out-of-sample estimate may be used).

– **Diversification index.** For a portfolio w the following (Herfindahl) index measures diversification (variations are typically norms $||\cdot||$ of w),

$$\mathrm{HI}(w) = \sum_{i=1}^{p} w_i^2 .$$

A portfolio based purely on Σ (or $\hat{\Sigma}$) is the one with minimum variance, i.e.,

$$\begin{aligned} &\min_w w^\top \Sigma w \\ &\text{s.t. } w^\top \mathrm{e} = 1 \\ &\quad (w \geq 0) \end{aligned} \tag{20}$$

for $\mathrm{e} = (1, \ldots, 1)^\top$ and with the no short sales constraint ($w \geq 0$) optional.

The solution may be stated explitly in the SSIM $\Sigma = \sigma_{\mathrm{M}}^2 \beta\beta^\top + \Delta$.

$$w = \frac{\Delta^{-1}(\mathrm{e} - \theta\beta)}{\mathrm{e}^\top \Delta^{-1}(\mathrm{e} - \theta\beta)}$$

where θ for the portolio allowing short sales is

$$\theta = \frac{\sum_{i=1}^{p} \beta_i / \delta_i^2}{1/\sigma_{\mathrm{M}}^2 + \sum_{i=1}^{p} \beta_i^2 / \delta_i^2} \tag{21}$$

and we replace $\sum_{i=1}^{p}$ with $\sum_{\theta\beta_i < 1}$ to obtain the solution with no short sales.

