Incoherence-Space Semantics

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Chapter 1

Incoherence Spaces

1.1 Moves and positions

Definition 1. A *move* over a language L is a signed sentence: either an assertion $+A$ (written `Move.assert` A) or a denial $-A$ (written `Move.deny` A). A *position* is a list of moves, `List (Move` L`)`.

Definition 2. An *incoherence space* on a language L is a set $\mathcal{I}$ of *incoherent* positions satisfying:

- The empty position is coherent: $[] \notin \mathcal{I}$.

Persistence ($\Gamma \in \mathcal{I} \Rightarrow \Gamma +\!\!+ \Delta \in \mathcal{I}$) is *not* assumed, enabling defeasible material incompatibilities.

1.2 Incoherence profiles and closure

Definition 3. The *incoherence profile* of a set X of positions is

$$X^{\perp} = \{\, \Gamma \mid \forall \Delta \in X,\ \Gamma +\!\!+ \Delta \in \mathcal{I} \,\}.$$

Definition 4. The *incoherence closure* is $X^{\perp\perp} = (X^{\perp})^{\perp}$: the set of positions that incompatibility-entail X.

Lemma 5. *$X \subseteq Y \Rightarrow Y^{\perp} \subseteq X^{\perp}$ (perp is antitone).*

Lemma 6. *$X \subseteq Y \Rightarrow X^{\perp\perp} \subseteq Y^{\perp\perp}$ (dperp is monotone).*

Definition 7. *Incompatibility entailment*: $\Delta \vDash_{\mathcal{I}} \Gamma$ when $\{\Gamma\}^{\perp} \subseteq \{\Delta\}^{\perp}$, i.e., everything incompatible with Γ is also incompatible with Δ.

Lemma 8. *Incompatibility entailment is reflexive.*

Lemma 9. *Incompatibility entailment is transitive.*

1.3 Fusion of positions

Definition 10. The *fusion* of two sets of positions is

$$X \mathbin{\dot\cup} Y = \{\, \Gamma +\!\!+ \Delta \mid \Gamma \in X,\ \Delta \in Y \,\}.$$

Lemma 11. *Fusion is associative:* $(X \dot{\cup} Y) \dot{\cup} Z = X \dot{\cup} (Y \dot{\cup} Z)$.

Lemma 12. $\{[]\} \dot{\cup} X = X$.

Lemma 13. $X \dot{\cup} \{[]\} = X$.

1.4 Structural conditions

1.4.1 Persistence

Definition 14. An incoherence space is *persistent* if $\Gamma \in \mathcal{I}$ implies $\Gamma +\!\!+ \Delta \in \mathcal{I}$ for all Δ. Intuitively: adding more moves to an incoherent position cannot restore coherence.

1.4.2 Exchange

Definition 15. An incoherence space satisfies *Exchange* if incoherence is invariant under reordering: $\Gamma +\!\!+ \Delta \in \mathcal{I} \Rightarrow \Delta +\!\!+ \Gamma \in \mathcal{I}$. Under Exchange, positions behave like multisets rather than lists.

Lemma 16. $\Gamma +\!\!+ \Delta \in \mathcal{I} \iff \Delta +\!\!+ \Gamma \in \mathcal{I}$.

Lemma 17. *Under Exchange,* $X \subseteq X^{\perp\perp}$ *(extensiveness of* $\perp\perp$*).*

Lemma 18. *Under Exchange,* $(X^{\perp\perp})^{\perp\perp} = X^{\perp\perp}$ *(idempotency of* $\perp\perp$*).*

Lemma 19. *Under Exchange, the intersection of two* $\perp\perp$*-closed sets is* $\perp\perp$*-closed.*

Lemma 20. *Under Exchange,* $X^{\perp}$ *is always* $\perp\perp$*-closed:* $(X^{\perp})^{\perp\perp} = X^{\perp}$.

1.4.3 Containment

Definition 21. *All-Containment*: any position containing both $+p$ and $-p$ for some p is incoherent.

Definition 22. *Only-Containment*: a position is incoherent only if it contains both $+p$ and $-p$ for some p.

Definition 23. *All-and-only Containment*: $\Gamma \in \mathcal{I}$ if and only if Γ contains both $+p$ and $-p$ for some atomic sentence p. This is the classical condition from Brandom's incompatibility semantics.

