Study of Prime Pairs

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Abstract

By using a new method with Sieves, we can find all kinds of prime pairs including the twin primes, and prove that for any natural number *k*, there are infinitely many pairs of primes that differ by *2k*. Also, by exploring the density of prime pairs that add together to equal to an even number, we prove that for every even number $n > 2$, there exists at least one pair of primes p and q such that $n = p + q$.

Contents

1. Introduction

A *prime number* (or a *prime*) is a natural number greater than 1 that has no positive divisors other than 1 and itself. We start with some basic notations.

We denote by N the set of all natural numbers, $P[X]$ the set of all prime numbers in the set of X, $P = P[N]$ the set of all prime numbers, $\{u_i\}$ a sequence with a general term u_i, lnx the natural logarithm of x, $\pi(x)$ the number of primes less than or equal to x, $P(x) = \{2, 3, 5, \ldots p\}$the set of all primes less than or equal to x, $gcd(a, b)$the greatest common divisor of a and b.

We say that a and b are *relatively prime* if $gcd(a, b) = 1$ ([2]).

It is well known that the *sieve of Eratosthenes* ([1]) has long been used in the study of prime numbers.

For a large positive number n, let $S = \{m, 1 < m \leq n\}$be the set of natural numbers great than 1 and less than or equal to n. Then we can use the sieve of Eratosthenes to find out all prime numbers in the set S such that:

$$(1)P[S] = S - \bigcup_{p \in P(\sqrt{n})} S_p, \tag{1}$$

$$whereS_p = \left\{mp, m \in Nandp \leq m \leq \frac{n}{p}\right\}, forallpP(\sqrt{n}). \tag{2}$$

We introduce *Euler ϕ- function,* $\phi(n)$, the number of natural numbers less than or equal to $n$that are relatively prime to n([2], ([1]).

For any large positive number x, let

$$N = \prod_{p \in P(\sqrt{x})} p, \tag{3}$$

the product of all primes less than or equal to $\sqrt{x}$;

and let

$$X(z) = \{m, 1 \leq m \leq zand \gcd\gcd(m, N) = 1\}, for1 < z \leq N, \tag{4}$$

the set of natural numbers less than or equal to z that are relatively prime to N.

Then from the definition of *ϕ- function,* we know that the size of $X(N)$ is:

$$\phi(N) = N \prod_{p \in P(\sqrt{x})} (1 - \frac{1}{p}). \tag{5}$$

Notice that all numbers in the set $X(x)$ are primes and $X(x) = P(x) - P\left(\sqrt{x}\right) + \{1\}$, therefore, thesize of $X(x)$ is:

$$\pi(x) - \pi\left(\sqrt{x}\right) + 1 = C_1 \frac{x}{N} \phi(N) = C_1 x \prod_{p \in P(\sqrt{x})} (1 - \frac{1}{p}), \tag{6}$$

for some constant C_1.

For Example: take $x = 100$, then $\sqrt{x} = 10$, $P\left(\sqrt{x}\right) = \{2, 3, 5, 7\}$, $N = 2 \cdot 3 \cdot 5 \cdot 7 = 210$,

$$X(210) = \{1, 11, 13, \ldots, 97, \ldots, 209\} = P(210) - P(10) + \{1, 121, 143, 169, 187, 209\} \tag{7}$$

$$\phi(210) = 210\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right) = 48, \tag{8}$$

$$X(100) = \{1, 11, 13, 17, \ldots, 97\} = P(100) - P(10) + \{1\}, \tag{9}$$

thesize of $X(100)$ is: $\pi(100) - \pi(10) + 1 = 25 - 4 + 1 = 22$, compare this with

$$C_1 \frac{100}{210}\phi(210) = C_1 \frac{100}{210} 48 \approx 22.86C_1 \tag{10}$$

we expect that $0 < C_1 < 1$.
So, if we use the Sieve to S to get $P[S]$as (1), then we have the following approximations of $\pi(n)$, the size of $P[S]$:

$$C_1 n \prod_{p\leq\sqrt{n}}\left(1-\frac{1}{p}\right) = C_1 n\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\ldots\left(1-\frac{1}{q}\right), 2 < 3 < \ldots < q \leq \sqrt{n}. \tag{11}$$

In fact, from ([1]), we know that C_1 is about $\frac{e^{\gamma}}{2} \approx 0.890536209\ldots$, where γ is Euler's constant,

$$and \prod_{p\in P(\sqrt{x})}\left(1-\frac{1}{p}\right) \sim \frac{2e^{-\gamma}}{\ln\ln x}, \tag{12}$$

so, we have

$$\pi(x) \sim \frac{e^{\gamma}}{2} x \prod_{p\in P(\sqrt{x})}\left(1-\frac{1}{p}\right) \sim \frac{x}{\ln\ln x}, \tag{13}$$

$$or \pi(x) \sim Li(x) = \int_2^x \frac{dt}{lnt}. \tag{14}$$

1. Prime Pairs

Now we study the prime pairs, that is, for $k \in N$ both $pandp + 2k$ are primes. We know that the *twin primes* are the special case when $k = 1$.

