單元 4: 指數與對數

4.1 指數函數

$$f(x) = a^x, \quad a > 0, \quad a \neq 1$$

必定通過點 (0,1)

$a > 1$

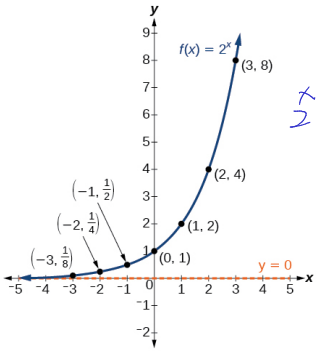
$\lim_{x\to\infty} a^x = \infty$ $\lim_{x\to-\infty} a^x = 0$

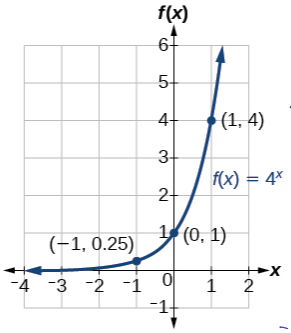
單調遞增函數

x 越右邊 , y 數值越大

定義域 $(-\infty, \infty)$

值域 $(0, \infty)$





$y = 0$ 為漸近線

$0 < a < 1$

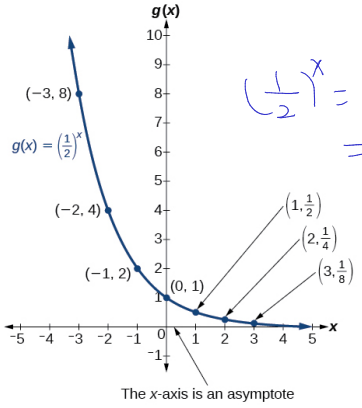
$\lim_{x\to\infty} a^x = 0$ $\lim_{x\to-\infty} a^x = \infty$

