CO370 Deterministic OR Models

LP Sensitivity Analysis

Let's begin with a short discussion of the connections between LP duality and the simplex method

LP in Standard Form

(Dual LP)

$$\begin{aligned} \min\ & c_1x_1 + \dots + c_nx_n \\ \text{s.t. } & a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ & \vdots \\ & a_{m1}x_1 + \dots + a_{mn}x_n = b_m \\ & x_1 \geq 0 \ \dots \ x_n \geq 0 \end{aligned} \quad \overset{\text{Dual LP}}{\Longrightarrow} \quad \begin{aligned} \max\ & y_1b_1 + \dots + y_mb_m \\ \text{s.t. } & y_1a_{11} + \dots + y_ma_{m1} \leq c_1 \\ & \vdots \\ & y_1a_{1n} + \dots + y_ma_{mn} \leq c_n \end{aligned}$$

(Primal LP)

(Dual variable y_i for each primal constraint i; dual constraint for each primal variable j.)

In matrix form, $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$, $A = [a_{ij}]$

$$\begin{aligned} \min\ & c^Tx \\ \text{s.t. } & Ax = b \\ & x \geq 0 \end{aligned} \quad \overset{\text{Dual LP}}{\Longrightarrow} \quad \begin{aligned} \max\ & y^Tb \\ \text{s.t. } & y^TA \leq c^T \end{aligned}$$

Weak Duality: If $\bar{x}$ is a solution to the primal LP and $\bar{y}$ is a solution to the dual LP, then

$$c^T\bar{x} \geq \bar{y}^T A\bar{x} = \bar{y}^T b. \quad \boxtimes$$

So the dual objective value gives a lower bound on the primal objective value, and viceversa.

(Strong) LP Duality Theorem (von Neumann 1947): If the primal and dual LPs both have feasible solutions, then $\exists$ an optimal soln. x^* to the primal LP and an optimal solution y^* to the dual LP and

$$c^T x^* = y^{*T} b. \quad \boxtimes$$

One way to prove the LP Duality Theorem is to show there is a version of the simplex method that terminates in a finite # of steps.

Simplex Tableau

In our discussion, we will assume the $m \times n$ matrix A has full row rank.

Let A_j denote the j^{th} column of A. So $A = [a_1 \cdots a_n]$

(3)

Let B be an ordered set of indices $(B_1, \dots, B_m)$ from $\{1, \dots, n\}$. The matrix $\mathbb{B} \equiv A_B$ consists of the columns $A_{B_1}, \dots, A_{B_M}$

$$A_B = [A_{B_1} \cdots A_{B_m}]$$

B is called a <u>basis header</u> and determines a <u>basis</u> if $\mathbb{B}$ is non-singular (that is, $\det \mathbb{B} \neq 0$).

Let N denote the <u>non-basic</u> indices $\{1, \dots, n\} \setminus B$.

We can write the LP as

$$\min \; c_B^T x_B + c_N^T x_N$$

$$\text{s.t.} \quad A_B x_B + A_N x_N = b$$

$$x_B \geq 0 \quad x_N \geq 0$$

and the dual LP is

$$\begin{aligned} \max \; & y^T b \\ \text{s.t.} \; & y^T A_B \leq c_B^T \\ & y^T A_N \leq c_N^T \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \max \; & b^T y \\ \text{s.t.} \; & A_B^T y \leq c_B \\ & A_N^T y \leq c_N \end{aligned}$$

④

The <u>basic solution</u> determined by B is

$$\bar{x}_B = B^{-1}b, \quad \bar{x}_N = 0$$

The simplex method proceeds from one basic solution to another, until a basic solution is found that can be proven to be an optimal solution for the LP model (or the LP is shown to be unbounded — we will assume we have an LP feasible soln.)

A basis header B is called <u>primal feasible</u> if

$B^{-1}b \geq 0$. That is, the basic solution is a feasible

solution to the LP.

