QAOA

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Wednesday - 20/3/2024 - Nikhil Londhe

Paper - A Review of QAOA and it Variants. Kostas Blekos - arXiv: 2306.09198v2

(1) Backround of Optimization problems (Sec 2) (In Content of Approximate Optimization Algorithm)

- Given a Combinatorial optimization problem defined on a n-bit string ($x = x_0, \dots, x_n$), where the goal is to maximize a objective function $C(x) : \{0,1\}^n \to \mathbb{R}_{\geq 0}$, appox optimization algorithm aims to find x^* such that approximation ratio α,

$$\alpha = \frac{C(x^*)}{C_{max}}$$

- Ideally we want $\alpha = 1$.
- We look for some optimality guarantee (lower bounds on the quality of the solution)
  - α - approximated algorithm $\Rightarrow$ $\alpha \leq 1$ for every instance of the problem
  - "hardness of approximation" - gap between $C(x^*)$ and C_{max} Cannot be reduced in poly time.

(2) QUBO - Qudratic Binary Uncostrained Optimization (Sec 2.2)

- QUBO (problem) belong to the NP-Complete class.
  - $\Rightarrow$ Any NP Complete problem can be mapped to QUBO
- A vector of unknowns $x = (x_0, \dots, x_n)$, $x \in \{0,1\}^n$ (Set of Decision variable)

& we have a square symmetric matrix $Q \in \mathbb{R}^{n \times n}$

And Cost function $C(x) = x^T Q x = \sum_{ij}^{n} Q_{ij} x_i x_j \xRightarrow{\text{Can be also written as}} \sum_{\substack{i,j \\ i \neq j \\ i<j \text{ or } i>j}} Q_{ij} x_i x_j + \sum_i Q_{ii} x_i$ $[x_i^2 = x_i]$ Binary variable

Because Q is square symmetric

OR More generally

$$\boxed{C(x) = \sum_{\substack{i,j \\ i \neq j \\ i<j \text{ or } i>j}}^{n} Q_{ij} x_i x_j + \sum_i^{n} Q_{ii} x_i + c}$$

c $\to$ Constant [Doesn't affect the solution]

So the aim is to find x^* st $\boxed{x^* = \arg\min_{x \in \{0,1\}^n} C(x)}$

or $\boxed{x^* = \arg\max_{x \in \{0,1\}^n} -C(x)}$

(3) Number Partitonig

$S = \{s_1, \dots, s_n\}$ $s_i \geq 0$

where we want to minimize d,

$$d = \left| \sum_{s_i \in A} s_i - \sum_{s_j \in A/S} s_j \right| \Rightarrow \boxed{d^2 = c^2 + 4x^T Q x} \quad c \Rightarrow \sum_{s_i \in S} s_i$$

$\therefore Q \Rightarrow$ $\boxed{Q_{ij} = \begin{cases} s_i(s_i - c) & \text{if } i = j \\ s_i s_j & \text{if } i \neq j \end{cases}}$

Expansion

$\Rightarrow (c - 2\sum_{i=1}^{n} s_i x_i)^2 \Rightarrow c^2 + 4\left[\sum_{i=1}^{n} s_i x_i\right]^2 - 4c\sum_{i=1}^{n} s_i x_i$

$$\Rightarrow \underbrace{c^2}_{\substack{\text{Constant Term} \\ \text{(Square of the Sum)} \\ C = \sum_{s_i \in S} s_i}} + 4\sum s_i^2 x_i + \underbrace{4\sum_{i,j=0} s_i s_j x_i x_j}_{\text{Qudratic Terms} - {}^nP_2} - 4c\sum s_i x_i \Rightarrow c^2 + 4\sum s_i s_j x_i x_j + 4\sum x_i s_i (s_i - c)$$

Linear Terms - n

(4) QUBO istances can be seen as 1-1 correspondance with Ising modles.

- Eg. mapping (Sec 2 - OG paper arXiv: 2401: 08989v1

Ising model $\Rightarrow$ $H(\sigma) = -\underbrace{\sum_{ij} J_{ij} \sigma_i \sigma_j}_{\text{Interaction}} - \underbrace{\sum h_i \sigma_i}_{\text{External mag field}}$, $\sigma_i \Rightarrow$ spin $\Rightarrow \{-1, 1\}$

$z_i = 2x_i - 1 \Leftrightarrow x_i = \frac{z_i + 1}{2}$

So we need to find H_c such that

$H_c |x\rangle = C(x) |x\rangle$

We have defind the Cost function above, so we just need to substitute to get H_c

$$C(x) = 4\sum_{i,j} s_i s_j x_i x_j + 4\sum_{i=1}^{n} s_i(s_i - c) x_i \text{ (Ignoring } c^2\text{)}$$

$S = [s_1, \dots, s_n] \Rightarrow$ array

$X = [x_1, \dots, x_n] \Rightarrow$ bit string $x_i \in \{0,1\}^n$

$$\therefore H_C = 4\sum_{i,j} s_i s_j \frac{(z_i + 1)}{2} \frac{(z_j + 1)}{2} + 4\sum_{i=1}^{n} s_i(s_i - c)\frac{(z_i + 1)}{2}$$

$$= \sum_{ij}^{n} s_i s_j z_i z_j + 2\sum_{ij}^{n} s_i s_j z_i + 2\sum_{i=1}^{n} s_i(s_i - c) z_i + \sum_{ij} s_i s_j + 2\sum_{ij} s_i$$

$$\boxed{H_C = \sum_{i,j} s_i s_j z_i z_j + 2\sum_{i=1}^{n}\left(s_i(s_i - c) + \sum_{j=1}^{n} s_i s_j\right) z_i} + \text{Constants}$$

or in general for

$$C(x) = \sum_{i,j=1}^{n} x_i Q_{ij} x_j + \sum_{i=1}^{n} c_i x_i$$

$$\boxed{H_C = \sum_{i,j=1}^{n} \frac{1}{4} Q_{ij} z_i z_j - \sum_{i=1}^{n} \frac{1}{2}\left(c_i + \sum_{j=1}^{n} Q_{ij}\right) z_i + \left(\sum_{ij=1}^{n} \frac{Q_{ij}}{4} + \sum_{i=1}^{n} \frac{c_i}{2}\right)}$$

(5) Adiabatic Theorem - Max Born & Vladmir Fock (1928)

" A physical system remains in its instaneous eigenstate if a given perturbation is acting slowly enough and there is a gap between the eigenvalue and the rest of the Hamiltonian's spectrum."

Eg. If we are in ground state $|0\rangle$ of $H_{t=0}$ then $t = T$ we will be in the ground state of $H_{t=T}$ $(|0'\rangle)$ given that the pertubation (Change) is slow.

Ideally $T = t_1 - t_0 \to \infty$