On the Prime Pairs

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Abstract

By studding the distribution of prime pairs, we proved that for any natural number k, there are infinitely many pairs of primes that differ by 2k. Also, by exploring the approximation of the number of prime pairs with the sum equal to an even number, we proved that every even number greater than 2 is the sum of two primes.

1. Prime Pairs and Twin Prime Conjecture

A *prime number* (or a *prime*) is a natural number greater than 1 that has no positive divisors other than 1 and itself. We start with some basic notations.

We denote by **P[X]** the set of all prime numbers in the set of X, **$\{u_n\}$** a sequence with a general term u_n, **ln(n)** the natural logarithm of n, and **π(n)** the number of primes that less than or equal to n.

It is well known that the sieve of Eratosthenes has long been used in the study of prime numbers.

To study the prime pairs (p, p+2k), we consider the sequences {m}, {2m-1}, and {2m-2k-1}. We use the sieve of Eratosthenes to {2m-1} and {2m-2k-1} and put the results to {m}. In another word, use the sieve of Eratosthenes to {m} twice.

See the following Table 1. for the examples of k=1, 4, 17, and 30:

k=1				k=4				k=17				k=30		
m	2m-1	2m+1		m	2m-1	2m+7		m	2m-1	2m+33		m	2m-1	2m+59
1	1	3		1	1	9		1	1	35		1	1	61
2	3	5		2	3	11		2	3	37		2	3	63
3	5	7		3	5	13		3	5	39		3	5	65
4	7	9		4	7	15		4	7	41		4	7	67
5	9	11		5	9	17		5	9	43		5	9	69
6	11	13		6	11	19		6	11	45		6	11	71
7	13	15		7	13	21		7	13	47		7	13	73
8	15	17		8	15	23		8	15	49		8	15	75
9	17	19		9	17	25		9	17	51		9	17	77
10	19	21		10	19	27		10	19	53		10	19	79
11	21	23		11	21	29		11	21	55		11	21	81
12	23	25		12	23	31		12	23	57		12	23	83
13	25	27		13	25	33		13	25	59		13	25	85
14	27	29		14	27	35		14	27	61		14	27	87
15	29	31		15	29	37		15	29	63		15	29	89
16	31	33		16	31	39		16	31	65		16	31	91
17	33	35		17	33	41		17	33	67		17	33	93
18	35	37		18	35	43		18	35	69		18	35	95
19	37	39		19	37	45		19	37	71		19	37	97
20	39	41		20	39	47		20	39	73		20	39	99
21	41	43		21	41	49		21	41	75		21	41	101
22	43	45		22	43	51		22	43	77		22	43	103
23	45	47		23	45	53		23	45	79		23	45	105
24	47	49		24	47	55		24	47	81		24	47	107
25	49	51		25	49	57		25	49	83		25	49	109
26	51	53		26	51	59		26	51	85		26	51	111
27	53	55		27	53	61		27	53	87		27	53	113
28	55	57		28	55	63		28	55	89		28	55	115
29	57	59		29	57	65		29	57	91		29	57	117
30	59	61		30	59	67		30	59	93		30	59	119
31	61	63		31	61	69		31	61	95		31	61	121
… …	… …	… …		… …	… …	… …		… …	… …	… …		… …	… …	… …

Table 1.

The following Table 2. shows the left prime pairs after the sieve of Eratosthenes:

k=1				k=4				k=17				k=30		
m	2m-1	2m+1		m	2m-1	2m+7		m	2m-1	2m+33		m	2m-1	2m+59
2	3	5		2	3	11		2	3	37		4	7	67
3	5	7		3	5	13		4	7	41		6	11	71
6	11	13		6	11	19		7	13	47		7	13	73
9	17	19		12	23	31		10	19	53		10	19	79
15	29	31		15	29	37		19	37	71		12	23	83
21	41	43		27	53	61		…	…	…		15	29	89
30	59	61		30	59	67						19	37	97
…	…	…		…	…	…						21	41	101
												22	43	103
												24	47	107
												27	53	113
												…	…	…

Table 2.

Before studying the counting of prime pairs, we first look at the counting of primes, that is, **π(n)** the number of primes that less than or equal to n.