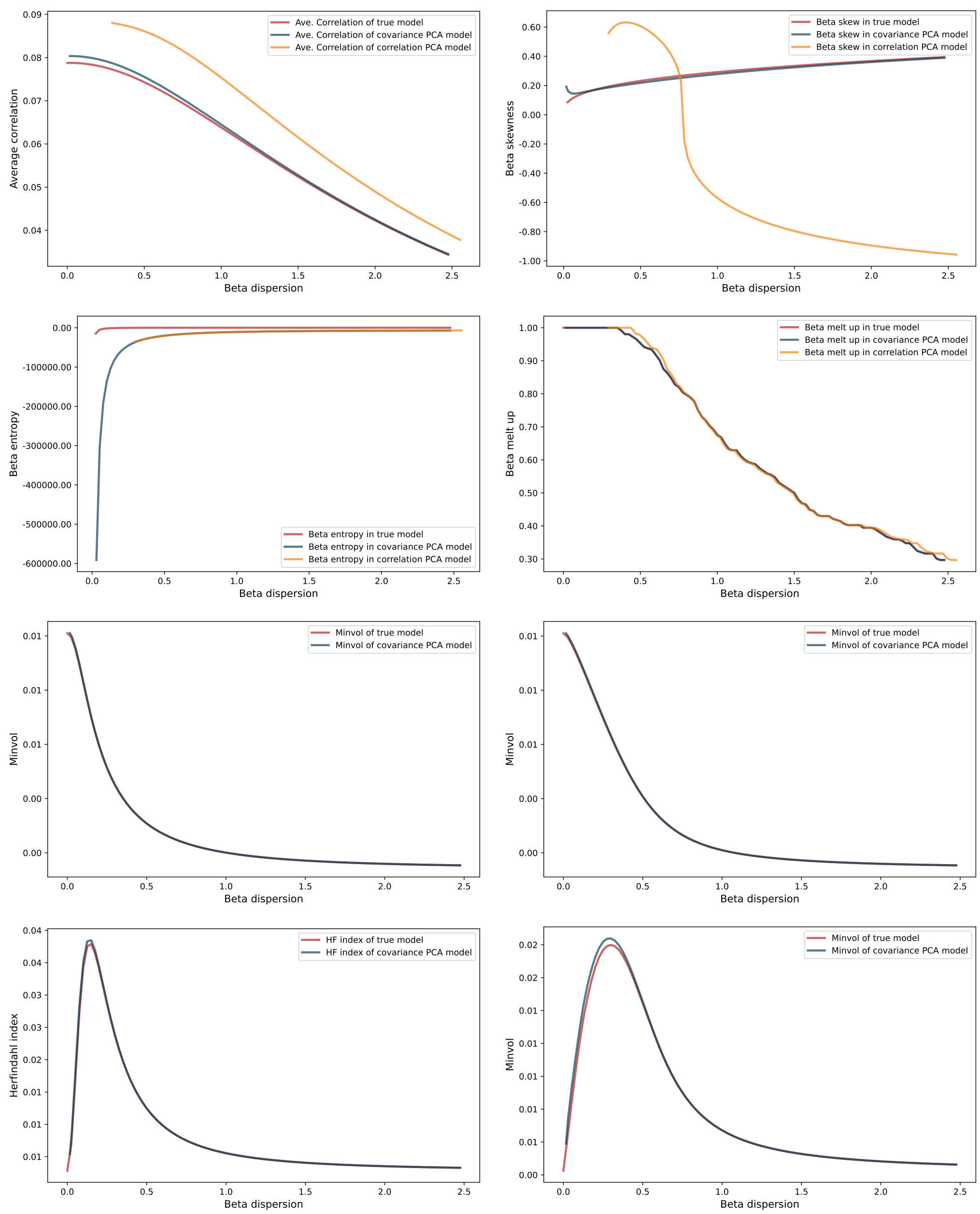

Figure 1. *Beta dispersion ($p = 512$) vs a.p. correlation, skewness, melt up, entropy, minimum volatility (long short and long only), and mininum volatility portfolio Herfindahl index (long short and long only).*

