# **Change of Basis**

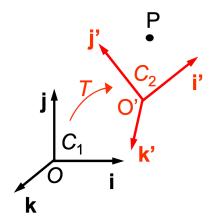
**Recall**: (Coordinate Systems) In homogeneous coordinate systems, a vector is denoted as:

$$egin{aligned} oldsymbol{v} = v_1 oldsymbol{a} + v_2 oldsymbol{b} + v_3 oldsymbol{c} 
ightarrow oldsymbol{v} = egin{bmatrix} oldsymbol{a} & oldsymbol{b} & oldsymbol{c} & O \end{bmatrix} egin{bmatrix} v_1 \ v_2 \ v_3 \ 0 \end{bmatrix} \end{aligned}$$

A point is

$$P=p_1oldsymbol{a}+p_2oldsymbol{b}+p_3oldsymbol{c}+O o P=\left[oldsymbol{a} egin{array}{cccc} oldsymbol{a} & oldsymbol{b} & oldsymbol{c} & O\end{array}
ight] egin{bmatrix} p_1 \ p_2 \ p_3 \ 1 \end{array}$$

## Transformations as a change of basis



**Question**: if we apply a transformation,  $M_1$ , to coordinate system  $C_1$ , and get  $C_2$ , what is the new position of P in  $C_2$ ?

We use  $[x,y,z,1]^T$  to represent P's posotion in  $C_1$ , and  $[x',y',z',1]^T$  to represent P's posotion in  $C_2$ .

#### Solution:

In coordinate system  $C_1$ :

$$P = x oldsymbol{i} + y oldsymbol{j} + z oldsymbol{k} + O = \left[ oldsymbol{i} \quad oldsymbol{j} \quad oldsymbol{k} \quad O 
ight] egin{bmatrix} x \ y \ z \ 1 \end{bmatrix}$$

In coordinate system  $C_2$ :

$$P = x'oldsymbol{i}' + y'oldsymbol{j}' + z'oldsymbol{k}' + O' = [oldsymbol{i}' \quad oldsymbol{j}' \quad oldsymbol{k}' \quad O' ] egin{bmatrix} x' \ y' \ z' \ 1 \end{bmatrix}$$

Since  $C_2$  is transformed from  $C_1$  with  $M_1$ ,

$$M_1 \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & O \end{bmatrix} = \begin{bmatrix} \mathbf{i}' & \mathbf{j}' & \mathbf{k}' & O' \end{bmatrix}$$
 (\*1)

Hence

$$egin{bmatrix} \left[m{i} & m{j} & m{k} & O 
ight] egin{bmatrix} x \ y \ z \ 1 \end{bmatrix} = M_1 \left[m{i} & m{j} & m{k} & O 
ight] egin{bmatrix} x' \ y' \ z' \ 1 \end{bmatrix}$$

Since  $\begin{bmatrix} i & j & k & O \end{bmatrix}$  is the <u>identity matrix</u>, we have

$$egin{bmatrix} x \ y \ z \ 1 \end{bmatrix} = M_1 egin{bmatrix} x' \ y' \ z' \ 1 \end{bmatrix} 
ightarrow P_{C_1} = M_1 P_{C_2}$$

l.e.,  $P_{C_2} = M_1^{-1} P_{C_1}$ 

**Question**: what is  $M_1$  with respect to the basis vectors?

Referring to Eq. (???), and since  $\begin{bmatrix} i & j & k & O \end{bmatrix}$  is the identity matrix:

$$M_1 = \left[egin{array}{cccc} m{i}' & m{j}' & m{k}' & O' \end{array}
ight] = \left[egin{array}{cccc} i_x' & j_x' & k_x' & O_x' \ i_y' & j_y' & k_y' & O_y' \ i_z' & j_z' & k_z' & O_z' \ 0 & 0 & 0 & 1 \end{array}
ight]$$

