Annals of Mathematics **00** (XXXX), 1–23
https://doi.org/00.1111/S0000-1111-22-3333-4

On the Prime Pairs

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Abstract

By utilizing our Sieve method, we introduce a fresh perspective on the study of prime pairs. Our methodology sheds light on their distribution, infinitude, and density. We see that the count of twin primes is determined by the difference of two different approximations of the count of primes: $Li(x)$ and $x/\ln x$.

Also, by exploring the approximation of the number of prime pairs with the sum equal to an even number, we discover that the number of prime pairs with the sum equal to $2n$ is related to the number of twin primes less than or equal to n.

Contents

AMS Classification: Primary: prime pairs; Secondary: twin primes.

1. **Introduction**

A *prime number* (or a *prime*) is a natural number greater than 1, that has no positive divisors other than 1 and itself.

We introduce the following notations:

- $\mathbb{N}$: The set of all natural numbers.
- $P[X]$: The set of all prime numbers in the set of X.
- $\mathbb{P}$: Equivalent to $P[\mathbb{N}]$, representing the set of all prime numbers.
- $\{u_i\}$: A sequence with a general term u_i.
- $\ln x$: The natural logarithm of x.
- $\pi(x)$: The number of primes less than or equal to x.
- $P(x)$: The set of all primes less than or equal to x, represented as $\{2, 3, 5, \ldots p\}$.
- $\gcd(a, b)$: The greatest common divisor of a and b.

We say that a and b are *relatively prime* if $\gcd(a, b) = 1$ ([1]).

The *Sieve of Eratosthenes* ([4]) has long been used in the study of prime numbers.

For a large positive number n, let $S = \{m, \quad 1 < m \le n\}$ be the set of natural numbers great than 1 and less than or equal to n. Then we can use the sieve of Eratosthenes to find all prime numbers in the set S such that:

$$P[S] = S - \bigcup_{p \in P(\sqrt{n})} S_p, \tag{1}$$

$$\text{where} \quad S_p = \left\{ mp \mid \quad m \in \mathbb{N} \quad \text{and} \quad p \le m \le \frac{n}{p} \right\} \quad \text{for all } p \in P(\sqrt{n}).$$

We introduce *Euler's* ϕ-function, $\phi(n)$, the number of natural numbers less than or equal to n that are relatively prime to n ([1], [4]).

For any large positive number x, let

$$N = \prod_{p \in P(\sqrt{x})} p \quad = \quad \prod_{p \le \sqrt{x}} p,$$

be the product of all primes less than or equal to $\sqrt{x}$.

$$\text{Let} \quad X(z) = \left\{ m \mid \quad 1 \le m \le z \quad \text{and} \quad \gcd(m, N) = 1 \right\} \quad \text{for } 1 < z \le N$$

be the set of natural numbers less than or equal to z that are relatively prime to N.

Since $\gcd(m, N) = \gcd(N - m, N)$, all numbers in $X(N)$ are symmetrical to $N/2$, that is, all numbers in $X(N)$ are in the form of $(N/2 - x, N/2 + x)$ for some x. Especially, we have $x = (N/2 - 1)$ so that $(1, N - 1)$ always in $X(N)$.

For Example: take $N = 2 \cdot 3 \cdot 5 = 30$, then

$$X(30) = \{1, 7, 11, 13, 17, 19, 23, 29\}$$

has 4 pairs: $(15 - 14, 15 + 14)$, $(15 - 8, 15 + 8)$, $(15 - 4, 15 + 4)$, $(15 - 2, 15 + 2)$.

Next take $N = 2 \cdot 3 \cdot 5 \cdot 7 = 210$, then

$$X(210) = \{1, 11, 13, 17, 19, 23, 29, 31, \ldots, 179, 181, 187, 191, 193, 197, 199, 209\}$$

has 24 pairs: $(105 - 104, 105 + 104)$, $(105 - 94, 105 + 94)$, $\ldots$, $(105 - 2, 105 + 2)$.

Here, we can see that all numbers in $X(30)$ except 7 are in $X(210)$, and the same will be: all numbers in $X(210)$ except those numbers multiple of 11 are in $X(2310)$, and so on:

$$X(30) - \{7\} \subset X(210),$$

$$X(210) - \{11m \mid \quad 1 \le m \le 19\} \subset X(2310), \quad \ldots,$$

in general, if $p_1, p_2, \ldots, p_{k-1}$ are first $k - 1$ primes, and p_k is the kth prime, then

$$X(p_1 p_2 \cdots p_{k-1}) - \left\{ m p_k \mid \quad 1 \le m \le \frac{p_1 p_2 \cdots p_{k-1}}{p_k} \right\} \subset X(p_1 p_2 \cdots p_{k-1} p_k).$$

From the definition of the ϕ-function, the size of $X(N)$ is given by:

$$\phi(N) = N \prod_{p \in P(\sqrt{x})} (1 - \frac{1}{p}). \tag{2}$$

And from the above discussion, we have:

$$\phi\left(\frac{N}{2}\right) = \frac{N}{2} \prod_{p \in P(\sqrt{x})} (1 - \frac{1}{p}).$$

Notice that all numbers in the set $X(x)$ are primes except 1, and

$$X(x) = P(x) - P\left(\sqrt{x}\right) + \{1\},$$

therefore, the size of $X(x)$ is :

$$\pi(x) - \pi\left(\sqrt{x}\right) + 1 = C_1 \frac{x}{N} \phi(N) = C_1 x \prod_{p \in P(\sqrt{x})} (1 - \frac{1}{p}), \tag{3}$$

for some constant C_1.

For Example: take $x = 100$, then $P\left(\sqrt{x}\right) = \{2, 3, 5, 7\}$, $N = 2 \cdot 3 \cdot 5 \cdot 7 = 210$,

$$X(210) = \{1, 11, 13, \dots, 209\} = P(210) - P(10) + \{1, 121, 143, 169, 187, 209\},$$

$$\phi(210) = 210\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{7}\right) = 48,$$

$$X(100) = \{1, 11, 13, 17, \dots, 97\} = P(100) - P(10) + \{1\},$$

the size of $X(100)$ is : $\pi(100) - \pi(10) + 1 = 25 - 4 + 1 = 22$, compare this with

$$C_1 \frac{100}{210}\phi(210) = C_1 \frac{100}{210} 48 \approx 22.86 C_1,$$

we expect that $0 < C_1 < 1$.

Notice that if we use the Sieve to S to get $P[S]$ as (1), then we have the following approximations of $\pi(n)$, the size of $P[S]$:

$$C_1 n \prod_{p \leq \sqrt{n}} \left(1 - \frac{1}{p}\right) = C_1 n \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \dots \left(1 - \frac{1}{q}\right), \quad 2 < 3 < \dots < q \leq \sqrt{n}. \tag{4}$$

In fact, from [4], we know that C_1 is about $e^{\gamma}/2 \approx 0.890536209\dots$, where γ is Euler's constant, and

$$\prod_{p \in P(\sqrt{x})} (1 - \frac{1}{p}) \sim \frac{2e^{-\gamma}}{\ln x}, \tag{5}$$

so, we have

$$\pi(x) \sim \frac{e^{\gamma}}{2} x \prod_{p \in P(\sqrt{x})} \left(1 - \frac{1}{p}\right) \sim \frac{x}{\ln x}, \tag{6}$$

$$\pi(x) \sim Li(x) = \int_2^x \frac{dt}{\ln t}. \tag{7}$$

2. **Prime Pairs**

Now we study the prime pairs: for $k \in \mathbb{N}$ both p and $p + 2k$ are primes. We know that twin primes are the special case of $k = 1$. [5]