Proposition 24. *All-and-only Containment implies Exchange.*

Proposition 25. *All-and-only Containment implies Persistence.*

Lemma 26. *perp*$(X \cup Y) = X^{\perp} \cap Y^{\perp}$. *This holds without any structural conditions.*

Lemma 27. *Under Containment,* $\emptyset^{\perp\perp} = \mathcal{I}$.

Lemma 28. *Under Containment,* $\mathcal{I}$ *is* $\perp\perp$*-closed.*

Lemma 29. *Under Containment,* $\mathcal{I} \subseteq X$ *for every* $\perp\perp$*-closed set* X.

Lemma 30. *Under Containment, for any* $\perp\perp$*-closed* X*:* $X \cap X^{\perp} = \mathcal{I}$.

Lemma 31. *Under Containment, for any* $\perp\perp$*-closed* X*:* $(X \cup X^{\perp})^{\perp\perp} = \top$.

Chapter 2

Closed Sets

2.1 The ClosedSet type

Definition 32. A *closed set* is a set of positions X satisfying $X^{\perp\perp} = X$. The type `ClosedSet` L requires only a bare `IncoherenceSpace`; the shape of closed sets varies with additional structural conditions.

Lemma 33. *Closed sets are partially ordered by set inclusion.*

2.2 Lattice structure (Exchange)

Lemma 34. *Under Exchange, any arbitrary intersection of $\perp\perp$-closed sets is $\perp\perp$-closed.*

Proposition 35. *Under Exchange, `ClosedSet` L is a lattice:*

- $X \sqcap Y = X \cap Y$ *(intersection of closed sets is closed).*
- $X \sqcup Y = (X \cup Y)^{\perp\perp}$ *(the closure of the union).*
- $\top = \mathcal{U}$ *(the universe, always closed under Exchange).*

2.3 Boolean algebra (Containment)

Definition 36. Under Containment, the complement of a closed set X is $X^{\perp}$ (which is always $\perp\perp$-closed).

Proposition 37. *Under Containment, `ClosedSet` L is a complemented lattice with:*

- $\perp = \mathcal{I}$ *(the incoherence set).*
- *The complement of X is $X^{\perp}$.*
- $X \sqcap X^{\perp} = \perp$ *and* $X \sqcup X^{\perp} = \top$.

Theorem 38. *Under Containment, the lattice of $\perp\perp$-closed sets is distributive.*

The proof uses Zorn's lemma to extend coherent sets of moves to maximal ones, then injects the lattice into a power-set Boolean algebra.

Corollary 39. *Under Containment, `ClosedSet` L is a complete lattice: arbitrary joins are $\perp\perp$-closures of unions; arbitrary meets are intersections.*

Chapter 3

Day Convolution

3.1 Congruence lemmas

Lemma 40. *Under Exchange,* $(X \mathbin{\dot\cup} Y^{\perp\perp})^{\perp\perp} = (X \mathbin{\dot\cup} Y)^{\perp\perp}$*: replacing* Y *by* $Y^{\perp\perp}$ *in the right slot of fusion does not change the closure.*

Lemma 41. *Under Exchange,* $(X^{\perp\perp} \mathbin{\dot\cup} Y)^{\perp\perp} = (X \mathbin{\dot\cup} Y)^{\perp\perp}$*: replacing* X *by* $X^{\perp\perp}$ *in the left slot of fusion does not change the closure.*

3.2 The tensor product

Definition 42. The *Day convolution tensor* of two closed sets is

$$X \otimes^c Y := (X \mathbin{\dot\cup} Y)^{\perp\perp}.$$

Definition 43. The *unit* for the Day convolution tensor is $\{[]\}^{\perp\perp}$.

Proposition 44. *The Day convolution tensor is associative:* $(X \otimes^c Y) \otimes^c Z = X \otimes^c (Y \otimes^c Z)$.

Proposition 45. $\mathbf{1} \otimes^c X = X$*: the unit is a left identity.*

Proposition 46. $X \otimes^c \mathbf{1} = X$*: the unit is a right identity.*

3.3 Monoid and quantale structure

Proposition 47. *Under Containment, the Day convolution tensor is commutative:* $X \otimes^c Y = Y \otimes^c X$.

Under Containment, incoherence depends only on the set *of moves present, so* $(X \mathbin{\dot\cup} Y)^{\perp\perp} = (Y \mathbin{\dot\cup} X)^{\perp\perp}$.