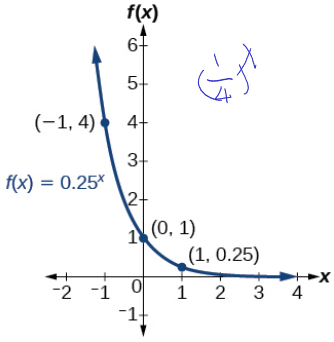
單調遞減函數

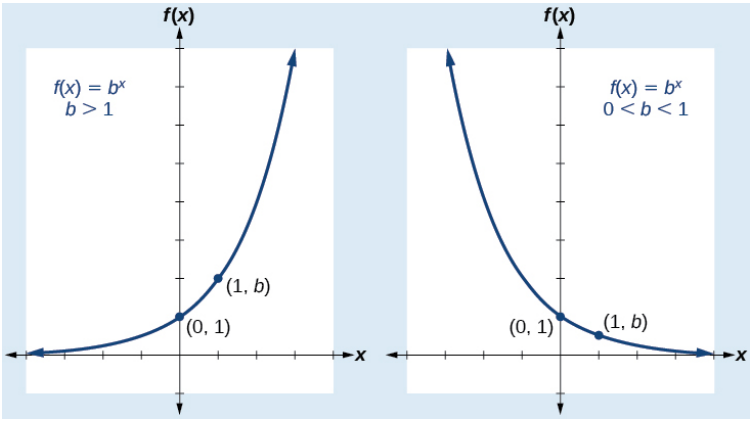
x 越右邊 , y 數值越小

定義域 $(-\infty, \infty)$

值域 $(0, \infty)$

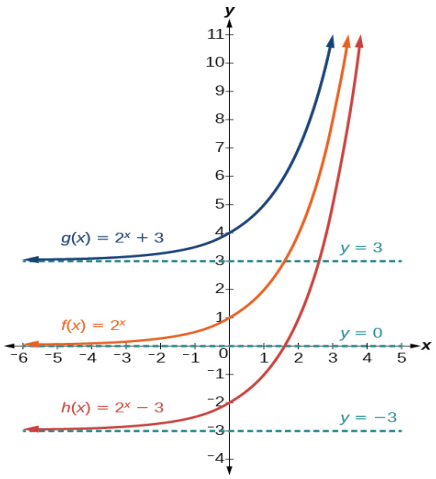






4.2 指數繪圖

上下位移

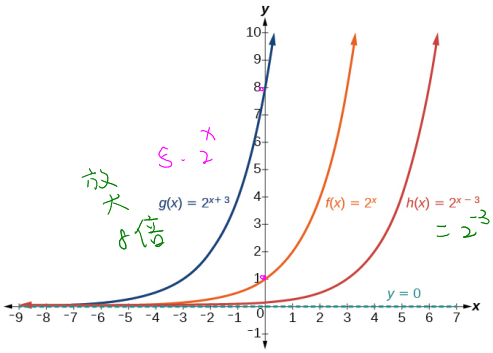


左右位移

2^x

2^{x+3} 往 左移 3格

2^{x-3} 往右移



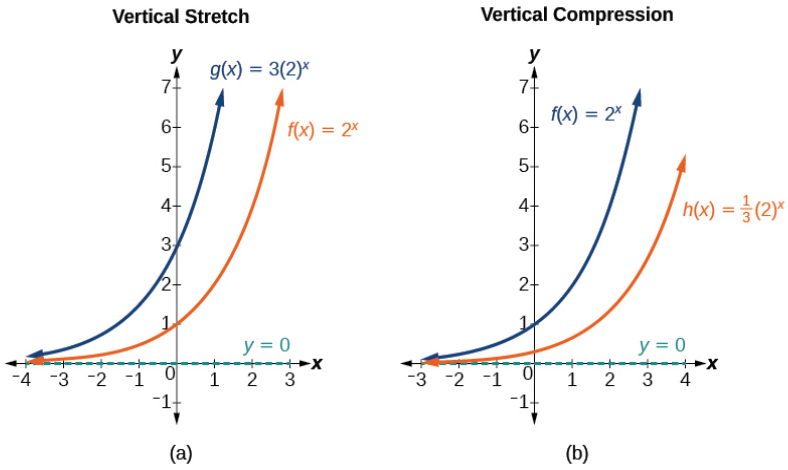
$2^{x+3} = 2^3 \cdot 2^x$

$= 8 \cdot 2^x$

放大8倍

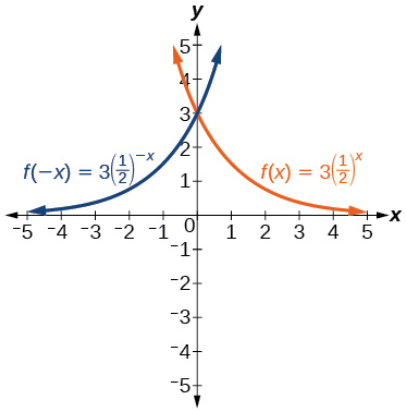
$= 2^{-3} \cdot 2^x = \frac{1}{8} 2^x$ 縮小成 $\frac{1}{8}$