To find all twin primes $(p, p+2)$ such that $p \leq n$, we only need to find the first prime p in the pair by using the Sieves to the set $S = \{m, 1 < m \leq n\}$as showing in the following set:

$$(2)\, P_1 P[S] = S - \bigcup_{p\in P(\sqrt{n})} S_p - \bigcup_{p\in P\left(\sqrt{n+2}\right)} S_p(2) = P[S] - \bigcup_{p\in P\left(\sqrt{n+2}\right)} S_p(2), \tag{15}$$

$$where S_p = \left\{mp, m \in N and p \leq m \leq \frac{n}{p}\right\} forall p P\left(\sqrt{n}\right), \tag{16}$$

$$and S_p(2) = \left\{mp - 2, m \in N and p \leq m \leq \frac{n+2}{p}\right\} forall p P(\sqrt{n+2}). \tag{17}$$

For Example: when $n = 100$, then $S = \{2, 3, \ldots, 99, 100\}$,

$S_2 = \{4, 6, 8, \ldots, 98, 100\}$, $S_3 = \{9, 12, 15, \ldots, 96, 99\}$,

$S_2(2) = \{2, 4, 6, \ldots, 96, 98, 100\}$, $S_3(2) = \{7, 10, 13, \ldots, 94, 97, 100\}$,

$S_5 = \{25, 30, 35, \ldots, 95, 100\}$, $S_7 = \{49, 56, 63, \ldots, 91, 98\}$,

$S_5(2) = \{23, 28, 33, \ldots, 93, 98\}$, $S_7(2) = \{47, 54, 61, \ldots, 89, 96\}$.

From

$$P_1P[S] = S - \{S_2 \cup S_2(2) \cup S_3 \cup S_3(2) \cup S_5 \cup S_5(2) \cup S_7 \cup S_7(2)\} = \{3, 5, 11, 17, 29, 41, 59, 71\}, \tag{18}$$

we find out all 8 twin primes less than 100:

$$\{(3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), (59, 61), (71, 73)\}. \tag{19}$$

See the following Table 1. for the illustration:

s	n	n+2	s	n	n+2	s	n	n+2	s	n	n+2	s	n	n+2
		3	21	21	23	41	41	43	61	61	63	81	81	83
2	2	4	22	22	24	42	42	44	62	62	64	82	82	84
3	3	5	23	23	25	43	43	45	63	63	65	83	83	85
4	4	6	24	24	26	44	44	46	64	64	66	84	84	86
5	5	7	25	25	27	45	45	47	65	65	67	85	85	87
6	6	8	26	26	28	46	46	48	66	66	68	86	86	88
7	7	9	27	27	29	47	47	49	67	67	69	87	87	89
8	8	10	28	28	30	48	48	50	68	68	70	88	88	90
9	9	11	29	29	31	49	49	51	69	69	71	89	89	91
10	10	12	30	30	32	50	50	52	70	70	72	90	90	92
11	11	13	31	31	33	51	51	53	71	71	73	91	91	93
12	12	14	32	32	34	52	52	54	72	72	74	92	92	94
13	13	15	33	33	35	53	53	55	73	73	75	93	93	95
14	14	16	34	34	36	54	54	56	74	74	76	94	94	96
15	15	17	35	35	37	55	55	57	75	75	77	95	95	97
16	16	18	36	36	38	56	56	58	76	76	78	96	96	98
17	17	19	37	37	39	57	57	59	77	77	79	97	97	99
18	18	20	38	38	40	58	58	60	78	78	80	98	98	100
19	19	21	39	39	41	59	59	61	79	79	81	99	99	
20	20	22	40	40	42	60	60	62	80	80	82	100	100	

Table 1.

Or from

$$P[S] = P(100) = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}, \tag{20}$$

and $S_2(2) = \{2\}, S_3(2) = \{7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97\}, S_5(2) = \{23, 53, 83\}, S_7(2) = \{47, 89\}$

we also have $P_1P[S] = P[S] - \{S_2(2) \cup S_3(2) \cup S_5(2) \cup S_7(2)\} = \{3, 5, 11, 17, 29, 41, 59, 71\}$.

See the following Table 2.

Table 2.

On the other hand, we can find the second prime in the pair by using the Sieve to the following set:

$$P[S](2) = \{p + 2, p \in P[S]\}, \tag{21}$$

$$asshowinginthisset : Q_1P[S] = P[S](2) - \bigcup_{p \in P\left(\sqrt{n+2}\right)} S_p, \tag{22}$$

$$whereS_p = \left\{mp, m \in Nandp \le m \le \frac{n+2}{p}\right\} forallpP\left(\sqrt{n+2}\right). \tag{23}$$

Take the above Example again: $n = 100$, $S = \{2, 3, \ldots, 99, 100\}$,
from $P[S](2) = \{4, 5, 7, 9, 13, 15, 19, 21, 25, 31, 33, 39, 43, 45, 49, 55, 61, 63, 69, 73, 75, 81, 85, 91, 99\}$,
and $S_2 = \{4\}, S_3 = \{9, 15, 21, 33, 39, 45, 63, 69, 75, 81, 99\}, S_5 = \{25, 45, 55, 75, 85\}, S_7 = \{49, 63, 91\}$,
we have $Q_1P[S] = P[S](2) - \{S_2 \cup S_3 \cup S_5 \cup S_7\} = \{5, 7, 13, 19, 31, 43, 61, 73\}$.
See the following Table 3.

p	p+2	p	p+2	p	p+2	p	p+2	p	p+2
2	4	13	15	31	33	53	55	73	75
3	5	17	19	37	39	59	61	79	81
5	7	19	21	41	43	61	63	83	85
7	9	23	25	43	45	67	69	89	91
11	13	29	31	47	49	71	73	97	99

Table 3.