To describe a version of the simplex tableau, introduce an objective variable $z = c^T x$. So the LP objective becomes min z and we can write the constraints as

$$B x_B + A_N x_N = b$$
$$c_B^T x_B + c_N^T x_N = z$$

Multiplying by B^{-1} we obtain

$$x_B + B^{-1} A_N x_N = B^{-1} b$$
$$c_B^T x_B + c_N^T x_N = z$$

To make it easy to check the basic solution is or is not optimal, we can multiply the first m constraints by c_B^T and subtract the sum from the z constraint

$$x_B + \mathbb{B}^{-1} A_N x_N = \mathbb{B}^{-1} b$$

$$0 + (c_N^T - c_B^T \mathbb{B}^{-1} A_N) x_N = z - c_B^T \mathbb{B}^{-1} b$$

NOTE: This tableau form is to understand the simplex method data. An efficient implementation of the simplex method will never actually compute $\mathbb{B}^{-1}$, but rather solve linear equation systems to obtain the information needed to move to the next basic solution.

From the tableau, we can read off that the basic solution is

$$(\bar{x}_B, \bar{x}_N) = (\mathbb{B}^{-1} b, 0)$$

and the objective value is $z = c_B^T \mathbb{B}^{-1} b$.

The crucial observation is that the feasible basic solution is optimal if

$$c_N^T - c_B^T \mathbb{B}^{-1} A_N \geq 0$$

This follows from the fact that currently $\bar{x}_N = 0$

6

and making the value of any variable in x_N positive can only increase the objective value z

$$z = c_B^T \mathbb{B}^{-1} b + \underbrace{(c_N^T - c_B^T \mathbb{B}^{-1} A_N)}_{\geq 0} \underbrace{x_N}_{\geq 0}$$

(If a component j in $c_N^T - c_B^T \mathbb{B}^{-1} A_N$ is negative, then the simplex method will consider ~~x_j as~~ j as a candidate to enter the basis header.)

Let's now connect the tableau information with LP duality

The dual LP is

$$\begin{aligned} & \max\ y^T b \\ \text{s.t.}\quad & y^T \mathbb{B} \leq c_B^T \\ & y^T A_N \leq c_N^T \end{aligned}$$

From the tableau, set $\bar{y} = c_B^T \mathbb{B}^{-1}$. Then

$$\bar{y}^T \mathbb{B} = c_B^T \mathbb{B}^{-1} B = c_B^T$$

So $\bar{y}$ satisfies the first constraint. And

$$\bar{y}^T A_N = c_B^T \mathbb{B}^{-1} A_N$$

So $\bar{y}$ satisfies the second constraint $\Longleftrightarrow$

$$c_N^T - c_B^T \mathbb{B}^{-1} A_N \geq 0 ..$$

Thus, the simplex method optimality condition is precisely that the dual solution $\bar{y}$ is feasible.

Note that the dual objective is

$$\bar{y}^T b = c_B^T B^{-1} b$$

which matches the primal objective. So the primal and dual solutions are both optimal.

The component j of $c_N^T - c_B^T B^{-1} A_N$ is called the <u>reduced cost</u> of the variable x_j. Another way of stating the optimality conditions is to say that all reduced costs must be non-negative. Note that the reduced cost of every basic variable is 0.

Sensitivity Analysis

We will use the simplex tableau and optimality conditions to discuss how small changes to the LP model can impact optimal solutions.

Adding a new variable x_{n+1}

Let's start with an easy case. Suppose we would like to introduce a new variable x_{n+1}, say representing a new product. The now x_{n+1} has associated data c_{n+1} and A_{n+1}. The modified LP model is

$$\begin{aligned} \min \quad & c^T x + c_{n+1} x_{n+1} \\ \text{s.t.} \quad & Ax + A_{n+1} x_{n+1} = b \\ & x \geq 0, \; x_{n+1} \geq 0 \end{aligned}$$

Since the number of rows m did not change, $\mathbb{B}$ remains a basis header and we add $n+1$ to the non-basic indices N.

The new basic solution $\bar{x}$ sets $\bar{x}_m = 0$ and keeps all other variables at their same value.

Since b has not changed we still have $\mathbb{B}^{-1} b \geq 0$, so the new basic solution is feasible.

