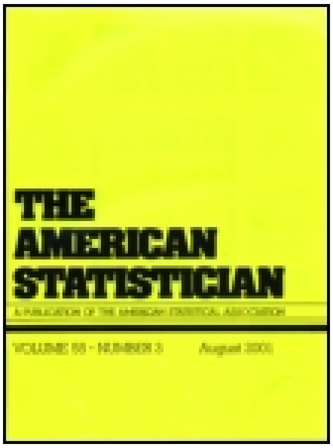

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A Reformulation of Weighted Least Squares Estimators

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A Reformulation of Weighted Least Squares Estimators

Jaechoul LEE

This article studies weighted, generalized, least squares estimators in simple linear regression with serially correlated errors. Closed-form expressions of weighted least squares estimators and variances are presented under some common stationary autocorrelation settings, a first-order autoregression and a first-order moving-average. These explicit expressions also have appealing applications, including an efficient weighted least squares computation method and a new sufficient and necessary condition on the equality of weighted least squares estimators and ordinary least squares estimators.

KEY WORDS: Autocorrelation; Linear trend; Ordinary least squares; Simple regression; Weighted least squares.

1. INTRODUCTION

This article considers the simple linear regression model

$$Y_t = \mu + \beta x_t + \varepsilon_t, \quad t = 1, 2, \ldots, n, \tag{1}$$

where μ is a location parameter, β is the slope for the fixed covariate x_t, and $\{\varepsilon_t\}$ are zero mean stationary, in time t, errors with autocovariances $\gamma(h) = \mathrm{cov}(\varepsilon_t, \varepsilon_{t+h})$ at lag h.

The slope parameter β has great importance in many applications using the time series linear regression model (cf., Zinde-Walsh and Galbraith 1991; Sun and Pantula 1999; Fomby and Vogelsang 2002; Roy, Falk, and Fuller 2004). The ordinary least squares estimator for β, denoted by $\hat{\beta}_{\mathrm{OLS}}$, is

$$\hat{\beta}_{\mathrm{OLS}} = \frac{\sum_{t=1}^{n} (x_t - \bar{x}) Y_t}{\sum_{t=1}^{n} (x_t - \bar{x})^2} := \sum_{t=1}^{n} w_t^{\mathrm{OLS}} Y_t;$$

here $\bar{x} = \sum_{t=1}^{n} x_t / n$. Lee and Lund (2004) obtain

$$\mathrm{var}(\hat{\beta}_{\mathrm{OLS}}) = \frac{\gamma(0) + 2\sum_{h=1}^{n-1} r_x(h)\gamma(h)}{\sum_{t=1}^{n} (x_t - \bar{x})^2} := \sum_{h=0}^{n-1} u_h^{\mathrm{OLS}} \gamma(h),$$

where $r_x(h)$ is the sample autocorrelation function of $\{x_t\}$, that is,

$$r_x(h) = \frac{\sum_{t=1}^{n-h} (x_t - \bar{x})(x_{t+h} - \bar{x})}{\sum_{t=1}^{n} (x_t - \bar{x})^2}.$$

The regression parameters are often estimated by using weighted, generalized, least squares, also called the Gauss-Markov or best linear unbiased, method. The resulting estimators, $\hat{\mu}_{\mathrm{WLS}}$ and $\hat{\beta}_{\mathrm{WLS}}$, are expressed as the following well-known general linear models form

$$(\hat{\mu}_{\mathrm{WLS}}, \hat{\beta}_{\mathrm{WLS}})' = (X_n' \Gamma_n^{-1} X_n)^{-1} X_n' \Gamma_n^{-1} Y^{(n)},$$
$$\mathrm{var}\big((\hat{\mu}_{\mathrm{WLS}}, \hat{\beta}_{\mathrm{WLS}})'\big) = \big(X_n' \Gamma_n^{-1} X_n\big)^{-1};$$

here $Y^{(n)} = (Y_1, \ldots, Y_n)'$, $X_n = (1^{(n)}, x^{(n)})$ with $1^{(n)} = (1, \ldots, 1)'$ and $x^{(n)} = (x_1, \ldots, x_n)'$, and Γ_n denotes the $n \times n$ invertible covariance matrix of $Y^{(n)}$ with $\gamma(|i-j|)$ as the (i, j)th component.

The weighted least squares estimators have minimum variance among all linear unbiased estimators, including the ordinary least squares estimators, in the presence of stationary errors with known autocovariances. Closed-form (nonmatrix) expressions for the best estimators, however, have been less explored. The use of these explicit expressions, if available, can be useful in understanding further details of the weighted least squares estimation.

The objectives of this study are (1) to reformulate the weighted least squares estimators and their variances for a simple linear regression model with autocorrelated errors into explicit expressions and (2) to present appealing applications of the reformulated expressions. In Section 2, explicit expressions for weighted least squares are presented under a general stationary autocorrelation structure. In Section 3, these explicit expressions are applied to a linear trend regression model in first-order autoregressive and first-order moving-average settings. In Section 4, a ‘short-cut’ method of computing weighted least squares is presented, together with a new necessary and sufficient condition on the error covariances which makes ordinary and weighted least squares estimators identical.