Let $N_1 = N - \{1\}$ be the set of natural numbers greater than 1. Then to find all twin primes we can find the first prime in the pair by using the Sieves to N_1 as shown in the following set:

$$(3)\, P_1P\,[N] = N_1 - \bigcup_{p\in P}\{N_p \cup N_p(2)\} = N_1 - \bigcup_{p\in P} N_p - \bigcup_{p\in P} N_p(2) = P - \bigcup_{p\in P} N_p(2), \tag{24}$$

$$where N_p = \{mp, m \in N and m \geq p\}\, for all p \in P, \tag{25}$$

$$and N_p\,(2) = \{mp - 2, m \in N and m \geq p\}\, for all p \in P. \tag{26}$$

Or we can find the second prime in the pair by using the Sieve to the following set:

$$P(2) = \{p + 2, p \in P\}\,, \tag{27}$$

$$such that Q_1P\,[N] = P\,(2) - \bigcup_{p\in P} N_p, \tag{28}$$

$$where N_p = \{mp, m \in N and m \geq p\}\, for all p \in P. \tag{29}$$

Similarly, for any $k \in N$, to find the general prime pairs $(p, p + 2k)$ such that $p \leq n$, we only need to find the first prime p in the pair by using the Sieves to $S = \{m, 1 < m \leq n\}$ as shown in the following set:

$$(4)\, P_kP\,[S] = S - \bigcup_{p\in P(\sqrt{n})} S_p - \bigcup_{p\in P\left(\sqrt{n+2k}\right)} S_p\,(2k) = P\,[S] - \bigcup_{p\in P\left(\sqrt{n+2k}\right)} S_p\,(2k)\,, \tag{30}$$

$$where S_p = \left\{mp, m \in N and p \leq m \leq \frac{n}{p}\right\} for all p P\left(\sqrt{n}\right), and \tag{31}$$

$$S_p\,(2k) = \left\{mp - 2k, m \in N and Max\left(p, \frac{1+2k}{p}\right) \leq m \leq \frac{n+2k}{p}\right\} for all p P(\sqrt{n+2k}). \tag{32}$$

Again, to find all prime pairs (p, p+2k) we only need to find the first prime in the pair by using the Sieves to $_1$ as showing in the following set:

$$(5)\, P_kP\,[N] = N_1 - \bigcup_{p\in P}\{N_p \cup N_p(2k)\} = N_1 - \bigcup_{p\in P} N_p - \bigcup_{p\in P} N_p(2k) = P - \bigcup_{p\in P} N_p(2k), \tag{33}$$

$$where N_p = \{mp, m \in N and m \geq p\}\, for all p \in P, \tag{34}$$

$$and N_p\,(2k) = \left\{mp - 2k, m \in N and m \geq Max\left(p, \frac{1+2k}{p}\right)\right\} for all p \in P. \tag{35}$$

Or we can find the second prime in the pair by using the Sieve to the following set:

$$P(2k) = \{p + 2k, p \in P\}\,, \tag{36}$$

$$suchthat Q_k P[N] = P(2k) - \bigcup_{p \in P} N_p, \tag{37}$$

$$where N_p = \{mp, m \in N and m \geq p\} forall p \in P. \tag{38}$$

Let $\pi_2(k;x)$ denote the number of prime pairs $(p, p+2k)$ such that $p \leq x$. We first look at the case $k = 1$, let $\pi_2(x) = \pi_2(1;x)$ denote the number of twin primes $(p, p+2)$ such that $p \leq x$. Let $P_2(x)$be the set of first primes $p$in the twin primes $(p, p+2)$ such that $p \leq x$.

Like *Euler* ϕ*- function,* $\phi(n)$, on the previous section, we introduce ϕ_2*- function,* $\phi_2(n)$**, the number of natural numbers** $m$less than or equal to $n$such that both $m$and $m+2$are relatively prime to n.

Again, for any large positive number x, let

$$N = \prod_{p \in P(\sqrt{x})} p, \tag{39}$$

$$and let X_2(z) = \{m, 1 \leq m \leq z, \gcd \gcd(m, N) = 1 and \gcd \gcd(m+2, N) = 1\}. \tag{40}$$

Then from the definition of ϕ_2*- function,* we know that the size of $X_2(N)$ is:

$$\phi_2(N) = \frac{1}{2} N \prod_{3 \leq p \leq \sqrt{x}} (1 - \frac{2}{p}). \tag{41}$$

Notice that all numbers in the set $X_2(x)$ are the first primes of twin prime pairs, and $X_2(x) = P_2(x) - P_2\left(\sqrt{x}\right)$,
therefore, thesize of $X_2(x)$ is:

$$\pi_2(x) - \pi_2\left(\sqrt{x}\right) = C_2 \frac{x}{N} \phi_2(N) = C_2 \frac{x}{N} \left\{ \frac{N}{2} \prod_{3 \leq p \leq \sqrt{x}} \left(1 - \frac{2}{p}\right) \right\} = \frac{C_2 x}{2} \prod_{3 \leq p \leq \sqrt{x}} \left(1 - \frac{2}{p}\right), \tag{42}$$

for some constant C_2.
Take the Exampleagain: $x = 100$, $\sqrt{x} = 10$, $P\left(\sqrt{x}\right) = \{2, 3, 5, 7\}$, $N = 210$,

$$X_2(210) = \{11, 17, 29, 41, 59, 71, 101, 107, 137, 149, 167, 179, 191, 197, 209\} = P_2(210) - P_2(10) + \{167, 209\} \tag{43}$$