To find all twin primes $(p, p + 2)$ such that $p \le n$, we only need to find the first prime p in the pair by using the Sieves to the set $S = \{m \mid \ 1 < m \le n\}$ as showing in the following set:

$$P_1P[S] = S - \bigcup_{p\in P(\sqrt{n})} S_p - \bigcup_{p\in P\left(\sqrt{n+2}\right)} S_p(2) = P[S] - \bigcup_{p\in P\left(\sqrt{n+2}\right)} S_p(2), \tag{8}$$

$$\text{where} \quad S_p = \left\{mp \mid \ m \in \mathbb{N} \text{ and } p \le m \le \frac{n}{p}\right\} \text{ for all } p \in P(\sqrt{n}),$$

$$\text{and} \quad S_p(2) = \left\{mp - 2 \mid \ m \in \mathbb{N} \text{ and } p \le m \le \frac{n+2}{p}\right\} \text{ for all } p \in P(\sqrt{n+2}).$$

For Example: when $n = 100$,

$S = \{ 2, 3, \dots, 99, 100\}$,
$S_2 = \{ 4, 6, 8, \dots, 98, 100\}$, $\quad S_2(2) = \{ 2, 4, 6, \dots, 96, 98, 100\}$,
$S_3 = \{ 9, 12, 15, \dots, 96, 99\}$, $\quad S_3(2) = \{ 7, 10, 13, \dots, 94, 97, 100\}$,
$S_5 = \{ 25, 30, 35, \dots, 95, 100\}$, $\quad S_5(2) = \{ 23, 28, 33, \dots, 93, 98\}$,
$S_7 = \{ 49, 56, 63, \dots, 91, 98\}$, $\quad S_7(2) = \{ 47, 54, 61, \dots, 89, 96\}$.

From

$$P_1P[S] = S - \{S_2 \cup S_2(2) \cup S_3 \cup S_3(2) \cup S_5 \cup S_5(2) \cup S_7 \cup S_7(2)\},$$

we get

$$P_1P[S] = \{3, 5, 11, 17, 29, 41, 59, 71\}.$$

Therefore, we find out all 8 twin primes less than 100:

$$\{(3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), (59, 61), (71, 73)\}.$$

See the following Table 1. for the illustration:

s	n	n+2	s	n	n+2	s	n	n+2	s	n	n+2	s	n	n+2
		3	21	21	23	41	41	43	61	61	63	81	81	83
2	2	4	22	22	24	42	42	44	62	62	64	82	82	84
3	3	5	23	23	25	43	43	45	63	63	65	83	83	85
4	4	6	24	24	26	44	44	46	64	64	66	84	84	86
5	5	7	25	25	27	45	45	47	65	65	67	85	85	87
6	6	8	26	26	28	46	46	48	66	66	68	86	86	88
7	7	9	27	27	29	47	47	49	67	67	69	87	87	89
8	8	10	28	28	30	48	48	50	68	68	70	88	88	90
9	9	11	29	29	31	49	49	51	69	69	71	89	89	91
10	10	12	30	30	32	50	50	52	70	70	72	90	90	92
11	11	13	31	31	33	51	51	53	71	71	73	91	91	93
12	12	14	32	32	34	52	52	54	72	72	74	92	92	94
13	13	15	33	33	35	53	53	55	73	73	75	93	93	95
14	14	16	34	34	36	54	54	56	74	74	76	94	94	96
15	15	17	35	35	37	55	55	57	75	75	77	95	95	97
16	16	18	36	36	38	56	56	58	76	76	78	96	96	98
17	17	19	37	37	39	57	57	59	77	77	79	97	97	99
18	18	20	38	38	40	58	58	60	78	78	80	98	98	100
19	19	21	39	39	41	59	59	61	79	79	81	99	99	
20	20	22	40	40	42	60	60	62	80	80	82	100	100	

Table 1.

Or we can start from the set P[S] = P(100) :

$$P[S] = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}$$

and the shifted sieve sets:

$$S_2(2) = \{2\},$$
$$S_3(2) = \{7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97\},$$
$$S_5(2) = \{23, 53, 83\},$$
$$S_7(2) = \{47, 89\},$$

to derive:

$$P_1P[S] = P[S] - \{S_2(2) \cup S_3(2) \cup S_5(2) \cup S_7(2)\} = \{3, 5, 11, 17, 29, 41, 59, 71\}.$$

See the following Table 2.

p	p+2		p	p+2		p	p+2		p	p+2		p	p+2
2	4		13	15		31	33		53	55		73	75
3	5		17	19		37	39		59	61		79	81
5	7		19	21		41	43		61	63		83	85
7	9		23	25		43	45		67	69		89	91
11	13		29	31		47	49		71	73		97	99

Table 2.

On the other hand, we can find the second prime in the pair by using the Sieve to the following set:

$$P[S](2) = \{p+2 \mid \ p \in P[S]\}, \tag{9}$$

as shown in this set:

$$Q_1 P[S] = P[S](2) - \bigcup_{p \in P(\sqrt{n+2})} S_p, \tag{10}$$

$$\text{where} \quad S_p = \left\{ mp \mid \ m \in \mathbb{N} \ \text{ and } \ p \le m \le \frac{n+2}{p} \right\} \quad \text{for all} \quad p \in P(\sqrt{n+2}).$$

Take the above Example again: $n = 100$, $S = \{2, 3, \ldots, 99, 100\}$, from

$$P[S](2) = \{4, 5, 7, 9, 13, 15, 19, 21, 25, 31, 33, 39, 43, 45, 49, 55, 61, 63, 69, 73, 75, 81, 85, 91, 99\}$$

and the sieve sets:

$$\begin{aligned}
S_2 &= \{4\}, \\
S_3 &= \{9, 15, 21, 33, 39, 45, 63, 69, 75, 81, 99\}, \\
S_5 &= \{25, 45, 55, 75, 85\}, \\
S_7 &= \{49, 63, 91\},
\end{aligned}$$

we have

$$Q_1 P[S] = P[S](2) - \{S_2 \cup S_3 \cup S_5 \cup S_7\} = \{5, 7, 13, 19, 31, 43, 61, 73\}.$$

See the following Table 3.

p	p+2		p	p+2		p	p+2		p	p+2		p	p+2
2	4		13	15		31	33		53	55		73	75
3	5		17	19		37	39		59	61		79	81
5	7		19	21		41	43		61	63		83	85
7	9		23	25		43	45		67	69		89	91
11	13		29	31		47	49		71	73		97	99

Table 3.

Let $\mathbb{N}_1 = \mathbb{N} - \{1\}$ be the set of natural numbers greater than 1. Then to find all twin primes we can find the first prime in the pair by using the Sieves to $\mathbb{N}_1$ as shown in the following set:

(11) $$P_1P[\mathbb{N}] = \mathbb{N}_1 - \bigcup_{p\in\mathbb{P}}\{N_p \cup N_p(2)\} = \mathbb{N}_1 - \bigcup_{p\in\mathbb{P}} N_p - \bigcup_{p\in\mathbb{P}} N_p(2) = \mathbb{P} - \bigcup_{p\in\mathbb{P}} N_p(2),$$

$$\text{where} \quad N_p = \{mp \mid \ m \in \mathbb{N} \ \text{ and } \ m \geq p\} \quad \text{for all } p \in \mathbb{P},$$

$$\text{and} \quad N_p(2) = \{mp - 2 \mid \ m \in \mathbb{N} \ \text{ and } \ m \geq p\} \quad \text{for all } p \in \mathbb{P}.$$

Or we can find the second prime in the pair by using the Sieve to the following set:

(12) $$\mathbb{P}(2) = \{p + 2 \mid \ p \in \mathbb{P}\},$$

such that

(13) $$Q_1P[\mathbb{N}] = \mathbb{P}(2) - \bigcup_{p\in P} N_p,$$

$$\text{where} \quad N_p = \{mp \mid \ m \in \mathbb{N} \ \text{ and } \ m \geq p\} \quad \text{for all } p \in \mathbb{P}.$$

