Bounded-Degree Spanning Trees

Problem Statement

Let $G = (V, E)$ be a connected undirected graph, and for each vertex $v \in V$ let $b_v \geq 0$ be an integer upper bound on its degree. The objective is to find a spanning tree T of G minimizing the maximum degree exceedance

$$\max_{v \in V}(d_T(v) - b_v),$$

where $d_T(v)$ denotes the degree of v in T.

Result

We present a local-search algorithm that either finds a feasible tree satisfying all bounds (i.e., $d_T(v) \leq b_v$ for all v), or else returns a tree whose maximum exceedance is at most one more than the optimal possible exceedance. Formally,

$$\max_{v \in V}(d_T(v) - b_v) \;\leq\; \mathrm{OPT}_{\mathrm{exc}} + 1,$$

where

$$\mathrm{OPT}_{\mathrm{exc}} = \min_{T' \in \text{ spanning tree}} \max_{v \in V}(d_{T'}(v) - b_v).$$

1 Setup and notation

- For a spanning tree T and vertex v define the *exceedance*

  $$e_T(v) := d_T(v) - b_v.$$

- For a fixed tree T let

  $$k := \max_{v \in V} e_T(v) \qquad (\text{so } k \geq 0).$$

- Define also

  $$D_{k-1} := \{v \in V :\, e_T(v) = k - 1\}, \qquad S := D_k \cup D_{k-1}.$$

  Note that D_k and D_{k-1} are disjoint, so $|S| = |D_k| + |D_{k-1}|$.

- Let F denote the set of edges of T that are incident to at least one vertex in S. Removing the $|F|$ edges of F from T splits the tree into exactly $|F| + 1$ connected components; call this collection of components $\mathcal{C}$.

2 Key lemma (lower bound on OPT)

Lemma 1. *Assume a local-reduction move is not available. Then for the current tree T with exceedance k we have*

$$\mathrm{OPT}_{\mathrm{exc}} \;\geq\; k-1.$$

Equivalently, if no local move is possible at level k, then any spanning tree T' must have some vertex of exceedance at least $k-1$.

Proof. We follow the combinatorial counting argument.

(1) Any spanning tree needs many incidences into S. The removal of the $|F|$ edges incident to S produced $|F|+1$ components (the set $\mathcal{C}$). To reconnect these components into a spanning tree one needs at least $|F|$ edges whose endpoints lie in different components of $\mathcal{C}$. By the assumption that no local-reduction move exists at level k, every edge that connects two distinct components of $\mathcal{C}$ must have at least one endpoint in S. Therefore, for any spanning tree T' we have that the total degree (incidence count) of vertices of S in T' is at least $|F|$:

$$\sum_{v\in S} d_{T'}(v) \;\geq\; |F|. \tag{1}$$

(2) Convert to exceedances. Subtract the bounds b_v from both sides of (1) to obtain

$$\sum_{v\in S} \big(d_{T'}(v)-b_v\big) \;\geq\; |F| - \sum_{v\in S} b_v.$$

Dividing by $|S|$ shows that the average exceedance of vertices in S in any spanning tree T' satisfies

$$\frac{1}{|S|}\sum_{v\in S} e_{T'}(v) \;\geq\; \frac{|F|-\sum_{v\in S} b_v}{|S|}. \tag{2}$$

Since the maximum exceedance of T' is at least its average exceedance on any nonempty subset, $\mathrm{OPT}_{\mathrm{exc}}$ is bounded below by the RHS of (2).

(3) Lower bound $|F|$ using the current tree T. In the current tree T the degree of each vertex in D_k equals b_v+k, and the degree of each vertex in D_{k-1} equals b_v+k-1. Therefore the sum of degrees of vertices in S, taken inside T, is

$$\sum_{v\in D_k}(b_v+k) \;+\; \sum_{v\in D_{k-1}}(b_v+k-1) \;=\; \sum_{v\in S} b_v \;+\; k|D_k| + (k-1)|D_{k-1}|.$$

This sum counts each edge of T with both endpoints in S twice; edges with exactly one endpoint in S are counted once. The number of edges inside S is at most $|S|-1$ because these edges form a forest on the vertex set S (they are a subgraph of the tree T). Thus the number of edges incident to S (which is $|F|$) satisfies

$$|F| \;\geq\; \sum_{v\in S} b_v \;+\; k|D_k| + (k-1)|D_{k-1}| \;-\; (|S|-1). \tag{3}$$

A small algebraic rearrangement yields

$$|F| - \sum_{v\in S} b_v \;\geq\; (k-1)|S| - |D_{k-1}| + 1. \tag{4}$$

(4) Combine (2) **and** (4)**.** Substituting (4) into (2) gives

$$\frac{1}{|S|}\sum_{v\in S} e_{T'}(v) \;\geq\; \frac{(k-1)|S| - |D_{k-1}| + 1}{|S|} \;=\; (k-1) - \frac{|D_{k-1}| - 1}{|S|}.$$

Since $|D_{k-1}| \leq |S| - 1$ (because D_k is nonempty), we have $0 \leq \frac{|D_{k-1}|-1}{|S|} < 1$. Therefore

$$\frac{1}{|S|}\sum_{v\in S} e_{T'}(v) \;>\; k-2 \qquad \text{and} \qquad \frac{1}{|S|}\sum_{v\in S} e_{T'}(v) \;\geq\; k-2+1 = k-1.$$

More importantly, the quantity on the left has ceiling at least $k-1$, which implies that the maximum exceedance in any spanning tree T' is at least $k-1$. In symbols,

$$\mathrm{OPT}_{\mathrm{exc}} \;\geq\; k-1.$$

This proves the lemma, giving the desired +1 guarantee.

$\square$