$$\phi_2(210) = \frac{210}{2}\left(1 - \frac{2}{3}\right)\left(1 - \frac{2}{5}\right)\left(1 - \frac{2}{7}\right) = 15, \tag{44}$$

$$X_2(100) = \{11, 17, 29, 41, 59, 71\} = P_2(100) - P_2(10), \tag{45}$$

thesize of $X_2(100)$ is: $\pi_2(100) - \pi_2(10) = 8 - 2 = 6$, compare this with

$$C_2 \frac{100}{210} \phi_2(210) = C_2 \frac{100}{210} 15 \approx 7.14 C_2, \tag{46}$$

we also expect that $0 < C_2 < 1$.
So, like before, if we use the Sieve to S to get $P_1 P[S]$as (2), then we have the following approximations of $\pi_2(n)$, the size of $P_1 P[S]$:

$$\frac{C_2}{2}n\prod_{3\le p\le\sqrt{n}}\left(1-\frac{2}{p}\right)=C_2n\left(1-\frac{1}{2}\right)\left(1-\frac{2}{3}\right)\left(1-\frac{2}{5}\right)\dots\left(1-\frac{2}{q}\right),2<3<\dots<q\le\sqrt{n}. \tag{47}$$

Now we prove the following ***Twin Prime Conjecture***

THEOREM 2.1. *There are infinitely many twin primes.*

Proof. Suppose that there is a finite list of twin primes, and q is the largest first prime of the last pair.

$$\text{Let} x=q^2 \text{ , and}$$

$$N=\prod_{p\le q}p \tag{48}$$

$$X_2(z)=\{m,1\le m\le z,\gcd\gcd(m,N)=1 and \gcd\gcd(m+2,N)=1\} \tag{49}$$

Then all elements in $X_2(x)$ are the first primes of twin primes, $X_2(x)=P_2(x)-P_2\left(\sqrt{x}\right)$, *and* are greater than q, a contradiction. Graphics Type 'SHAPE' is not supported yet. Please insert it as image.

By using the similar argument of Theorem 1.1, we have the following general result:

THEOREM 2.2. *For any natural number k, there are infinitely many pairs of primes that differ by 2k.*

Notice that the above Theorem 1.2 can be interpreted in a different way as follows:

THEOREM 2.3. *Every even number is the difference of two primes and there are an infinite number of such pairs of primes.*

On the other hand, we want to know if every even number greater than 2 is the sum of two primes, which is also known as the *Goldbach Conjecture* and will be explored in the following section.

Now in order to use the following ([1]):

$$(6)\prod_{p\le\sqrt{x}}(1-\frac{1}{p})\sim\frac{2e^{-\gamma}}{\ln\ln x}, \tag{50}$$

we need to do some calculations from $\prod(1-\frac{2}{p})$to $\prod(1-\frac{1}{p})$:

$$\frac{1}{2}\prod_{3\le p\le\sqrt{x}}(1-\frac{2}{p})=\frac{1\cdot1\cdot3\cdots(q-2)}{2\cdot3\cdot5\cdots q}=\frac{1\cdot2\cdot4\cdots(q-1)}{2\cdot3\cdot5\cdots q}\frac{1\cdot1\cdot3\cdots(q-2)}{1\cdot2\cdot4\cdots(q-1)} \tag{51}$$

$$=\prod_{p\le\sqrt{x}}(1-\frac{1}{p})\frac{1\cdot1\cdot3\cdots(q-2)}{1\cdot2\cdot4\cdots(q-1)}\frac{2\cdot3\cdot5\cdots q}{1\cdot2\cdot4\cdots(q-1)}\frac{1\cdot2\cdot4\cdots(q-1)}{2\cdot3\cdot5\cdots q} \tag{52}$$

$$=2\left(\prod_{3\le p\le\sqrt{x}}\frac{p(p-2)}{(p-1)^2}\right)\left(\prod_{p\le\sqrt{x}}(1-\frac{1}{p})\right)^2. \tag{53}$$

Thus, by (6) we have

$$\pi_2(x)-\pi_2\left(\sqrt{x}\right)=\frac{C_2x}{2}\prod_{3\le p\le\sqrt{x}}\left(1-\frac{2}{p}\right)\sim\frac{8C_2}{e^{2\gamma}}\left(\prod_{p\ge3}\frac{p(p-2)}{(p-1)^2}\right)\frac{x}{(\ln\ln x)^2}. \tag{54}$$

If let

$$A = \frac{8C_2}{e^{2\gamma}}\left(\prod_{p\geq 3}\frac{p(p-2)}{(p-1)^2}\right), \tag{55}$$

then

$$\pi_2(x) \sim A\frac{x}{(\ln\ln x)^2}. \tag{56}$$

or

$$(7)\,\pi_2(x) \sim A\int_2^x \frac{dt}{(\ln\ln t)^2} = A\left(Li(x) - \frac{x}{\ln\ln x} + \frac{2}{\ln\ln 2} - Li(2)\right), \tag{57}$$

since

$$\int \frac{dt}{(\ln\ln t)^2} = Li(t) - \frac{t}{\ln\ln t} + C. \tag{58}$$

From above (7) we can see that the count of twin primes is determined by the difference of two different approximations of the count of primes, $Li(x)\,and\,\frac{x}{\ln\ln x}$.

