

# On the Prime Pairs

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## Abstract

By using the method of Sieves, we can find all kinds of prime pairs including the twin primes, and thus we prove that for any natural number  $k$ , there are infinitely many pairs of primes that differ by  $2k$ . We discover that the distribution of prime pairs has the similar pattern as the distribution of primes. Also, by exploring the approximation of the number of prime pairs with the sum equal to an even number, we prove that every even number greater than 2 are the sum of different two prime pairs.

## 1. Prime Pairs

A *prime number* (or a *prime*) is a natural number greater than 1 that has no positive divisors other than 1 and itself. We start with some basic notations.

We denote by  $\mathbb{N}$  the set of all natural numbers,  $P[X]$  the set of all prime numbers in the set of  $X$ ,  $\mathbb{P} = P[\mathbb{N}]$  the set of all prime numbers,  $\{u_i\}$  a sequence with a general term  $u_i$ ,  $\ln x$  the natural logarithm of  $x$ ,  $\pi(x)$  the number of primes less than or equal to  $x$ ,  $P(x) = \{2, 3, 5, \dots p\}$  be the set of all primes less than or equal to  $x$ ,  $\gcd(a, b)$  the greatest common divisor of  $a$  and  $b$ .

We say that  $a$  and  $b$  are *relatively prime* if  $\gcd(a, b) = 1$ .

It is well known that the sieve of Eratosthenes has long been used in the study of prime numbers.

Let  $S = \{m, 1 < m \leq n\}$  be the set of natural numbers great than 1 and less than or equal to  $n$ . Then we can use the sieve of Eratosthenes to find out all prime numbers in the set  $S$  such that:

$$P[S] = S - \bigcup_{p \in P(\sqrt{n})} S_p,$$

$$\text{where } S_p = \left\{ mp, \quad m \in \mathbb{N} \text{ and } p \leq m \leq \frac{n}{p} \right\}, \quad \text{for all } p \in P(\sqrt{n}).$$

Now to find all twin primes  $(p, p+2)$  such that  $p \leq n$ , we only need to find the first prime  $p$  in the pair by using the Sieves to the set  $S$  as showing in the following set:

$$(1) \quad P_1 P[S] = S - \bigcup_{p \in P(\sqrt{n})} S_p - \bigcup_{p \in P(\sqrt{n+2})} S_p(2) = P[S] - \bigcup_{p \in P(\sqrt{n+2})} S_p(2),$$

$$\text{where } S_p = \left\{ mp, \quad m \in \mathbb{N} \text{ and } p \leq m \leq \frac{n}{p} \right\} \quad \text{for all } p \in P(\sqrt{n}),$$

$$\text{and } S_p(2) = \left\{ mp - 2, \quad m \in \mathbb{N} \text{ and } p \leq m \leq \frac{n+2}{p} \right\} \quad \text{for all } p \in P(\sqrt{n+2}).$$

For Example: when  $n = 100$ , then  $S = \{2, 3, \dots, 99, 100\}$ ,

$$S_2 = \{4, 6, 8, \dots, 98, 100\},$$

$$S_3 = \{9, 12, 15, \dots, 96, 99\},$$

$$S_2(2) = \{2, 4, 6, \dots, 96, 98, 100\},$$

$$S_3(2) = \{7, 10, 13, \dots, 94, 97, 100\},$$

$$S_5 = \{25, 30, 35, \dots, 95, 100\},$$

$$S_7 = \{49, 56, 63, \dots, 91, 98\},$$

$$S_5(2) = \{23, 28, 33, \dots, 93, 98\},$$

$$S_7(2) = \{47, 54, 61, \dots, 89, 96\}.$$

From

$$P_1 P[S] = S - \{S_2 \cup S_2(2) \cup S_3 \cup S_3(2) \cup S_5 \cup S_5(2) \cup S_7 \cup S_7(2)\} = \{3, 5, 11, 17, 29, 41, 59, 71\},$$

we find out all 8 twin primes less than 100:

$$\{(3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), (59, 61), (71, 73)\}.$$

Or from  $P[S] = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}$ ,

and  $S_2(2) = \{2\}$ ,  $S_3(2) = \{7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97\}$ ,  $S_5(2) = \{23, 53, 83\}$ ,  $S_7(2) = \{47, 89\}$ ,

we also have  $P_1 P[S] = P[S] - \{S_2(2) \cup S_3(2) \cup S_5(2) \cup S_7(2)\} = \{3, 5, 11, 17, 29, 41, 59, 71\}.$

On the other hand, we can find the second prime in the pair by using the Sieve to the following set:

$$P[S](2) = \{p + 2, \quad p \in P[S]\},$$

$$\text{as showing in this set: } Q_1 P[S] = P[S](2) - \bigcup_{p \in P(\sqrt{n+2})} S_p,$$

$$\text{where } S_p = \left\{ mp, \quad m \in \mathbb{N} \text{ and } p \leq m \leq \frac{n+2}{p} \right\} \quad \text{for all } p \in P(\sqrt{n+2}).$$

Take the above Example again:  $n = 100$ ,  $S = \{2, 3, \dots, 99, 100\}$ ,

$$\text{from } P[S](2) = \{4, 5, 7, 9, 13, 15, 19, 21, 25, 31, 33, 39, 43, 45, 49, 55, 61, 63, 69, 73, 75, 81, 85, 91, 99\},$$

and  $S_2 = \{4\}$ ,  $S_3 = \{9, 15, 21, 33, 39, 45, 63, 69, 75, 81, 99\}$ ,  $S_5 = \{25, 45, 55, 75, 85\}$ ,  $S_7 = \{49, 63, 91\}$ ,

we have  $Q_1 P[S] = P[S](2) - \{S_2 \cup S_3 \cup S_5 \cup S_7\} = \{5, 7, 13, 19, 31, 43, 61, 73\}.$

