MATH2715 Workshop 7 Questions

Only question **Q4** counts towards the continuous assessment for this module.

Q1. Suppose $X_i \overset{\text{ind}}{\sim} \mathrm{N}(0, \sigma^2)$ and σ^2 is estimated using

$$T = \frac{1}{n}\sum_{i=1}^{n} X_i^2.$$

Obtain the mean of T, the bias of T, the variance of T, and the mean square error of T.

Hint: To obtain $\mathrm{E}[T^2]$ notice that $\displaystyle\left(\sum_{i=1}^{n} Y_i\right)^2 = \left(\sum_{i=1}^{n} Y_i\right)\left(\sum_{j=1}^{n} Y_j\right) = \sum_{i=1}^{n} Y_i^2 + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ i\neq j}}^{n} Y_i Y_j.$

Remember that if Y_i and Y_j are independent, then $\mathrm{E}[Y_iY_j] = \mathrm{E}[Y_i]\mathrm{E}[Y_j]$. How many terms are in the summation with $i \neq j$? Now put $Y_i = X_i^2$.

Hint: If $X \sim \mathrm{N}(\mu, \sigma^2)$, then $\mathrm{E}[(X-\mu)^4] = 3\sigma^4$ (see lecture 3 notes).

Q2. If $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables with a $\mathrm{N}(\mu, \sigma_0^2)$ distribution with known variance σ_0^2, obtain the maximum likelihood estimator for μ.

Q3. The time interval X between successive feedings of a certain type of insect has an exponential distribution with mean proportional to a positive measured characteristic z of the insect. Thus the probability density function of X for an insect with characteristic z is

$$f_X(x; z, \theta) = \frac{1}{\theta z}\exp\left\{-\frac{x}{\theta z}\right\}, \quad x > 0.$$

Suppose we observe n independent insects, observing time intervals $X_1, X_2, \ldots, X_n$ with associated characteristics $z_1, z_2, \ldots, z_n$.

(a) Find the maximum likelihood estimator $\widehat{\theta}$ of θ and show that it is unbiased.

Hint: $X_i \overset{\text{ind}}{\sim} \text{exponential}(\lambda_i = 1/(\theta z_i))$.

(b) Show that $\mathrm{Var}[\widehat{\theta}] = \theta^2/n$.

Hint: The z_i are known constants. Only θ is an unknown parameter to be estimated.

Q4. **(WILL BE MARKED)** An individual taken from a very large biological population is of type A with probability $\theta = \frac{1}{2}(1+\phi)$ and of type B with probability $\frac{1}{2}(1-\phi)$ where $0 \le \phi \le 1$, all individuals being independent.

(a) Suppose that X denotes the number of type A individuals in a random sample of size n. What is the probability $\mathrm{pr}\{X = x; \theta\}$ that $X = x$? (Write your answer as a function of ϕ.)

Hint: Individuals are independent and either of type A or type B.

(b) Find the maximum likelihood estimator $\widehat{\phi}$ of ϕ based upon x, and show that it is unbiased.

Hint: To obtain $\mathrm{E}[\widehat{\phi}]$ recall the distribution of X. The likelihood function is $L(\theta; x) = \mathrm{pr}\{X = x; \theta\}$. Write this as a function of ϕ.

(c) Find $\mathrm{Var}[\widehat{\phi}]$.