垂直拉伸



底變倒數 ⇒ **指數變號**

$2^x \Rightarrow \left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}$



底變倒數

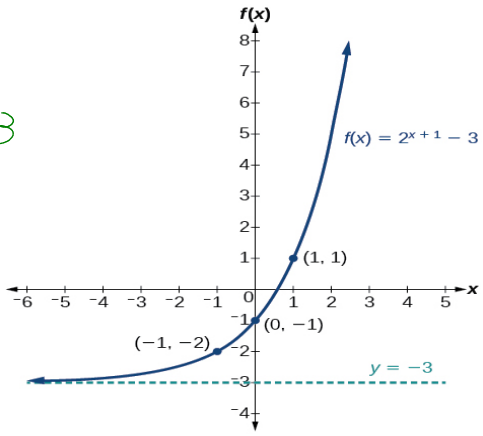
左右翻

即指數變號

例 1: 畫出 $y = 2^{x+1} - 3$ 的函數圖形

2^x 放大2倍 (左移1格) $\Rightarrow$ 2^{x+1}

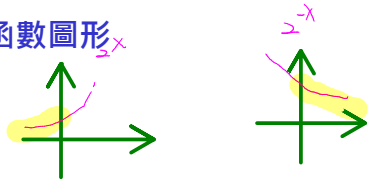
下移3格 $2^{x+1} - 3$



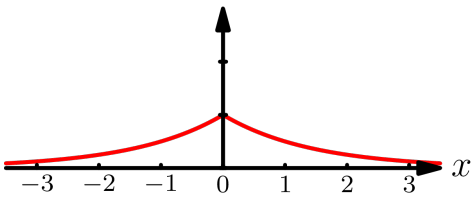
例 2: 畫出 $y = 2^{-|x|}$ 的函數圖形

$x \geq 0$ $y = 2^{-x}$

$x < 0$ $y = 2^{-(-x)} = 2^x$



$f(x) = \left(\frac{1}{2}\right)^{|x|} = 2^{-|x|}$



4.3 自然指數

尤拉數 e 為工程中常用無理數≈ 2.718281828

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots$

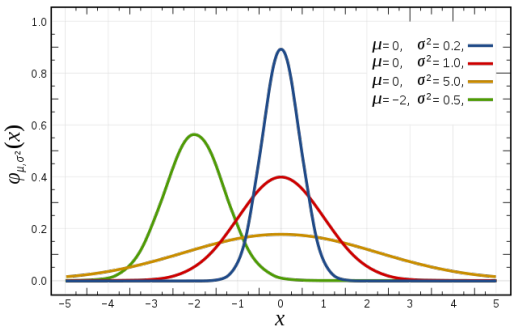
重要的應用

機率上的常態分佈

$$f_x(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

μ: 平均值

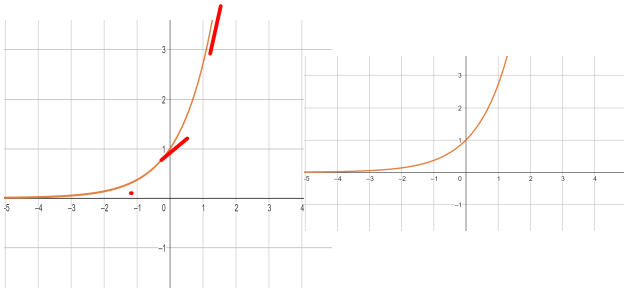
σ: 標準差



微積分

$$\frac{d}{dx} e^x = e^x$$

以e 為底的指數其微分為自己



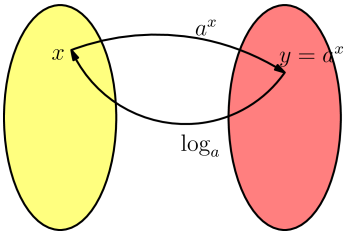
4.4 對數函數

對數與與指數互為反函數

$\log_b(c) = a$ means $b^a = c$

to

$$f(x) = \log_a x, \quad a > 0, \quad a \neq 1$$



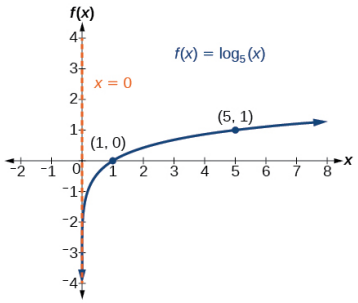
必定通過點 (1,0)