Let  $\mathbb{N}_1 = \mathbb{N} - \{1\}$  be the set of natural numbers great than 1, then to find all twin primes we can find the first prime in the pair by using the Sieves to  $\mathbb{N}_1$  as showing in the following set:

$$(2) \quad P_1 P[\mathbb{N}] = \mathbb{N}_1 - \bigcup_{p \in \mathbb{P}} \{N_p \cup N_p(2)\} = \mathbb{N}_1 - \bigcup_{p \in \mathbb{P}} N_p - \bigcup_{p \in \mathbb{P}} N_p(2) = \mathbb{P} - \bigcup_{p \in \mathbb{P}} N_p(2),$$

where  $N_p = \{mp, \quad m \in \mathbb{N} \text{ and } m \geq p\}$  for all  $p \in \mathbb{P}$ ,  
and  $N_p(2) = \{mp - 2, \quad m \in \mathbb{N} \text{ and } m \geq p\}$  for all  $p \in \mathbb{P}$ .

Or we can find the second prime in the pair by using the Sieve to the following set:

$$\mathbb{P}(2) = \{p + 2, \quad p \in \mathbb{P}\},$$

such that  $Q_1 P[\mathbb{N}] = \mathbb{P}(2) - \bigcup_{p \in \mathbb{P}} N_p,$

where  $N_p = \{mp, \quad m \in \mathbb{N} \text{ and } m \geq p\}$  for all  $p \in \mathbb{P}$ .

Similarly, for any  $k \in \mathbb{N}$ , to find the general prime pairs  $(p, p+2k)$  such that  $p \leq n$ , we only need to find the first prime  $p$  in the pair by using the Sieves to the set  $S$  as showing in the following set:

$$(3) \quad P_k P[S] = S - \bigcup_{p \in P(\sqrt{n})} S_p - \bigcup_{p \in P(\sqrt{n+2k})} S_p(2k) = P[S] - \bigcup_{p \in P(\sqrt{n+2k})} S_p(2k)$$

where  $S_p = \left\{ mp, \quad m \in \mathbb{N} \text{ and } p \leq m \leq \frac{n}{p} \right\}$  for all  $p \in P(\sqrt{n})$ , and  
 $S_p(2k) = \left\{ mp - 2k, m \in \mathbb{N} \text{ and } \text{Max}\left(p, \frac{1+2k}{p}\right) \leq m \leq \frac{n+2k}{p} \right\}$  for all  $p \in P(\sqrt{n+2k})$ .

Again, to find all prime pairs  $(p, p+2k)$  we only need to find the first prime in the pair by using the Sieves to  $\mathbb{N}_1$  as showing in the following set:

$$(4) \quad P_k P[\mathbb{N}] = \mathbb{N}_1 - \bigcup_{p \in \mathbb{P}} \{N_p \cup N_p(2k)\} = \mathbb{N}_1 - \bigcup_{p \in \mathbb{P}} N_p - \bigcup_{p \in \mathbb{P}} N_p(2k) = \mathbb{P} - \bigcup_{p \in \mathbb{P}} N_p(2k),$$

where  $N_p = \{mp, \quad m \in \mathbb{N} \text{ and } m \geq p\}$  for all  $p \in \mathbb{P}$ ,  
and  $N_p(2k) = \left\{ mp - 2k, \quad m \in \mathbb{N} \text{ and } m \geq \text{Max}\left(p, \frac{1+2k}{p}\right) \right\}$  for all  $p \in \mathbb{P}$ .

Or we can find the second prime in the pair by using the Sieve to the following set:

$$\mathbb{P}(2k) = \{p + 2k, \quad p \in \mathbb{P}\},$$

such that  $Q_k P[\mathbb{N}] = \mathbb{P}(2k) - \bigcup_{p \in \mathbb{P}} N_p,$

where  $N_p = \{mp, \quad m \in \mathbb{N} \text{ and } m \geq p\}$  for all  $p \in \mathbb{P}$ .

It is obvious that the procedure of the sieve in (2) and (4) must go through all  $p \in \mathbb{P}$ . If there is an  $n$  such that  $P_1 P[\mathbb{N}] = \mathbb{N}_1 - \bigcup_{p \in P(n)} \{N_p \cup N_p(2)\}$ , let  $W = \{(2 \cdot 3 \cdot 5 \cdots p)k, k \in \mathbb{N} \text{ and } p \in P(n)\}$ , then  $W$  is an infinite subset of  $\mathbb{N}_1$  and  $W \cap \{\bigcup_{p \in P(n)} (N_p \cup N_p(2))\} = \emptyset$  is empty. So we can continue the procedure of the sieve on  $W$  with those primes greater than  $n$ , that is,  $W - \bigcup_{p \in \mathbb{P}-P(n)} \{N_p \cup N_p(2)\}$  to produce the new first prime in the pair. Therefore  $P_1 P[\mathbb{N}] \neq \mathbb{N}_1 - \bigcup_{p \in P(n)} \{N_p \cup N_p(2)\}$  for any  $n$ , and thus the procedure of the sieve in (2) and (4) must go through all  $p \in \mathbb{P}$ . In other words, any finite of  $\{N_p \cup N_p(2)\}$  cannot cover  $\mathbb{N}_1$ , that is,  $\mathbb{N}_1 - \bigcup_{p \in P(n)} \{N_p \cup N_p(2)\}$  is infinite for any  $n$ .

To prove the infinity of  $P_k P[\mathbb{N}]$ , we introduce *Euler  $\phi$ -function*,  $\phi(n)$ , the number of natural numbers less than or equal to  $n$  that are relatively prime to  $n$ .