To check if the solution is optimal, we compute the reduced cost for x_{n+1}

$$\hat{c}_{n+1} = c_n - c_B^T \mathbb{B}^{-1} A_{n+1}$$

If $\hat{c}_{n+1} \geq 0$, then we have an optimal solution. (That is, we can keep $x_{n+1} = 0$)

If $\hat{c}_{n+1} < 0$, then we may be able to improve the solution. We would continue the simplex method, starting with $n+1$ entering the basic set.

Note: It may well take more than one simplex pivot to reach a new optimal solution, but it is typically a small number of pivots (and much faster than solving the problem from scratch).

Changing a right-hand-side value b_i

We now consider changing a RHS value b_i, replacing ~~b_i by~~ b_i by $b_i + \Delta$ for some value Δ. To write this in matrix form, let e_i denote the vector with 1 in the i^{th} component and 0 in every other component. So we have $b \to b + \Delta e_i$

The change to b does not alter the reduced costs of the variables, so the optimality conditions hold. But the current basic solution may no longer be feasible. To maintain feasiblity, we need

$$\mathbb{B}^{-1}(b + \Delta e_i) \geq 0.$$

That is, $\mathbb{B}^{-1}b + \Delta \mathbb{B}^{-1}e_i = \bar{x}_B + \Delta \mathbb{B}^{-1}e_i \geq 0.$

Let $d = \mathbb{B}^{-1}e_i$, that is, d is the i^{th} column of $\mathbb{B}^{-1}$.

So the feasibility condition is

$$\bar{x}_B + \Delta d \geq 0$$

Writing the components of $d^T = (d_1, \dots, d_m)$

we can write all the feasibility condition as

$$\bar{x}_{B_j} + \Delta d_j \geq 0 \quad \text{for } j = 1, \dots, m$$

This allows us to find the range of values Δ such that the current basic solution remains optimal.

For each j such that $d_j > 0$ we need

$$\Delta \geq \frac{-\bar{x}_{B_j}}{d_j} \qquad (*)$$

and for each j such that $d_j < 0$ we need

$$\Delta \leq \frac{-\bar{x}_{B_j}}{d_j} \qquad (**)$$

We can obtain the range of allowed values of Δ by taking the max of $(*)$ and the min of $(**)$

If Δ is outside of this range, then the current basic solution is not feasible and we would need further work with the simplex method to restore feasibility

Changing an objective coefficient c_j

Like in the RHS case, suppose c_j is replaced by $c_j + \Delta$.

In this case, primal feasibility is not impacted, so we need only to focus on the optimality conditions

If j is not in the basic set B, then c_B is not changed.

In this case, we only need to examine the reduced cost of variable x_j

$$\hat{c}_j = c_j - c_B^T B^{-1} A_j$$

The reduced cost changes to

$$\hat{\hat{c}}_j = c_j + \Delta - c_B^T B^{-1} A_j = \hat{c}_j + \Delta$$

So the current basis is optimal if $\hat{c}_j + \Delta \geq 0$,

that is, $\Delta \geq -\hat{c}_j$. (From this you can see

(12)

why the term "reduced cost" is used.

If j is included in the basic set B, then changing c_j will change c_B and possibly impact the reduced cost of one or more of the ~~ba~~ non-basic variables N.

Let k be the index in B of variable j, that is, $B_k = j$. Then $c_B \to c_B + \Delta e_k$. For each non-basic index ℓ, the ~~reduced~~ optimality condition is

$$c_\ell - (c_B + \Delta e_k)^T B^{-1} A_\ell \geq 0$$

In other words

$$c_\ell - c_B^T B^{-1} A_\ell - \Delta e_k^T B^{-1} A_\ell \geq 0$$

This can be written as

$$\hat{c}_\ell - \Delta f_{k\ell} \geq 0$$

where $\hat{c}_\ell$ is the old reduced cost for ℓ and $f_{k\ell}$ is the k^{th} entry in the vector $B^{-1} A_\ell$.

(13)

The inequalities $\Delta f_{k\ell} \leq \hat{c}_\ell$ let us compute the range of Δ such that the current basis is still optimal.