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Upper bounds for some graph energies

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ABSTRACT

A general inequality for non-negative real numbers is proven. Based on it, upper bounds for (ordinary) graph energy, minimum dominating energy, minimum covering energy, Laplacian-energy-like invariant, Laplacian energy, Randić energy, and incidence energy are obtained.

1. Introduction

Let $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$, be a simple graph with $n = |V|$ vertices and $m = |E|$ edges. If the vertices v_i and v_j are adjacent, then we write $v_i \sim v_j$. Denote by d_i the degree (number of first neighbors) of the vertex v_i, and assume that $d_1 \geq d_2 \geq \cdots \geq d_n$. Some well known properties of the vertex degrees are [7]:

$$\sum_{i=1}^{n} d_i = 2m \qquad \text{and} \qquad \sum_{i=1}^{n} d_i^2 = \sum_{v_i \sim v_j} (d_i + d_j) = M_1$$

where M_1 is the first Zagreb index [17,21].

The adjacency matrix $\mathbf{A} = (a_{ij})$ of the graph G is defined as

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of $\mathbf{A}$, denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, are referred to as the (ordinary) eigenvalues of the graph G [7]. Some of their well known properties are:

$$\sum_{i=1}^{n} \lambda_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n} d_i = 2m.$$

Denote by $|\lambda_1^*| \geq |\lambda_2^*| \geq \cdots \geq |\lambda_n^*|$ a non-increasing sequence of absolute values of the eigenvalues of G. The graph invariant $E = E(G)$, called energy of G, is defined to be the sum of the absolute values of the eigenvalues of G [20,27], i.e.,

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i| = \sum_{i=1}^{n} |\lambda_i^*|.$$

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A subset $\mathcal{D}$ of V is said to be a dominating set of G if every vertex of $V \setminus \mathcal{D}$ is adjacent to some vertex in $\mathcal{D}$. Any dominating set with minimum cardinality is called a minimum dominating set. Let $\mathcal{D}$ be a minimum dominating set of the graph G. The minimum dominating adjacency matrix of G, denoted by $\mathbf{A}_{\mathcal{D}} = (a_{ij}^{\mathcal{D}})$, is the $n \times n$ matrix defined as

$$a_{ij}^{\mathcal{D}} = \begin{cases} 1 & \text{if } v_i \sim v_j \\ 1 & \text{if } i = j\,,\ v_i \in \mathcal{D} \\ 0 & \text{otherwise.} \end{cases}$$

The minimum dominating eigenvalues of the graph G, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, are the eigenvalues of $\mathbf{A}_{\mathcal{D}}$. The following equalities are valid for them [24]:

$$\sum_{i=1}^{n} \alpha_i = |\mathcal{D}| \qquad \text{and} \qquad \sum_{i=1}^{n} \alpha_i^2 = 2m + |\mathcal{D}|.$$

Let $|\alpha_1^*| \geq |\alpha_2^*| \geq \cdots \geq |\alpha_n^*|$, $\alpha_1 = |\alpha_1| = |\alpha_1^*|$, be a non-increasing sequence of absolute values of the minimum dominating eigenvalues of G. The minimum dominating energy of the graph G, $E_{\mathcal{D}} = E_{\mathcal{D}}(G)$, is defined as [24,38]

$$E_{\mathcal{D}} = E_{\mathcal{D}}(G) = \sum_{i=1}^{n} |\alpha_i| = \sum_{i=1}^{n} |\alpha_i^*|.$$

A subset C of V, $C \subset V$, is called a covering set of G if every edge of G is incident to at least one vertex of G. Any covering set with minimum cardinality is called a minimum covering set. Let C be a minimum covering set of the graph G. The minimum covering matrix of G is the matrix defined by $\mathbf{A}_C = (a_{ij}^C)$, where

$$a_{ij}^{C} = \begin{cases} 1 & v_i \sim v_j \\ 1 & i = j, v_i \in C \\ 0 & \text{otherwise.} \end{cases}$$

The minimum covering eigenvalues of G, denoted by $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$, are the eigenvalues of $\mathbf{A}_C$. The following equalities are valid for β_i, $i = 1, 2, \ldots, n$ [1]

$$\sum_{i=1}^{n} \beta_i = |C| \qquad \text{and} \qquad \sum_{i=1}^{n} \beta_i^2 = 2m + |C|.$$