### **Successive transformations**

If we transform  $C_2$  to  $C_3$  with  $M_2$ . We cansider the  $[i' \ j' \ k' \ O']$  as the identity matrix in the section transforamtion, then,

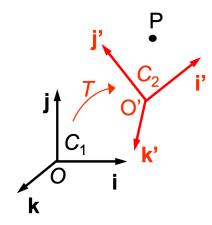
$$P_{C_2} = M_2 P_{C_3}$$

Therefore,

$$P_{C_1} = M_1 P_{C_2} = M_1 M_2 P_{C_3}$$

*Note*:  $[i'' \quad j'' \quad k'' \quad O'']$  of  $C_3$  is the matrix representation in the  $C_2$ .

### Coordinate system transformation with rotation and translation



Since the  $m{i}'$  ,  $m{j}'$  , and  $m{k}'$  are orthoginal, the rotation from  $C_1$  to  $C_2$  is

$$R = egin{bmatrix} i'_x & j'_x & k'_x & 0 \ i'_y & j'_y & k'_y & 0 \ i'_z & j'_z & k'_z & 0 \ 0 & 0 & 0 & 1 \end{bmatrix}$$

The translation from O to O' is

$$T = egin{bmatrix} 1 & 0 & 0 & O_x' \ 0 & 1 & 0 & O_y' \ 0 & 0 & 1 & O_z' \ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus,

$$M_1 = TR = egin{bmatrix} i_x' & j_x' & k_x' & O_x' \ i_y' & j_y' & k_y' & O_y' \ i_z' & j_z' & k_z' & O_z' \ 0 & 0 & 0 & 1 \end{bmatrix}$$

P's posotion in  $C_2$  ,  $[x^\prime,y^\prime,z^\prime,1]^T$  , can be calculated with the inverse of the transformations

$$P_{C_2} = M_1^{-1} P_{C_1} = R^{-1} T^{-1} P_{C_1} = egin{bmatrix} i_x' & j_x' & k_x' & 0 \ i_y' & j_y' & k_y' & 0 \ i_z' & j_z' & k_z' & 0 \ 0 & 0 & 0 & 1 \end{bmatrix}^T egin{bmatrix} 1 & 0 & 0 & -O_x' \ 0 & 1 & 0 & -O_y' \ 0 & 0 & 1 & -O_z' \ 0 & 0 & 0 & 1 \end{bmatrix} P_{C_1}$$

## **Model-View Transformation**

Given eye point  $P_{eye}$ , reference point  $P_{ref}$ , and up vector  $\mathbf{v}_{up}$ , build  $M_{cam}$  for model-view transformation.

$$egin{aligned} oldsymbol{k}' &= rac{P_{eye} - P_{ref}}{|P_{eye} - P_{ref}|} \ oldsymbol{i}' &= rac{oldsymbol{v}_{up} imes oldsymbol{k}'}{|oldsymbol{v}_{up} imes oldsymbol{k}'|} \ oldsymbol{j}' &= oldsymbol{k}' imes oldsymbol{i}' \end{aligned}$$

The new origin  $O'=P_{eye}$ . Thus,

$$M_{cam} = egin{bmatrix} i'_x & j'_x & k'_x & O'_x \ i'_y & j'_y & k'_y & O'_y \ i'_z & j'_z & k'_z & O'_z \ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transform from WCS (word coordinate system) to VCS (view coordinate system)

$$P_{vcs} = M_{cam}^{-1} P_{wcs} = egin{bmatrix} i'_x & j'_x & k'_x & 0 \ i'_y & j'_y & k'_y & 0 \ i'_z & j'_z & k'_z & 0 \ 0 & 0 & 0 & 1 \end{bmatrix}^T egin{bmatrix} 1 & 0 & 0 & -O'_x \ 0 & 1 & 0 & -O'_y \ 0 & 0 & 1 & -O'_z \ 0 & 0 & 0 & 1 \end{bmatrix} P_{wcs}$$