Let S = {1, 2, 3, 4, 5, …, n} be the set of all the natural number less than or equal to n, and P = {2, 3, 5, 7, …, p} be the set of all primes less than or equal to $\sqrt{n}$. By using the sieve of Eratosthenes to the set S we have P[S] = S – {$S_2 \cup S_3 \cup S_5 \cup S_7 \cup \ldots \cup S_p$}, where S_i = {k*i, i≤ k≤ n/i} for all i ∈P, and π(n) is the size of the set P[S]. Therefore, we have the following approximations of **π(n)**:

$$n\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\ldots\left(1-\frac{1}{p}\right);$$

from the prime number theorem (PNT)

$$\pi(n) \sim \frac{n}{ln\,(n)}$$

or

$$\pi(n) \sim Li(n) = \int_2^n \frac{dx}{ln\,x}\,.$$

Now consider the prime pairs, let **π_2(k; n)** denote the number of prime pairs (p, p+2k) that less than or equal to n. Then from Table 1. we can see that π_2(k; n) is the size of corresponding (after the sieve) subset of {m}. Notice that for each $3 \le p \le \sqrt{n}$, there are at most two corresponding p-sequences sieved out from {m} (some numbers may be marked more than once in {m}).

We first look at the case k=1, let **π_2(n)** = π_2(1; n) denote the number of twin primes that less than or equal to n. Then we have the following approximations of π_2(n):

$$\frac{n}{2}\left(1-\frac{2}{3}\right)\left(1-\frac{2}{5}\right)\left(1-\frac{2}{7}\right)\dots\left(1-\frac{2}{p}\right);$$

by using the prime number theorem (PNT) twice

$$\pi_2(n) \sim A\,\frac{\frac{n}{ln(n)}}{ln\left[\frac{n}{ln(n)}\right]} \sim B\,\frac{\pi(n)}{ln\ \pi(n)}$$

or

$$\pi_2(n) \sim C\int_2^n \frac{dx}{ln\ Li(x)} \sim D\int_2^n \frac{dx}{ln\ \pi(x)}$$

or

$$\pi_2(n) \sim E\int_2^n \frac{dx}{ln\,x(ln\,x - ln\,ln\,x)}$$

where A, B, C, D and E are some corresponding constants between 1 and 2.

From Table 1, we can see k=30 has factors 3 and 5, then only one 3-sequence and one 5-sequence sieved out from {m}. Therefore, we have

1) When k only has factor 2 less than $\sqrt{n}$, then

$$\pi_2(k;n) \sim \pi_2(n);$$

2) When k has prime factors $2 < p, \dots, q \le \sqrt{n}$, then

$$\pi_2(k;n) \sim \frac{(p-1)\dots(q-1)}{(p-2)\dots(q-2)}\,\pi_2(n).$$

About the constants A, B, C, D and E, we have the following Table 3 showing some examples of the choices of A and B with A=1.265 and B=1.188. But whether they are the best choices still unknow.

				a=y/ln(y)	b=x/ln(x)			
n	$\pi_2(n)$	$x=\pi(n)$	y=n/ln(n)	c=1.265a	d=1.188b	$y-\pi(n)$	$c-\pi_2(n)$	$d-\pi_2(n)$
1000	35	168	145	37	39	-23	2	4
5000	126	669	587	116	123	-82	-10	-3
10000	205	1229	1086	195	207	-143	-10	2
50000	705	5133	4621	688	722	-512	-17	17
100000	1224	9592	8686	1202	1256	-906	-22	32
500000	4565	41538	38103	4535	4691	-3435	-30	126
1000000	8169	78498	72382	8121	8364	-6116	-48	195
5000000	32463	348513	324150	32071	32798	-24363	-392	335
10000000	58980	664579	620421	58395	59531	-44158	-585	551
100000000	440312	5761455	5428681	439487	444488	-332774	-825	4176
10000000000	27412679	455052511	434294482	27412702	27412591	-20758029	23	-88

Table 3.

From above discussion of the distribution of prime pairs, we have the following general result:

THEOREM 1.1. *For any natural number k, there are infinitely many pairs of primes that differ by 2k.*

Notice that ***Twin Prime Conjecture*** is the case k=1 and the above Theorem 1.1 can also be interpreted in a different way as follows:

THEOREM 1.2. *Every even number is the difference of two primes and there are infinite of such pairs of primes.*

On the other hand, we want to know if every even number greater than 2 is the sum of two primes, which is the so called Goldbach conjecture and will be explored in the following section.