For $\pi_2(k;n)$**,** if k has factors 2< p ≤ $\sqrt{n}$, or $\sqrt{n}$ <p ≤ $\sqrt{n+2k}$then only one p-sequence sieved out from S. Therefore, we have

1. When k has no other prime factor less than $\sqrt{n}$ except 2, then

$$\pi_2(k;n) \sim A_k\pi_2(n) \tag{59}$$

2. When k has prime factors 2 < p, . . ., q ≤ $\sqrt{n}$, then

$$\pi_2(k;n) \sim \frac{(p-1)\ldots(q-1)}{(p-2)\ldots(q-2)}A_k\pi_2(n) \tag{60}$$

where A_k is some constant corresponds to k.

Regarding the constant A, we have the following Table 4. showing some examples of the choice of A with A=1.32038. But whether this is the best choice is still unknown.

		y= n/ln(n)		A=1.32038			A=1.32038	
n	π_2(n)	Li(n)-y	a	b=Aa	b-π_2(n)	c	d=Ac	d-π_2(n)
1,000	35	31	35	46	11	21	28	-7
5,000	126	96	99	131	5	69	91	-35
10,000	205	159	162	214	9	118	156	-49
50,000	705	544	547	722	17	427	564	-141
100,000	1,224	942	946	1249	25	754	996	-228
500,000	4,565	3502	3505	4628	63	2904	3834	-731
1,000,000	8,169	6244	6247	8248	79	5239	6918	-1251
5,000,000	32,463	24487	24490	32336	-127	21015	27747	-4716
10,000,000	58,980	44496	44500	58756	-224	38492	50824	-8156
100,000,000	440,312	333529	333532	440389	77	294706	389124	-51188
10,000,000,000	27,412,679	20761132	20761136	27412589	-90	18861170	24903911	-2508768

Table 4.

Here $a = \int_2^n \frac{dt}{(\ln\ln t)^2}, andc = \frac{n}{(\ln\ln n)^2}.$

Prime pairs can be generalized to ***prime* k-tuples**, $(p_1, p_2, p_3, \cdots p_k)$,patterns in the differences between more than two prime numbers.
Like the prime pairs, we can study the infinitude and density of prime k-tuples.

1. Prime Pairs with sum to an Even Number

Now, we look at those prime pairs with a sum equal to an even number. Again, we use the Sieve of Eratosthenes to find such prime pairs.
For a large number $nN, consider\ all\ pairs$ (a, b)such that $a + b = 2n$, with $a = 2, 3, \ldots, n$, and $b = 2n-2, 2n-3, ..., n$. We use the Sieve to $T = \{m, mNand2 \leq m \leq 2n-2\}$and mark down those numbers being sieved out, then reverse the second half $\{n, n+1, \ldots, 2n-2\}$ and pass all the corresponding marks from $\{a, 2 \leq a \leq n\}$ and $\{b, 2n-2 \geq b \geq n\}$ to $\{s, 2 \leq s \leq n\} = S$. After sieving out all marked down numbers from S, we have all the prime pairs (p, q)such that $p + q = 2n$. See the following Table 5. (here we already sieve out all even numbers) and Table 6. for examples.

$60 = a + b$			$62 = a + b$			$64 = a + b$			$66 = a + b$			$68 = a + b$		
s	**a**	**b**	**s**	**a**	**b**	**s**	**a**	**b**	**s**	**a**	**b**	**s**	**a**	**b**
3	3	57	3	3	59	3	3	61	3	3	63	3	3	65
5	5	55	5	5	57	5	5	59	5	5	61	5	5	63
7	7	53	7	7	55	7	7	57	7	7	59	7	7	61
9	9	51	9	9	53	9	9	55	9	9	57	9	9	59
11	11	49	11	11	51	11	11	53	11	11	55	11	11	57
13	13	47	13	13	49	13	13	51	13	13	53	13	13	55
15	15	45	15	15	47	15	15	49	15	15	51	15	15	53
17	17	43	17	17	45	17	17	47	17	17	49	17	17	51
19	19	41	19	19	43	19	19	45	19	19	47	19	19	49
21	21	39	21	21	41	21	21	43	21	21	45	21	21	47
23	23	37	23	23	39	23	23	41	23	23	43	23	23	45
25	25	35	25	25	37	25	25	39	25	25	41	25	25	43
27	27	33	27	27	35	27	27	37	27	27	39	27	27	41
29	29	31	29	29	33	29	29	35	29	29	37	29	29	39
			31	31	31	31	31	33	31	31	35	31	31	37
									33	33	33	33	33	35

Table 5.

After the Sieves, we have the following left prime pairs:														
$60 = p + q$			$62 = p + q$			$64 = p + q$			$66 = p + q$			$68 = p + q$		
s	***p***	***q***	***s***	***p***	***q***	***s***	***p***	***q***	***s***	***p***	***q***	***s***	***p***	***q***
7	7	53	3	3	59	3	3	61	5	5	61	7	7	61
13	13	47	19	19	43	5	5	59	7	7	59	31	31	37
17	17	43	31	31	31	11	11	53	13	13	53			
19	19	41				17	17	47	19	19	47			
23	23	37				23	23	41	23	23	43			
29	29	31							29	29	37			

Table 6.