For any large positive number  $x$ , let

$$N = \prod_{p \leq \sqrt{x}} p$$

and

$$X(n) = \{m, \quad 1 \leq m \leq n \text{ and } \gcd(m, N) = 1\}.$$

Then we have ([1]):

1) The size of  $X(N)$  is

$$\phi(N) = N \prod_{p \leq \sqrt{x}} \left(1 - \frac{1}{p}\right)$$

2) The size of  $X(x)$  is

$$\pi(x) - \pi(\sqrt{x}) \sim \frac{e^\gamma}{2} x \prod_{p \leq \sqrt{x}} \left(1 - \frac{1}{p}\right) \sim \frac{x}{\ln x}$$

where  $\gamma$  is Euler's constant; or

$$\pi(x) \sim Li(x) = \int_2^x \frac{dt}{\ln t}.$$

To consider the prime pairs, let  $\pi_2(k; x)$  denote the number of prime pairs  $(p, p+2k)$  such that  $p \leq x$ . We first look at the case  $k=1$ , let  $\pi_2(x) = \pi_2(1; x)$  denote the number of twin primes such that the first prime in the pair is less than or equal to  $x$ .

If let

$$X_2(n) = \{m, \quad 1 \leq m \leq n, \quad \gcd(m, N) = 1 \text{ and } \gcd(m+2, N) = 1\},$$

then we have ([1]):

1) The size of  $X_2(N)$  is

$$\frac{1}{2} N \prod_{3 \leq p \leq \sqrt{x}} (1 - \frac{2}{p})$$

2) The size of  $X_2(x)$  is

$$\pi_2(x) - \pi_2(\sqrt{x}) \sim \frac{C}{2} x \prod_{3 \leq p \leq \sqrt{x}} (1 - \frac{2}{p})$$

for some constant C.

In order to use the following ([1] THEOREM 429)

$$\prod_{p \leq \sqrt{x}} (1 - \frac{1}{p}) \sim \frac{2e^{-\gamma}}{\ln x},$$

we need to do some calculations from  $\prod(1 - \frac{2}{p})$  to  $\prod(1 - \frac{1}{p})$ :

$$\begin{aligned} \frac{1}{2} \prod_{3 \leq p \leq \sqrt{x}} (1 - \frac{2}{p}) &= \frac{1 \cdot 1 \cdot 3 \cdots (p-2)}{2 \cdot 3 \cdot 5 \cdots p} = \frac{1 \cdot 2 \cdot 4 \cdots (p-1)}{2 \cdot 3 \cdot 5 \cdots p} \frac{1 \cdot 1 \cdot 3 \cdots (p-2)}{1 \cdot 2 \cdot 4 \cdots (p-1)} \\ &= \frac{1 \cdot 2 \cdot 4 \cdots (p-1)}{2 \cdot 3 \cdot 5 \cdots p} \frac{1 \cdot 1 \cdot 3 \cdots (p-2)}{1 \cdot 2 \cdot 4 \cdots (p-1)} \frac{1 \cdot 2 \cdot 4 \cdots (p-1)}{2 \cdot 3 \cdot 5 \cdots p} \frac{2 \cdot 3 \cdot 5 \cdots p}{1 \cdot 2 \cdot 4 \cdots (p-1)} \\ &= 2 \left( \prod_{3 \leq p \leq \sqrt{x}} \frac{p(p-2)}{(p-1)^2} \right) \left( \prod_{p \leq \sqrt{x}} (1 - \frac{1}{p}) \right)^2. \end{aligned}$$

Thus,

$$\pi_2(x) - \pi_2(\sqrt{x}) \sim \frac{8C}{e^{2\gamma}} \left( \prod_{p \geq 3} \frac{p(p-2)}{(p-1)^2} \right) \frac{x}{(\ln x)^2}.$$

If let

$$A = \frac{8C}{e^{2\gamma}} \left( \prod_{p \in \mathbb{P}, p \geq 3} \frac{p(p-2)}{(p-1)^2} \right),$$

then

$$(5) \quad \pi_2(x) \sim A \frac{x}{(\ln x)^2}.$$

or

$$(6) \quad \pi_2(x) \sim A \int_2^x \frac{dt}{(\ln t)^2} = A \left( Li(x) - \frac{x}{\ln x} + \frac{2}{\ln 2} - Li(2) \right),$$

since

$$\int \frac{dt}{(\ln t)^2} = Li(t) - \frac{t}{\ln t} + C .$$

From above (6) we can see that the count of twin primes is determined by the difference of two different approximations of the count of primes.

For  $\pi_2(k; n)$ , if  $k$  has factors  $2 < p \leq \sqrt{n}$ , or  $\sqrt{n} < p \leq \sqrt{n+2k}$  then only one  $p$ -sequence sieved out from  $S$ . Therefore, we have

1) When  $k$  has no other prime factor less than  $\sqrt{n}$  except 2, then

$$\pi_2(k; n) \sim A_k \pi_2(n)$$

2) When  $k$  has prime factors  $2 < p, \dots, q \leq \sqrt{n}$ , then

$$\pi_2(k; n) \sim \frac{(p-1) \dots (q-1)}{(p-2) \dots (q-2)} A_k \pi_2(n)$$

where  $A_k$  is some constant corresponds to  $k$ .

Regarding the constant  $A$ , we have the following Table 1 showing some examples of the choice of  $A$  with  $A=1.32038$ . But whether this is the best choice is still unknown.

		$y = n/\ln(n)$		$A=1.32038$			$A=1.32038$	
$n$	$\pi_2(n)$	$Li(n)-y$	$a$	$b=Aa$	$b-\pi_2(n)$	$c$	$d=Ac$	$d-\pi_2(n)$
1,000	35	31	35	46	11	21	28	-7
5,000	126	96	99	131	5	69	91	-35
10,000	205	159	162	214	9	118	156	-49
50,000	705	544	547	722	17	427	564	-141
100,000	1,224	942	946	1249	25	754	996	-228
500,000	4,565	3502	3505	4628	63	2904	3834	-731
1,000,000	8,169	6244	6247	8248	79	5239	6918	-1251
5,000,000	32,463	24487	24490	32336	-127	21015	27747	-4716
10,000,000	58,980	44496	44500	58756	-224	38492	50824	-8156
100,000,000	440,312	333529	333532	440389	77	294706	389124	-51188
10,000,000,000	27,412,679	20761132	20761136	27412589	-90	18861170	24903911	-2508768

Table 1.