Similarly, for any $k \in \mathbb{N}$, to find the general prime pairs $(p, p + 2k)$ such that $p \leq n$, we only need to find the first prime p in the pair by using the Sieves to $S = \{m \mid \ 1 < m \leq n\}$ as shown in the following set:

(14) $$P_kP[S] = S - \bigcup_{p\in P(\sqrt{n})} S_p - \bigcup_{p\in P\left(\sqrt{n+2k}\right)} S_p(2k) = P[S] - \bigcup_{p\in P\left(\sqrt{n+2k}\right)} S_p(2k),$$

$$\text{where} \quad S_p = \left\{mp \mid \ m \in \mathbb{N} \ \text{ and } \ p \leq m \leq \frac{n}{p}\right\} \quad \text{for all } p \in P(\sqrt{n}), \quad \text{and}$$

$$S_p(2k) = \left\{mp - 2k \mid \ m \in \mathbb{N} \ \text{ and } \ \max\left(p, \frac{1 + 2k}{p}\right) \leq m \leq \frac{n + 2k}{p}\right\} \quad \text{for all } p \in P(\sqrt{n + 2k}).$$

Again, to find all prime pairs $(p, p + 2k)$ we only need to find the first prime in the pair by using the Sieves to $\mathbb{N}_1$ as shown in the following set:

$$P_k P[\mathbb{N}] = \mathbb{N}_1 - \bigcup_{p\in\mathbb{P}} \{N_p \cup N_p(2k)\} = \mathbb{N}_1 - \bigcup_{p\in\mathbb{P}} N_p - \bigcup_{p\in\mathbb{P}} N_p(2k) = \mathbb{P} - \bigcup_{p\in\mathbb{P}} N_p(2k), \tag{15}$$

$$\text{where} \quad N_p = \{mp \mid \ m \in \mathbb{N} \ \text{ and } \ m \geq p\} \quad \text{for all } p \in \mathbb{P}, \quad \text{and}$$

$$N_p(2k) = \left\{mp - 2k \mid \ m \in \mathbb{N} \ \text{ and } \ m \geq \max\left(p, \frac{1+2k}{p}\right)\right\} \quad \text{for all } p \in \mathbb{P}.$$

Or we can find the second prime in the pair by using the Sieve to the following set:

$$\mathbb{P}(2k) = \{p + 2k \mid \quad p \in \mathbb{P}\}, \tag{16}$$

such that:

$$Q_k P[N] = \mathbb{P}(2k) - \bigcup_{p\in\mathbb{P}} N_p, \tag{17}$$

$$\text{where} \quad N_p = \{mp \mid \ m \in \mathbb{N} \ \text{ and } \ m \geq p\} \quad \text{for all } p \in \mathbb{P}.$$

Notice that the procedure of the sieve in (11) and (15) must go through all $p \in \mathbb{P}$.

Because if there is an n such that

$$P_1 P[\mathbb{N}] = \mathbb{N}_1 - \bigcup_{p\in P(n)} \{N_p \cup N_p(2)\},$$

$$\text{then let} \quad W = \{m(2 \times 3 \times \cdots \times p) - 1 \mid \quad m \in \mathbb{N} \ \text{ and } \ p \in P(n)\},$$

we have W is an infinite subset of $\mathbb{N}_1$ and

$$W \cap \Big(\bigcup_{p\in P(n)} \{N_p \cup N_p(2)\} \Big) = \varnothing,$$

$$\text{that is,} \quad \mathbb{N}_1 - \bigcup_{p\in P(n)} \{N_p \cup N_p(2)\} \quad \text{is infinite for any } n.$$

Therefore, we can continue the procedure of the sieve on W with those primes greater than n such that

$$W - \bigcup_{p>n} \{N_p \cup N_p(2)\},$$

to produce the new first primes in the pair of twin primes.

And thus

$$P_1 P[\mathbb{N}] \neq \mathbb{N}_1 - \bigcup_{p\in P(n)} \{N_p \cup N_p(2)\} \quad \text{for any } n,$$

and so the procedure of the sieve in (11) and (15) must go through all $p \in \mathbb{P}$.

Let $\pi_2(k;x)$ denote the number of prime pairs $(p, p+2k)$ such that $p \le x$. For the case $k=1$, let $\pi_2(x) = \pi_2(1;x)$ denote the number of twin primes $(p, p+2)$ such that $p \le x$. Let $P_2(x)$ be the set of first primes p in the twin primes $(p, p+2)$ such that $p \le x$.

Similar to the Euler ϕ-function, $\phi(n)$, introduced in the previous section, we define the ϕ_2-function, $\phi_2(n)$, as the number of natural numbers m less than or equal to n such that both m and $m+2$ are relatively prime to n.

For any large positive number x, let

$$N = \prod_{p \in P(\sqrt{x})} p, \quad \text{and} \tag{18}$$

$$X_2(z) = \left\{ m \mid \; 1 \le m \le z, \;\; \gcd(m, N) = 1 \;\; \text{and} \;\; \gcd(m+2, N) = 1 \right\}. \tag{19}$$

$$\text{Then,} \quad X_2(N) = \{ m \mid \;\; m \in X(N) \;\; \text{and} \;\; (m+2) \in X(N) \}.$$

From the previous section, we know that all numbers in $X(N)$ are symmetrical to $N/2$, so numbers in $X_2(N)$ are symmetrical to $(N/2-1)$ except one pair $(1, N-1)$ because 1 is no longer in $X_2(N)$ while $(N-1)$ is always in $X_2(N)$ for any N.

For Example: take $N = 2 \cdot 3 \cdot 5 = 30$, then

$$X_2(30) = \{11, 17, 29\}$$

has 1 pairs: $(14-3, 14+3)$, and number $(30-1)$.

Next for $N = 2 \cdot 3 \cdot 5 \cdot 7 = 210$,

$$X_2(210) = \{11, 17, 29, 41, 59, 71, 101, 107, 137, 149, 167, 179, 191, 197, 209\}$$

has 7 pairs: $(104-93, 104+93)$, $(104-87, 104+87)$, $\ldots$, $(104-3, 104+3)$ and number $(210-1)$.

Again, we see that all numbers in $X_2(30)$ are in $X_2(210)$, and so on:

$$X_2(30) \subset X_2(210),$$

$$X_2(210) - \{11m, \;\; (11m-2) \mid \;\; 1 \le m \le 19\} \subset X_2(2310), \quad \ldots,$$

in general, if $p_1, p_2, \ldots, p_{k-1}$ are first $k-1$ primes, and p_k is the kth prime, then

$$X_2(p_1 p_2 \cdots p_{k-1}) - \left\{ m p_k, \;\; (m p_k - 2) \mid \;\; 1 \le m \le \frac{p_1 p_2 \cdots p_{k-1}}{p_k} \right\} \subset X_2(p_1 p_2 \cdots p_{k-1} p_k).$$

From the definition of ϕ_2-function, we know that the size of $X_2(N)$ is:

$$\phi_2(N) = \frac{1}{2}N \prod_{3 \le p \le \sqrt{x}} (1 - \frac{2}{p}). \tag{20}$$

Notice that all numbers in the set $X_2(x)$ are the first primes of twin prime pairs.
Moreover,

$$X_2(x) = P_2(x) - P_2(\sqrt{x}) \tag{21}$$

Thus, the size of $X_2(x)$ is $\pi_2(x) - \pi_2(\sqrt{x})$:

$$C_2 \frac{x}{N} \phi_2(N) = C_2 \frac{x}{N} \left\{ \frac{N}{2} \prod_{3 \le p \le \sqrt{x}} \left(1 - \frac{2}{p}\right) \right\} = \frac{C_2 x}{2} \prod_{3 \le p \le \sqrt{x}} \left(1 - \frac{2}{p}\right), \tag{22}$$

for some constant C_2.