The above Sieve process also can be expressed in the following way.
Like before, to find all the prime pairs (p, q)such that $p + q = 2n$, we only need to find the first prime p in the pair by using the Sieves to the set $S = \{s, 2 \leq s \leq n\}$ as shown in the following set:

$$(8)\, PP[S] = S - \bigcup_{p \in P(\sqrt{n})} S_p - \bigcup_{p \in P(\sqrt{2n})} S_p(x_p) = P[S] - \bigcup_{p \in P(\sqrt{2n})} S_p(x_p) \tag{61}$$

$$whereS_p = \left\{mp, m \in Nandp \leq m \leq \frac{n}{p}\right\} forallpP\left(\sqrt{n}\right), \tag{62}$$

$$andS_p\left(x_p\right) = \left\{mp + x_p, m \in Nandm \leq \frac{n - x_p}{p}, forsome - p < x_p < p\right\} forallpP(\sqrt{2n}). \tag{63}$$

Notice that x_p varies depending on the n, but we always have $x_2 = 0$.
For example, in the Table 2: when $n = 30$, then $x_2 = x_3 = x_5 = 0$ and $x_7 = 4$;
$n = 32$, then $x_2 = 0, x_3 = x_7 = 1$ and $x_5 = 4$;
$n = 33$, then $x_2 = x_3 = 0, x_5 = 1$ and $x_7 = 3$;
$n = 34$, then $x_2 = 0, x_3 = 2, x_5 = -2$ and $x_7 = -2$.
For $n = 31$,

$$S = \{a\} = \{2, 3, \ldots, 30, 31\}, \{b\} = \{60, 59, \ldots, 32.31\}, T = \{a\} \cup \{b\} = \{2, 3, \ldots, 31, \ldots, 60\}, \tag{64}$$

since $x_2 = 0, x_3 = x_5 = 2$, and $x_7 = 6$ we have
$S_2 = \{4, 6, \ldots, 30\}$, $S_3 = \{9, 12, \ldots, 27, 30\}$,
$S_2(0) = \{2, 4, \ldots, 30\}$, $S_3(2) = \{5, 8, \ldots, 23, 26, 29\}$,
$S_5 = \{25\}$,
$S_5(2) = \{7, 12, 17, 22, 27\}$, $S_7(6) = \{13, 20, 27\}$.
From $PP[S] = S - \{S_2 \cup S_2(0) \cup S_3 \cup S_3(2) \cup S_5 \cup S_5(2) \cup S_7(6)\} = \{3, 19, 31\}$,
we find out all 3 prime pairs:
$\{(3, 59), (19, 43), (31, 31)\}$ such that $62 = 3 + 59 = 19 + 43 = 31 + 31$.

For $n \geq 2$, let $\eta(2n)$ denote the number of prime pairs with the sum equal to $2n$, and we can call it the ***Goldbach number*** since it comes from the Goldbach conjecture.
For example, $\eta(4) = \eta(6) = \eta(8) = 1, \eta(10) = 2, \eta(12) = 1, \ldots\ldots$
from Table 3, we have $\eta(60) = 6, \eta(62) = 3, \eta(64) = 5, \eta(66) = 6, \eta(68) = 2$.
Since $\eta(2n)$ is the size of the set PP[S] from (8), then

1. like $\pi_2(k; n)$, $\eta(2n)$ also varies with n depending on the factors of n. If $2 < p \leq \sqrt{n}$ is a factor of n, or $\sqrt{n} < p \leq \sqrt{2n}$,then only one p-sequence sieved out from S.

   For example, $n = 30$ has factors 3 and 5, and $\sqrt{30} < 7 \leq \sqrt{60}$,so only one 3-sequence, 5-sequence, and 7-sequence sieved out, that $\eta(60) = 6$ (see Table 5).

2. Also, from Table 5 we can see that when $n = 32$, some elements from 5-sequence and 7-sequence are merged into 3-sequence so that less elements are sieved out.

While other than 1), 2) above, in general for each prime $3 \leq p \leq \sqrt{n}$, there are at most two p-sequences are sieved out from S, and therefore, we havethe following approximations of $\eta(2n)$:

$$\frac{C}{2}n\left\{\prod_{3 \leq p \leq \sqrt{n}}\left(1 - \frac{2}{p}\right)\right\}\left\{\prod_{\sqrt{n} < q \leq \sqrt{2n}}\left(1 - \frac{1}{q}\right)\right\} \sim G\frac{n}{(\ln\ln n)^2}, \tag{65}$$

where C and G are some corresponding constants.
For 1) when n has prime factors $2 < p, \ldots, q \leq \sqrt{n}$, then

$$\eta(2n) \sim \frac{(p-1)\ldots(q-1)}{(p-2)\ldots(q-2)}G\frac{n}{(\ln\ln n)^2} > G\frac{n}{(\ln\ln n)^2}; \tag{66}$$

For 2) when some elements from p-sequence are merged into other q-sequence, then

$$\eta(2n) > G\frac{n}{(\ln\ln n)^2}. \tag{67}$$

Therefore, when n is large, we have the following:

$$\eta(2n) \geq G\frac{n}{(\ln\ln n)^2} > 1. \tag{68}$$

The appendix shows some examples of $\eta(2n)$ and $\frac{n}{(\ln\ln n)^2}$ such that $\eta(2n) > \frac{n}{(\ln\ln n)^2}$ with only a few exceptions.
So, we have the following:
THEOREM 3.1. *For any even number* n > 2, *there exists* $\eta(n) \geq 1$ *of prime pairs p and q such that* $n = p + q$.
This is much stronger than the following *Goldbach's conjecture*.
THEOREM 3.2. *Every even number greater than 2 can be written as the sum of two primes.*

References

[1] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 6th ed. Oxford University Press, 2008.