Let $|\beta_1^*| \geq |\beta_2^*| \geq \cdots \geq |\beta_n^*|$, $\beta_1 = |\beta_1| = |\beta_1^*|$, be a non-increasing sequence of absolute values of the minimum covering eigenvalues of G. The minimum covering energy of the graph G, $E_C = E_C(G)$, is defined as [1]

$$E_C = E_C(G) = \sum_{i=1}^{n} |\beta_i| = \sum_{i=1}^{n} |\beta_i^*|.$$

Let $\mathbf{D} = \mathrm{diag}\{d_1, d_2, \ldots, d_n\}$. Then $\mathbf{L} = \mathbf{D} - \mathbf{A}$ is the Laplacian matrix of G. The eigenvalues of $\mathbf{L}$, denoted by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$, are the Laplacian eigenvalues of the graph G. Some of their well known properties are [15]:

$$\sum_{i=1}^{n-1} \mu_i = \sum_{i=1}^{n} d_i = 2m \qquad \text{and} \qquad \sum_{i=1}^{n-1} \mu_i^2 = \sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i = M_1 + 2m.$$

Let $\gamma_i = \mu_i - \frac{2m}{n}$, $i = 1, 2, \ldots, n$, and $|\gamma_1^*| \geq |\gamma_2^*| \geq \cdots \geq |\gamma_n^*|$, $|\gamma_1^*| = \frac{2m}{n}$, be a sequence of non-increasing values of absolute values of γ_i. The Laplacian energy of G is defined as [22]

$$LE = LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| = \sum_{i=1}^{n} |\gamma_i^*|.$$

Since for several reasons the concept of Laplacian energy was not fully satisfactory, a seemingly different quantity, named Laplacian-energy-like invariant, LEL, has been put forward [30], defined as

$$LEL = LEL(G) = \sum_{i=1}^{n-1} \sqrt{\mu_i}.$$

The matrix $\mathbf{L}^+ = \mathbf{D} + \mathbf{A}$ is the signless Laplacian matrix of the graph G. Its eigenvalues, denoted by $\mu_1^+ \geq \mu_2^+ \geq \cdots \geq \mu_n^+ \geq 0$, are referred to as the signless Laplacian eigenvalues of G [8]. Some of their well known properties are

$$\sum_{i=1}^{n} \mu_i^+ = 2m \qquad \text{and} \qquad \sum_{i=1}^{n} \left(\mu_i^+\right)^2 = M_1 + 2m.$$

The signless Laplacian eigenvalues are encountered in the theory of the so-called incidence energy, IE [12,18]. The incidence energy, although defined in a completely different manner [23], satisfies the relation

$$IE = IE(G) = \sum_{i=1}^{n} \sqrt{\mu_i^+}.$$

Assume that the graph G has no isolated vertices. Then the matrix $\mathbf{D}$ is nonsingular and $\mathbf{D}^{-1/2}$ is well defined. The matrix $\mathbf{L}^* = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2} = \mathbf{I} - \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$ is the normalized Laplacian matrix of G. Its eigenvalues, $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n = 0$, are the normalized Laplacian eigenvalues of the graph G. The following relations hold [6]

$$\sum_{i=1}^{n-1} \delta_i = n \qquad \text{and} \qquad \sum_{i=1}^{n-1} \delta_i^2 = n + 2R_{-1},$$

where $R_{-1} = \sum_{v_i \sim v_j} \frac{1}{d_i d_j}$ is the general Randić index of G [5].

The normalized Laplacian energy $NLE = NLE(G)$ has been defined as [5]

$$NLE = NLE(G) = \sum_{i=1}^{n} |\delta_i - 1|.$$

The matrix $\mathbf{R} = (r_{ij})$ defined as

$$r_{ij} = \begin{cases} 1/\sqrt{d_i d_j} & \text{if } v_i \sim v_j \\ 0 & \text{otherwise} \end{cases}$$

is referred to as the Randić matrix of the underlying graph G. For its eigenvalues $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$, $\rho_1 = 1$, the following is valid [4,28]

$$\sum_{i=1}^{n} \rho_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{n} \rho_i^2 = 2R_{-1}.$$

The Randić energy of the graph G has been defined as [3,4]

$$RE = RE(G) = \sum_{i=1}^{n} |\rho_i| = \sum_{i=1}^{n} |\rho_i^*|,$$

where $|\rho_1^*| \geq |\rho_2^*| \geq \cdots \geq |\rho_n^*|$, $\rho_1 = |\rho_1| = |\rho_1^*|$, is a non-increasing sequence of absolute values of the eigenvalues of $\mathbf{R}$.