Consider the example where $x = 100$, $P(\sqrt{x}) = \{2, 3, 5, 7\}$, and $N = 210$,

$$X_2(210) = \{11, 17, 29, 41, 59, 71, 101, 107, 137, 149, 167, 179, 191, 197, 209\},$$
$$\text{or} \quad X_2(210) = P_2(210) - P_2(10) + \{167, 209\},$$

$$\phi_2(210) = \frac{210}{2}\left(1 - \frac{2}{3}\right)\left(1 - \frac{2}{5}\right)\left(1 - \frac{2}{7}\right) = 15,$$

$$X_2(100) = \{11, 17, 29, 41, 59, 71\} = P_2(100) - P_2(10),$$

the size of $X_2(100)$ is: $\pi_2(100) - \pi_2(10) = 8 - 2 = 6$, compare this with

$$C_2 \frac{100}{210} \phi_2(210) = C_2 \frac{100}{210} 15 \approx 7.14 C_2,$$

we also expect that $0 < C_2 < 1$.

So, like before, if we use the Sieve to S to get $P_1 P[S]$ as (8), then we have the following approximations of $\pi_2(n)$, the size of $P_1 P[S]$:

$$\frac{C_2}{2} n \prod_{3 \le p \le \sqrt{n}} \left(1 - \frac{2}{p}\right), \qquad \text{or} \tag{23}$$

$$C_2 n \left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{3}\right)\left(1 - \frac{2}{5}\right)\dots\left(1 - \frac{2}{q}\right), \quad 2 < 3 < 5 < \dots < q \le \sqrt{n}. \tag{24}$$

Now we prove the following *Twin Prime Conjecture*.

THEOREM 2.1. *There are infinitely many twin primes.*

Proof. Suppose that there is a finite list of twin primes, and q is the largest first prime of the last pair.

Let $x = q^2$, and

$$N = \prod_{p \le q} p \tag{25}$$

$$X_2(z) = \left\{ m \quad | \quad 1 \le m \le z, \quad \gcd(m, N) = 1 \quad \text{and} \quad \gcd(m+2, N) = 1 \right\}. \tag{26}$$

Then all elements in $X_2(x) = P_2(x) - P_2\left(\sqrt{x}\right)$ are the first primes of twin primes and are greater than q, a contradiction. □

By using the similar argument of Theorem 2.1, we have the following general result:

THEOREM 2.2. *For any natural number k, there are infinitely many pairs of primes that differ by 2k.*

Notice that the above Theorem 2.2 can also be interpreted as follows theorem regarding the difference of two primes according to the work of Hardy and Littlewood ([3]):

THEOREM 2.3. *Every even number is the difference of two primes, and there are an infinite number of such pairs of primes.*

On the other hand, we want to know if every even number greater than 2 is the sum of two primes, which is also known as the *Goldbach Conjecture* ([2]) and will be explored in the following section.

Now, in order to use the following ([4]):

$$\prod_{p \le \sqrt{x}} (1 - \frac{1}{p}) \sim \frac{2e^{-\gamma}}{\ln x}, \tag{27}$$

we need to do some calculations

$$\text{from} \quad \prod (1 - \frac{2}{p}) \quad \text{to} \quad \prod (1 - \frac{1}{p}) :$$

$$\frac{1}{2} \prod_{3 \le p \le \sqrt{x}} (1 - \frac{2}{p}) = \frac{1 \cdot 1 \cdot 3 \cdots (q-2)}{2 \cdot 3 \cdot 5 \cdots q} = \frac{1 \cdot 2 \cdot 4 \cdots (q-1)}{2 \cdot 3 \cdot 5 \cdots q} \frac{1 \cdot 1 \cdot 3 \cdots (q-2)}{1 \cdot 2 \cdot 4 \cdots (q-1)}$$

$$= \prod_{p\le\sqrt{x}}(1-\frac{1}{p})\frac{1\cdot 1\cdot 3\cdots(q-2)}{1\cdot 2\cdot 4\cdots(q-1)}\frac{2\cdot 3\cdot 5\cdots q}{1\cdot 2\cdot 4\cdots(q-1)}\frac{1\cdot 2\cdot 4\cdots(q-1)}{2\cdot 3\cdot 5\cdots q}$$

$$= 2\left(\prod_{3\le p\le\sqrt{x}}\frac{p(p-2)}{(p-1)^2}\right)\left(\prod_{p\le\sqrt{x}}(1-\frac{1}{p})\right)^2.$$

Thus, by (27) we have

$$\pi_2(x)-\pi_2\left(\sqrt{x}\right) = \frac{C_2 x}{2}\prod_{3\le p\le\sqrt{x}}\left(1-\frac{2}{p}\right) \sim \frac{8C_2}{e^{2\gamma}}\left(\prod_{p\ge 3}\frac{p(p-2)}{(p-1)^2}\right)\frac{x}{(\ln x)^2}. \tag{28}$$

If let

$$A = \frac{8C_2}{e^{2\gamma}}\left(\prod_{p\ge 3}\frac{p(p-2)}{(p-1)^2}\right), \tag{29}$$

then

$$\pi_2(x) \sim A\frac{x}{(\ln x)^2}, \tag{30}$$

or

$$\pi_2(x) \sim A\int_2^x \frac{dt}{(\ln t)^2} = A\left(Li(x)-\frac{x}{\ln x}+\frac{2}{\ln 2}-Li(2)\right) \tag{31}$$

$$\text{since}\qquad \int\frac{dt}{(\ln t)^2} = Li(t)-\frac{t}{\ln t}+C.$$

From above (31) we can see that the count of twin primes is determined by the difference of two different approximations of the count of primes:

$$Li(x)\qquad\text{and}\qquad \frac{x}{\ln x}.$$

.

For $\pi_2(k;n)$**,** if k has factors 2< p $\le\sqrt{n}$, or $\sqrt{n}$ <p $\le\sqrt{n+2k}$ then only one p-sequence sieved out from S. Therefore, we have:

1). when k has no other prime factor less than $\sqrt{n}$ except 2, then

$$\pi_2(k;n) \sim A_k\pi_2(n); \tag{32}$$

2). when k has prime factors 2 < p, . . ., q $\le\sqrt{n}$, then

$$\pi_2(k;n) \sim \frac{(p-1)\dots(q-1)}{(p-2)\dots(q-2)}A_k\pi_2(n), \tag{33}$$

where A_k is some constant corresponds to k.

Regarding the constant A, we have the following Table 4. showing some examples of the choice of A with A=1.32038. But whether this is the best choice is still unknown.

(Note: since

$$\left(\prod_{p\geq 3}\frac{p\,(p-2)}{(p-1)^2}\right)\approx 0.66016118\ldots,$$

so if

$$C_2=\frac{e^{2\gamma}}{4}\approx 0.7930547\ldots,$$

where γ is Euler's constant, then A is about 1.320...)

		y= n/ln(n)		A=1.32038			A=1.32038	
n	π_2(n)	Li(n)-y	a	b=Aa	b-π_2(n)	c	d=Ac	d-π_2(n)
1,000	35	31	35	46	11	21	28	-7
5,000	126	96	99	131	5	69	91	-35
10,000	205	159	162	214	9	118	156	-49
50,000	705	544	547	722	17	427	564	-141
100,000	1,224	942	946	1249	25	754	996	-228
500,000	4,565	3502	3505	4628	63	2904	3834	-731
1,000,000	8,169	6244	6247	8248	79	5239	6918	-1251
5,000,000	32,463	24487	24490	32336	-127	21015	27747	-4716
10,000,000	58,980	44496	44500	58756	-224	38492	50824	-8156
100,000,000	440,312	333529	333532	440389	77	294706	389124	-51188
10,000,000,000	27,412,679	20761132	20761136	27412589	-90	18861170	24903911	-2508768

Table 4.

$$\text{here}\qquad a=\int_2^n\frac{dt}{(\ln t)^2},\qquad\text{and}\qquad c=\frac{n}{(\ln n)^2}.$$

Prime pairs can be generalized to *prime k-tuples*, $(p_1,p_2,p_3,\cdots p_k)$, which are patterns in the differences between more than two prime numbers.

