Change of Basis

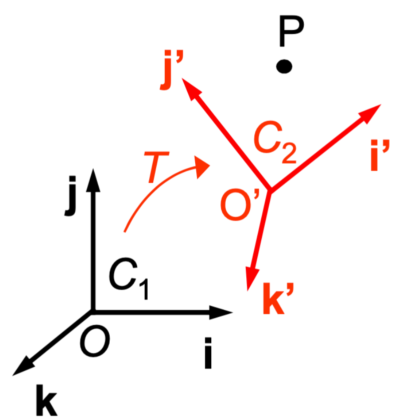
Recall: (Coordinate Systems) In homogeneous coordinate systems, a vector is denoted as:

$$\boldsymbol{v} = v_1\boldsymbol{a} + v_2\boldsymbol{b} + v_3\boldsymbol{c} \rightarrow \boldsymbol{v} = [\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{c} \quad O]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}$$

A point is

$$P = p_1\boldsymbol{a} + p_2\boldsymbol{b} + p_3\boldsymbol{c} + O \rightarrow P = [\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{c} \quad O]\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix}$$

Transformations as a change of basis



Question: if we apply a transformation, M_1, to coordinate system C_1, and get C_2, what is the new position of P in C_2?

We use $[x, y, z, 1]^T$ to represent P's posotion in C_1, and $[x', y', z', 1]^T$ to represent P's posotion in C_2.

Solution:

In coordinate system C_1:

$$P = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} + O = [\boldsymbol{i} \quad \boldsymbol{j} \quad \boldsymbol{k} \quad O]\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

In coordinate system C_2:

$$P = x'\boldsymbol{i}' + y'\boldsymbol{j}' + z'\boldsymbol{k}' + O' = [\,\boldsymbol{i}' \quad \boldsymbol{j}' \quad \boldsymbol{k}' \quad O'\,] \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

Since C_2 is transformed from C_1 with M_1,

$$M_1\,[\,\boldsymbol{i} \quad \boldsymbol{j} \quad \boldsymbol{k} \quad O\,] = [\,\boldsymbol{i}' \quad \boldsymbol{j}' \quad \boldsymbol{k}' \quad O'\,] \tag{*1}$$

Hence

$$[\,\boldsymbol{i} \quad \boldsymbol{j} \quad \boldsymbol{k} \quad O\,] \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = M_1\,[\,\boldsymbol{i} \quad \boldsymbol{j} \quad \boldsymbol{k} \quad O\,] \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

Since $[\,\boldsymbol{i} \quad \boldsymbol{j} \quad \boldsymbol{k} \quad O\,]$ is the identity matrix, we have

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = M_1 \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} \rightarrow P_{C_1} = M_1 P_{C_2}$$

I.e., $P_{C_2} = M_1^{-1} P_{C_1}$

Question: what is M_1 with respect to the basis vectors?

Referring to Eq. $(???)$, and since $[\,\boldsymbol{i} \quad \boldsymbol{j} \quad \boldsymbol{k} \quad O\,]$ is the identity matrix:

$$M_1 = [\,\boldsymbol{i}' \quad \boldsymbol{j}' \quad \boldsymbol{k}' \quad O'\,] = \begin{bmatrix} i'_x & j'_x & k'_x & O'_x \\ i'_y & j'_y & k'_y & O'_y \\ i'_z & j'_z & k'_z & O'_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Successive transformations

If we transform C_2 to C_3 with M_2. We cansider the $[\,\boldsymbol{i}' \quad \boldsymbol{j}' \quad \boldsymbol{k}' \quad O'\,]$ as the identity matrix in the section transforamtion, then,

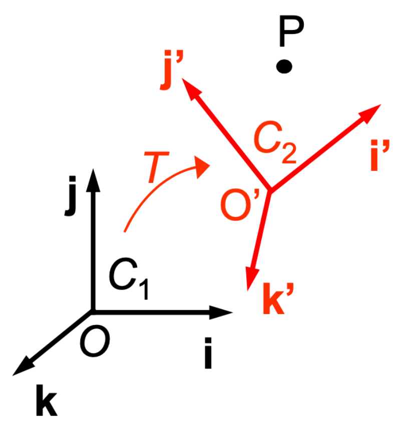
$$P_{C_2} = M_2 P_{C_3}$$

Therefore,

$$P_{C_1} = M_1 P_{C_2} = M_1 M_2 P_{C_3}$$

Note: $[\,\boldsymbol{i}'' \quad \boldsymbol{j}'' \quad \boldsymbol{k}'' \quad O''\,]$ of C_3 is the matrix representation in the C_2.

Coordinate system transformation with rotation and translation



Since the $\boldsymbol{i}'$, $\boldsymbol{j}'$, and $\boldsymbol{k}'$ are orthoginal, the rotation from C_1 to C_2 is

$$R = \begin{bmatrix} i'_x & j'_x & k'_x & 0 \\ i'_y & j'_y & k'_y & 0 \\ i'_z & j'_z & k'_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The translation from O to O' is

$$T = \begin{bmatrix} 1 & 0 & 0 & O'_x \\ 0 & 1 & 0 & O'_y \\ 0 & 0 & 1 & O'_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus,

$$M_1 = TR = \begin{bmatrix} i'_x & j'_x & k'_x & O'_x \\ i'_y & j'_y & k'_y & O'_y \\ i'_z & j'_z & k'_z & O'_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

P's posotion in C_2, $[x', y', z', 1]^T$, can be calculated with the inverse of the transformations

$$P_{C_2} = M_1^{-1} P_{C_1} = R^{-1} T^{-1} P_{C_1} = \begin{bmatrix} i'_x & j'_x & k'_x & 0 \\ i'_y & j'_y & k'_y & 0 \\ i'_z & j'_z & k'_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & -O'_x \\ 0 & 1 & 0 & -O'_y \\ 0 & 0 & 1 & -O'_z \\ 0 & 0 & 0 & 1 \end{bmatrix} P_{C_1}$$

Model-View Transformation

Given eye point P_{eye}, reference point P_{ref}, and up vector $\boldsymbol{v}_{up}$, build M_{cam} for model-view transformation.

$$
\begin{aligned}
\boldsymbol{k}' &= \frac{P_{eye} - P_{ref}}{|P_{eye} - P_{ref}|} \\
\boldsymbol{i}' &= \frac{\boldsymbol{v}_{up} \times \boldsymbol{k}'}{|\boldsymbol{v}_{up} \times \boldsymbol{k}'|} \\
\boldsymbol{j}' &= \boldsymbol{k}' \times \boldsymbol{i}'
\end{aligned}
$$

The new origin $O' = P_{eye}$. Thus,

$$
M_{cam} = \begin{bmatrix} i'_x & j'_x & k'_x & O'_x \\ i'_y & j'_y & k'_y & O'_y \\ i'_z & j'_z & k'_z & O'_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Transform from WCS (word coordinate system) to VCS (view coordinate system)

$$
P_{vcs} = M_{cam}^{-1} P_{wcs} = \begin{bmatrix} i'_x & j'_x & k'_x & 0 \\ i'_y & j'_y & k'_y & 0 \\ i'_z & j'_z & k'_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & -O'_x \\ 0 & 1 & 0 & -O'_y \\ 0 & 0 & 1 & -O'_z \\ 0 & 0 & 0 & 1 \end{bmatrix} P_{wcs}
$$