The fact that for all graphs, the normalized Laplacian energy and Randić energy coincide, has been immediately recognized, cf. [10,11,16]. At this point, it is worth noting that all the above defined energies are related with the entropy of the underlying graph [13,26].

In this paper we prove a general inequality for non-negative real numbers, and then use it to establish upper bounds for the graph energies E, $E_{\mathcal{D}}$, E_C, LEL, IE, LE, and RE. In most cases, our results improve the inequalities that earlier have been reported in the literature.

2. Main results

Let $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ be real non-negative numbers. For the sums P and Q defined as

$$P = \sum_{i=1}^{n} a_i^2 \qquad \text{and} \qquad Q = \sum_{i=1}^{n} a_i$$

we prove the following result:

Theorem 1. *For arbitrary real numbers k_1 and k_2 with the properties*

$$a_1 \geq k_1 \geq \sqrt{\frac{P}{n}} \qquad \text{and} \qquad \sqrt{\frac{P}{n}} \geq k_2 \geq a_n$$

the following is valid

$$Q \leq \min\left\{ k_1 + \sqrt{(n-1)(P-k_1^2)},\ k_2 + \sqrt{(n-1)(P-k_2^2)},\ \sqrt{nP - \frac{n}{2}(a_1 - a_n)^2} \right\}. \tag{1}$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Proof. In [32], a class $\mathcal{P}_n(b_1, b_2)$ of real polynomials of the form

$$P_n(x) = x^n + b_1 x^{n-1} + b_2 x^{n-2} + c_3 x^{n-3} + \cdots + c_n$$

was considered, where b_1 and b_2 are fixed real numbers. It was proven that for the roots $x_1 \ge x_2 \ge \cdots \ge x_n$ of that class of polynomials, the following inequalities are valid

$$x_1 \le \bar{x} + \frac{1}{n}\sqrt{(n-1)\Delta} \tag{2}$$

$$x_n \ge \bar{x} - \frac{1}{n}\sqrt{(n-1)\Delta} \tag{3}$$

$$x_1 - x_n \le \sqrt{\frac{2\Delta}{n}} \tag{4}$$

where

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \text{and} \qquad \Delta = n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2. \tag{5}$$

Consider now the polynomial

$$P_n(x) = \prod_{i=1}^{n}(x-a_i) = x^n + b_1x^{n-1} + b_2x^{n-2} + c_3x^{n-3} + \cdots + c_n$$

where $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$ are given real numbers. Since

$$b_1 = -\sum_{i=1}^{n} a_i = -Q \qquad \text{and} \qquad b_2 = \frac{1}{2}\left[\left(\sum_{i=1}^{n} a_i\right)^2 - \sum_{i=1}^{n} a_i^2\right] = \frac{1}{2}(Q^2 - P)$$

the above polynomial belongs to the class $\mathcal{P}_n\big(-Q, \frac{1}{2}(Q^2-P)\big)$. According to (3), for $x_i = a_i,\ i = 1, 2, \ldots, n,$ we have

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} a_i = \frac{Q}{n} \qquad \text{and} \qquad \Delta = n\sum_{i=1}^{n} a_i^2 - \left(\sum_{i=1}^{n} a_i\right)^2 = nP - Q^2. \tag{6}$$