2. Prime Pairs of an Even Number and Goldbach conjecture

For any even number n >12, consider all pairs (a, b) such that a + b = n, a =1, 2, …, n/2, and b = n/2, (n/2)+1, …., n-1. The following Table 4. Shows some examples:

n=60=a+b			n=62=a+b			n=64=a+b			n=66=a+b			n=68=a+b		
m	a	b	m	a	b	m	a	b	m	a	b	m	a	b
1	1	59	1	1	61	1	1	63	1	1	65	1	1	67
2	2	58	2	2	60	2	2	62	2	2	64	2	2	66
…	…	…	…	…	…	…	…	…	…	…	…	…	…	…
30	30	30	30	30	32	30	30	34	30	30	36	30	30	38
			31	31	31	31	31	33	31	31	35	31	31	37
						32	32	32	32	32	34	32	32	36
									33	33	33	33	33	35
												34	34	34

Table 4.

Like before, we use the sieve of Eratosthenes to {a} and {b} and put the results to {m}, see the following Table 5 and Table 6 for the examples. (Note that we first take out all even numbers so that the size of {m} reduce to n/4.)

60=a+b			62=a+b			64=a+b			66=a+b			68=a+b		
m	a	b	m	a	b	m	a	b	m	a	b	m	a	b
1	1	59	1	1	61	1	1	63	1	1	65	1	1	67
3	3	57	3	3	59	3	3	61	3	3	63	3	3	65
5	5	55	5	5	57	5	5	59	5	5	61	5	5	63
7	7	53	7	7	55	7	7	57	7	7	59	7	7	61
9	9	51	9	9	53	9	9	55	9	9	57	9	9	59
11	11	49	11	11	51	11	11	53	11	11	55	11	11	57
13	13	47	13	13	49	13	13	51	13	13	53	13	13	55
15	15	45	15	15	47	15	15	49	15	15	51	15	15	53
17	17	43	17	17	45	17	17	47	17	17	49	17	17	51
19	19	41	19	19	43	19	19	45	19	19	47	19	19	49
21	21	39	21	21	41	21	21	43	21	21	45	21	21	47
23	23	37	23	23	39	23	23	41	23	23	43	23	23	45
25	25	35	25	25	37	25	25	39	25	25	41	25	25	43
27	27	33	27	27	35	27	27	37	27	27	39	27	27	41
29	29	31	29	29	33	29	29	35	29	29	37	29	29	39
			31	31	31	31	31	33	31	31	35	31	31	37
									33	33	33	33	33	35

Table 5.

After A-Sieve, we have the following left prime pairs:														
7	7	53	3	3	59	3	3	61	5	5	61	7	7	61
13	13	47	19	19	43	5	5	59	7	7	59	31	31	37
17	17	43	31	31	31	11	11	53	13	13	53			
19	19	41				17	17	47	19	19	47			
23	23	37				23	23	41	23	23	43			
29	29	31							29	29	37			

Table 6.

Let **η(n)** denote the number of prime pairs with the sum equal to n, and we can call it ***Goldbach number*** since it comes from the Goldbach conjecture.

Like π_2(k; n), η(n) also varies with n depending on the factors of n. For example: in the above Table 4, we can see n=60 has factor 3 and 5, then only one 3-sequence and one 5-sequence sieved out from {m}, while in general for each prime 3≤ p≤$\sqrt{\text{n}}$, there are at most two corresponding p-sequences sieved out from {m} (some numbers may be marked more than once in {m}).

Therefore, we have the following approximations of η(n) like π_2(k; n):

1) When n only has factor 2 less than $\sqrt{\text{n}}$, then

$$\eta(n) \sim \frac{n}{4}\left(1-\frac{2}{3}\right)\left(1-\frac{2}{5}\right)\left(1-\frac{2}{7}\right)\dots\left(1-\frac{2}{p}\right)$$

by the prime number theorem (PNT)

$$\eta(n) \sim \frac{G}{2}\ \frac{\frac{n}{ln(n)}}{ln\left[\frac{n}{ln(n)}\right]} \sim \frac{F}{2}\ \pi_2(n)\ ;$$

2) When n has prime factors 2, p, …, q less than $\sqrt{\text{n}}$, then

$$\eta(n) \sim \frac{(p-1)\dots(q-1)}{(p-2)\dots(q-2)}\ \frac{G}{2}\ \frac{\frac{n}{ln(n)}}{ln\left[\frac{n}{ln(n)}\right]}\ .$$

where G, F are some corresponding constants between 1 and 2.