2. EXPLICIT EXPRESSIONS FOR WEIGHTED LEAST SQUARES

The general linear model forms can be reformulated into explicit expressions. Indeed, we have that

$$\hat{\beta}_{\mathrm{WLS}} = \frac{\sum_{t=1}^{n} \big(m_{\cdot,\cdot} \sum_{s=1}^{n} x_s m_{s,t} - m_{\cdot,t} \sum_{s=1}^{n} x_s m_{s,\cdot}\big) Y_t}{\sum_{t=1}^{n} \big(m_{\cdot,\cdot} \sum_{s=1}^{n} x_s m_{s,t} - m_{\cdot,t} \sum_{s=1}^{n} x_s m_{s,\cdot}\big)(x_t - \bar{x})}$$
$$:= \frac{\sum_{t=1}^{n} v_t Y_t}{\sum_{t=1}^{n} v_t (x_t - \bar{x})}, \tag{2}$$

here $m_{s,t}$ is the (s, t)th entry of M_n^{-1}, where $M_n = \Gamma_n/\sigma^2$. The dot notation indicates column or row summation: $m_{s,\cdot} = \sum_{t=1}^{n} m_{s,t}$, $m_{\cdot,t} = \sum_{s=1}^{n} m_{s,t}$, and $m_{\cdot,\cdot} = \sum_{t=1}^{n} \sum_{s=1}^{n} m_{s,t}$. The weight series $\{v_t\}$ is determined by both the covariate series $\{x_t\}$ and the elements of M_n^{-1}.

Taking variance in (2) results in the following closed-form expression

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$$\operatorname{var}(\hat{\beta}_{\mathrm{WLS}})=\frac{\gamma(0)+2\sum_{h=1}^{n-1} r_v(h)\gamma(h)}{r_{v,x}^2\sum_{t=1}^{n}(x_t-\bar{x})^2}, \tag{3}$$

where

$$r_v(h)=\frac{\sum_{t=1}^{n-h} v_t v_{t+h}}{\sum_{t=1}^{n} v_t^2},\quad r_{v,x}=\frac{\sum_{t=1}^{n} v_t(x_t-\bar{x})}{\sqrt{\sum_{t=1}^{n} v_t^2\sum_{t=1}^{n}(x_t-\bar{x})^2}}.$$

Note that $r_v(h)$ can be viewed as the sample autocorrelation function of $\{v_t\}$, and $r_{v,x}$ as the sample cross-correlation function of $\{v_t\}$ and $\{x_t\}$, in that the unbiasedness of $\hat{\beta}_{\mathrm{WLS}}$ gives $\bar{v}=\sum_{t=1}^{n} v_t/n=0$. Indeed, we have $|r_v(h)|\le 1$ and $|r_{v,x}|\le 1$.

The weighted least squares variance in (3) has an expression akin to the ordinary least squares variance. It is worth emphasizing that the expressions, presented above, for weighted least squares estimator and variance can be used under any general stationary error settings if the elements of M_n^{-1} are determined. In particular, for the regression model with causal $(p,\ q)$-th order autoregressive moving-average errors, the inverted variance-covariance matrix in Haddad (2004) can be used to explicitly evaluate v_t. If $\{\varepsilon_t\}$ is uncorrelated with constant variance, then simply $v_t=x_t-\bar{x}$.

Interestingly, the weighted least squares estimator and variance can be easily related to the ordinary least squares estimator and variance; that is,

$$\hat{\beta}_{\mathrm{WLS}}=\hat{\beta}_{\mathrm{OLS}}\frac{s_x}{r_{v,x}s_v}\left\{\frac{\sum_{t=1}^{n} v_t Y_t}{\sum_{t=1}^{n}(x_t-\bar{x})Y_t}\right\},$$

$$\operatorname{var}(\hat{\beta}_{\mathrm{WLS}})=\operatorname{var}(\hat{\beta}_{\mathrm{OLS}})\frac{1}{r_{v,x}^2}\left\{\frac{1+2\sum_{h=1}^{n-1} r_v(h)\rho(h)}{1+2\sum_{h=1}^{n-1} r_x(h)\rho(h)}\right\},$$

where s_x and s_v are the sample standard deviations of $\{x_t\}$ and $\{v_t\}$, and $\rho(h)=\gamma(h)/\gamma(0)$.

3. LINEAR TREND REGRESSION WITH AUTOCORRELATED ERRORS

This section focuses on explicit formulations for the weighted least squares estimation in the simple linear regression model (1) with the covariate $x_t=t$ replaced. Here, explicit expressions of the weighted least squares trend estimator and variance are derived for simple but important stationary autocorrelated error structures, a first-order autoregression and a first-order moving-average. Some new facts are also pointed out along the way.

3.1 Model 1

Consider estimation in regression model (1) with linear trend coefficient β under the first-order autoregressive error setup, such that

$$\varepsilon_t=\phi\varepsilon_{t-1}+Z_t,\quad |\phi|<1,$$

where Z_t is white noise with mean zero and variance σ^2. The corresponding autocovariance function is $\gamma(h)=\sigma^2\phi^h/(1-\phi^2)$. Using the general explicit expressions in (2) and (3), we have that

- Weighted least squares estimator:

$$\hat{\beta}_{\mathrm{WLS}}=\frac{\{(1-\phi)(1-\bar{t})-\phi\}Y_1+(1-\phi)^2\sum_{t=2}^{n-1}(t-\bar{t})Y_t+\{(1-\phi)(n-\bar{t})+\phi\}Y_n}{\sum_{t=1}^{n}(t-\bar{t})^2\left\{(1-\phi)^2+\frac{6}{n}\phi-\frac{6(n-1)}{n(n+1)}\phi^2\right\}}$$
$$:=\sum_{t=1}^{n} w_t^{\mathrm{AR1}}Y_t. \tag{4}$$