Now, for any real k_1 with the property $a_1 \ge k_1$, according to (2),

$$k_1 \le a_1 \le \frac{Q}{n} + \frac{1}{n}\sqrt{(n-1)(nP-Q^2)}$$

i.e.,

$$n\,k_1 - Q \le \sqrt{(n-1)(nP-Q^2)}. \tag{7}$$

Under the assumption of Theorem 1, $k_1 \ge \sqrt{\frac{P}{n}}$. According to Jensen's inequality (see for example [36]),

$$P = \sum_{i=1}^{n} a_i^2 \ge \frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2 = \frac{Q^2}{n}$$

i.e.,

$$\frac{Q}{n} \le \sqrt{\frac{P}{n}}.$$

From the above it follows that

$$k_1 - \frac{Q}{n} \ge k_1 - \sqrt{\frac{P}{n}} \ge 0.$$

According to this and inequality (7), we conclude that

$$Q \le k_1 + \sqrt{(n-1)(P-k_1^2)}, \tag{8}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

For any k_2 with the property $k_2 \ge a_n$, according to (3), the following is valid

$$k_2 \ge a_n \ge \frac{Q}{n} - \frac{1}{n}\sqrt{(n-1)\Delta}$$

i.e.,

$$Q - n\,k_2 \le \sqrt{(n-1)(nP-Q^2)}.$$

Since $\sqrt{P/n} \geq k_2$, it follows that

$$k_2 \leq \sqrt{\frac{P}{n}} \leq a_1 \leq \frac{Q}{n} + \frac{1}{n}\sqrt{(n-1)(nP-Q^2)}$$

wherefrom we obtain

$$n\,k_2 - Q \leq \sqrt{(n-1)(nP-Q^2)}.$$

This means that

$$|Q - n\,k_2| \leq \sqrt{(n-1)(nP-Q^2)}$$

which implies

$$Q \leq k_2 + \sqrt{(n-1)(P-k_2^2)} \tag{9}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

According to (4), for $x_i = a_i$, $i = 1, 2, \ldots, n,$ we have that

$$a_1 - a_n \leq \sqrt{\frac{2(nP-Q^2)}{n}}$$

i.e.,

$$Q \leq \sqrt{nP - \frac{n}{2}\,(a_1 - a_n)^2}. \tag{10}$$

Equality in (10) holds if and only if $a_1 = a_2 = \cdots = a_n$ or when $n = 2$ for any $a_1 \geq a_2 \geq 0$.

Finally, the inequality (1) is obtained from (8)–(10). □

Setting $a_i = |\lambda_i^*|,\ a_i = |\gamma_i^*|,\ a_i = |\delta_i^*|,\ a_i = |\mu_i^*|,\ a_i = \sqrt{\mu_i},\ a_i = |\rho_i^*|,$ and $a_i = \sqrt{\beta_i},$ we arrive at the following corollaries of Theorem 1.

Corollary 1. *Let G be a simple graph with n vertices and m edges. Then, for any k_1 and k_2 with the properties $\lambda_1 \geq k_1 \geq \sqrt{2m/n}$ and $\sqrt{2m/n} \geq k_2 \geq |\lambda_n^*|$,*

$$E \leq \min\left\{k_1+\sqrt{(n-1)(2m-k_1^2)}\,,\ k_2+\sqrt{(n-1)(2m-k_2^2)},\ \sqrt{2mn-\frac{n}{2}\,(|\lambda_1^*|-|\lambda_n^*|)^2}\right\}. \tag{11}$$

Equality holds if and only if $G \cong \overline{K}_n$ or $G \cong \frac{n}{2}\,K_2$ for even n.

Remark 1. The inequalities

$$E \leq \sqrt{2mn - \frac{n}{2}\,(|\lambda_1^*| - |\lambda_n^*|)^2}$$

and

$$E \leq k_1 + \sqrt{(n-1)(2m-k_1^2)}$$

were proven in [35] and [34], respectively.