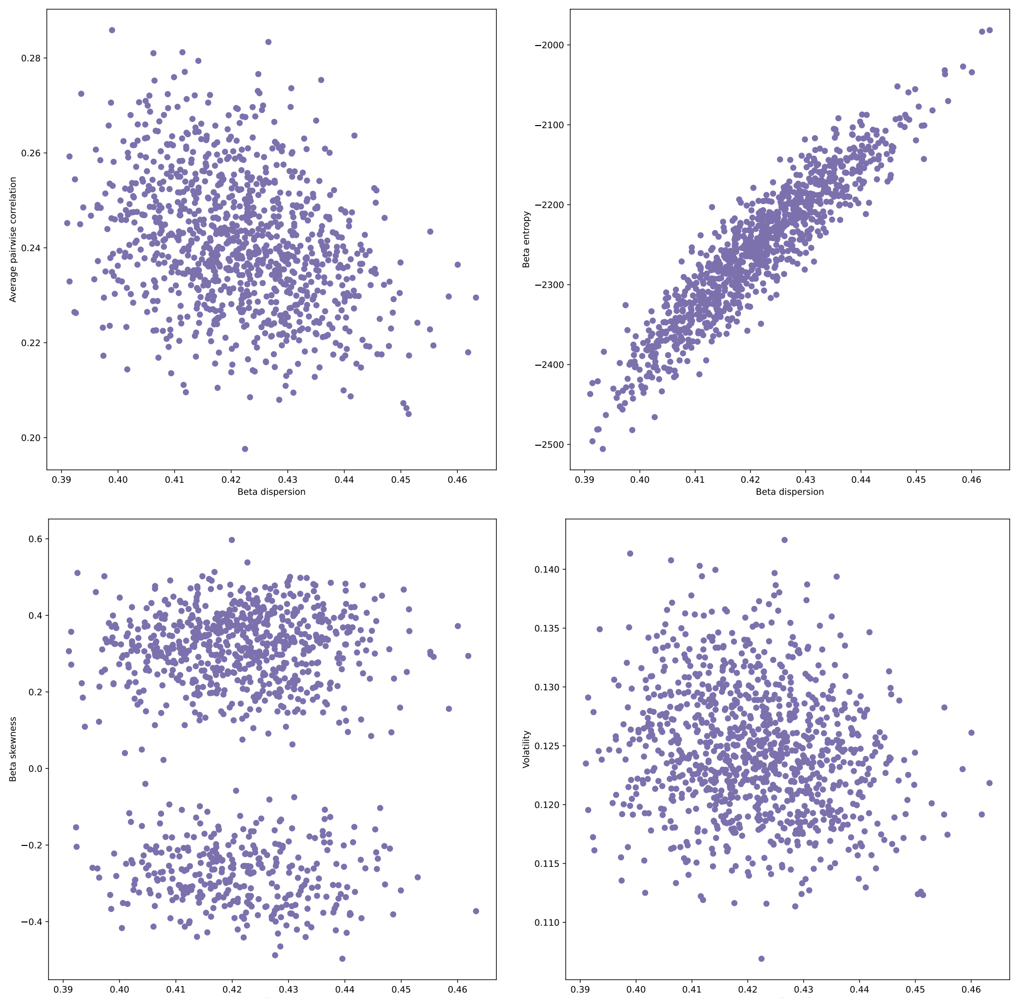

Figure 2. *Sample scatter plots for covariance PCA.* ($p = 512, n = 128$)*. Model* $(\mathrm{d}^2(\beta), \sigma_{\mathrm{M}}, ave(\delta)) = (0.4, 16, 0.4)$.