- Variance expression 1:

$$\operatorname{var}(\hat{\beta}_{\mathrm{WLS}})=\frac{\gamma(0)+2\sum_{h=1}^{n-1} s_h^{\mathrm{AR1}}\gamma(h)}{\sum_{t=1}^{n}(t-\bar{t})^2\left[\dfrac{n(n+1)(n-1)\left\{(1-\phi)^2+\frac{6}{n}\phi-\frac{6(n-1)}{n(n+1)}\phi^2\right\}^2}{6\{(n-1)(1-\phi)+2\phi\}^2+(n-1)(n-2)(n-3)(1-\phi)^4}\right]}$$
$$:=\sum_{h=0}^{n-1} u_h^{\mathrm{AR1}}\gamma(h), \tag{5}$$

where

$$s_h^{\mathrm{AR1}}=\begin{cases}(1-\phi)^2[6\{(n-1)(1-\phi)+2\phi\}\\ \quad\times(n-2h-1)+(1-\phi)^2\\ \quad\times[(n-1)(n-2)(n-3)\\ \quad-\{3(n-2)^2-1\}h+2h^3]]/C, & 1\le h\le n-3,\\ -6(1-\phi)^2(n-3)\\ \quad\times\{(n-1)(1-\phi)+2\phi\}/C, & h=n-2,\\ -3\{(n-1)(1-\phi)+2\phi\}^2/C, & h=n-1\end{cases}$$

with $C=6\{(n-1)(1-\phi)+2\phi\}^2+(n-1)(n-2)(n-3)(1-\phi)^4$.

- Variance expression 2:

$$\operatorname{var}(\hat{\beta}_{\mathrm{WLS}})=\frac{\sigma^2}{\sum_{t=1}^{n}(t-\bar{t})^2\left\{(1-\phi)^2+\frac{6}{n}\phi-\frac{6(n-1)}{n(n+1)}\phi^2\right\}}. \tag{6}$$

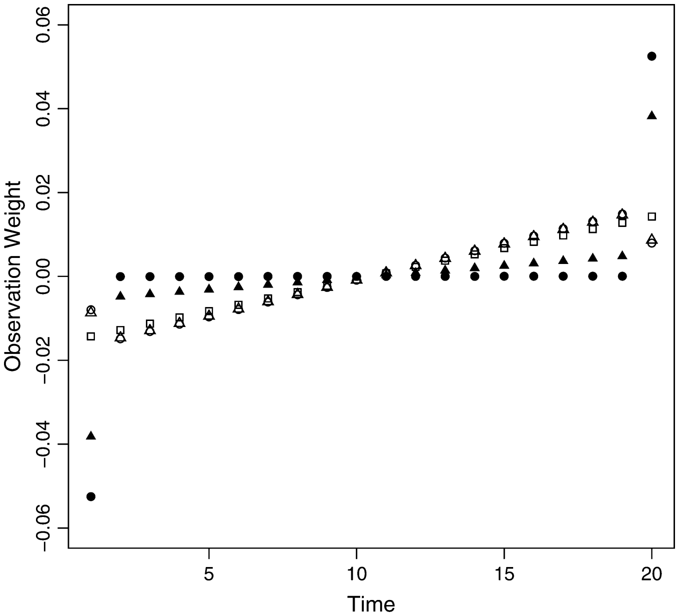

Figure 1. The first-order autoregressive observation weights on the weighted least squares trend estimator against time for $n = 20$. Each symbol represents $\bullet = w_t^{\mathrm{AR1}}$ with $\phi = 0.99$, $\blacktriangle = w_t^{\mathrm{AR1}}$ with $\phi = 0.80$, $\square = w_t^{\mathrm{AR1}}$ with $\phi = 0$ (or $\square = w_t^{\mathrm{OLS}}$), $\triangle = w_t^{\mathrm{AR1}}$ with $\phi = -0.80$, and $\bigcirc = w_t^{\mathrm{AR1}}$ with $\phi = -0.99$.

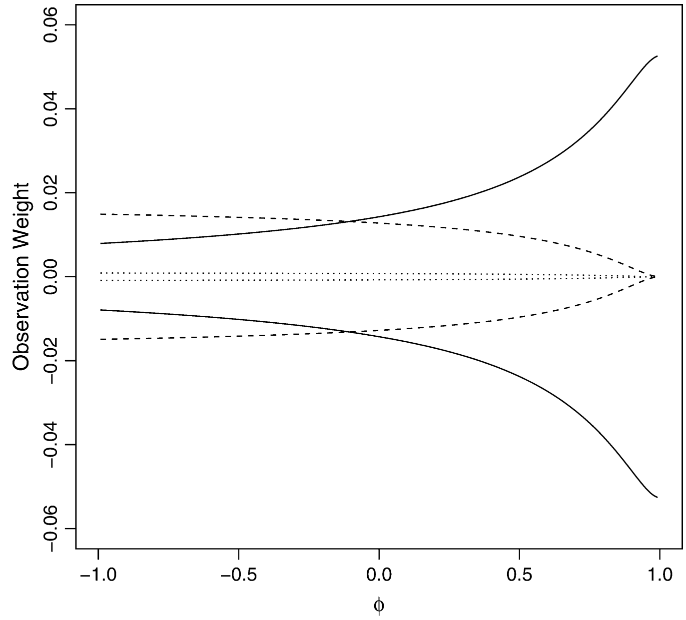

Figure 2. The first-order autoregressive observation weights on the weighted least squares trend estimator against ϕ for $n = 20$. Arranged in order from top to bottom at $\phi = 0.8$, we display w_n^{AR1} (solid), w_{n-1}^{AR1} (dashed), $w_{\bar{t}+0.5}^{\mathrm{AR1}}$ (dotted), $w_{\bar{t}-0.5}^{\mathrm{AR1}}$ (dotted), w_2^{AR1} (dashed), and w_1^{AR1} (solid).

