STAT 547T Homework 1

Instructions: Typeset your solutions and submit them as a PDF via Canvas.

Grading: Problems are equally weighted. One problem will be fully graded, and the others will be graded on a binary scale (full points or no points) based on effort.

Problem 1

Suppose that we test $H_0 : T(F_X) \in \Theta_0$ using a random variable $X \sim F_X$. Let

$$\mathcal{F}_0 = \{F : T(F) \in \Theta_0\}$$

and

$$p(x) = \sup_{F_X \in \mathcal{F}_0} \mathbb{P}_{F_X}(S(X) \geq S(x))$$

for some scalar-valued test statistic function $S(\cdot)$.

Show that $p(X)$ is a valid p-value, i.e. that

$$\sup_{F_X \in \mathcal{F}_0} \mathbb{P}_{F_X}(p(X) \leq \alpha) \leq \alpha, \quad \forall\ 0 \leq \alpha \leq 1.$$

Problem 2

Suppose that we test $H_{0j} : T_j(F_X) \in \Theta_0$ for all $j = 1, 2, \ldots, M$ using a random variable $X \sim F_X$. Define the following notation:

	H_{0j} retained	H_{0j} rejected	Total
H_{0j} true	$\mathrm{TN}(X)$	$\mathrm{FD}(X)$	M_0
H_{0j} false	$\mathrm{FN}(X)$	$\mathrm{TD}(X)$	M_1
Total	$N(X)$	$R(X)$	M

1. Show that if

$$\mathbb{P}_{F_X}(\text{Reject } H_{0j} \text{ using } X) = \alpha, \quad \text{for all } F_X \in \mathcal{F}_{0j} \text{ and } 0 \le \alpha \le 1 \text{ and } j = 1, 2, \ldots, M,$$

then

$$\mathbb{E}_{F_X}\left[\frac{\text{FD}(X)}{M_0}\right] = \alpha, \quad \text{for all } 0 \le \alpha \le 1.$$

2. Let $\Lambda_1 = \{j : H_{0j} \text{ false}\}$, and define

$$\text{Power}_j = \mathbb{P}_{F_X}(\text{Reject } H_{0j} \text{ using } X), \quad \text{for all } j \in \Lambda_1.$$

Show that

$$\mathbb{E}_{F_X}\left[\frac{\text{TD}(X)}{M_1}\right] = \frac{\sum\limits_{j \in \Lambda_1} \text{Power}_j}{|\Lambda_1|}.$$

Problem 3

In the same setting as Problem 2, let

$$\text{FWER} = \mathbb{P}_{F_X}(\text{FD}(X) \ge 1)$$

and

$$\text{FDP}(X) = \begin{cases} \frac{\text{FD}(X)}{R(X)}, & \text{if } R(X) > 0, \\ 0, & \text{otherwise.} \end{cases}, \quad \text{FDR} = \mathbb{E}_{F_X}[FDP(X)].$$

Show that

$$\text{FWER} \ge \text{FDR},$$

for any choice of F_X.