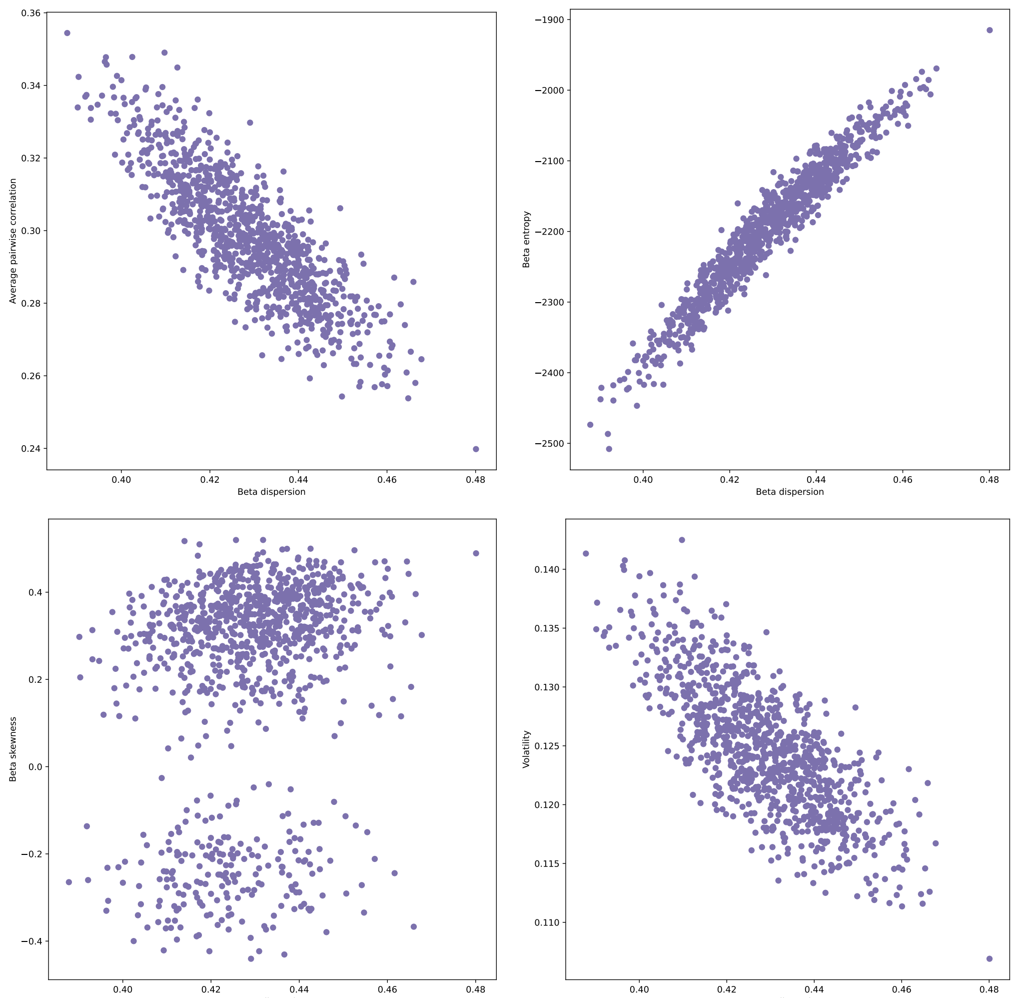

Figure 3. *Sample scatter plots for correlation PCA. (*$p = 512, n = 128$*). Model* $(\mathrm{d}^2(\beta), \sigma_{\mathrm{M}}, ave(\delta)) = (0.4, 16, 0.4)$.

6. Empirics

1. High beta dispersion is a sign of financial stress.
2. Mean beta is mean-reverting (business cycles).
3. In crisis all correlations **do not** go to one (anti-Markowitz).
4. Melt-up and beta dispersion are associated with market bubbles.
5. Skewness, entropy, etc.