- Relation to ordinary least squares estimator:

$$\hat{\beta}_{\mathrm{WLS}} = \frac{1}{\left\{1 + \dfrac{6\phi}{n(1-\phi)^2} - \dfrac{6(n-1)\phi^2}{n(n+1)(1-\phi)^2}\right\}} \times \left[\hat{\beta}_{\mathrm{OLS}} + \frac{6\phi\{(n-1)(1-\phi)+2\}}{n(n+1)(n-1)(1-\phi)^2}(Y_n - Y_1)\right].$$

Several interesting properties of weighted least squares estimation can be obtained from these explicit expressions. Figure 1 presents a plot of the weights $\{w_t^{\mathrm{AR1}}\}$ as in (4) against time t, with $w_t^{\mathrm{AR1}} = w_t^{\mathrm{OLS}}$ for $\phi = 0$, when $n = 20$. Notice that as observations are more positively correlated, relatively heavier weights are put on the first and last observations. This result implies that in positively strongly correlated time series regressions, even great changes among $Y_2, \ldots, Y_{n-1}$ cause little change to $\hat{\beta}_{\mathrm{WLS}}$. However, the weighted least squares estimate can be seriously vulnerable to even small changes in Y_1 and Y_n, more so than for the ordinary least squares counterpart. While several previous studies have discussed the effect of Y_1 on $\hat{\beta}_{\mathrm{WLS}}$ (cf., Poirier 1978; Maeshiro 1979; Park and Mitchell 1980), the effect of Y_n has received much less attention.

The vulnerability of $\hat{\beta}_{\mathrm{WLS}}$ to first and last observations is reinforced by considering the monotonicity of $\{w_t^{\mathrm{AR1}}\}$ as follows: for fixed n, (1) w_1^{AR1} is monotone-decreasing in ϕ with convergence to $-(n-1)^{-1}$ as $\phi \to 1$ and approaching $-3\{n(n-1)\}^{-1}$ as $\phi \to -1$; (2) w_t^{AR1} is monotone-increasing in ϕ for $1 < t < \bar{t}$ and decreasing in ϕ for $\bar{t} < t \leq n-1$, and converges to 0 as $\phi \to 1$ and to $12(t-\bar{t})\{n(n-1)(n-2)\}^{-1}$ as $\phi \to -1$; (3) w_n^{AR1} is monotone-increasing in ϕ and converges to $(n-1)^{-1}$ as $\phi \to 1$ and to $3\{n(n-1)\}^{-1}$ as $\phi \to -1$. Figure 2 graphically shows this monotonicity and convergence of w_t^{AR1} when $n = 20$. Incorporating these properties of w_t^{AR1} into (4) results in

$$\hat{\beta}_{\mathrm{WLS}} \to \begin{cases} \dfrac{Y_n - Y_1}{n-1}, & \text{as } \phi \to 1, \\ \hat{\beta}_{\mathrm{OLS}(1,n)} + \dfrac{3}{n}\left(\dfrac{Y_n - Y_1}{n-1}\right), & \text{as } \phi \to -1, \end{cases} \tag{7}$$

where $\hat{\beta}_{\mathrm{OLS}(1,n)}$ is the ordinary least squares trend estimator with Y_1 and Y_n excluded. See Bloomfield and Nychka (1992) for further statistical issues in climatology.

Some properties of the autocovariance weights in (5) are intriguing. Detailed calculations give $\sum_{h=1}^{n-1} s_h^{\mathrm{AR1}} = -1/2$ and $\sum_{h=0}^{n-1} u_h^{\mathrm{AR1}} = 0$. Also, we have that for fixed n, (1) as $\phi \to 1$,

$$u_h^{\mathrm{AR1}} \to \begin{cases} 2(n-1)^{-2}, & h = 0, \\ 0, & 1 \leq h \leq n-2, \\ -2(n-1)^{-2}, & h = n-1, \end{cases}$$

and (2) as $\phi \to -1$,

$$u_h^{\mathrm{AR1}} \to \begin{cases} 6(2n-5)\{n(n-1)^2(n-2)\}^{-1}, & h = 0, \\ 24\{n(n-1)(n-2) \\ \quad -(3n^2 - 6n - 1)h + 2h^3\} \\ \quad \times\{n(n-1)(n-2)\}^{-2}, & 1 \leq h \leq n-3, \\ -72(n-3)\{n^2(n-1)^2(n-2)\}^{-1}, & h = n-2, \\ -18\{n(n-1)\}^{-2}, & h = n-1. \end{cases}$$

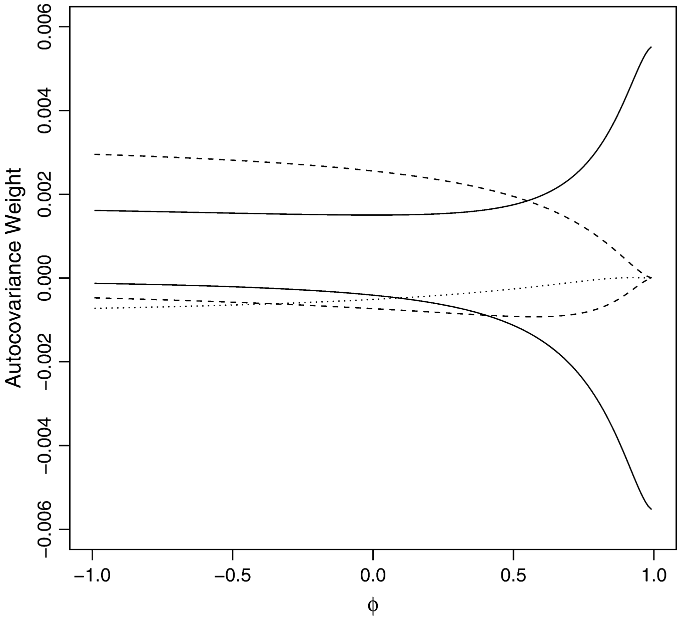

Figure 3. The first-order autoregressive variance and autocovariance weights on the variance of the weighted least squares trend estimator against ϕ for $n = 20$. From top to bottom at $\phi = 0.8$, we present u_0^{AR1} (solid), u_1^{AR1} (dashed), $u_{n/2-1}^{\mathrm{AR1}}$ (dotted), u_{n-2}^{AR1} (dashed), and u_{n-1}^{AR1} (solid).

