Change of Basis

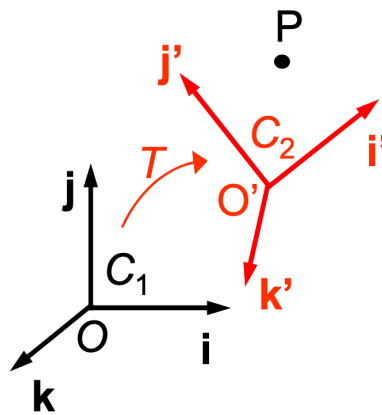**Recall**: (Coordinate Systems) In homogeneous coordinate systems, a vector is denoted as:

$$v = v_1 a + v_2 b + v_3 c \rightarrow v = \begin{bmatrix} a & b & c & O \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}$$

A point is

$$P = p_1 a + p_2 b + p_3 c + O \rightarrow P = \begin{bmatrix} a & b & c & O \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix}$$

Transformations as a change of basis

Question: if we apply a transformation, M_1, to coordinate system C_1, and get C_2, what is the new position of P in C_2?

We use $[x, y, z, 1]^T$ to represent P's posotion in C_1, and $[x', y', z', 1]^T$ to represent P's posotion in C_2.

Solution:

In coordinate system C_1:

$$P = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} + O = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & O \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

In coordinate system C_2:

$$P = x'\boldsymbol{i}' + y'\boldsymbol{j}' + z'\boldsymbol{k}' + O' = \begin{bmatrix} \boldsymbol{i}' & \boldsymbol{j}' & \boldsymbol{k}' & O' \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

Since C_2 is transformed from C_1 with M_1,

Label 'eq:m4' multiply defined

Hence

$$\begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & O \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = M_1 \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & O \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

Since $\begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & O \end{bmatrix}$ is the <u>identity matrix</u>, we have

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = M_1 \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} \rightarrow P_{C_1} = M_1 P_{C_2}$$

I.e., $P_{C_2} = M_1^{-1} P_{C_1}$

Question: what is M_1 with respect to the basis vectors?

Referring to Eq. (???), and since $\begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & O \end{bmatrix}$ is the identity matrix:

$$M_1 = \begin{bmatrix} \boldsymbol{i}' & \boldsymbol{j}' & \boldsymbol{k}' & O' \end{bmatrix} = \begin{bmatrix} i'_x & j'_x & k'_x & O'_x \\ i'_y & j'_y & k'_y & O'_y \\ i'_z & j'_z & k'_z & O'_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Successive transformations

If we transform C_2 to C_3 with M_2. We cansider the $\begin{bmatrix} \boldsymbol{i'} & \boldsymbol{j'} & \boldsymbol{k'} & O' \end{bmatrix}$ as the identity matrix in the section transforamtion, then,

$$P_{C_2} = M_2 P_{C_3}$$

Therefore,

$$P_{C_1} = M_1 P_{C_2} = M_1 M_2 P_{C_3}$$

Note: $\begin{bmatrix} \boldsymbol{i''} & \boldsymbol{j''} & \boldsymbol{k''} & O'' \end{bmatrix}$ of C_3 is the matrix representation in the C_2.

Coordinate system transformation with rotation and translation
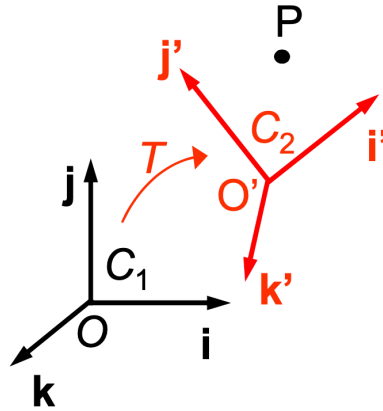
Since the $\boldsymbol{i'}$, $\boldsymbol{j'}$, and $\boldsymbol{k'}$ are orthoginal, the rotation from C_1 to C_2 is

$$R = \begin{bmatrix} i'_x & j'_x & k'_x & 0 \\ i'_y & j'_y & k'_y & 0 \\ i'_z & j'_z & k'_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The translation from O to O' is

$$T = \begin{bmatrix} 1 & 0 & 0 & O'_x \\ 0 & 1 & 0 & O'_y \\ 0 & 0 & 1 & O'_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus,

$$M_1 = TR = \begin{bmatrix} i'_x & j'_x & k'_x & O'_x \\ i'_y & j'_y & k'_y & O'_y \\ i'_z & j'_z & k'_z & O'_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

P's posotion in C_2, $[x', y', z', 1]^T$, can be calculated with the inverse of the transformations

$$P_{C_2} = M_1^{-1} P_{C_1} = R^{-1} T^{-1} P_{C_1} = \begin{bmatrix} i'_x & j'_x & k'_x & 0 \\ i'_y & j'_y & k'_y & 0 \\ i'_z & j'_z & k'_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & -O'_x \\ 0 & 1 & 0 & -O'_y \\ 0 & 0 & 1 & -O'_z \\ 0 & 0 & 0 & 1 \end{bmatrix} P_{C_1}$$

Model-View Transformation

Given eye point P_{eye}, reference point P_{ref}, and up vector $\boldsymbol{v}_{up}$, build M_{cam} for model-view transformation.

$$\boldsymbol{k}' = \frac{P_{eye} - P_{ref}}{|P_{eye} - P_{ref}|}$$
$$\boldsymbol{i}' = \frac{\boldsymbol{v}_{up} \times \boldsymbol{k}'}{|\boldsymbol{v}_{up} \times \boldsymbol{k}'|}$$
$$\boldsymbol{j}' = \boldsymbol{k}' \times \boldsymbol{i}'$$

The new origin $O' = P_{eye}$. Thus,

$$M_{cam} = \begin{bmatrix} i'_x & j'_x & k'_x & O'_x \\ i'_y & j'_y & k'_y & O'_y \\ i'_z & j'_z & k'_z & O'_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transform from WCS (word coordinate system) to VCS (view coordinate system)

$$P_{vcs} = M_{cam}^{-1} P_{wcs} = \begin{bmatrix} i'_x & j'_x & k'_x & 0 \\ i'_y & j'_y & k'_y & 0 \\ i'_z & j'_z & k'_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & -O'_x \\ 0 & 1 & 0 & -O'_y \\ 0 & 0 & 1 & -O'_z \\ 0 & 0 & 0 & 1 \end{bmatrix} P_{wcs}$$