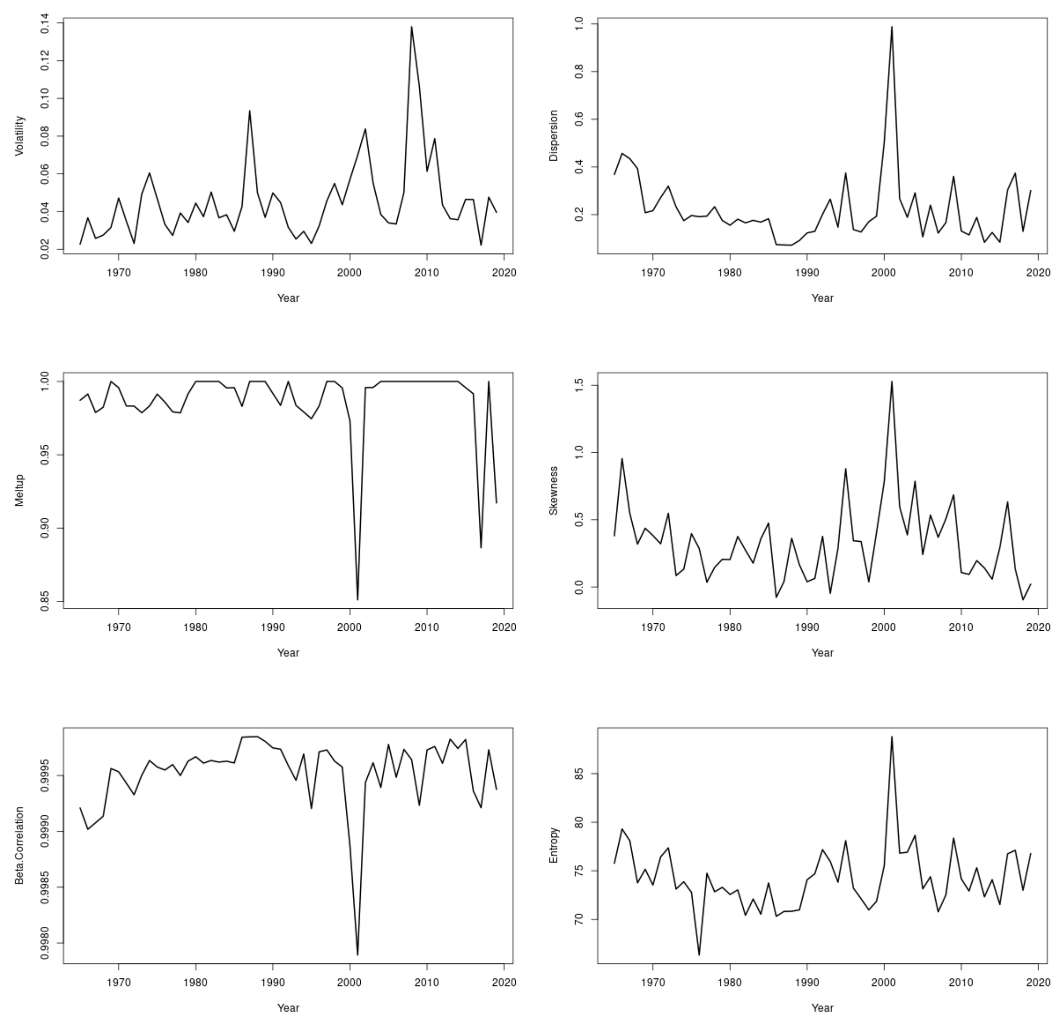

Figure 4. *S&P 500 data from 1965-2019.*

7. Regime models

8. Multi-index and graphical models

9. Asset return distributions

A. Beta dispersion vs Skew, Melt up, Correlation, Entropy plots

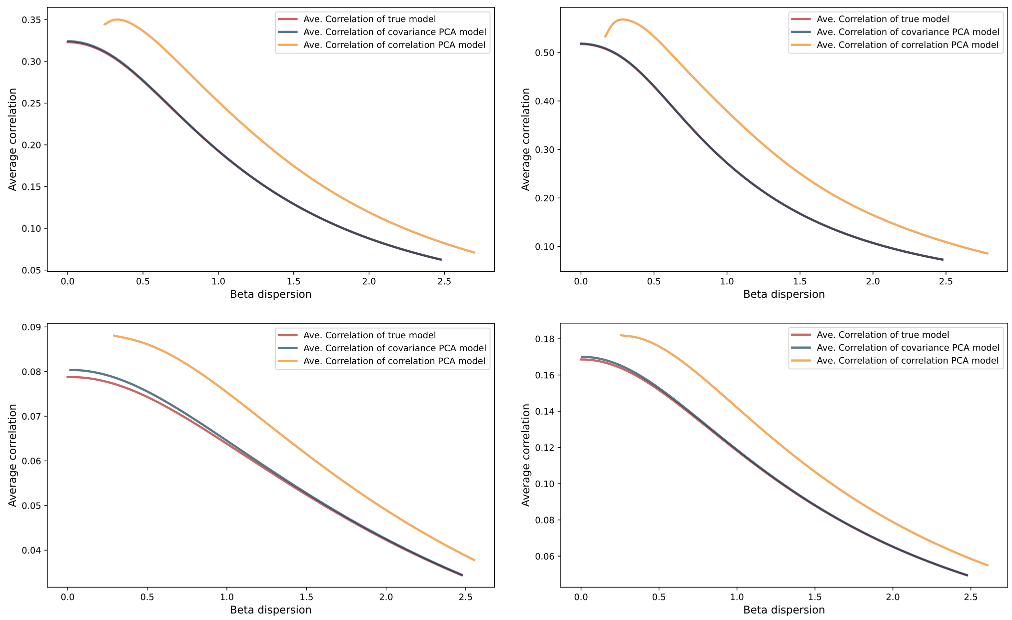

Figure 5. *Average pairwise correlation vs Beta dispersion* ($p = 512$). *Top left:* $(\sigma_{\mathrm{M}}, ave(\delta)) = (16, 25)$. *Top right:* $(\sigma_{\mathrm{M}}, ave(\delta)) = (25, 25)$. *Bottom left:* $(\sigma_{\mathrm{M}}, ave(\delta)) = (16, 60)$. *Bottom left:* $(\sigma_{\mathrm{M}}, ave(\delta)) = (25, 60)$. *Volatility units are percent annualized.*