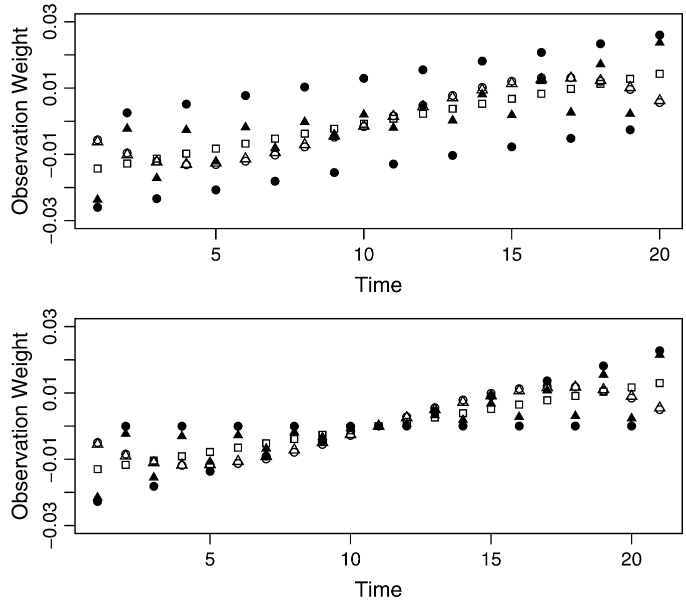

Figure 4. The first-order moving-average observation weights on the weighted least squares trend estimator against time for $n = 20$ (top) and $n = 21$ (bottom). Each symbol represents ● $= w_t^{\mathrm{MA1}}$ with $\theta = 0.99$, ▲$= w_t^{\mathrm{MA1}}$ with $\theta = 0.80$, □ $= w_t^{\mathrm{MA1}}$ with $\theta = 0$ (or □$= w_t^{\mathrm{OLS}}$), △ $= w_t^{\mathrm{MA1}}$ with $\theta = –0.80$, and ○ $= w_t^{\mathrm{MA1}}$ with $\theta = –0.99$.

Figure 3 compares $\{u_t^{\mathrm{AR1}}\}$ for various ϕ when $n = 20$. For larger ϕ, u_0^{AR1} and u_{n-1}^{AR1} get larger, suggesting that $\gamma(0)$ and $\gamma(n-1)$ take a relatively more significant role in evaluating the weighted least squares variance. This is as expected from (7).

The weighted least squares variance expression in (6) also provides a useful approximation: for large n,

$$\mathrm{var}(\hat{\beta}_{\mathrm{WLS}}) \approx \frac{\sigma^2}{(1-\phi)^2 \sum_{t=1}^{n} (t-\bar{t})^2},$$

which is in turn identical to the asymptotic variance of the ordinary least squares estimator (cf., Grenander 1954). When n is fixed, $\mathrm{var}(\hat{\beta}_{\mathrm{WLS}}) \approx \sigma^2/(n-1)$ if $\phi \approx 1$, and $\mathrm{var}(\hat{\beta}_{\mathrm{WLS}}) \approx 3\sigma^2/\{n(n-1)(n-2)\}$ if $\phi \approx -1$.

3.2 Model 2

Suppose that $\{\varepsilon_t\}$ is a first-order moving-average model as follows

$$\varepsilon_t = Z_t + \theta Z_{t-1}, \quad |\theta| < 1.$$

Use of the autocovariance function $\gamma(0) = (1+\theta^2)\sigma^2$, $\gamma(1) = \theta\sigma^2$, and $\gamma(h) = 0$ for $h \geq 2$, and the inverse of the covariance matrix of $\{\varepsilon_t\}$ as in Shaman (1969) reveals that

- Weighted least squares estimator:

$$\hat{\beta}_{\mathrm{WLS}} = \frac{\sum_{t=1}^{n} \left[\{1-(-\theta)^{n+1}\}t - \{1-(-\theta)^{t}+(-\theta)^{n+1-t}-(-\theta)^{n+1}\}\bar{t}\right] Y_t}{\sum_{t=1}^{n} (t-\bar{t})^2 \left\{1-(-\theta)^{n+1} + \frac{6\theta\left(1-(-\theta)^{n+1}\right)}{n(1+\theta)^2} + \frac{6(n+1)\theta^2\left(1-(-\theta)^{n-1}\right)}{n(n-1)(1+\theta)^2}\right\}}$$

$$:= \sum_{t=1}^{n} w_t^{\mathrm{MA1}} Y_t.$$

- Variance expression 1:

$$\mathrm{var}(\hat{\beta}_{\mathrm{WLS}}) = u_0^{\mathrm{MA1}}\gamma(0) + u_1^{\mathrm{MA1}}\gamma(1),$$

where

$$u_0^{\mathrm{MA1}} = \frac{1 - \frac{1}{(1-\theta)(1+\theta)^2}\sum_{i=1}^{2n+5} \psi_i^{(0)}(-\theta)^i}{\sum_{t=1}^{n} (t-\bar{t})^2 \left\{1-(-\theta)^{n+1} + \frac{6\theta\left(1-(-\theta)^{n+1}\right)}{n(1+\theta)^2} + \frac{6(n+1)\theta^2\left(1-(-\theta)^{n-1}\right)}{n(n-1)(1+\theta)^2}\right\}^2},$$

$$u_1^{\mathrm{MA1}} = \frac{2\left\{1 - \frac{1}{(1-\theta)(1+\theta)^2}\sum_{i=0}^{2n+5} \psi_i^{(1)}(-\theta)^i\right\}}{\sum_{t=1}^{n} (t-\bar{t})^2 \left\{1-(-\theta)^{n+1} + \frac{6\theta\left(1-(-\theta)^{n+1}\right)}{n(1+\theta)^2} + \frac{6(n+1)\theta^2(1-(-\theta)^{n-1})}{n(n-1)(1+\theta)^2}\right\}^2}.$$