[2] G. E. Andrews, *Number Theory,* 1st Dover ed. Dover Publications, 1994.

Appendix

In the following table, $\eta(N)$ is the *Goldbach number*, and

$$g = \frac{N/2}{(\ln\ln(N/2))^2} \tag{69}$$

n	N=12n	η(N)	g	η(N)-g	N=12n+2	η(N)	g	η(N)-g	N=12n+4	η(N)	g	η(N)-g
1	12	1	1.87	-0.9	14	2	1.85	0.2	16	2	1.85	0.1
2	24	3	1.94	1.1	26	3	1.98	1.0	28	2	2.01	0.0
3	36	4	2.15	1.8	38	2	2.19	-0.2	40	3	2.23	0.8
4	48	5	2.38	2.6	50	4	2.41	1.6	52	3	2.45	0.6
5	60	6	2.59	3.4	62	3	2.63	0.4	64	5	2.66	2.3
6	72	6	2.80	3.2	74	5	2.84	2.2	76	5	2.87	2.1
7	84	8	3.01	5.0	86	5	3.04	2.0	88	4	3.07	0.9
8	96	7	3.20	3.8	98	3	3.24	-0.2	100	6	3.27	2.7
9	108	8	3.39	4.6	110	6	3.42	2.6	112	7	3.46	3.5
10	120	12	3.58	8.4	122	4	3.61	0.4	124	5	3.64	1.4
11	132	9	3.76	5.2	134	6	3.79	2.2	136	5	3.82	1.2
12	144	11	3.94	7.1	146	6	3.97	2.0	148	5	3.99	1.0
13	156	11	4.11	6.9	158	5	4.14	0.9	160	8	4.17	3.8
14	168	13	4.28	8.7	170	9	4.31	4.7	172	6	4.33	1.7
15	180	14	4.44	9.6	182	6	4.47	1.5	184	8	4.50	3.5
16	192	11	4.61	6.4	194	7	4.63	2.4	196	9	4.66	4.3
17	204	14	4.77	9.2	206	7	4.79	2.2	208	7	4.82	2.2
18	216	13	4.93	8.1	218	7	4.95	2.0	220	9	4.98	4.0
19	228	12	5.08	6.9	230	9	5.11	3.9	232	7	5.13	1.9
20	240	18	5.24	12.8	242	8	5.26	2.7	244	9	5.29	3.7
21	252	16	5.39	10.6	254	9	5.41	3.6	256	8	5.44	2.6
22	264	16	5.54	10.5	266	8	5.56	2.4	268	9	5.59	3.4
23	276	16	5.68	10.3	278	7	5.71	1.3	280	14	5.73	8.3
24	288	17	5.83	11.2	290	10	5.85	4.1	292	8	5.88	2.1
25	300	21	5.97	15.0	302	9	6.00	3.0	304	10	6.02	4.0
26	312	17	6.12	10.9	314	9	6.14	2.9	316	10	6.16	3.8
27	324	20	6.26	13.7	326	7	6.28	0.7	328	10	6.31	3.7
28	336	19	6.40	12.6	338	9	6.42	2.6	340	11	6.45	4.6
29	348	16	6.54	9.5	350	13	6.56	6.4	352	10	6.58	3.4
30	360	22	6.67	15.3	362	8	6.70	1.3	364	14	6.72	7.3
31	372	18	6.81	11.2	374	10	6.83	3.2	376	11	6.86	4.1
32	384	19	6.95	12.1	386	12	6.97	5.0	388	9	6.99	2.0
33	396	21	7.08	13.9	398	7	7.10	-0.1	400	14	7.12	6.9
34	408	20	7.21	12.8	410	13	7.24	5.8	412	11	7.26	3.7
35	420	30	7.34	22.7	422	11	7.37	3.6	424	12	7.39	4.6
36	432	19	7.48	11.5	434	13	7.50	5.5	436	11	7.52	3.5
37	444	21	7.61	13.4	446	12	7.63	4.4	448	13	7.65	5.4
38	456	24	7.73	16.3	458	9	7.76	1.2	460	16	7.78	8.2
39	468	23	7.86	15.1	470	15	7.88	7.1	472	13	7.91	5.1
40	480	29	7.99	21.0	482	11	8.01	3.0	484	14	8.03	6.0
41	492	22	8.12	13.9	494	13	8.14	4.9	496	13	8.16	4.8
42	504	26	8.24	17.8	506	15	8.26	6.7	508	14	8.28	5.7
43	516	23	8.37	14.6	518	11	8.39	2.6	520	17	8.41	8.6
44	528	25	8.49	16.5	530	14	8.51	5.5	532	17	8.53	8.5
45	540	30	8.61	21.4	542	9	8.64	0.4	544	13	8.66	4.3
46	552	23	8.74	14.3	554	11	8.76	2.2	556	11	8.78	2.2
47	564	21	8.86	12.1	566	12	8.88	3.1	568	13	8.90	4.1
48	576	26	8.98	17.0	578	12	9.00	3.0	580	18	9.02	9.0
49	588	26	9.10	16.9	590	14	9.12	4.9	592	11	9.14	1.9
50	600	30	9.22	20.8	602	11	9.24	1.8	604	13	9.26	3.7
51	612	23	9.34	13.7	614	15	9.36	5.6	616	17	9.38	7.6
52	624	30	9.46	20.5	626	12	9.48	2.5	628	16	9.50	6.5
53	636	27	9.58	17.4	638	15	9.60	5.4	640	18	9.62	8.4
54	648	26	9.70	16.3	650	19	9.72	9.3	652	15	9.73	5.3