Like the prime pairs, we can study the infinitude and density of prime k-tuples.

3. **Prime Pairs Summing to an Even Number**

Next, we consider prime pairs whose sum equals an even number. Again, we employ the Sieve of Eratosthenes to identify such prime pairs.

For a large number $n \in \mathbb{N}$, consider all pairs (a, b) satisfying $a + b = 2n$. Here, a ranges from 2 to n, while b is taken from the set $\{2n-2, 2n-3, \ldots, n\}$. Using the Sieve to the set

$$T = \{m \mid \quad m \in \mathbb{N} \quad \text{and} \quad 2 \le m \le 2n-2\}$$

to identify numbers that are sieved out. Following this, we reverse the second half of the set $\{n, n+1, \ldots, 2n-2\}$. The corresponding marks from sets $\{a \mid \ 2 \le a \le n\}$ and $\{b \mid \ 2n-2 \ge b \ge n\}$ are then transferred to the set S, where

$$S = \{s \mid \quad 2 \le s \le n\}.$$

After sieving out all marked down numbers from S, we have all the prime pairs (p, q) such that $p + q = 2n$.

See the following Table 5. (note all even numbers have been sieved out already) and Table 6 for examples.

60=a+b			62=a+b			64=a+b			66=a+b			68=a+b		
k	a	b	k	a	b	k	a	b	k	a	b	k	a	b
1	1	59	1	1	61	1	1	63	1	1	65	1	1	67
3	3	57	3	3	59	3	3	61	3	3	63	3	3	65
5	5	55	5	5	57	5	5	59	5	5	61	5	5	63
7	7	53	7	7	55	7	7	57	7	7	59	7	7	61
9	9	51	9	9	53	9	9	55	9	9	57	9	9	59
11	11	49	11	11	51	11	11	53	11	11	55	11	11	57
13	13	47	13	13	49	13	13	51	13	13	53	13	13	55
15	15	45	15	15	47	15	15	49	15	15	51	15	15	53
17	17	43	17	17	45	17	17	47	17	17	49	17	17	51
19	19	41	19	19	43	19	19	45	19	19	47	19	19	49
21	21	39	21	21	41	21	21	43	21	21	45	21	21	47
23	23	37	23	23	39	23	23	41	23	23	43	23	23	45
25	25	35	25	25	37	25	25	39	25	25	41	25	25	43
27	27	33	27	27	35	27	27	37	27	27	39	27	27	41
29	29	31	29	29	33	29	29	35	29	29	37	29	29	39
			31	31	31	31	31	33	31	31	35	31	31	37
									33	33	33	33	33	35

Table 5.

After the Sieves, we have the following left prime pairs:														
$60 = p + q$			$62 = p + q$			$64 = p + q$			$66 = p + q$			$68 = p + q$		
s	p	q	s	p	q	s	p	q	s	p	q	s	p	q
7	7	53	3	3	59	3	3	61	5	5	61	7	7	61
13	13	47	19	19	43	5	5	59	7	7	59	31	31	37
17	17	43	31	31	31	11	11	53	13	13	53			
19	19	41				17	17	47	19	19	47			
23	23	37				23	23	41	23	23	43			
29	29	31							29	29	37			

Table 6.

The above Sieve process also can be expressed in the following way.

Like before, to find all the prime pairs (p, q) such that $p+q = 2n$, we only need to find the first prime p in the pair by using the Sieves to the set $S = \{s \mid \ 2 \leq s \leq n\}$ as shown in the following set:

$$PP[S] = S - \bigcup_{p\in P(\sqrt{n})} S_p - \bigcup_{p\in P(\sqrt{2n})} S_p(x_p) = P[S] - \bigcup_{p\in P(\sqrt{2n})} S_p(x_p) \tag{34}$$

$$\text{where} \quad S_p = \left\{mp \mid \ m \in \mathbb{N} \ \text{ and } \ p \leq m \leq \frac{n}{p}\right\} \text{ for all } p \in P(\sqrt{n}), \quad \text{and}$$

$$S_p(x_p) = \left\{mp + x_p \mid \ m \in \mathbb{N} \ \text{ and } \ m \leq \frac{n - x_p}{p}, \text{ for some } -p < x_p < p\right\} \text{ for all } p \in P(\sqrt{2n}).$$

Notice that x_p varies depending on the n, but we always have $x_2 = 0$.

For example, in Table 2:

$$n = 30, \quad \text{then } x_2 = x_3 = x_5 = 0, \quad \text{and} \quad x_7 = 4;$$
$$n = 32, \quad \text{then } x_2 = 0, \ x_3 = x_7 = 1, \quad \text{and} \quad x_5 = 4;$$
$$n = 33, \quad \text{then } x_2 = x_3 = 0, \ x_5 = 1, \quad \text{and} \quad x_7 = 3;$$
$$n = 34, \quad \text{then } x_2 = 0, \ x_3 = 2, \ x_5 = -2, \quad \text{and} \quad x_7 = -2.$$

For $n = 31$ in Table 2,

$$S = \{a\} = \{2, 3, \ldots, 30, 31\},$$
$$\{b\} = \{60, 59, \ldots, 32, 31\},$$
$$T = \{a\} \cup \{b\} = \{2, 3, \ldots, 31, \ldots, 60\},$$

Since $x_2 = 0$, $x_3 = x_5 = 2$, and $x_7 = 6$, we have

$$\begin{array}{ll} S_2 = \{4, 6, \ldots, 30\}, & S_2(0) = \{2, 4, \ldots, 30\}, \\ S_3 = \{9, 12, \ldots, 27, 30\}, & S_3(2) = \{5, 8, \ldots, 23, 26, 29\}, \\ S_5 = \{25\}, & S_5(2) = \{7, 12, 17, 22, 27\}, \\ & S_7(6) = \{13, 20, 27\}. \end{array}$$

From

$$P[S] = S - (S_2 \cup S_2(0) \cup S_3 \cup S_3(2) \cup S_5 \cup S_5(2) \cup S_7(6)) = \{3, 19, 31\},$$

We find all 3 prime pairs:

$$\{(3, 59), (19, 43), (31, 31)\}$$

such that $62 = 3 + 59 = 19 + 43 = 31 + 31$.

For $n \geq 2$, let $\eta(2n)$ denote the number of prime pairs with the sum equal to $2n$. We can call it the ***Goldbach number*** since it comes from Goldbach's conjecture.

For example,

$$\eta(4) = \eta(6) = \eta(8) = 1,\ \eta(10) = 2,\ \eta(12) = 1, \ldots$$

From Table 6, we have

$$\eta(60) = 6,\ \eta(62) = 3,\ \eta(64) = 5,\ \eta(66) = 6,\ \eta(68) = 2.$$

Since $\eta(2n)$ is the size of the set PP$[S]$ from (34), then

1). Like $\pi_2(k; n)$, $\eta(2n)$ also varies with n depending on the factors of n. If $2 < p \leq \sqrt{n}$ is a factor of n, or $\sqrt{n} < p \leq \sqrt{2n}$, then only one p-sequence is sieved out from S.

For example, $n = 30$ has factors 3 and 5, and $\sqrt{30} < 7 \leq \sqrt{60}$, so only one 3-sequence, one 5-sequence, and one 7-sequence are sieved out, that $\eta(60) = 6$ (see Table 5).