- Variance expression 2:

$$\mathrm{var}(\hat{\beta}_{\mathrm{WLS}}) = \frac{\sigma^2\left\{(1+\theta)^2 - \frac{1}{(1-\theta)(1+\theta)^2}\sum_{i=1}^{2n+7} \tau_i(-\theta)^i\right\}}{\sum_{t=1}^{n} (t-\bar{t})^2 \left\{1-(-\theta)^{n+1} + \frac{6\theta(1-(-\theta)^{n+1})}{n(1+\theta)^2} + \frac{6(n+1)\theta^2(1-(-\theta)^{n-1})}{n(n-1)(1+\theta)^2}\right\}^2}.$$

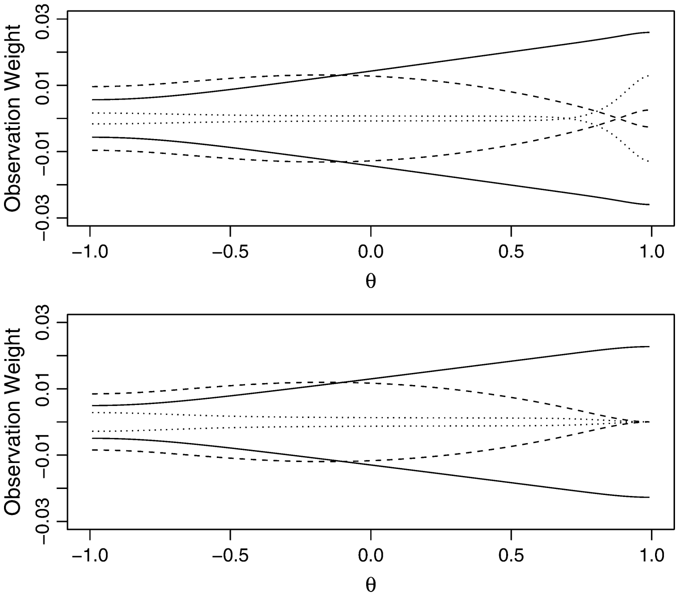

Figure 5. The first-order moving-average observation weights on the weighted least squares trend estimator against θ for $n = 20$ (top) and $n = 21$ (bottom). Displayed in order from top to bottom at $\theta = 0.8$ are w_n^{MA1} (solid), w_{n-1}^{MA1} (dashed), $w_{\bar{t}+0.5}^{\mathrm{MA1}}$ (dotted), $w_{\bar{t}-0.5}^{\mathrm{MA1}}$ (dotted), w_2^{MA1} (dashed), and w_1^{MA1} (solid).

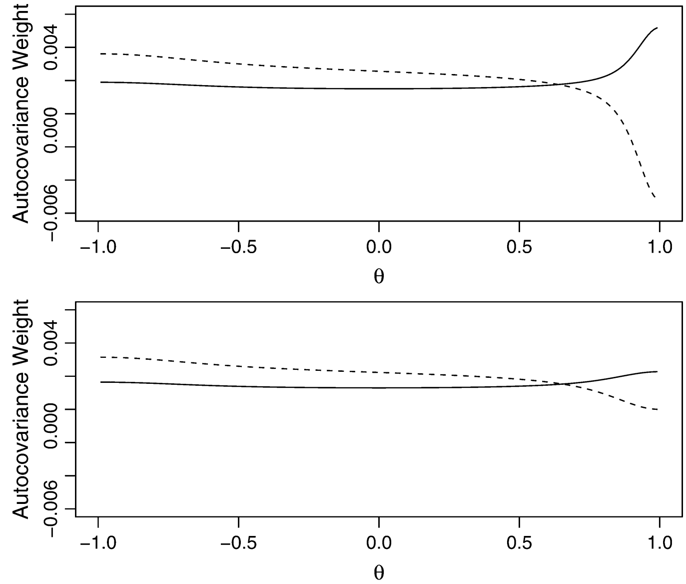

Figure 6. The first-order moving-average variance and autocovariance weights on the variance of the weighted least squares trend estimator against θ for $n = 20$ (top) and $n = 21$ (bottom). Presented are u_0^{MA1} (solid) and u_1^{MA1} (dashed).

- Relation to ordinary least squares estimator:

$$\hat{\beta}_{\mathrm{WLS}} = \frac{1}{\left\{1 + \dfrac{6\theta}{n(1+\theta)^2} + \dfrac{6(n+1)\theta^2(1-(-\theta)^{n-1})}{n(n-1)(1+\theta)^2(1-(-\theta)^{n+1})}\right\}} \times \left[\hat{\beta}_{\mathrm{OLS}} + \frac{6\sum_{t=1}^{n}\{(-\theta)^t - (-\theta)^{n+1-t}\}Y_t}{n(n-1)(1-(-\theta)^{n+1})}\right].$$

For nonzero constants $\psi_i^{(0)}$, $\psi_i^{(1)}$, and τ_i, refer to the appendix.

