1.4 Ricorrenze

- $T(n) = 2T\left(\frac{n}{2}\right) + O(\sqrt{n})$

Verifichiamo se il metodo dell'esperto è applicabile

$a = 2$

$b = 2$

$f(n) = O(\sqrt{n})$

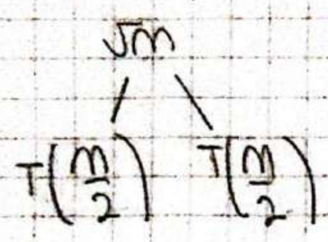
$n^{\log_2 2} = n$

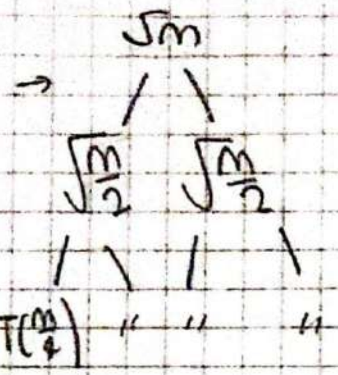
$\hookrightarrow f(n) = O(n^{\log_2 2 - \varepsilon})$, $\varepsilon = \frac{1}{2}$

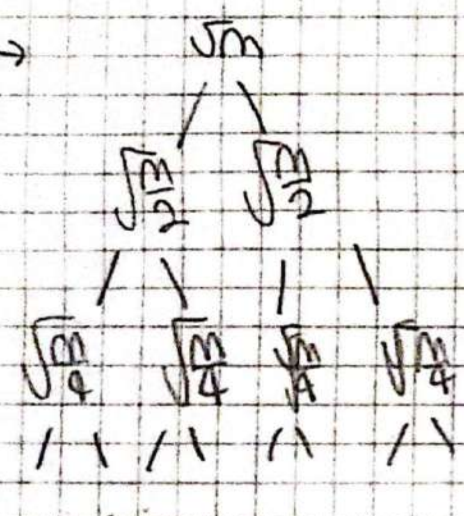
$= O(\sqrt{n})$

quindi, $T(n) = \Theta(n)$

Con l'albero di ricorsione:







n° nodi i-livello $\quad 2^i$

costo per nodo $\quad \sqrt{\frac{n}{2^i}}$

costo per livello $\quad 2^i\sqrt{\frac{n}{2^i}} = 2^i \frac{\sqrt{n}}{2^{i/2}}$

$= \sqrt{n \cdot 2^i}$

costo k-esimo $\quad \sqrt{\frac{n}{2^k}} = 1 \Rightarrow \frac{n}{2^k} = 1$

$\Rightarrow 2^k = n \Rightarrow k = \log_2 n$

calcolando il costo

$$T(n) = \sum_{i=0}^{\log_2 n - 1} \sqrt{2^i n} + \sqrt{2^{\log n}}\, T(1)$$

$$= \sqrt{n} \sum_{i=0}^{\log n - 1} \sqrt{2}^i + \underbrace{\sqrt{n}\, T(1)}_{\Theta(\sqrt{n})} = \sqrt{n}\left(\frac{1 - \overbrace{\sqrt{n}}^{2^{\log_2 \sqrt{n}}}}{\underbrace{1-2}_{-1}}\right) + \Theta(\sqrt{n})$$

$$= \sqrt{n}(\sqrt{n} - 1) + \Theta(\sqrt{n}) = n - \sqrt{n} + \Theta(\sqrt{n}) = \Theta(n)$$

Verifichiamo se la soluzione è corretta con il metodo di sostituzione

$$T(n) = 2T\left(\frac{n}{2}\right) + O(\sqrt{n})$$

assumendo sia valida per $\frac{n}{2}$, cioè che

$$T\left(\frac{n}{2}\right) \leq \frac{c}{2} n$$

dimostrare che

$$T(n) \leq \cancel{2} \frac{c}{\cancel{2}} n + O(\sqrt{n})$$

$$\leq cn + c_1 \sqrt{n} = \Theta(n)$$

- $T(n) = T(\sqrt{n}) + \Theta(\log \log n)$

applico cambio di variabile

$\hookrightarrow \log n = m \rightarrow n = 2^m$

$$T(2^m) = T(2^{m/2}) + \Theta(\log m)$$

$* \; T(2^m) = S(m)$

$$S(m) = S(m/2) + \Theta(\log m)$$

costruiamo l'albero di ricorsione

$$c \log m \quad \rightarrow \quad c \log m \quad \rightarrow \quad c \log m$$

$$S\left(\frac{m}{2}\right) \qquad c \log \frac{m}{2} \qquad c \log \frac{m}{2}$$