Here  $a = \int_2^n \frac{dt}{(\ln t)^2}$ , and  $c = \frac{n}{(\ln n)^2}$ .

Prime pairs can be generalized to **prime  $k$ -tuples**,  $(p_1, p_2, p_3, \dots, p_k)$ , patterns in the differences between more than two prime numbers.

Like the prime pairs, we can study the infinitude and density of prime  $k$ -tuples.

Now, we try to prove the following ***Twin Prime Conjecture***:

**THEOREM 1.1.** *There are infinitely many twin primes.*

*Proof.* Suppose that there are only finite twin primes, and let  $q$  be the largest first prime of the pair.

Let  $x = q^2$ , and

$$N = \prod_{p \leq q} p$$

$$X_2(n) = \{ m, \quad 1 \leq m \leq n, \quad \gcd(m, N) = 1 \text{ and } \gcd(m + 2, N) = 1 \}$$

Then all elements in  $X_2(x)$  are greater than  $q$  and also the first prime of the twin primes, a contradiction. □

Similar argument of Theorem 1.1, we have the following general result:

**THEOREM 1.2.** *For any natural number  $k$ , there are infinitely many pairs of primes that differ by  $2k$ .*

Notice that the above Theorem 1.2 can be interpreted in a different way as follows:

**THEOREM 1.3.** *Every even number is the difference of two primes and there are infinite of such pairs of primes.*

On the other hand, we want to know if every even number greater than 2 is the sum of two primes, which is also known as the Goldbach conjecture and will be explored in the following section.

## 2. Prime Pairs sum to an Even Number and Goldbach conjecture

For any  $3 \leq n \in \mathbb{N}$ , consider all pairs  $(a, b)$  such that  $a + b = 2n$ , with  $a = 2, \dots, n$ , and  $b = 2n - 2, \dots, n$ .

$2n-3, \dots, n$ . We use the sieve of Eratosthenes and mark down those numbers being sieved out to  $T = \{m, m \in \mathbb{N} \text{ and } 2 \leq m \leq 2n-2\}$ , then reverse the second half  $\{n, n+1, \dots, 2n-2\}$  and put all the marks from  $\{a, 2 \leq a \leq n\}$  and  $\{b, 2n-2 \geq b \geq n\}$  to  $\{s, 2 \leq s \leq n\} = S$ . See the following Table 2 (here we already sieve out all even numbers) and Table 3 for examples.

60=a+b			62=a+b			64=a+b			66=a+b			68=a+b		
s	a	b	s	a	b	s	a	b	s	a	b	s	a	b
3	3	57	3	3	59	3	3	61	3	3	63	3	3	65
5	5	55	5	5	57	5	5	59	5	5	61	5	5	63
7	7	53	7	7	55	7	7	57	7	7	59	7	7	61
9	9	51	9	9	53	9	9	55	9	9	57	9	9	59
11	11	49	11	11	51	11	11	53	11	11	55	11	11	57
13	13	47	13	13	49	13	13	51	13	13	53	13	13	55
15	15	45	15	15	47	15	15	49	15	15	51	15	15	53
17	17	43	17	17	45	17	17	47	17	17	49	17	17	51
19	19	41	19	19	43	19	19	45	19	19	47	19	19	49
21	21	39	21	21	41	21	21	43	21	21	45	21	21	47
23	23	37	23	23	39	23	23	41	23	23	43	23	23	45
25	25	35	25	25	37	25	25	39	25	25	41	25	25	43
27	27	33	27	27	35	27	27	37	27	27	39	27	27	41
29	29	31	29	29	33	29	29	35	29	29	37	29	29	39
			31	31	31	31	31	33	31	31	35	31	31	37
									33	33	33	33	33	35

Table 2.

After the Sieves, we have the following left prime pairs:														
60=a+b			62=a+b			64=a+b			66=a+b			68=a+b		
s	a	b	s	a	b	s	a	b	s	a	b	s	a	b
7	7	53	3	3	59	3	3	61	5	5	61	7	7	61
13	13	47	19	19	43	5	5	59	7	7	59	31	31	37
17	17	43	31	31	31	11	11	53	13	13	53			
19	19	41				17	17	47	19	19	47			
23	23	37				23	23	41	23	23	43			
29	29	31							29	29	37			

Table 3.



Thus, we can get the following set of the first prime in the pair with the sum equal to  $2n$ :

$$(7) \quad PP[S] = S - \bigcup_{p \in P(\sqrt{n})} S_p - \bigcup_{p \in P(\sqrt{2n})} S_p(x_p) = P[S] - \bigcup_{p \in P(\sqrt{2n})} S_p(x_p)$$

where  $S_p = \left\{ mp, \quad m \in \mathbb{N} \text{ and } p \leq m \leq \frac{n}{p} \right\}$  for all  $p \in P(\sqrt{n})$ ,

and  $S_p(x_p) = \left\{ mp + x_p, \quad m \in \mathbb{N} \text{ and } m \leq \frac{n - x_p}{p}, \right.$

$\left. \text{for some } -p < x_p < p \right\}$  for all  $p \in P(\sqrt{2n})$ .

Notice that  $x_p$  varies depending on the  $n$ , but we always have  $x_2 = 0$ .