The edge effect on the weighted least squares estimate is also apparent. Figure 4 plots w_t^{MA1} against t when $n = 20$ and $n = 21$. Surprisingly, the different patterns of w_t^{MA1} are caused not only by θ, but also by whether n is even or odd. When n is odd and θ is close to 1, $w_t^{\mathrm{MA1}} \approx 0$ for even time points t, and as a result, $\hat{\beta}_{\mathrm{WLS}}$ greatly depends on the observations recorded at odd times. Figure 5 indicates that Y_1 and Y_n seriously affect the weighted least squares estimate particularly for θ close to 1.

Figure 6 shows the sample size effect, whether n is even or odd, on the variance and autocovariance weights. For large θ close to 1, u_0^{MA1} is much bigger when n is even than when n is odd. More interestingly, $u_1^{\mathrm{MA1}} \approx 0$ for odd n. This implies that, in the linear trend plus first-order moving-average error regression with an odd series length, the weighted least square variance is largely determined by $\gamma(0)$ if the series is strongly positive-correlated. This result is perhaps rather surprising, but can be explained as follows. For n odd, as shown in Figure 4, the observations during even time points are negligible in obtaining $\hat{\beta}_{\mathrm{WLS}}$. The autocovariance $\gamma(1)$, which is the covariance for even and next odd observations, has therefore little effect on the weighted least squares variance.

4. APPLICATIONS

4.1 Example 1

A 'short-cut' weighted least squares computing method can be developed for simple linear regressions under stationary autocorrelated error settings. As a motivating example, we consider the regression model (1) with $x_t = t$ in the presence of general stationary autocorrelated errors. The v_t as in (2) then satisfies $v_{n+1-t} = -v_t$ for $t = 1, \ldots, m$, where m is set to $n/2$ if n is even and to $(n-1)/2$ if n is odd. As a result, we have that

$$\hat{\beta}_{\mathrm{WLS}} = \frac{\sum_{t=1}^{m} v_{n+1-t}(Y_{n+1-t} - Y_t)}{\sum_{t=1}^{n} v_t(t - \bar{t})}. \tag{8}$$

Note that this expression is based on, if n is even, the $(n/2)$ "matched end-point differences" $Y_n - Y_1, \ldots, Y_{m+1} - Y_m$, and if n is odd, the $(n-1)/2$ differences $Y_n - Y_1, \ldots, Y_{m+2} - Y_m$.

In fact, if the covariate x_t is antisymmetric about $\bar{x}$, the alternative weighted least squares expression in (8) holds. For example, trend, periodic, and trend periodic input variables (cf., Kramer 1980), including $x_t = t$, $x_t = \sin(2\pi t/T)$, and $x_t = t + \sin(2\pi t/T)$, $t = 1, 2, \ldots, gT - 1$ with T as period of the series and g as the number of the cycles, produce $v_{n+1-t} = -v_t$.

To elaborate on this differenced series based model, we consider the following linear model, for even n (a similar exposition can be easily made also for odd n)

$$Y_d^{(m)} = A_d x^{(n)} \beta + \varepsilon_d^{(m)};$$

here, $Y_d^{(m)} = (Y_n - Y_1, \ldots, Y_{m+1} - Y_m)'$, $A_d = [-e(1), \ldots, -e(m), e(m), \ldots, e(1)]$ is an $m \times n$ "end-point difference matching" matrix with m dimensional column vector

$e(l) = \left[1_{\{i=l\}}\right]_{i=1}^{m}$, where $1_{\{i\,=\,l\}}$ is an indicator returning 1 when $i = l$, and $x^{(n)}$ is an antisymmetric input vector. Note that $Y_d^{(m)} = A_d Y^{(n)}$ and $\varepsilon_d^{(m)} = A_d \varepsilon^{(n)}$. With $C_d = \mathrm{var}\left(Y_d^{(m)}\right) = A_d \Gamma_n A_d^{'}$, we have

$$\hat{\beta}_{\mathrm{WLS}} = \left(x^{(n)\prime} A_d^{'} C_d^{-1} A_d x^{(n)}\right)^{-1} x^{(n)\prime} A_d^{'} C_d^{-1} Y_d^{(m)},$$
$$\mathrm{var}\left(\hat{\beta}_{\mathrm{WLS}}\right) = \left(x^{(n)\prime} A_d^{'} C_d^{-1} A_d x^{(n)}\right)^{-1}.$$

These results agree with the weighted least squares expressions in (2) and (3). An appealing point is that in the matched end-point differenced model, the reduced $m \times m$ dimensional C_d, instead of the full size $n \times n$ matrix Γ_n, is inverted for the weighted least squares estimate and variance. Also note that this affordable computing method can be applied to any stationary covariance structure setting. One can appreciate these results for an efficient and fast calculation in applications with large size data.

In the case where x_t is symmetric about $\bar{x}$, including $x_t = \pm\cos(2\pi t/T)$, $t = 1, 2, \ldots, gT - 1$, an efficient weighted least squares computing method can be similarly developed. For such x_t, we obtain that $v_{n+1-t} = v_t$ for $t = 1, \ldots, m$. The $(n + 1)/2$ dimensional (if n is odd; even n cases are similar) "matched end-point summations" $Y_1 + Y_n, \ldots, Y_m + Y_{m+2}, Y_{m+1}$ produce the equivalent weighted least squares results. That is, applying the $(m + 1) \times n$ end-point summation matching transform $A_s = [e(1), \ldots, e(m), e(m + 1), e(m), \ldots, e(1)]$ to $Y^{(n)}$, X_n, and $\varepsilon^{(n)}$, respectively, we rewrite the linear model as

$$Y_s^{(m+1)} = A_s X_n (\mu, \beta)^{'} + \varepsilon_s^{(m+1)},$$

where $Y_s^{(m+1)} = A_s Y^{(n)}$ and $\varepsilon_s^{(m+1)} = A_s \varepsilon^{(n)}$. The covariance matrix of $Y_s^{(m+1)}$ is then $C_s = \mathrm{var}\left(Y_s^{(m+1)}\right) = A_s \Gamma_n A_s^{'}$, and therefore, we have that

$$(\hat{\mu}_{\mathrm{WLS}}, \hat{\beta}_{\mathrm{WLS}})' = (X_n^{'} A_s^{'} C_s^{-1} A_s X_n)^{-1} X_n^{'} A_s^{'} C_s^{-1} Y_s^{(m+1)},$$
$$\mathrm{var}((\hat{\mu}_{\mathrm{WLS}}, \hat{\beta}_{\mathrm{WLS}})') = (X_n^{'} A_s^{'} C_s^{-1} A_s X_n)^{-1}.$$

Again, this end-point matching method inverts the lower $(m + 1) \times (m + 1)$ dimensional C_s.