$$S\left(\frac{m}{4}\right) \qquad c \log \frac{m}{4}$$

$$\ldots$$

$$S(1)$$

Dimostriamo con il metodo di sostituzione della variabile di appoggio

$S(m) = S\left(\frac{m}{2}\right) + \Theta(\log m)$

assumendo sia valido per $m/2$, vale

$S\left(\frac{m}{2}\right) \leq c \log^2 \frac{m}{2}$

Vediamo

$S(m) \leq c \log^2 \frac{m}{2} + \Theta(\log m)$

$\leq c \log^2 m - c \log^2 2 + c_1 \log m$

$\leq c \log^2 m - c + c_1 \log m$

$\leq c \log^2 m + c_1 \log m \quad$ per $c \geq 1$

$= O(\log^2 m)$

(3) $T(n) = 10T(n/3) + 17n^{1.2}$

Verifichiamo col teorema dell'esperto

$a = 10$, $b = 3 \rightarrow n^{\log_b a} = n^{\log_3 10} \simeq n^{2.10}$

$f(n) = 17n^{1.2} = O(n^{2.1-\varepsilon})$ con $\varepsilon = 0.9$

si ricade nel I caso

$\Downarrow$

$T(n) = \Theta(n^{2.1})$

- albero delle ricorrenze

$17n^{1.2}$

/

$T\left(\frac{n}{3}\right)$ ... $\times 10$

$17n^{1.2}$

/

$17\left(\frac{n}{3}\right)^{1.2}$ ... $\times 10$

...

$T(1)$

costo di ogni nodo: $17\left(\frac{n}{3^i}\right)^{1.2}$

n° nodi per livello: 10^i

costo per livello: $17 \cdot 10^i \left(\frac{n}{3^i}\right)^{1.2}$

$$T(n) = \sum_{i=0}^{\log n - 1} 10^i \, 17 \left(\frac{n}{3^i}\right)^{1.2} + T(1)\left(\frac{10}{3^{1.2}}\right)^{\log_3 n}$$

$$= 17 n^{1.2} \sum_{i=0}^{\log n - 1} \left(\frac{10}{3^{1.2}}\right)^i + T(1)\, n^{\log \frac{10}{3^{1.2}} = 0.9}$$

$$= 17 n^{1.2} \left[\frac{1 - \left(\frac{10}{3^{1.2}}\right)^{\log n}}{1 - \frac{10}{3^{1.2}}} \right] + \Theta(n^{0.9})$$

$$= \Theta(n^{1.2})\,\Theta(n^{0.9}) + \Theta(n^{0.9}) = \Theta(n^{2.1})$$

④ $T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2}{3}n\right) + cn$

$$\begin{tikzcd}[column sep=small]
 & cn \arrow[dl, dash] \arrow[dr, dash] & \\
T\left(\frac{n}{3}\right) & & T\left(\frac{2}{3}n\right)
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & & & cn \arrow[dll, dash] \arrow[drr, dash] & & & \\
 & \frac{cn}{3} \arrow[dl, dash] \arrow[dr, dash] & & & & \frac{2cn}{3} \arrow[dl, dash] \arrow[dr, dash] & \\
T\left(\frac{n}{9}\right) & & T\left(\frac{2}{9}n\right) & & T\left(\frac{2n}{9}\right) & & T\left(\frac{4}{9}n\right)
\end{tikzcd}$$

$\frac{n}{3^i}$ costo dei rami sx $\qquad \left(\frac{2}{3}\right)^i n$ costo dei rami dx

a che profondità la complessità diventa costante?

$h \Rightarrow \frac{n}{3^k} = 1 \rightarrow \log_3 n = h$

$k \Rightarrow \left(\frac{2}{3}\right)^k n = 1 \Longrightarrow k = \log_{\frac{3}{2}} n$

↳ fino al ramo h costo x livello
è cm, dopo sarà $\leq cm$

$$\sum_{i=0}^{\log_{3/2} m} c_i \leq \sum_{i=0}^{\log_{3/2} m} cm = \left(\log_{3/2} m + 1\right) cm$$

$$\Longrightarrow \Theta\left(m \log_{3/2} m\right)$$

n° nodi per livello $\quad 1$

costo per nodo $\quad c \log \frac{n}{2^i}$

costo per livello $\quad c \log \frac{n}{2^i}$

costo costante $\quad c \log \frac{n}{2^k} = 1$

$\rightarrow \frac{n}{2^k} = 2 \rightarrow 2^{k+1} = n$

$\rightarrow 2^k = \frac{n}{2} \rightarrow k = \log_2 \frac{n}{2} = \log_2 n - 1$

$$S(n) = \sum_{i=0}^{\log_2 n - 1} c \underbrace{\log \frac{n}{2^i}}_{\log_2 n - i}$$

$$= c \sum_{i=0}^{\log_2 n - 1} \log_2 n - i = c \underbrace{\log_2 n \sum_{i=0}^{\log_2 n - 1} 1}_{} - c \underbrace{\sum_{i=0}^{\log_2 n - 1} i}_{\frac{1}{2} \log_2 n (\log_2 n - 1)}$$

$$= c \log_2 n (\log_2 n - 1) - \frac{c}{2} \log_2 n (\log_2 n - 1)$$

$$= \frac{c}{2} \log_2 n (\log_2 n - 1)$$

$$= \frac{c}{2} (\log_2^2 n - \log_2 n) = \Theta(\log^2 n)$$

$\hookrightarrow T(n) = \Theta(\log^2 \log n)$