2). Also, from Table 5, we can see that when $n = 32$, some elements from the 5-sequence and 7-sequence are merged into the 3-sequence so that fewer elements are sieved out.

While other than 1) and 2) above, in general for each prime $3 \leq p \leq \sqrt{n}$, there are at most two p-sequences that are sieved out from S, and therefore, we have the following approximations of $\eta(2n)$:

$$\frac{C}{2} n \left\{ \prod_{3 \leq p \leq \sqrt{n}} \left(1 - \frac{2}{p}\right) \right\} \left\{ \prod_{\sqrt{n} < q \leq \sqrt{2n}} \left(1 - \frac{1}{q}\right) \right\} \sim G \frac{n}{(\ln n)^2}, \tag{35}$$

where C and G are some corresponding constants.

For 1) when n has prime factors $2 < p, \ldots, q \leq \sqrt{n}$, then

$$\eta\,(2n) \geq \frac{(p-1)\ldots(q-1)}{(p-2)\ldots(q-2)} G \frac{n}{(\ln n)^2} > G \frac{n}{(\ln n)^2}; \tag{36}$$

For 2) when some elements from p-sequence are merged into other q-sequence, then

$$\eta(2n) > G\frac{n}{(\ln n)^2}. \tag{37}$$

Therefore, when n is large, we have the following:

$$\eta(2n) \geq G\frac{n}{(\ln n)^2} > 1. \tag{38}$$

The appendix shows some examples of $\eta(2n)$ and $\frac{n}{(\ln n)^2}$ such that $\eta(2n) > \frac{n}{(\ln n)^2}$ with only a few exceptions.

So, we have the following:

THEOREM 3.1. *For any even number $n > 2$, there exists $\eta(n) \geq 1$ of prime pairs p and q such that $n = p + q$.*

This is much stronger than the following *Goldbach's conjecture*.

THEOREM 3.2. *Every even number greater than 2 can be written as the sum of two primes.*

Notice that from (38), we have

$$\eta(2n) \geq G\frac{n}{(\ln n)^2} \sim H\pi_2(n), \tag{39}$$

for some constant H.

Therefore, the number of prime pairs with the sum equal to $2n$ is related to the number of twin primes less than or equal to n. See the following some Examples:

$100 = 2 \times 2 \times 5 \times 5$	$\eta(100) = 6,$	$\pi_2(50) = 6$
$560 = 2 \times 2 \times 2 \times 2 \times 5 \times 7$	$\eta(560) = 18,$	$\pi_2(280) = 18$
$940 = 2 \times 2 \times 5 \times 47$	$\eta(940) = 24,$	$\pi_2(470) = 24$
$4660 = 2 \times 2 \times 5 \times 233$	$\eta(4660) = 69,$	$\pi_2(2330) = 70$
$9470 = 2 \times 5 \times 947$	$\eta(9470) = 124,$	$\pi_2(4735) = 122$
$11630 = 2 \times 5 \times 1163$	$\eta(11630) = 145,$	$\pi_2(5815) = 140$
$18928 = 2 \times 2 \times 2 \times 2 \times 7 \times 13 \times 13$	$\eta(18928) = 213,$	$\pi_2(9464) = 199$
$210 = 2 \times 3 \times 5 \times 7$	$\eta(210) = 19,$	$\pi_2(105) = 9$
$4620 = 2 \times 2 \times 3 \times 5 \times 7 \times 11$	$\eta(4620) = 190,$	$\pi_2(2310) = 70$
$18942 = 2 \times 3 \times 7 \times 11 \times 41$	$\eta(18942) = 428,$	$\pi_2(9471) = 199$

From the above examples and (36), we can see that when n has factor 3, then $\eta(2n) \geq 2\pi_2(n)$.

Acknowledgments

Thank you to the referees for carefully reading the manuscript and making useful suggestions.

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Appendix

In the following table, $\eta(N)$ is the *Goldbach number*, and

$$g = \frac{N/2}{(\ln(N/2))^2}$$

n	N=12n	η(N)	g	η(N)-g	N=12n+2	η(N)	g	η(N)-g	N=12n+4	η(N)	g	η(N)-g
1	12	1	1.87	-0.9	14	2	1.85	0.2	16	2	1.85	0.1
2	24	3	1.94	1.1	26	3	1.98	1.0	28	2	2.01	0.0
3	36	4	2.15	1.8	38	2	2.19	-0.2	40	3	2.23	0.8
4	48	5	2.38	2.6	50	4	2.41	1.6	52	3	2.45	0.6
5	60	6	2.59	3.4	62	3	2.63	0.4	64	5	2.66	2.3
6	72	6	2.80	3.2	74	5	2.84	2.2	76	5	2.87	2.1
7	84	8	3.01	5.0	86	5	3.04	2.0	88	4	3.07	0.9
8	96	7	3.20	3.8	98	3	3.24	-0.2	100	6	3.27	2.7
9	108	8	3.39	4.6	110	6	3.42	2.6	112	7	3.46	3.5
10	120	12	3.58	8.4	122	4	3.61	0.4	124	5	3.64	1.4
11	132	9	3.76	5.2	134	6	3.79	2.2	136	5	3.82	1.2
12	144	11	3.94	7.1	146	6	3.97	2.0	148	5	3.99	1.0
13	156	11	4.11	6.9	158	5	4.14	0.9	160	8	4.17	3.8
14	168	13	4.28	8.7	170	9	4.31	4.7	172	6	4.33	1.7
15	180	14	4.44	9.6	182	6	4.47	1.5	184	8	4.50	3.5
16	192	11	4.61	6.4	194	7	4.63	2.4	196	9	4.66	4.3
17	204	14	4.77	9.2	206	7	4.79	2.2	208	7	4.82	2.2
18	216	13	4.93	8.1	218	7	4.95	2.0	220	9	4.98	4.0
19	228	12	5.08	6.9	230	9	5.11	3.9	232	7	5.13	1.9
20	240	18	5.24	12.8	242	8	5.26	2.7	244	9	5.29	3.7
21	252	16	5.39	10.6	254	9	5.41	3.6	256	8	5.44	2.6
22	264	16	5.54	10.5	266	8	5.56	2.4	268	9	5.59	3.4
23	276	16	5.68	10.3	278	7	5.71	1.3	280	14	5.73	8.3
24	288	17	5.83	11.2	290	10	5.85	4.1	292	8	5.88	2.1
25	300	21	5.97	15.0	302	9	6.00	3.0	304	10	6.02	4.0
26	312	17	6.12	10.9	314	9	6.14	2.9	316	10	6.16	3.8
27	324	20	6.26	13.7	326	7	6.28	0.7	328	10	6.31	3.7
28	336	19	6.40	12.6	338	9	6.42	2.6	340	11	6.45	4.6
29	348	16	6.54	9.5	350	13	6.56	6.4	352	10	6.58	3.4
30	360	22	6.67	15.3	362	8	6.70	1.3	364	14	6.72	7.3
31	372	18	6.81	11.2	374	10	6.83	3.2	376	11	6.86	4.1
32	384	19	6.95	12.1	386	12	6.97	5.0	388	9	6.99	2.0
33	396	21	7.08	13.9	398	7	7.10	-0.1	400	14	7.12	6.9
34	408	20	7.21	12.8	410	13	7.24	5.8	412	11	7.26	3.7
35	420	30	7.34	22.7	422	11	7.37	3.6	424	12	7.39	4.6
36	432	19	7.48	11.5	434	13	7.50	5.5	436	11	7.52	3.5
37	444	21	7.61	13.4	446	12	7.63	4.4	448	13	7.65	5.4
38	456	24	7.73	16.3	458	9	7.76	1.2	460	16	7.78	8.2
39	468	23	7.86	15.1	470	15	7.88	7.1	472	13	7.91	5.1
40	480	29	7.99	21.0	482	11	8.01	3.0	484	14	8.03	6.0
41	492	22	8.12	13.9	494	13	8.14	4.9	496	13	8.16	4.8
42	504	26	8.24	17.8	506	15	8.26	6.7	508	14	8.28	5.7
43	516	23	8.37	14.6	518	11	8.39	2.6	520	17	8.41	8.6
44	528	25	8.49	16.5	530	14	8.51	5.5	532	17	8.53	8.5

Table 7.