Proposition 48. *Under Exchange,* $(\texttt{ClosedSet}\ L, \otimes^c, \mathbf{1})$ *is a monoid.*

Proposition 49. *Under Containment,* $(\texttt{ClosedSet}\ L, \otimes^c, \mathbf{1})$ *is a commutative monoid.*

Theorem 50. *Under Containment,* $(\texttt{ClosedSet}\ L, \otimes^c)$ *is a commutative unital quantale: a complete lattice in which* $\otimes^c$ *distributes over all joins.*

$$X \otimes^c \bigsqcup_i Y_i = \bigsqcup_i (X \otimes^c Y_i).$$

This is the main algebraic result of the formalization.

Chapter 4

Ternary Frames

4.1 Abstract ternary frames

Definition 51. A *ternary frame* on W is a ternary relation $R : W \to W \to W \to \mathsf{Prop}$. We read $R(x, y, z)$ as "z is a possible result of combining x and y".

Definition 52. The *multiplicative tensor* of two sets $P, Q \subseteq W$ is

$$P \otimes Q = \{\, x \mid \exists y \in P,\ \exists z \in Q,\ R(y, z, x) \,\}.$$

Definition 53. The *left residual* is $P \multimap Q = \{\, x \mid \forall y \in P,\ \forall z,\ R(x, y, z) \Rightarrow z \in Q \,\}$.

Proposition 54. *Left residuation:* $P \otimes Q \subseteq S \iff P \subseteq Q \multimap S$.

Definition 55. A ternary frame is *commutative* if $R(x, y, z) \Rightarrow R(y, x, z)$.

Proposition 56. *Under commutativity of R, tensor is commutative:* $P \otimes Q = Q \otimes P$.

Definition 57. A ternary frame is *associative* if $(R(x, y, z) \wedge R(z, u, v)) \iff (\exists w, R(y, u, w) \wedge R(x, w, v))$.

Proposition 58. *Under associativity of R, tensor is associative:* $(P \otimes Q) \otimes S = P \otimes (Q \otimes S)$.

4.2 Positions as a ternary frame

Proposition 59. *Positions form a preorder under incompatibility entailment:*

$$\Gamma \leq \Delta \iff \{\Gamma\}^{\perp} \subseteq \{\Delta\}^{\perp}.$$

This matches the blog's $p \leq_{IE} q$ iff $\{p\}^{\perp} \subseteq \{q\}^{\perp}$.

Proposition 60. *Positions form a ternary frame with*

$$R(\Gamma, \Delta, \Theta) \iff \Theta \vDash_{\mathcal{I}} \Gamma +\!\!+ \Delta \iff \{\Gamma +\!\!+ \Delta\}^{\perp} \subseteq \{\Theta\}^{\perp}.$$

In words: "Θ is achievable from combining Γ and Δ".

Lemma 61. *Under Containment,* $\{\Gamma +\!\!+ \Delta\}^{\perp} = \{\Delta +\!\!+ \Gamma\}^{\perp}$.

Proposition 62. *Under Containment, the ternary frame on positions is commutative.*

4.3 Connection theorems

Theorem 63. *The fusion is contained in the tensor product:* $X \mathbin{\dot\cup} Y \subseteq X \otimes Y$.

For $\Theta = \Gamma \mathbin{+\!\!+} \Delta \in X \mathbin{\dot\cup} Y$*, we have* $R(\Gamma, \Delta, \Gamma \mathbin{+\!\!+} \Delta)$ *by reflexivity. This holds for bare* `IncoherenceSpace`.

Theorem 64. *Under Exchange, the tensor product is contained in the Day convolution:* $X \otimes Y \subseteq X \otimes^c Y$.

For $\Theta \in X \otimes Y$*, antitonicity gives* $(X \dot\cup Y)^\perp \subseteq \{\Gamma \mathbin{+\!\!+} \Delta\}^\perp \subseteq \{\Theta\}^\perp$*, so* $\Theta \in (X \dot\cup Y)^{\perp\perp} = X \otimes^c Y$.

Theorem 65. *Under Exchange, the Day convolution equals the* $\perp\perp$*-closure of the tensor:*

$$X \otimes^c Y = (X \otimes Y)^{\perp\perp}.$$

This is the main connection theorem: the semantic tensor is $(X \mathbin{\dot\cup} Y)^{\perp\perp} = (X \otimes Y)^{\perp\perp}$.

4.4 Incoherence spaces as ternary frames

The results above combine into the following picture: every incoherence space canonically yields a ternary frame whose worlds are positions and whose quantale of propositions is the lattice of closed sets.