定義域 $(0, \infty)$ $x > 0$

值域 $(-\infty, \infty)$

$a > 1$

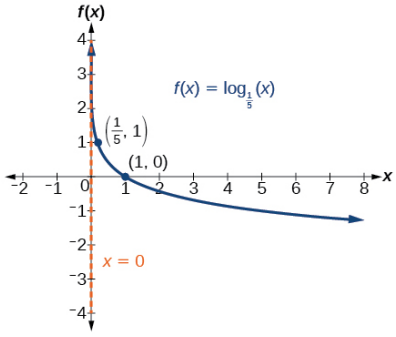
$\lim_{x\to\infty} \log_a x = \infty$ $\lim_{x\to 0^+} \log_a x = -\infty$



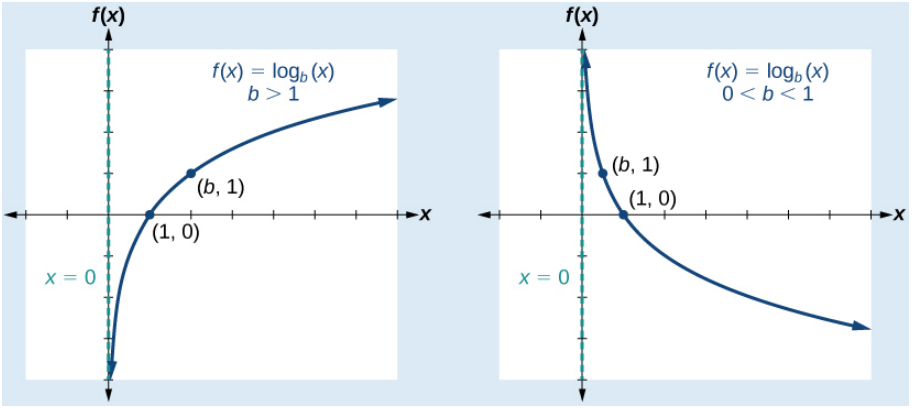
單調遞增函數

$0 < a < 1$

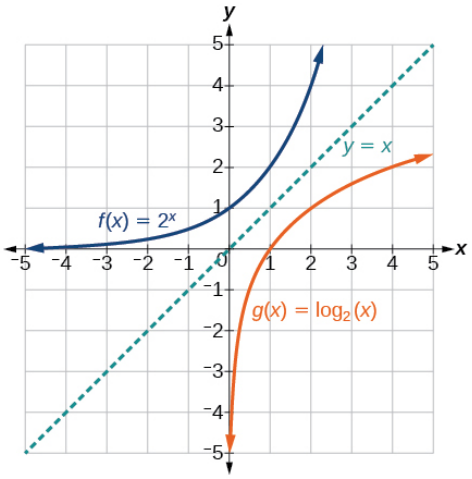
$\lim_{x\to\infty} \log_a x = -\infty$ $\lim_{x\to 0^+} \log_a x = \infty$