n	N=12n+6	η(N)	g	η(N)-g	N=12n+8	η(N)	g	η(N)-g	N=12n+10	η(N)	g	η(N)-g
1	18	2	1.86	0.1	20	2	1.89	0.1	22	3	1.91	1.1
2	30	3	2.05	1.0	32	2	2.08	-0.1	34	4	2.12	1.9
3	42	4	2.27	1.7	44	3	2.30	0.7	46	4	2.34	1.7
4	54	5	2.49	2.5	56	3	2.52	0.5	58	4	2.56	1.4
5	66	6	2.70	3.3	68	2	2.73	-0.7	70	5	2.77	2.2
6	78	7	2.91	4.1	80	4	2.94	1.1	82	5	2.97	2.0
7	90	9	3.11	5.9	92	4	3.14	0.9	94	5	3.17	1.8
8	102	8	3.30	4.7	104	5	3.33	1.7	106	6	3.36	2.6
9	114	10	3.49	6.5	116	6	3.52	2.5	118	6	3.55	2.5
10	126	10	3.67	6.3	128	3	3.70	-0.7	130	7	3.73	3.3
11	138	8	3.85	4.2	140	7	3.88	3.1	142	8	3.91	4.1
12	150	12	4.02	8.0	152	4	4.05	-0.1	154	8	4.08	3.9
13	162	10	4.19	5.8	164	5	4.22	0.8	166	6	4.25	1.7
14	174	11	4.36	6.6	176	7	4.39	2.6	178	7	4.42	2.6
15	186	13	4.53	8.5	188	5	4.55	0.4	190	8	4.58	3.4
16	198	13	4.69	8.3	200	8	4.72	3.3	202	9	4.74	4.3
17	210	19	4.85	14.2	212	6	4.87	1.1	214	8	4.90	3.1
18	222	11	5.00	6.0	224	7	5.03	2.0	226	7	5.06	1.9
19	234	15	5.16	9.8	236	9	5.18	3.8	238	9	5.21	3.8
20	246	16	5.31	10.7	248	6	5.34	0.7	250	9	5.36	3.6
21	258	14	5.46	8.5	260	10	5.49	4.5	262	9	5.51	3.5
22	270	19	5.61	13.4	272	7	5.64	1.4	274	11	5.66	5.3
23	282	16	5.76	10.2	284	8	5.78	2.2	286	12	5.81	6.2
24	294	19	5.90	13.1	296	8	5.93	2.1	298	11	5.95	5.0
25	306	15	6.05	9.0	308	8	6.07	1.9	310	12	6.09	5.9
26	318	15	6.19	8.8	320	11	6.21	4.8	322	11	6.24	4.8
27	330	24	6.33	17.7	332	6	6.35	-0.4	334	11	6.38	4.6
28	342	17	6.47	10.5	344	10	6.49	3.5	346	9	6.51	2.5
29	354	20	6.61	13.4	356	9	6.63	2.4	358	10	6.65	3.3
30	366	18	6.74	11.3	368	8	6.77	1.2	370	14	6.79	7.2
31	378	23	6.88	16.1	380	13	6.90	6.1	382	10	6.92	3.1
32	390	27	7.01	20.0	392	11	7.04	4.0	394	11	7.06	3.9
33	402	17	7.15	9.9	404	11	7.17	3.8	406	13	7.19	5.8
34	414	21	7.28	13.7	416	10	7.30	2.7	418	11	7.32	3.7
35	426	21	7.41	13.6	428	9	7.43	1.6	430	14	7.45	6.5
36	438	21	7.54	13.5	440	13	7.56	5.4	442	14	7.58	6.4
37	450	26	7.67	18.3	452	12	7.69	4.3	454	12	7.71	4.3
38	462	27	7.80	19.2	464	12	7.82	4.2	466	13	7.84	5.2
39	474	23	7.93	15.1	476	14	7.95	6.1	478	11	7.97	3.0
40	486	23	8.05	14.9	488	9	8.07	0.9	490	19	8.10	10.9
41	498	23	8.18	14.8	500	13	8.20	4.8	502	15	8.22	6.8
42	510	32	8.30	23.7	512	11	8.33	2.7	514	13	8.35	4.7
43	522	24	8.43	15.6	524	11	8.45	2.6	526	15	8.47	6.5
44	534	22	8.55	13.4	536	13	8.57	4.4	538	14	8.59	5.4
45	546	30	8.68	21.3	548	11	8.70	2.3	550	19	8.72	10.3
46	558	22	8.80	13.2	560	18	8.82	9.2	562	14	8.84	5.2
47	570	32	8.92	23.1	572	11	8.94	2.1	574	15	8.96	6.0
48	582	25	9.04	16.0	584	12	9.06	2.9	586	12	9.08	2.9
49	594	27	9.16	17.8	596	12	9.18	2.8	598	15	9.20	5.8
50	606	27	9.28	17.7	608	13	9.30	3.7	610	19	9.32	9.7
51	618	26	9.40	16.6	620	17	9.42	7.6	622	16	9.44	6.6
52	630	41	9.52	31.5	632	10	9.54	0.5	634	13	9.56	3.4
53	642	25	9.64	15.4	644	17	9.66	7.3	646	15	9.68	5.3