From the above 1) and 2), when n >12 we have

$$\eta(n) > \frac{1}{3} \frac{\frac{n}{ln(n)}}{ln\left[\frac{n}{ln(n)}\right]} > 1 .$$

This indicates that for any even number n >12 there exists at lease one pair of primes p and q such that n = p + q, and therefore it proved the so called ***Goldbach conjecture***.

THEOREM 2.1. *Every even number greater than 2 can be written as the sum of two primes.*

3. Further Study

Prime pairs can be generalized to ***prime k-tuples***, (p_1,p_2,p_3,...,p_k), patterns in the differences between more than two prime numbers.

Like the prime pairs, if we take different combinations of sequences, then we can study the infinitude and density of prime k-tuples.

For example (see Table 7.): If we consider sequences {m}, {2m-1}, {2m+1}, and {2m+5}, then we can discuss the distribution and infiniteness of prime 3-tuples (p, p+2, p+6).

If consider sequences {m}, {2m-1}, {2m+1}, {2m+5}, and {2m+7} then we can discuss the distribution and infiniteness of prime 4-tuples (p, p+2, p+6, p+8).

… …

Let $\boldsymbol{\pi_k(n)}$ denote the number of prime k-tuples that less than or equal to n. Then we can expect

$$\pi_k(n) \sim C_k \frac{\pi_{k-1}(n)}{ln\ \pi_{k-1}(n)}$$

for some constant C_k.

Here we can set

$$\pi(n) = \pi_1(n).$$

The study of more general prime k-tuples could be the future work.

(p, p+2, p+6)				(p, p+4, p+6)				(p, p+6, p+8)				(p, p+2, p+6, p+8)				
m	2m-1	2m+1	2m+5	m	2m-1	2m+3	2m+5	m	2m-1	2m+5	2m+7	m	2m-1	2m+1	2m+5	2m+7
1	1	3	7	1	1	5	7	1	1	7	9	1	1	3	7	9
2	3	5	9	2	3	7	9	2	3	9	11	2	3	5	9	11
3	5	7	11	3	5	9	11	3	5	11	13	3	5	7	11	13
4	7	9	13	4	7	11	13	4	7	13	15	4	7	9	13	15
5	9	11	15	5	9	13	15	5	9	15	17	5	9	11	15	17
6	11	13	17	6	11	15	17	6	11	17	19	6	11	13	17	19
7	13	15	19	7	13	17	19	7	13	19	21	7	13	15	19	21
8	15	17	21	8	15	19	21	8	15	21	23	8	15	17	21	23
9	17	19	23	9	17	21	23	9	17	23	25	9	17	19	23	25
10	19	21	25	10	19	23	25	10	19	25	27	10	19	21	25	27
11	21	23	27	11	21	25	27	11	21	27	29	11	21	23	27	29
12	23	25	29	12	23	27	29	12	23	29	31	12	23	25	29	31
13	25	27	31	13	25	29	31	13	25	31	33	13	25	27	31	33
14	27	29	33	14	27	31	33	14	27	33	35	14	27	29	33	35
15	29	31	35	15	29	33	35	15	29	35	37	15	29	31	35	37
16	31	33	37	16	31	35	37	16	31	37	39	16	31	33	37	39
17	33	35	39	17	33	37	39	17	33	39	41	17	33	35	39	41
18	35	37	41	18	35	39	41	18	35	41	43	18	35	37	41	43
19	37	39	43	19	37	41	43	19	37	43	45	19	37	39	43	45
20	39	41	45	20	39	43	45	20	39	45	47	20	39	41	45	47
21	41	43	47	21	41	45	47	21	41	47	49	21	41	43	47	49
22	43	45	49	22	43	47	49	22	43	49	51	22	43	45	49	51
23	45	47	51	23	45	49	51	23	45	51	53	23	45	47	51	53
24	47	49	53	24	47	51	53	24	47	53	55	24	47	49	53	55
25	49	51	55	25	49	53	55	25	49	55	57	25	49	51	55	57
26	51	53	57	26	51	55	57	26	51	57	59	26	51	53	57	59
27	53	55	59	27	53	57	59	27	53	59	61	27	53	55	59	61
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...

Table 7.

References

Prime number - Wikipedia. https://en.wikipedia.org/wiki/Prime_number. (The page was last edited on 13 May 2021)