45	540	30	8.61	21.4	542	9	8.64	0.4	544	13	8.66	4.3
46	552	23	8.74	14.3	554	11	8.76	2.2	556	11	8.78	2.2
47	564	21	8.86	12.1	566	12	8.88	3.1	568	13	8.90	4.1
48	576	26	8.98	17.0	578	12	9.00	3.0	580	18	9.02	9.0
49	588	26	9.10	16.9	590	14	9.12	4.9	592	11	9.14	1.9
50	600	30	9.22	20.8	602	11	9.24	1.8	604	13	9.26	3.7
51	612	23	9.34	13.7	614	15	9.36	5.6	616	17	9.38	7.6
52	624	30	9.46	20.5	626	12	9.48	2.5	628	16	9.50	6.5
53	636	27	9.58	17.4	638	15	9.60	5.4	640	18	9.62	8.4
54	648	26	9.70	16.3	650	19	9.72	9.3	652	15	9.73	5.3
55	660	41	9.81	31.2	662	13	9.83	3.2	664	16	9.85	6.1
56	672	30	9.93	20.1	674	15	9.95	5.1	676	16	9.97	6.0
57	684	27	10.05	17.0	686	16	10.06	5.9	688	15	10.08	4.9
58	696	28	10.16	17.8	698	14	10.18	3.8	700	23	10.20	12.8
59	708	24	10.28	13.7	710	16	10.30	5.7	712	17	10.31	6.7
60	720	39	10.39	28.6	722	14	10.41	3.6	724	13	10.43	2.6
61	732	29	10.50	18.5	734	15	10.52	4.5	736	19	10.54	8.5
62	744	31	10.62	20.4	746	18	10.64	7.4	748	19	10.66	8.3
63	756	33	10.73	22.3	758	15	10.75	4.2	760	21	10.77	10.2
64	768	29	10.84	18.2	770	26	10.86	15.1	772	17	10.88	6.1
65	780	44	10.96	33.0	782	14	10.98	3.0	784	17	10.99	6.0
66	792	34	11.07	22.9	794	16	11.09	4.9	796	14	11.11	2.9
67	804	31	11.18	19.8	806	16	11.20	4.8	808	14	11.22	2.8
68	816	32	11.29	20.7	818	17	11.31	5.7	820	20	11.33	8.7
69	828	33	11.40	21.6	830	21	11.42	9.6	832	21	11.44	9.6
70	840	50	11.51	38.5	842	18	11.53	6.5	844	17	11.55	5.5
71	852	30	11.62	18.4	854	20	11.64	8.4	856	19	11.66	7.3
72	864	32	11.73	20.3	866	16	11.75	4.3	868	21	11.77	9.2
73	876	36	11.84	24.2	878	14	11.86	2.1	880	23	11.88	11.1
74	888	37	11.95	25.1	890	23	11.97	11.0	892	19	11.98	7.0
75	900	48	12.06	35.9	902	15	12.07	2.9	904	15	12.09	2.9
76	912	31	12.16	18.8	914	20	12.18	7.8	916	18	12.20	5.8
77	924	46	12.27	33.7	926	18	12.29	5.7	928	17	12.31	4.7
78	936	35	12.38	22.6	938	18	12.40	5.6	940	24	12.42	11.6
79	948	33	12.49	20.5	950	25	12.50	12.5	952	23	12.52	10.5
80	960	45	12.59	32.4	962	16	12.61	3.4	964	17	12.63	4.4
81	972	32	12.70	19.3	974	17	12.72	4.3	976	18	12.73	5.3
385	4620	190	38.51	151.5	4622	54	38.52	15.5	4624	52	38.53	13.5
386	4632	105	38.58	66.4	4634	67	38.60	28.4	4636	58	38.61	19.4
387	4644	112	38.66	73.3	4646	58	38.67	19.3	4648	68	38.68	29.3
388	4656	114	38.73	75.3	4658	55	38.74	16.3	4660	69	38.76	30.2
389	4668	111	38.81	72.2	4670	74	38.82	35.2	4672	56	38.83	17.2
390	4680	162	38.88	123.1	4682	52	38.89	13.1	4684	62	38.90	23.1
391	4692	116	38.95	77.0	4694	55	38.97	16.0	4696	58	38.98	19.0
392	4704	129	39.03	90.0	4706	59	39.04	20.0	4708	63	39.05	23.9
393	4716	112	39.10	72.9	4718	58	39.11	18.9	4720	79	39.13	39.9
788	9456	194	66.04	128.0	9458	100	66.05	33.9	9460	140	66.06	73.9
789	9468	185	66.10	118.9	9470	124	66.11	57.9	9472	99	66.13	32.9
790	9480	255	66.17	188.8	9482	98	66.18	31.8	9484	91	66.19	24.8
968	11616	235	77.32	157.7	11618	106	77.33	28.7	11620	177	77.34	99.7
969	11628	230	77.38	152.6	11630	145	77.39	67.6	11632	111	77.40	33.6
970	11640	290	77.44	212.6	11642	105	77.45	27.5	11644	111	77.46	33.5
1576	18912	314	112.84	201.2	18914	188	112.85	75.2	18916	158	112.85	45.1
1577	18924	334	112.89	221.1	18926	159	112.90	46.1	18928	213	112.91	100.1
1578	18936	305	112.95	192.1	18938	159	112.96	46.0	18940	215	112.97	102.0

Table 8.