For example, in the Table 2: when  $n=30$ , then  $x_2 = x_3 = x_5 = 0$  and  $x_7 = 4$ ;  
 $n=32$ , then  $x_2 = 0$ ,  $x_3 = x_7 = 1$  and  $x_5 = 4$ ;  
 $n=33$ , then  $x_2 = x_3 = 0$ ,  $x_5 = 1$  and  $x_7 = 3$ ;  
 $n=34$ , then  $x_2 = 0$ ,  $x_3 = 2$ ,  $x_5 = -2$  and  $x_7 = -2$ .  
For  $n=31$ , then  $S=\{a\}=\{2,3,4,\dots,30,31\}$ ,  $\{b\}=\{60,59,\dots,32,31\}$ ,  $T=\{a\} \cup \{b\}=\{2,3,\dots,31,32,\dots,59,60\}$ ,  
since  $x_2 = 0$ ,  $x_3 = x_5 = 2$ , and  $x_7 = 6$  we have

$$\begin{aligned} S_2 &= \{4, 6, \dots, 30\}, & S_2(0) &= \{2, 4, 6, \dots, 30\}, \\ S_3 &= \{9, 12, \dots, 27, 30\}, & S_3(2) &= \{5, 8, 11, \dots, 26, 29\}, \\ S_5 &= \{25\}, & S_5(2) &= \{7, 12, 17, 22, 27\}, \\ & & S_7(6) &= \{13, 20, 27\}. \end{aligned}$$

From  $PP[S] = S - \{S_2 \cup S_3 \cup S_5\} - \{S_2(0) \cup S_3(2) \cup S_5(2) \cup S_7(6)\} = \{3, 19, 31\}$ ,  
we find out all 3 prime pairs:  $\{(3, 59), (19, 43), (31, 31)\}$  such that  $62 = 3+59=19+43=31+31$ .

For  $n \geq 2$ , let  $\eta(2n)$  denote the number of prime pairs with the sum equal to  $2n$ , and we can call it **Goldbach number** since it comes from the Goldbach conjecture.

$\eta(2n)$  is the size of the set  $PP[S]$ , like  $\pi_2(k; n)$ ,  $\eta(2n)$  also varies with  $n$  depending on the factors of  $n$ . If  $n$  has factors  $2 < p \leq \sqrt{n}$ , or  $\sqrt{n} < p \leq \sqrt{2n}$  then only one  $p$ -sequence sieved out from  $S$ . While in general for each prime  $3 \leq p \leq \sqrt{n}$ , there are at most two corresponding  $p$ -sequences sieved out from  $S$ .

Therefore, we have the following approximations of  $\eta(2n)$  like  $\pi_2(k; n)$ :

- 1) When  $2n$  has no other prime factor less than  $\sqrt{2n}$  except 2, then we have the following approximations of  $\eta(2n)$ :

$$\eta(2n) \sim G \frac{n}{(\ln n)^2};$$

- 2) When  $2n$  has prime factors  $2 < p, \dots, q$  less than  $\sqrt{2n}$ , then

$$\eta(2n) \sim \frac{(p-1) \dots (q-1)}{(p-2) \dots (q-2)} G \frac{n}{(\ln n)^2}$$

where  $G$  is some corresponding constant.

For example, from the Table 3, we have  $\eta(60)=6$ ,  $\eta(62)=3$ ,  $\eta(64)=5$ ,  $\eta(66)=6$ ,  $\eta(68)=2$ .

From the above 1) and 2), we have the following:

**THEOREM 2.1.** *For any even number  $n > 2$ , there exists  $\eta(n) \geq 1$  of prime pairs  $p$  and  $q$  such that  $n = p + q$ .*

This is much stronger than the following ***Goldbach's conjecture***.

**THEOREM 2.2.** *Every even number greater than 2 can be written as the sum of two primes.*

The appendix shows some examples of  $\eta(2n)$  and  $\frac{n}{(\ln n)^2}$  such that  $\eta(2n) \geq \frac{n}{(\ln n)^2}$  with only a few exceptions.

## References

- [1] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 6<sup>th</sup> ed. Oxford University Press, 2008.
- [2] G. E. Andrews, *Number Theory*, 1<sup>st</sup> Dover ed. Dover Publications, 1994.

## Appendix

In the following table,  $\eta(N)$  is the *Goldbach number*, and

$$g = \frac{N/2}{(\ln (N/2))^2}$$

n	N=12n	$\eta(N)$	g	$\eta(N)-g$	N=12n+2	$\eta(N)$	g	$\eta(N)-g$	N=12n+4	$\eta(N)$	g	$\eta(N)-g$
1	12	1	1.87	-0.9	14	2	1.85	0.2	16	2	1.85	0.1
2	24	3	1.94	1.1	26	3	1.98	1.0	28	2	2.01	0.0
3	36	4	2.15	1.8	38	2	2.19	-0.2	40	3	2.23	0.8
4	48	5	2.38	2.6	50	4	2.41	1.6	52	3	2.45	0.6
5	60	6	2.59	3.4	62	3	2.63	0.4	64	5	2.66	2.3
6	72	6	2.80	3.2	74	5	2.84	2.2	76	5	2.87	2.1
7	84	8	3.01	5.0	86	5	3.04	2.0	88	4	3.07	0.9
8	96	7	3.20	3.8	98	3	3.24	-0.2	100	6	3.27	2.7
9	108	8	3.39	4.6	110	6	3.42	2.6	112	7	3.46	3.5
10	120	12	3.58	8.4	122	4	3.61	0.4	124	5	3.64	1.4
11	132	9	3.76	5.2	134	6	3.79	2.2	136	5	3.82	1.2
12	144	11	3.94	7.1	146	6	3.97	2.0	148	5	3.99	1.0
13	156	11	4.11	6.9	158	5	4.14	0.9	160	8	4.17	3.8
14	168	13	4.28	8.7	170	9	4.31	4.7	172	6	4.33	1.7
15	180	14	4.44	9.6	182	6	4.47	1.5	184	8	4.50	3.5
16	192	11	4.61	6.4	194	7	4.63	2.4	196	9	4.66	4.3
17	204	14	4.77	9.2	206	7	4.79	2.2	208	7	4.82	2.2
18	216	13	4.93	8.1	218	7	4.95	2.0	220	9	4.98	4.0
19	228	11	5.08	5.9	230	9	5.11	3.9	232	7	5.13	1.9
20	240	18	5.24	12.8	242	8	5.26	2.7	244	9	5.29	3.7
21	252	16	5.39	10.6	254	9	5.41	3.6	256	8	5.44	2.6
22	264	16	5.54	10.5	266	8	5.56	2.4	268	9	5.59	3.4
23	276	16	5.68	10.3	278	7	5.71	1.3	280	14	5.73	8.3
24	288	17	5.83	11.2	290	10	5.85	4.1	292	8	5.88	2.1
25	300	21	5.97	15.0	302	9	6.00	3.0	304	10	6.02	4.0
26	312	17	6.12	10.9	314	9	6.14	2.9	316	10	6.16	3.8
27	324	20	6.26	13.7	326	7	6.28	0.7	328	10	6.31	3.7
28	336	19	6.40	12.6	338	9	6.42	2.6	340	11	6.45	4.6