Since

$$\lambda_1 \geq \sqrt{\frac{\sum_{i=1}^n t_i^2}{\sum_{i=1}^n d_i^2}} \geq \sqrt{\frac{M_1}{n}} \geq \frac{2m}{n} \geq \sqrt{\frac{2m}{n}}$$

where t_i is the 2-degree of the vertex v_i (see [42]), for $k_1 = \sqrt{2m/n}$ we obtain the classical McClelland bound [9,20,27]

$$E \leq \sqrt{2mn}$$

whereas $k_1 = 2m/n$ yields the Koolen–Moulton estimate [20,25,27]

$$E \leq \frac{2m}{n} + \frac{1}{n}\sqrt{2m(n-1)(n^2-2m)}.$$

For $k_1 = \sqrt{M_1/n},$ we obtain

$$E \leq \sqrt{\frac{M_1}{n}} + \sqrt{(n-1)\left(2m - \frac{M_1}{n}\right)}$$

which was proven in [41]. For $k_1 = \left[\sum_{i=1}^n t_i^2 / \sum_{i=1}^n d_i^2\right]^{1/2}$,

$$E \le \sqrt{\frac{\sum_{i=1}^n t_i^2}{\sum_{i=1}^n d_i^2}} + \sqrt{(n-1)\left(2m - \frac{\sum_{i=1}^n t_i^2}{\sum_{i=1}^n d_i^2}\right)}$$

which was proven in [40].

In [14], the inequality $\lambda_1 \ge \sqrt{d_1}$ was proven. Since $\sqrt{d_1} \ge \sqrt{M_1/n}$, for $k_1 = \sqrt{d_1}$, from (11) it follows

$$E \le \sqrt{d_1} + \sqrt{(n-1)(2m-d_1)}.$$

Equality holds if and only if $G \cong \bar{K}_n$, or $G \cong \frac{n}{2} K_2$ for even n.

Note finally, that for $k_1 = |\lambda_1^*| = \lambda_1$ and $k_2 = |\lambda_n^*|$, from (11) it follows

$$E \le \min\left\{\lambda_1 + \sqrt{(n-1)(2m-\lambda_1^2)},\ |\lambda_n^*| + \sqrt{(n-1)(2m-|\lambda_n^*|^2)}\right\}.$$

Corollary 2. *Let $G = (V, E)$ be a simple graph with n vertices and m edges, and let $C \subset V$ be its minimum covering set. Then for any k_1 and k_2 with the properties $|\beta_1^*| \ge k_1 \ge \sqrt{(2m+|C|)/n}$ and $\sqrt{(2m+|C|)/n} \ge k_2 \ge |\beta_n^*|$, it holds*

$$E_C \le \min\left\{k_1 + \sqrt{(n-1)(2m+|C|-k_1^2)},\ k_2 + \sqrt{(n-1)(2m+|C|-k_2^2)},\right.$$
$$\left.\sqrt{n\left[2m+|C|-0.5\,(|\beta_1^*|-|\beta_n^*|)^2\right]}\right\}. \tag{12}$$

Equality holds if and only if $G \cong \overline{K}_n$, or $G \cong \frac{n}{2} K_2$ when n is even.

Remark 2. For $k_1 = \sqrt{(2m+|C|)/n}$, from (12) it follows

$$E_C \le \sqrt{n(2m+|C|)} \tag{13}$$

a bound earlier reported in [1]. If $2m + |C| \ge n$, then

$$\beta_1 \ge \frac{2m+|C|}{n} \ge \sqrt{\frac{2m+|C|}{n}}.$$

For $k_1 = (2m+|C|)/n$, from (12) it follows

$$E_C \le \frac{2m+|C|}{n} + \sqrt{(n-1)\left(2m+|C| - \left(\frac{2m+|C|}{n}\right)^2\right)}.$$

Since $\frac{1}{2}\,(|\beta_1^*| - |\beta_n^*|)^2 \ge 0$, the inequality

$$E_C \le \sqrt{n\left(2m+|C| - \frac{1}{2}\,(|\beta_1^*|-|\beta_n^*|)^2\right)}$$

from (12), is stronger than (13).

Corollary 3. *Let G be a simple graph with n vertices and m edges. Then, for any real k_1 and k_2 with the properties $\sqrt{\mu_1} \ge k_1 \ge \sqrt{2m/(n-1)}$ and $\sqrt{2m/(n-1)} \ge k_2 \ge \sqrt{\mu_{n-1}}$,*

$$LEL \le \min\left\{k_1 + \sqrt{(n-2)(2m-k_1^2)},\ k_2 + \sqrt{(n-2)(2m-k_2^2)},\right.$$
$$\left.\sqrt{(n-1)\left(2m - \frac{1}{2}\,(\sqrt{\mu_1}-\sqrt{\mu_{n-1}})^2\right)}\right\}. \tag{14}$$

Equality holds if and only if $G \cong K_n$ or $G \cong \overline{K}_n$.