4.2 Example 2

There is a wealth of literature on the necessary and sufficient conditions that ordinary least estimators be equal to weighted least squares estimators (cf., McElroy 1967; Balestra 1970; Bloomfield and Watson 1975; Baksalary 1988; Puntanen and Styan 1989). An explicit form of a new necessary and sufficient condition on $\{\varepsilon_t\}$ and $\{x_t\}$ for $\hat{\beta}_{\mathrm{OLS}} = \hat{\beta}_{\mathrm{WLS}}$ can be established. As implied by the closed-form expressions in (2), $\hat{\beta}_{\mathrm{OLS}} = \hat{\beta}_{\mathrm{WLS}}$, if and only if

$$v_t := m_{\cdot,\cdot} \sum_{s=1}^{n} x_s m_{s,t} - m_{\cdot,t} \sum_{s=1}^{n} x_s m_{s,\cdot} \propto x_t - \bar{x}. \qquad (9)$$

As a simple example, consider an identical variance and equal nonnegative autocorrelated error structure where the covariance matrix is $\Gamma_n = \sigma^2[(1 - \rho)I_n + \rho J_n]$. It is well known that $\hat{\beta}_{\mathrm{OLS}} = \hat{\beta}_{\mathrm{WLS}}$ (see McElroy 1967). The new condition (9) also verifies $\hat{\beta}_{\mathrm{OLS}} = \hat{\beta}_{\mathrm{WLS}}$, since

$$v_t = \frac{n(x_t - \bar{x})}{(1 - \rho)(1 - \rho + n\rho)} \propto x_t - \bar{x}.$$

APPENDIX: ALGEBRAIC DETAILS

For the linear trend plus first-order moving-average error regression model in Section 3, the weighted least squares trend estimator has variance with the nonzero $\psi_i^{(0)}$ and $\psi_i^{(1)}$ as follows:

$$\psi_1^{(0)} = \frac{12}{n},\ \psi_2^{(0)} = -\frac{6(n+5)}{n(n-1)},\ \psi_3^{(0)} = -\frac{6(n+1)}{n(n-1)},$$
$$\psi_{n+1}^{(0)} = \psi_{n+4}^{(0)} = \frac{4(2n^2+4n+3)}{n(n-1)},$$
$$\psi_{n+2}^{(0)} = \psi_{n+3}^{(0)} = -\frac{4(2n^2+4n-9)}{n(n-1)},$$
$$\psi_{2n+2}^{(0)} = -\frac{(n+2)(n+3)}{n(n-1)},\ \psi_{2n+3}^{(0)} = \frac{(n-10)(n+3)}{n(n-1)},$$
$$\psi_{2n+4}^{(0)} = 1 + \frac{12}{n},\ \psi_{2n+5}^{(0)} = -1,$$

and

$$\psi_0^{(1)} = \frac{3}{n},\ \psi_1^{(1)} = \psi_2^{(1)} = \frac{3(n-5)}{n(n-1)},$$
$$\psi_3^{(1)} = -\frac{3(3n+5)}{n(n-1)},\ \psi_n^{(1)} = \psi_{n+5}^{(1)} = \frac{3(n+1)^2}{n(n-1)},$$
$$\psi_{n+1}^{(1)} = \frac{1}{2}\psi_{n+2}^{(1)} = \frac{1}{2}\psi_{n+3}^{(1)} = \psi_{n+4}^{(1)} = -\psi_{2n+3}^{(1)}$$
$$= -\psi_{2n+4}^{(1)} = -\frac{(n-3)(n+5)}{n(n-1)},$$
$$\psi_{2n+2}^{(1)} = -\frac{(n+3)(n+5)}{n(n-1)},\ \psi_{2n+5}^{(1)} = -1 + \frac{3}{n}.$$

The only nonzero τ_i in the second expression of var $(\hat{\beta}_{\mathrm{WLS}})$ are

$$\tau_1 = \frac{6}{n},\ \tau_2 = -\tau_4 = -\frac{12}{n-1},$$
$$\tau_3 = \frac{12}{n(n-1)},\ \tau_5 = -\frac{6(n+1)}{n(n-1)},$$
$$\tau_{n+1} = \tau_{n+6} = \frac{2(n^2+2n+3)}{n(n-1)},$$
$$\tau_{n+2} = \tau_{n+5} = -\frac{6(n^2+2n-1)}{n(n-1)},$$
$$\tau_{n+3} = \tau_{n+4} = 4 + \frac{12}{n},\ \tau_{2n+2} = -\frac{(n+2)(n+3)}{n(n-1)},$$
$$\tau_{2n+3} = \frac{3(n+3)}{n-1},\ \tau_{2n+4} = -\frac{2(n-3)(n+2)}{n(n-1)},$$
$$\tau_{2n+5} = -\frac{2(n+5)}{n-1},\ \tau_{2n+6} = 3 + \frac{6}{n},\ \tau_{2n+7} = -1.$$

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