29	348	16	6.54	9.5	350	13	6.56	6.4	352	10	6.58	3.4
30	360	22	6.67	15.3	362	8	6.70	1.3	364	14	6.72	7.3
31	372	18	6.81	11.2	374	10	6.83	3.2	376	11	6.86	4.1
32	384	19	6.95	12.1	386	12	6.97	5.0	388	9	6.99	2.0
33	396	21	7.08	13.9	398	7	7.10	-0.1	400	14	7.12	6.9
34	408	20	7.21	12.8	410	13	7.24	5.8	412	11	7.26	3.7
35	420	30	7.34	22.7	422	11	7.37	3.6	424	12	7.39	4.6
36	432	19	7.48	11.5	434	13	7.50	5.5	436	11	7.52	3.5
37	444	21	7.61	13.4	446	12	7.63	4.4	448	13	7.65	5.4
38	456	24	7.73	16.3	458	9	7.76	1.2	460	16	7.78	8.2
39	468	23	7.86	15.1	470	15	7.88	7.1	472	13	7.91	5.1
40	480	29	7.99	21.0	482	11	8.01	3.0	484	14	8.03	6.0
41	492	22	8.12	13.9	494	13	8.14	4.9	496	13	8.16	4.8
42	504	26	8.24	17.8	506	15	8.26	6.7	508	14	8.28	5.7
43	516	23	8.37	14.6	518	11	8.39	2.6	520	17	8.41	8.6
44	528	25	8.49	16.5	530	14	8.51	5.5	532	17	8.53	8.5
45	540	30	8.61	21.4	542	9	8.64	0.4	544	13	8.66	4.3
46	552	23	8.74	14.3	554	11	8.76	2.2	556	11	8.78	2.2
47	564	21	8.86	12.1	566	12	8.88	3.1	568	13	8.90	4.1
48	576	26	8.98	17.0	578	12	9.00	3.0	580	18	9.02	9.0
49	588	26	9.10	16.9	590	14	9.12	4.9	592	11	9.14	1.9
50	600	30	9.22	20.8	602	11	9.24	1.8	604	13	9.26	3.7
51	612	23	9.34	13.7	614	15	9.36	5.6	616	17	9.38	7.6
52	624	30	9.46	20.5	626	12	9.48	2.5	628	16	9.50	6.5
53	636	27	9.58	17.4	638	15	9.60	5.4	640	18	9.62	8.4
54	648	26	9.70	16.3	650	19	9.72	9.3	652	15	9.73	5.3
55	660	41	9.81	31.2	662	13	9.83	3.2	664	16	9.85	6.1
56	672	30	9.93	20.1	674	15	9.95	5.1	676	16	9.97	6.0
57	684	27	10.05	17.0	686	16	10.06	5.9	688	15	10.08	4.9
58	696	28	10.16	17.8	698	14	10.18	3.8	700	23	10.20	12.8
59	708	24	10.28	13.7	710	16	10.30	5.7	712	17	10.31	6.7
60	720	39	10.39	28.6	722	14	10.41	3.6	724	13	10.43	2.6
61	732	29	10.50	18.5	734	15	10.52	4.5	736	19	10.54	8.5
62	744	31	10.62	20.4	746	18	10.64	7.4	748	19	10.66	8.3
63	756	33	10.73	22.3	758	15	10.75	4.2	760	21	10.77	10.2
64	768	29	10.84	18.2	770	26	10.86	15.1	772	17	10.88	6.1
65	780	44	10.96	33.0	782	14	10.98	3.0	784	17	10.99	6.0
66	792	34	11.07	22.9	794	16	11.09	4.9	796	14	11.11	2.9
67	804	31	11.18	19.8	806	16	11.20	4.8	808	14	11.22	2.8
68	816	32	11.29	20.7	818	17	11.31	5.7	820	20	11.33	8.7