單調遞減函數



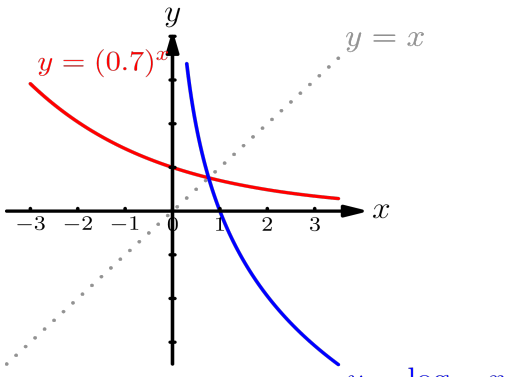
對數與與指數互為反函數



$\log_a x$

$a > 1$

指數與對數函數對稱於直線 y=x

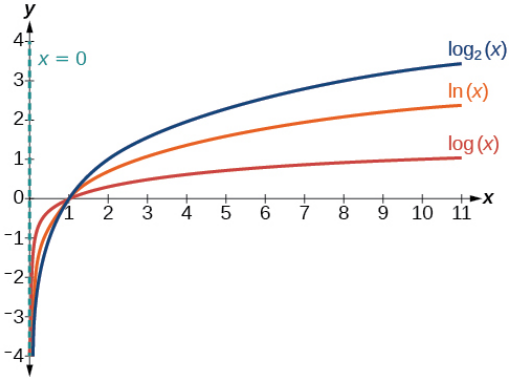


$0 < a < 1$

4.5 對數函數繪圖

$\ln(x) = \log_e x$

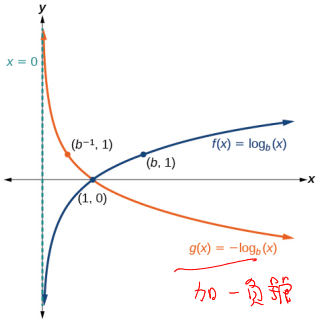
$\log(x) = \log_{10}(x)$



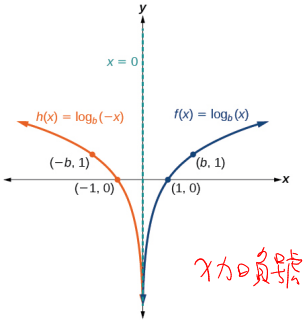
$g(x) = -\log_b(x)$

$= \log_b x^{-1}$

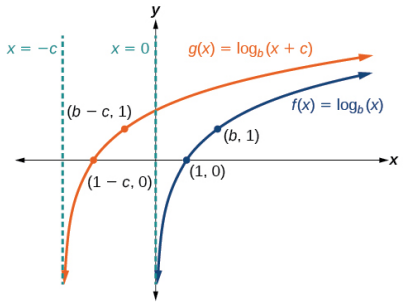
$= \log_b \frac{1}{x}$

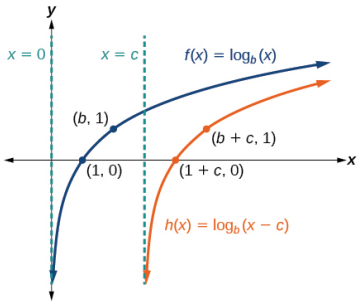


加一負號



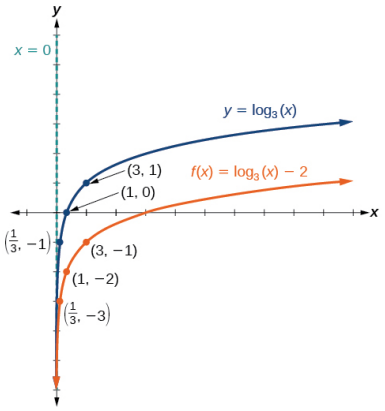
x加負號





$\log_3(x) - 2 = \log_3(x) - \log_3 9$

$= \log_3\left(\frac{x}{9}\right)$



例 1: 畫出 $y = \log_2(2x)$ 的函數圖形

$\log_2(2x) = \log_2(2) + \log_2(x)$

$= 1 + \log_2(x)$

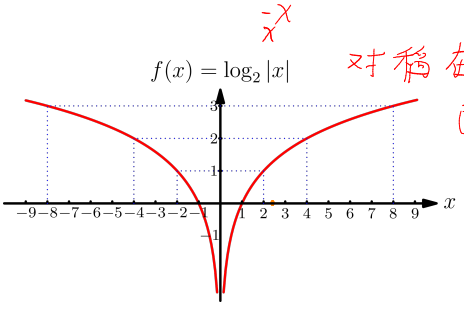
例 2: 畫出 $y = \log_2|x|$ 的函數圖形

$x \geq 0$ $\log_2 x$

$x < 0$ $\log_2(-x)$

畫一回

另一对对稱



對稱在 x=0

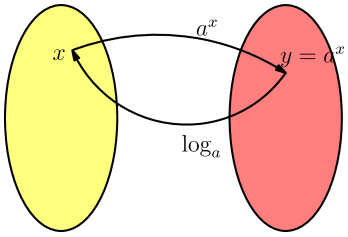
即 y軸

4.6 指數與對數的換底公式

指數換底

$$a^x = b^{x \log_b a}$$

底為a 變成底為 b



a^x 經過底數b的對數再經過指數b的對數

$$a^x = b^{\log_b a^x}$$

$2^x = e^{(\ln 2)x}$

$\frac{d}{dx}[2^x] = e^{(\ln 2)\cdot x} \cdot \frac{d}{dx}[(\ln 2)x]$

$= (e^{\ln 2})^x \cdot \ln 2$

$= 2^x \cdot \ln 2$

對數換底

$$\log_a M = \frac{\log_b M}{\log_b a}$$

$$x = \log_a M \quad \Rightarrow \quad a^x = M$$