Corollary 66. *Let* L *be a type with* `[IsContainment L]`*. Then:*

1. *Positions* $\mathsf{List}(\mathsf{Move}\ L)$ *carry a commutative ternary frame structure with* $R(\Gamma, \Delta, \Theta) \iff \Theta \vDash_{\mathcal{J}} \Gamma \mathbin{+\!\!+} \Delta$.
2. *The closed sets* $\mathsf{ClosedSet}\ L$ *form a commutative unital quantale under Day convolution* $\otimes^c$.
3. *The ternary tensor of closed sets (computed via* R*) agrees with the Day convolution:* $X \otimes Y = (X \otimes^c Y)$*, i.e.,* $X \otimes Y \subseteq X \otimes^c Y$ *and* $X \otimes^c Y = (X \otimes Y)^{\perp\perp}$.

In short: every incoherence space is a ternary frame, and the incoherence-space semantics of that frame is exactly the quantale $\mathsf{ClosedSet}\ L$.

Chapter 5

Sequent Calculi: NMMS and NMMS$^{\text{ctr}}$

This chapter formalizes two sequent calculi for the logic of incoherence spaces, following Hlobil & Brandom [?] Chapter 3: the *Non-Monotonic Material Sequent* calculus (NMMS) and its multiset variant NMMS$^{\text{ctr}}$.

5.1 The formula type

Both calculi operate over a type of formulas parameterized by an atomic type α.

Definition 67 (Formulas)**.** The type $\mathsf{Formula}\ \alpha$ is generated by:

$$A, B ::= p \mid A \wedge B \mid A \vee B \mid \neg A \mid A \to B$$

where $p : \alpha$ ranges over atomic formulas. The conditional $\to$ is a *primitive* connective, not defined as $\neg A \vee B$.

5.2 NMMS: the set-based calculus

Sequents $\Gamma \vdash \Delta$ in NMMS have *sets* (Lean: `Finset`) of formulas on each side. Exchange and contraction are built in for free.

The calculus is parameterized by a *base consequence relation* $\mathsf{base} : \mathsf{Finset}\ (\mathsf{Formula}\ \alpha) \to \mathsf{Finset}\ (\mathsf{Formula}\ \alpha) \to \mathsf{Prop}$, which provides the axiom instances. For the Containment soundness theorem, we instantiate base with the Containment relation $(\Gamma \cap \Delta) \neq \emptyset$.

Definition 68 (NMMS)**.** The **NMMS** derivability predicate is the smallest relation satisfying:

- [**Ax**] $\Gamma \vdash \Delta$ if $\mathsf{base}\ \Gamma\ \Delta$.
- [**L**$\wedge$] If $\Gamma, A, B \vdash \Delta$ then $\Gamma, A \wedge B \vdash \Delta$.
- [**L**$\vee$] If $\Gamma, A \vdash \Delta$ and $\Gamma, B \vdash \Delta$ and $\Gamma, A, B \vdash \Delta$ then $\Gamma, A \vee B \vdash \Delta$.
- [**L**$\to$] If $\Gamma \vdash \Delta, A$ and $\Gamma, B \vdash \Delta$ and $\Gamma, B \vdash \Delta, A$ then $\Gamma, A \to B \vdash \Delta$.
- [**L**$\neg$] If $\Gamma \vdash A, \Delta$ then $\Gamma, \neg A \vdash \Delta$.

- [**R**$\wedge$] If $\Gamma \vdash \Delta, A$ and $\Gamma \vdash \Delta, B$ and $\Gamma \vdash \Delta, A, B$ then $\Gamma \vdash \Delta, A \wedge B$.
- [**R**$\vee$] If $\Gamma \vdash \Delta, A, B$ then $\Gamma \vdash \Delta, A \vee B$.
- [**R**$\to$] If $\Gamma, A \vdash \Delta, B$ then $\Gamma \vdash \Delta, A \to B$.
- [**R**$\neg$] If $\Gamma, A \vdash \Delta$ then $\Gamma \vdash \Delta, \neg A$.

The rules [**L**$\vee$], [**L**$\to$], and [**R**$\wedge$] each have a *third "mixed" premiss* that distinguishes NMMS from NMMS$^{\text{ctr}}$.

5.3 NMMS$^{\text{ctr}}$: the multiset-based calculus

Sequents in NMMS$^{\text{ctr}}$ use *multisets* (`Multiset`) on each side, giving exchange but not contraction. The rules use the Ketonen style: the three-premiss rules of NMMS become two-premiss rules.