69	828	33	11.40	21.6	830	21	11.42	9.6	832	21	11.44	9.6
70	840	50	11.51	38.5	842	18	11.53	6.5	844	17	11.55	5.5
71	852	30	11.62	18.4	854	20	11.64	8.4	856	19	11.66	7.3
72	864	32	11.73	20.3	866	16	11.75	4.3	868	21	11.77	9.2
73	876	36	11.84	24.2	878	14	11.86	2.1	880	23	11.88	11.1
74	888	37	11.95	25.1	890	23	11.97	11.0	892	19	11.98	7.0
75	900	48	12.06	35.9	902	15	12.07	2.9	904	15	12.09	2.9
76	912	31	12.16	18.8	914	20	12.18	7.8	916	18	12.20	5.8
77	924	46	12.27	33.7	926	18	12.29	5.7	928	17	12.31	4.7
78	936	35	12.38	22.6	938	18	12.40	5.6	940	24	12.42	11.6
79	948	33	12.49	20.5	950	25	12.50	12.5	952	23	12.52	10.5
80	960	45	12.59	32.4	962	16	12.61	3.4	964	17	12.63	4.4
81	972	32	12.70	19.3	974	17	12.72	4.3	976	18	12.73	5.3
385	4620	190	38.51	151.5	4622	54	38.52	15.5	4624	52	38.53	13.5
386	4632	105	38.58	66.4	4634	67	38.60	28.4	4636	58	38.61	19.4
387	4644	112	38.66	73.3	4646	58	38.67	19.3	4648	68	38.68	29.3
388	4656	114	38.73	75.3	4658	55	38.74	16.3	4660	69	38.76	30.2
389	4668	111	38.81	72.2	4670	74	38.82	35.2	4672	56	38.83	17.2
390	4680	162	38.88	123.1	4682	52	38.89	13.1	4684	62	38.90	23.1
391	4692	116	38.95	77.0	4694	55	38.97	16.0	4696	58	38.98	19.0
392	4704	129	39.03	90.0	4706	59	39.04	20.0	4708	63	39.05	23.9
393	4716	112	39.10	72.9	4718	58	39.11	18.9	4720	79	39.13	39.9
788	9456	194	66.04	128.0	9458	100	66.05	33.9	9460	140	66.06	73.9
789	9468	185	66.10	118.9	9470	124	66.11	57.9	9472	99	66.13	32.9
790	9480	255	66.17	188.8	9482	98	66.18	31.8	9484	91	66.19	24.8
968	11616	235	77.32	157.7	11618	106	77.33	28.7	11620	177	77.34	99.7
969	11628	230	77.38	152.6	11630	145	77.39	67.6	11632	111	77.40	33.6
970	11640	290	77.44	212.6	11642	105	77.45	27.5	11644	111	77.46	33.5
1576	18912	314	112.84	201.2	18914	188	112.85	75.2	18916	158	112.85	45.1
1577	18924	334	112.89	221.1	18926	159	112.90	46.1	18928	213	112.91	100.1
1578	18936	305	112.95	192.1	18938	159	112.96	46.0	18940	215	112.97	102.0

n	N=12n+6	$\eta(N)$	g	$\eta(N)-g$	N=12n+8	$\eta(N)$	g	$\eta(N)-g$	N=12n+10	$\eta(N)$	g	$\eta(N)-g$
1	18	2	1.86	0.1	20	2	1.89	0.1	22	3	1.91	1.1
2	30	3	2.05	1.0	32	2	2.08	-0.1	34	4	2.12	1.9
3	42	4	2.27	1.7	44	3	2.30	0.7	46	4	2.34	1.7
4	54	5	2.49	2.5	56	3	2.52	0.5	58	4	2.56	1.4
5	66	6	2.70	3.3	68	2	2.73	-0.7	70	5	2.77	2.2
6	78	7	2.91	4.1	80	4	2.94	1.1	82	5	2.97	2.0
7	90	9	3.11	5.9	92	4	3.14	0.9	94	5	3.17	1.8
8	102	8	3.30	4.7	104	5	3.33	1.7	106	6	3.36	2.6
9	114	10	3.49	6.5	116	6	3.52	2.5	118	6	3.55	2.5
10	126	10	3.67	6.3	128	3	3.70	-0.7	130	7	3.73	3.3
11	138	8	3.85	4.2	140	7	3.88	3.1	142	8	3.91	4.1
12	150	12	4.02	8.0	152	4	4.05	-0.1	154	8	4.08	3.9
13	162	10	4.19	5.8	164	5	4.22	0.8	166	6	4.25	1.7
14	174	11	4.36	6.6	176	7	4.39	2.6	178	7	4.42	2.6
15	186	13	4.53	8.5	188	5	4.55	0.4	190	8	4.58	3.4
16	198	13	4.69	8.3	200	8	4.72	3.3	202	9	4.74	4.3
17	210	19	4.85	14.2	212	6	4.87	1.1	214	8	4.90	3.1
18	222	11	5.00	6.0	224	7	5.03	2.0	226	7	5.06	1.9
19	234	15	5.16	9.8	236	9	5.18	3.8	238	9	5.21	3.8
20	246	16	5.31	10.7	248	6	5.34	0.7	250	9	5.36	3.6
21	258	14	5.46	8.5	260	10	5.49	4.5	262	9	5.51	3.5
22	270	19	5.61	13.4	272	7	5.64	1.4	274	11	5.66	5.3
23	282	16	5.76	10.2	284	8	5.78	2.2	286	12	5.81	6.2
24	294	19	5.90	13.1	296	8	5.93	2.1	298	11	5.95	5.0
25	306	15	6.05	9.0	308	8	6.07	1.9	310	12	6.09	5.9
26	318	15	6.19	8.8	320	11	6.21	4.8	322	11	6.24	4.8
27	330	23	6.33	16.7	332	6	6.35	-0.4	334	10	6.38	3.6
28	342	17	6.47	10.5	344	10	6.49	3.5	346	9	6.51	2.5
29	354	20	6.61	13.4	356	9	6.63	2.4	358	10	6.65	3.3
30	366	18	6.74	11.3	368	8	6.77	1.2	370	14	6.79	7.2
31	378	23	6.88	16.1	380	13	6.90	6.1	382	10	6.92	3.1
32	390	27	7.01	20.0	392	11	7.04	4.0	394	11	7.06	3.9
33	402	17	7.15	9.9	404	11	7.17	3.8	406	13	7.19	5.8
34	414	21	7.28	13.7	416	10	7.30	2.7	418	11	7.32	3.7
35	426	21	7.41	13.6	428	9	7.43	1.6	430	14	7.45	6.5
36	438	21	7.54	13.5	440	13	7.56	5.4	442	14	7.58	6.4
37	450	26	7.67	18.3	452	12	7.69	4.3	454	12	7.71	4.3
38	462	27	7.80	19.2	464	12	7.82	4.2	466	13	7.84	5.2
39	474	23	7.93	15.1	476	14	7.95	6.1	478	11	7.97	3.0