兩邊取對數b $\quad \log_b a^x = \log_b M$

$$x \log_b a = \log_b M \quad \Rightarrow \quad x = \frac{\log_b M}{\log_a M}$$

4.7 指數與對數的微積分

指數微分

$$\frac{d}{dx} e^x = e^x$$

$\lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x}$

$$\frac{d}{dx} a^x = (\ln a)\, a^x$$

手癢度

$a^x = e^{(\ln a)\cdot x}$

$\frac{d}{dx}[a^x] = e^{(\ln a)\cdot x} \cdot \frac{d}{dx}[(\ln a)x]$

$= a^x \cdot \ln a$

對數微分

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

$$\frac{d}{dx} \log_a x = \frac{1}{\ln a} \cdot \frac{1}{x}$$

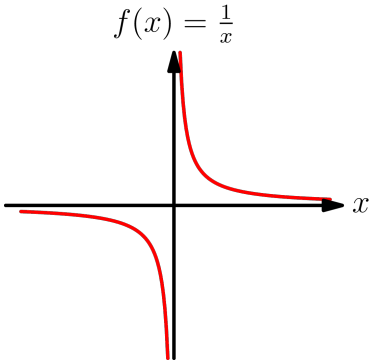
$\log_a x = \frac{\ln x}{\ln a}$

$y = \ln x$

$e^y = x$ 兩邊微分

$e^y dy = dx$

$\frac{dy}{dx} = e^{-y} = \frac{1}{e^y} = \frac{1}{x}$



指數積分

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{1}{\ln a} a^x + C$$

積分結果為對數

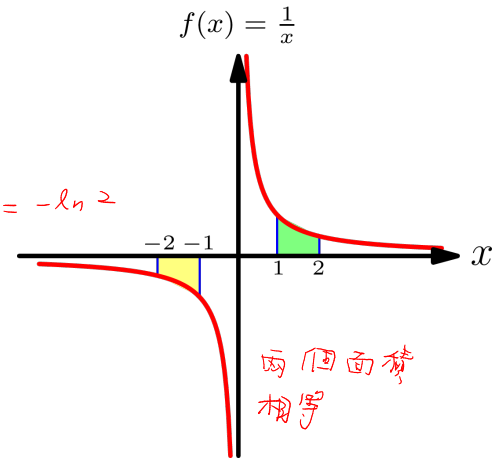
$$\int \frac{1}{x} dx = \ln|x| + C$$

y=1/x 的積分為對數

$$\int_2^3 \frac{1}{x} dx = \left[\ln x\right]_1^2 = \ln 2 - \ln 1 = \ln 2$$

$$\int_{-3}^{-2} \frac{1}{x} dx = \left[\ln|x|\right]_{-2}^{-1} = \ln 1 - \ln 2 = -\ln 2$$

倒數未限制為正



對數積分

$\int \ln x \, dx$ 分部積分 $\int u\,dv = uv - \int v\,du$

$u = \ln x \quad du = \frac{1}{x} dx$

$dv = dx \quad v = x$

$x \cdot \ln x - \int x \cdot \frac{1}{x} dx$

$= x \cdot \ln x - x + C$