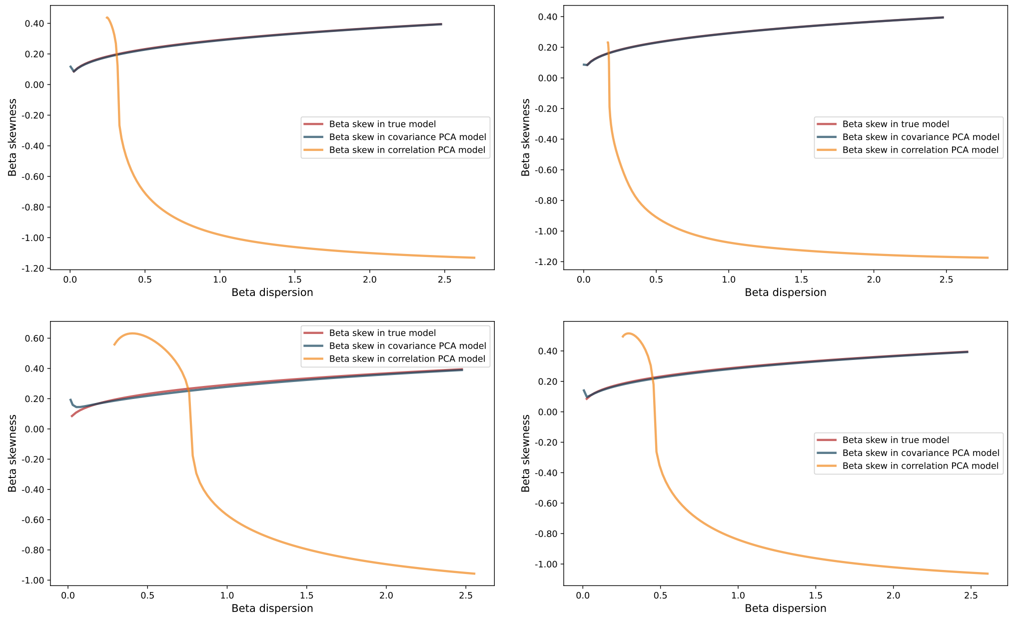

Figure 6. *Beta Skewness vs Beta dispersion ($p = 512$). Top left:* $(\sigma_{\mathrm{M}}, ave(\delta)) = (16, 25)$. *Top right:* $(\sigma_{\mathrm{M}}, ave(\delta)) = (25, 25)$. *Bottom left:* $(\sigma_{\mathrm{M}}, ave(\delta)) = (16, 60)$. *Bottom left:* $(\sigma_{\mathrm{M}}, ave(\delta)) = (25, 60)$. *Volatility units are percent annualized.*