40	486	23	8.05	14.9	488	9	8.07	0.9	490	19	8.10	10.9
41	498	23	8.18	14.8	500	13	8.20	4.8	502	15	8.22	6.8
42	510	32	8.30	23.7	512	11	8.33	2.7	514	13	8.35	4.7
43	522	24	8.43	15.6	524	11	8.45	2.6	526	15	8.47	6.5
44	534	22	8.55	13.4	536	13	8.57	4.4	538	14	8.59	5.4
45	546	30	8.68	21.3	548	11	8.70	2.3	550	19	8.72	10.3
46	558	22	8.80	13.2	560	18	8.82	9.2	562	14	8.84	5.2
47	570	32	8.92	23.1	572	11	8.94	2.1	574	15	8.96	6.0
48	582	25	9.04	16.0	584	12	9.06	2.9	586	12	9.08	2.9
49	594	27	9.16	17.8	596	12	9.18	2.8	598	15	9.20	5.8
50	606	27	9.28	17.7	608	13	9.30	3.7	610	19	9.32	9.7
51	618	26	9.40	16.6	620	17	9.42	7.6	622	16	9.44	6.6
52	630	41	9.52	31.5	632	10	9.54	0.5	634	13	9.56	3.4
53	642	25	9.64	15.4	644	17	9.66	7.3	646	15	9.68	5.3
54	654	29	9.75	19.2	656	13	9.77	3.2	658	19	9.79	9.2
55	666	31	9.87	21.1	668	11	9.89	1.1	670	21	9.91	11.1
56	678	28	9.99	18.0	680	20	10.01	10.0	682	16	10.03	6.0
57	690	39	10.10	28.9	692	11	10.12	0.9	694	18	10.14	7.9
58	702	31	10.22	20.8	704	17	10.24	6.8	706	19	10.26	8.7
59	714	37	10.33	26.7	716	14	10.35	3.6	718	15	10.37	4.6
60	726	32	10.45	21.6	728	14	10.47	3.5	730	20	10.49	9.5
61	738	29	10.56	18.4	740	17	10.58	6.4	742	18	10.60	7.4
62	750	39	10.68	28.3	752	14	10.69	3.3	754	16	10.71	5.3
63	762	30	10.79	19.2	764	16	10.81	5.2	766	17	10.83	6.2
64	774	32	10.90	21.1	776	16	10.92	5.1	778	15	10.94	4.1
65	786	30	11.01	19.0	788	15	11.03	4.0	790	21	11.05	10.0
66	798	38	11.12	26.9	800	21	11.14	9.9	802	16	11.16	4.8
67	810	39	11.24	27.8	812	18	11.25	6.7	814	18	11.27	6.7
68	822	29	11.35	17.7	824	15	11.36	3.6	826	20	11.38	8.6
69	834	32	11.46	20.5	836	18	11.47	6.5	838	17	11.49	5.5
70	846	32	11.57	20.4	848	15	11.58	3.4	850	25	11.60	13.4
71	858	39	11.68	27.3	860	18	11.69	6.3	862	17	11.71	5.3
72	870	46	11.79	34.2	872	15	11.80	3.2	874	19	11.82	7.2
73	882	39	11.89	27.1	884	21	11.91	9.1	886	18	11.93	6.1
74	894	34	12.00	22.0	896	20	12.02	8.0	898	19	12.04	7.0
75	906	34	12.11	21.9	908	15	12.13	2.9	910	31	12.15	18.9
76	918	35	12.22	22.8	920	23	12.24	10.8	922	19	12.25	6.7
77	930	43	12.33	30.7	932	17	12.34	4.7	934	20	12.36	7.6
78	942	34	12.43	21.6	944	18	12.45	5.5	946	20	12.47	7.5
79	954	37	12.54	24.5	956	19	12.56	6.4	958	22	12.58	9.4

80	966	45	12.65	32.4	968	17	12.66	4.3	970	27	12.68	14.3
81	978	35	12.75	22.2	980	26	12.77	13.2	982	17	12.79	4.2
385	4626	108	38.55	69.5	4628	54	38.56	15.4	4630	74	38.57	35.4
386	4638	105	38.62	66.4	4640	68	38.63	29.4	4642	64	38.65	25.4
387	4650	151	38.69	112.3	4652	54	38.71	15.3	4654	58	38.72	19.3
388	4662	135	38.77	96.2	4664	66	38.78	27.2	4666	54	38.79	15.2
389	4674	119	38.84	80.2	4676	68	38.86	29.1	4678	57	38.87	18.1
390	4686	127	38.92	88.1	4688	50	38.93	11.1	4690	94	38.94	55.1
391	4698	113	38.99	74.0	4700	75	39.00	36.0	4702	57	39.02	18.0
392	4710	146	39.06	106.9	4712	54	39.08	14.9	4714	52	39.09	12.9
393	4722	107	39.14	67.9	4724	55	39.15	15.8	4726	59	39.16	19.8
788	9462	198	66.07	131.9	9464	116	66.08	49.9	9466	91	66.09	24.9
789	9474	190	66.14	123.9	9476	96	66.15	29.9	9478	111	66.16	44.8
790	9486	210	66.20	143.8	9488	87	66.21	20.8	9490	142	66.22	75.8
968	11622	239	77.35	161.6	11624	114	77.36	36.6	11626	112	77.37	34.6
969	11634	262	77.41	184.6	11636	112	77.42	34.6	11638	121	77.43	43.6
970	11646	218	77.47	140.5	11648	134	77.48	56.5	11650	155	77.49	77.5
1576	18918	314	112.86	201.1	18920	239	112.87	126.1	18922	155	112.88	42.1
1577	18930	420	112.92	307.1	18932	161	112.93	48.1	18934	162	112.94	49.1
1578	18942	428	112.98	315.0	18944	168	112.99	55.0	18946	157	112.99	44.0