n	N=12n+6	η(N)	g	η(N)-g	N=12n+8	η(N)	g	η(N)-g	N=12n+10	η(N)	g	η(N)-g
1	18	2	1.86	0.1	20	2	1.89	0.1	22	3	1.91	1.1
2	30	3	2.05	1.0	32	2	2.08	-0.1	34	4	2.12	1.9
3	42	4	2.27	1.7	44	3	2.30	0.7	46	4	2.34	1.7
4	54	5	2.49	2.5	56	3	2.52	0.5	58	4	2.56	1.4
5	66	6	2.70	3.3	68	2	2.73	-0.7	70	5	2.77	2.2
6	78	7	2.91	4.1	80	4	2.94	1.1	82	5	2.97	2.0
7	90	9	3.11	5.9	92	4	3.14	0.9	94	5	3.17	1.8
8	102	8	3.30	4.7	104	5	3.33	1.7	106	6	3.36	2.6
9	114	10	3.49	6.5	116	6	3.52	2.5	118	6	3.55	2.5
10	126	10	3.67	6.3	128	3	3.70	-0.7	130	7	3.73	3.3
11	138	8	3.85	4.2	140	7	3.88	3.1	142	8	3.91	4.1
12	150	12	4.02	8.0	152	4	4.05	-0.1	154	8	4.08	3.9
13	162	10	4.19	5.8	164	5	4.22	0.8	166	6	4.25	1.7
14	174	11	4.36	6.6	176	7	4.39	2.6	178	7	4.42	2.6
15	186	13	4.53	8.5	188	5	4.55	0.4	190	8	4.58	3.4
16	198	13	4.69	8.3	200	8	4.72	3.3	202	9	4.74	4.3
17	210	19	4.85	14.2	212	6	4.87	1.1	214	8	4.90	3.1
18	222	11	5.00	6.0	224	7	5.03	2.0	226	7	5.06	1.9
19	234	15	5.16	9.8	236	9	5.18	3.8	238	9	5.21	3.8
20	246	16	5.31	10.7	248	6	5.34	0.7	250	9	5.36	3.6
21	258	14	5.46	8.5	260	10	5.49	4.5	262	9	5.51	3.5
22	270	19	5.61	13.4	272	7	5.64	1.4	274	11	5.66	5.3
23	282	16	5.76	10.2	284	8	5.78	2.2	286	12	5.81	6.2
24	294	19	5.90	13.1	296	8	5.93	2.1	298	11	5.95	5.0
25	306	15	6.05	9.0	308	8	6.07	1.9	310	12	6.09	5.9
26	318	15	6.19	8.8	320	11	6.21	4.8	322	11	6.24	4.8
27	330	24	6.33	17.7	332	6	6.35	-0.4	334	11	6.38	4.6
28	342	17	6.47	10.5	344	10	6.49	3.5	346	9	6.51	2.5
29	354	20	6.61	13.4	356	9	6.63	2.4	358	10	6.65	3.3
30	366	18	6.74	11.3	368	8	6.77	1.2	370	14	6.79	7.2
31	378	23	6.88	16.1	380	13	6.90	6.1	382	10	6.92	3.1
32	390	27	7.01	20.0	392	11	7.04	4.0	394	11	7.06	3.9
33	402	17	7.15	9.9	404	11	7.17	3.8	406	13	7.19	5.8
34	414	21	7.28	13.7	416	10	7.30	2.7	418	11	7.32	3.7
35	426	21	7.41	13.6	428	9	7.43	1.6	430	14	7.45	6.5
36	438	21	7.54	13.5	440	13	7.56	5.4	442	14	7.58	6.4
37	450	26	7.67	18.3	452	12	7.69	4.3	454	12	7.71	4.3
38	462	27	7.80	19.2	464	12	7.82	4.2	466	13	7.84	5.2
39	474	23	7.93	15.1	476	14	7.95	6.1	478	11	7.97	3.0
40	486	23	8.05	14.9	488	9	8.07	0.9	490	19	8.10	10.9
41	498	23	8.18	14.8	500	13	8.20	4.8	502	15	8.22	6.8
42	510	32	8.30	23.7	512	11	8.33	2.7	514	13	8.35	4.7
43	522	24	8.43	15.6	524	11	8.45	2.6	526	15	8.47	6.5
44	534	22	8.55	13.4	536	13	8.57	4.4	538	14	8.59	5.4
45	546	30	8.68	21.3	548	11	8.70	2.3	550	19	8.72	10.3
46	558	22	8.80	13.2	560	18	8.82	9.2	562	14	8.84	5.2
47	570	32	8.92	23.1	572	11	8.94	2.1	574	15	8.96	6.0
48	582	25	9.04	16.0	584	12	9.06	2.9	586	12	9.08	2.9
49	594	27	9.16	17.8	596	12	9.18	2.8	598	15	9.20	5.8

Table 9.

50	606	27	9.28	17.7	608	13	9.30	3.7	610	19	9.32	9.7
51	618	26	9.40	16.6	620	17	9.42	7.6	622	16	9.44	6.6
52	630	41	9.52	31.5	632	10	9.54	0.5	634	13	9.56	3.4
53	642	25	9.64	15.4	644	17	9.66	7.3	646	15	9.68	5.3
54	654	29	9.75	19.2	656	13	9.77	3.2	658	19	9.79	9.2
55	666	31	9.87	21.1	668	11	9.89	1.1	670	21	9.91	11.1
56	678	28	9.99	18.0	680	20	10.01	10.0	682	16	10.03	6.0
57	690	39	10.10	28.9	692	11	10.12	0.9	694	18	10.14	7.9
58	702	31	10.22	20.8	704	17	10.24	6.8	706	19	10.26	8.7
59	714	37	10.33	26.7	716	14	10.35	3.6	718	15	10.37	4.6
60	726	32	10.45	21.6	728	14	10.47	3.5	730	20	10.49	9.5
61	738	29	10.56	18.4	740	17	10.58	6.4	742	18	10.60	7.4
62	750	39	10.68	28.3	752	14	10.69	3.3	754	16	10.71	5.3
63	762	30	10.79	19.2	764	16	10.81	5.2	766	17	10.83	6.2
64	774	32	10.90	21.1	776	16	10.92	5.1	778	15	10.94	4.1
65	786	30	11.01	19.0	788	15	11.03	4.0	790	21	11.05	10.0
66	798	38	11.12	26.9	800	21	11.14	9.9	802	16	11.16	4.8
67	810	39	11.24	27.8	812	18	11.25	6.7	814	18	11.27	6.7
68	822	29	11.35	17.7	824	15	11.36	3.6	826	20	11.38	8.6
69	834	32	11.46	20.5	836	18	11.47	6.5	838	17	11.49	5.5
70	846	32	11.57	20.4	848	15	11.58	3.4	850	25	11.60	13.4
71	858	39	11.68	27.3	860	18	11.69	6.3	862	17	11.71	5.3
72	870	46	11.79	34.2	872	15	11.80	3.2	874	19	11.82	7.2
73	882	39	11.89	27.1	884	21	11.91	9.1	886	18	11.93	6.1
74	894	34	12.00	22.0	896	20	12.02	8.0	898	19	12.04	7.0
75	906	34	12.11	21.9	908	15	12.13	2.9	910	31	12.15	18.9
76	918	35	12.22	22.8	920	23	12.24	10.8	922	19	12.25	6.7
77	930	43	12.33	30.7	932	17	12.34	4.7	934	20	12.36	7.6
78	942	34	12.43	21.6	944	18	12.45	5.5	946	20	12.47	7.5
79	954	37	12.54	24.5	956	19	12.56	6.4	958	22	12.58	9.4
80	966	45	12.65	32.4	968	17	12.66	4.3	970	27	12.68	14.3
81	978	35	12.75	22.2	980	26	12.77	13.2	982	17	12.79	4.2
385	4626	108	38.55	69.5	4628	54	38.56	15.4	4630	74	38.57	35.4
386	4638	105	38.62	66.4	4640	68	38.63	29.4	4642	64	38.65	25.4
387	4650	151	38.69	112.3	4652	54	38.71	15.3	4654	58	38.72	19.3
388	4662	135	38.77	96.2	4664	66	38.78	27.2	4666	54	38.79	15.2
389	4674	119	38.84	80.2	4676	68	38.86	29.1	4678	57	38.87	18.1
390	4686	127	38.92	88.1	4688	50	38.93	11.1	4690	94	38.94	55.1
391	4698	113	38.99	74.0	4700	75	39.00	36.0	4702	57	39.02	18.0
392	4710	146	39.06	106.9	4712	54	39.08	14.9	4714	52	39.09	12.9
393	4722	107	39.14	67.9	4724	55	39.15	15.8	4726	59	39.16	19.8
788	9462	198	66.07	131.9	9464	116	66.08	49.9	9466	91	66.09	24.9
789	9474	190	66.14	123.9	9476	96	66.15	29.9	9478	111	66.16	44.8
790	9486	210	66.20	143.8	9488	87	66.21	20.8	9490	142	66.22	75.8
968	11622	239	77.35	161.6	11624	114	77.36	36.6	11626	112	77.37	34.6
969	11634	262	77.41	184.6	11636	112	77.42	34.6	11638	121	77.43	43.6
970	11646	218	77.47	140.5	11648	134	77.48	56.5	11650	155	77.49	77.5
1576	18918	314	112.86	201.1	18920	239	112.87	126.1	18922	155	112.88	42.1
1577	18930	420	112.92	307.1	18932	161	112.93	48.1	18934	162	112.94	49.1
1578	18942	428	112.98	315.0	18944	168	112.99	55.0	18946	157	112.99	44.0

Table 10.