Remark 3. For $k_1 = \sqrt{1+d_1}$, from (14) we deduce

$$LEL \le \sqrt{1+d_1} + \sqrt{(n-2)(2m-1-d_1)}$$

which earlier was obtained in [30].

For $k_1 = \sqrt{\mu_1}$ and $k_2 = \sqrt{\mu_{n-1}}$, from (14) we get

$$LEL \le \min\left\{\sqrt{\mu_1} + \sqrt{(n-2)(2m-\mu_1)},\ \sqrt{\mu_{n-1}} + \sqrt{(n-2)(2m-\mu_{n-1})},\right.$$

$$\sqrt{(n-1)\left[2m-\frac{1}{2}\left(\sqrt{\mu_1}-\sqrt{\mu_{n-1}}\right)^2\right]}\Bigg\}.$$

For $k_1 = \sqrt{2m/(n-1)}$, from (14) it follows

$$LEL \le \sqrt{2m(n-1)} \tag{15}$$

which was earlier reported in [30]. Since $\frac{1}{2}(\sqrt{\mu_1}-\sqrt{\mu_{n-1}})^2 \ge 0$, the inequality

$$LEL \le \sqrt{(n-1)\left(2m-\frac{1}{2}(\sqrt{\mu_1}-\sqrt{\mu_{n-1}})^2\right)}$$

from (14) is stronger than (15).

Corollary 4. *Let G be a simple graph of order $n \ge 2$ with m edges. Then, for any real k_1 and k_2 with the properties $1 \ge k_1 \ge \sqrt{2R_{-1}/n}$ and $\sqrt{2R_{-1}/n} \ge k_2 \ge |\rho_n^*|$,*

$$RE \le \min\left\{k_1+\sqrt{(n-1)(2R_{-1}-k_1^2)},\ k_2+\sqrt{(n-1)(2R_{-1}-k_2^2)},\right.$$
$$\left.\sqrt{2nR_{-1}-\frac{n}{2}(1-|\rho_n^*|)^2}\right\}. \tag{16}$$

Equality in (16) holds if and only if $G \cong K_n$.

Remark 4. For $k_1 = 1$, from (16) it follows that

$$RE \le \min\left\{1+\sqrt{(n-1)(2R_{-1}-1)},\ \sqrt{n\left[2R_{-1}-\frac{1}{2}(1-|\rho_n^*|)^2\right]}\right\}. \tag{17}$$

The inequality

$$RE \le 1+\sqrt{(n-1)(2R_{-1}-1)} \tag{18}$$

was proven in [2] (see also [29,33]). Since $R_{-1} \le \frac{n}{2d_n}$ (see [37]), inequality (18), is reduced to the earlier known [10] (see also [33])

$$RE \le 1+\sqrt{\frac{(n-1)(n-d_n)}{d_n}}.$$

For $k_1 = \sqrt{2R_{-1}/n}$, from (16) we obtain

$$RE \le \sqrt{2nR_{-1}} \tag{19}$$

which is an estimate earlier reported in [4,5]. Since $\frac{1}{2}\left(1-|\rho_n^*|\right)^2 \ge 0$, our inequality

$$RE \le \sqrt{n\left(2R_{-1}-\frac{1}{2}\left(1-|\rho_n^*|\right)^2\right)}$$

from (16), is stronger than (19).

Corollary 5. *Let G be a simple graph with n vertices and m edges. Then, for any real k_1 and k_2 with the properties $\sqrt{\mu_1^+} \ge k_1 \ge \sqrt{2m/n}$ and $\sqrt{2m/n} \ge k_2 \ge \sqrt{\mu_n^+}$, it holds*

$$IE \le \min\left\{k_1+\sqrt{(n-1)(2m-k_1^2)},\ k_2+\sqrt{(n-1)(2m-k_2^2)},\right.$$
$$\left.\sqrt{n\left[2m-\frac{1}{2}\left(\sqrt{\mu_1^+}-\sqrt{\mu_n^+}\right)^2\right]}\right\}. \tag{20}$$

Remark 5. The inequality

$$IE \le k_1+\sqrt{(n-1)(2m-k_1^2)}$$

was proven in [19]. For $k_1 = \sqrt{\mu_1^+}$ and $k_1 = \sqrt{1+d_1}$, from (20) we obtain

$$IE \le \sqrt{\mu_1^+}+\sqrt{(n-1)(2m-\mu_1^+)}$$

and

$$IE \le \sqrt{1+d_1} + \sqrt{(n-1)(2m-1-d_1)} \tag{21}$$

which were also proven in [19]. The inequality (21) was independently obtained in [31].