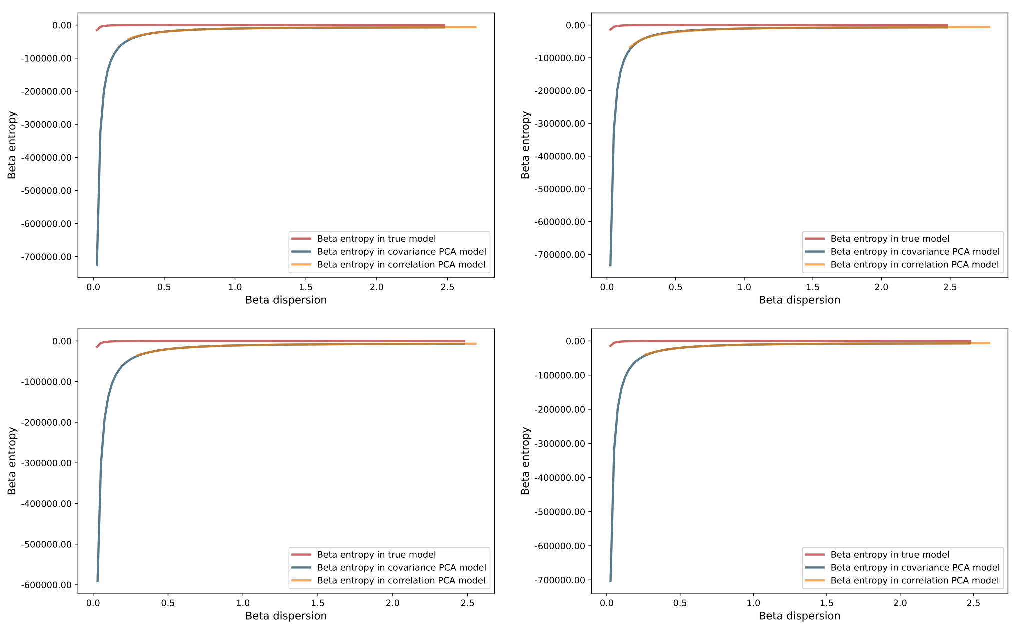

Figure 7. *Beta Skewness vs Beta dispersion ($p = 512$). Top left:* $(\sigma_{\mathrm{M}}, ave(\delta)) = (16, 25)$. *Top right:* $(\sigma_{\mathrm{M}}, ave(\delta)) = (25, 25)$. *Bottom left:* $(\sigma_{\mathrm{M}}, ave(\delta)) = (16, 60)$. *Bottom left:* $(\sigma_{\mathrm{M}}, ave(\delta)) = (25, 60)$. *Volatility units are percent annualized.*

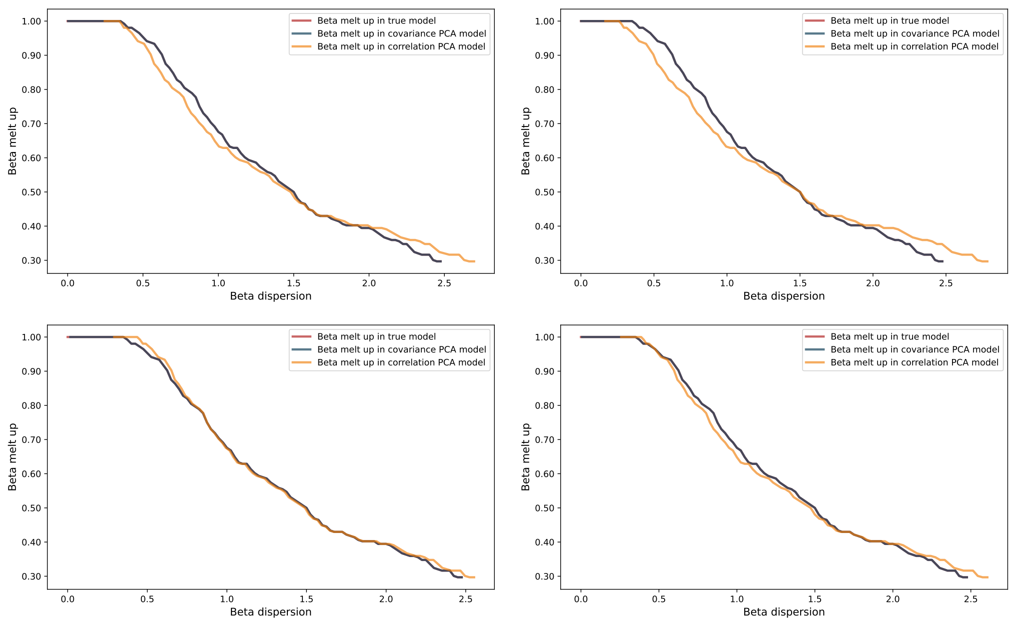

Figure 8. *Beta dispersion vs Beta Meltup* ($p = 512$)*. Top left:* $(\sigma_{\mathrm{M}}, ave(\delta)) = (16, 25)$. *Top right:* $(\sigma_{\mathrm{M}}, ave(\delta)) = (25, 25)$. *Bottom left:* $(\sigma_{\mathrm{M}}, ave(\delta)) = (16, 60)$. *Bottom left:* $(\sigma_{\mathrm{M}}, ave(\delta)) = (25, 60)$. *Volatility units are percent annualized.*