Definition 69 (NMMS$^{\text{ctr}}$)**.** The **NMMS$^{\text{ctr}}$** derivability predicate is defined exactly as NMMS except:

- Sequents are multisets: $\Gamma, \Delta : \mathsf{Multiset}\ (\mathsf{Formula}\ \alpha)$.
- [**L**$\vee$] has only *two* premisses: $\Gamma + \{A\} \vdash \Delta$ and $\Gamma + \{B\} \vdash \Delta$ (no mixed third premiss).
- [**L**$\to$] has only *two* premisses: $\Gamma \vdash \Delta + \{A\}$ and $\Gamma + \{B\} \vdash \Delta$.
- [**R**$\wedge$] has only *two* premisses: $\Gamma \vdash \Delta + \{A\}$ and $\Gamma \vdash \Delta + \{B\}$.
- All other rules are unchanged.

5.4 Soundness and Completeness: Theorem 76

The central result of Chapter 3 of [?] is the soundness and completeness of NMMS with respect to *b-models*: implication-space semantics (ISS) models that fit the base consequence relation.

The semantic framework

Fix a *base* $\mathfrak{B} = \langle \mathfrak{L}_{\mathfrak{B}}, \vdash_{\mathfrak{B}} \rangle$ consisting of a base vocabulary $\mathfrak{L}_{\mathfrak{B}}$ and a base consequence relation $\vdash_{\mathfrak{B}}$ on the base lexicon.

An implication-space model $\mathbf{M} = \langle C, \cdot \rangle$ **fits** the base $\mathfrak{B}$ (is a *b-model*) if for every atomic sequent $\langle \Gamma, \Delta \rangle \in \vdash_{\mathfrak{B}}$, the sequent $\Gamma \Vdash \Delta$ holds in C ([?], Def. 75, p. 221).

The **b-validity** relation $\vdash^b$ is defined by:

$$\Gamma \vdash^b \Delta \iff \text{for every b-model } \mathbf{M},\ \mathbf{M} \vDash \Gamma \vdash \Delta.$$

In Lean, the semantic notions are captured by:

Definition 70 (Consequence space)**.** A **consequence space** is an ISS model (placeholder for the full ISS definition).

Definition 71 (Satisfaction)**.** $\mathsf{Satisfies}\ M\ \Gamma\ \Delta$ holds when $M \vDash \Gamma \vdash \Delta$.

Definition 72 (Fits base)**.** A model M **fits** the base base if $\forall \Gamma\, \Delta,\ \mathsf{base}\ \Gamma\ \Delta \Rightarrow \mathsf{Satisfies}\ M\ \Gamma\ \Delta$.

Definition 73 (b-validity)**.** $\mathsf{bValid}\ \mathsf{base}\ \Gamma\ \Delta$ holds iff $\Gamma \vdash \Delta$ is satisfied in every model fitting base: $\forall M,\ \mathsf{FitsBase}\ \mathsf{base}\ M \Rightarrow \mathsf{Satisfies}\ M\ \Gamma\ \Delta$.

Theorem 76

Theorem 74 (NMMS Soundness (Theorem 76, $\to$))**.** *Every sequent derivable in* NMMS base *is b-valid:*

$$\mathsf{NMMS}\ \mathsf{base}\ \Gamma\ \Delta\ \Rightarrow\ \mathsf{bValid}\ \mathsf{base}\ \Gamma\ \Delta.$$

*The proof proceeds by induction on the derivation tree; each logical rule is shown to preserve b-validity (Props. 51 and 53 in [**?**]).*

Theorem 75 (NMMS Completeness (Theorem 76, $\leftarrow$))**.** *Every b-valid sequent is derivable in* NMMS base*:*

$$\mathsf{bValid}\ \mathsf{base}\ \Gamma\ \Delta\ \Rightarrow\ \mathsf{NMMS}\ \mathsf{base}\ \Gamma\ \Delta.$$

The proof uses a canonical model (principal b-model) construction.

Theorem 76 (Theorem 76 (RLLR, p. 222))**.** *For any base* base *and sentences* Γ*,* Δ *in the logically extended lexicon:*

$$\mathsf{NMMS}\ \mathsf{base}\ \Gamma\ \Delta\ \longleftrightarrow\ \mathsf{bValid}\ \mathsf{base}\ \Gamma\ \Delta.$$

NMMS derivability coincides exactly with b-validity (soundness and completeness).