For $k_1 = \sqrt{1+d_1+1/(d_1-1)}, \quad (d_1 \ge 2)$, from (20) we obtain

$$IE \le \sqrt{1+d_1+\frac{1}{d_1-1}} + \sqrt{(n-1)\Big(2m-1-d_1-\frac{1}{d_1-1}\Big)}$$

a result earlier communicated in [39].

For $k_1 = \sqrt{2m/n}$, from (20) we get the previously known bound [23]:

$$IE \le \sqrt{2mn}. \tag{22}$$

Since $\frac{1}{2}\,(\sqrt{\mu_1^+} - \sqrt{\mu_n^+})^2 \ge 0$, our estimate

$$IE \le \sqrt{n\Big(2m - \frac{1}{2}\,(\sqrt{\mu_1^+} - \sqrt{\mu_n^+})^2\Big)}$$

deduced from (20), is stronger than (22).

For $k_1 = \sqrt{M_1/m}$, from (20) we get an earlier known bound [42]

$$IE \le \sqrt{\frac{M_1}{m}} + \sqrt{(n-1)\Big(2m - \frac{M_1}{m}\Big)}.$$

Remark 6. If G is a bipartite graph, then $\mu_n^+ = 0$ [8]. Thus, for bipartite graphs instead of (20), the following inequality may be used

$$IE \le \min\Big\{k_1 + \sqrt{(n-2)(2m-k_1^2)},\ k_2 + \sqrt{(n-2)(2m-k_2^2)},$$
$$\sqrt{(n-1)\Big[2m - \frac{1}{2}\Big(\sqrt{\mu_1^+} - \sqrt{\mu_{n-1}^+}\Big)^2\Big]}\Big\}$$

where $\sqrt{\mu_1^+} \ge k_1 \ge \sqrt{2m/(n-1)}$ and $\sqrt{2m/(n-1)} \ge k_2 \ge \sqrt{\mu_{n-1}^+}$.

Corollary 6. *Let G be a simple connected graph with n vertices and m edges. Then, for any real k_1 and k_2 with the properties $2m/n \ge k_1 \ge \sqrt{2M/n}$ and $\sqrt{2M/n} \ge k_2 \ge |\gamma_n^*|$, it holds*

$$LE \le \min\Big\{k_1 + \sqrt{(n-1)(2M-k_1^2)},\ k_2 + \sqrt{(n-1)(2M-k_2^2)},$$
$$\sqrt{n\Big(2M - \frac{1}{2}\left(|\gamma_1^*| - |\gamma_n^*|\right)^2\Big)}\Big\} \tag{23}$$

where

$$M = \frac{1}{2}\sum_{i=1}^{n}\Big(\mu_i - \frac{2m}{n}\Big)^2 = m + \frac{1}{2}\sum_{i=1}^{n}\Big(d_i - \frac{2m}{n}\Big)^2.$$

Equality holds if and only if $G \cong K_n$.

Remark 7. For $k_1 = 2m/n$ and $k_1 = \sqrt{M/n}$, from (23) we obtain, respectively,

$$LE \le \frac{2m}{n} + \sqrt{(n-1)\Big[2M - \Big(\frac{2m}{n}\Big)^2\Big]}$$

and

$$LE \le \sqrt{2nM} \tag{24}$$

which both were proven in [22]. Since $\frac{1}{2}\left(|\gamma_1^*| - |\gamma_n^*|\right)^2 \ge 0$, the inequality

$$LE \le \sqrt{n\Big(2M - \frac{1}{2}\left(|\gamma_1^*| - |\gamma_n^*|\right)^2\Big)}$$

which follows from (23), is stronger than (24).

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