Perfect and Pretty Good State Transfer in Weighted Discrete Quantum Walks

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Quantum Computing

Faster computers. Very horrifying. All the damage computers have made will happen faster. On the other hand the good results from Computing will also work faster.

[3, 5, 6, 7]

Continuous Model

Let M be the Hermitian matrix associated to a graph X. The transistion matrix

$$U(t)$$

for a continuous quantum walk on X relative to M at time t is

$$U(t) = e^{itM}$$

[2]

Let $a \neq b$. If $U(t)e_a = \gamma e_b$ for some $\gamma \in \mathbb{C}$ then we say there is perfect state transfer (PST) from a to b at time t.

[2]

Discrete Model

We now look at the model where time is discrete.

$$(N_t)_{u,(a,b)} = \begin{cases} w_{ab}, & u = a, \\ 0, & u \neq a, \end{cases}$$

$$U = R(2N_t^* N_t - I)$$

[2]

We are interested in two types of state transfer. We say a quantum walk on X admits *perfect state transfer* (PST) from a to b at time $t \in \mathbb{Z}$ if there is a unimodular number $\gamma \in \mathbb{C}$ such that

$$U^t x_a = \gamma x_b.$$

[2]

Fair Example $K_{1,3}$

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Unfair (Weighted) Result C_4

Next we look at a quantum walk with unfair coins on C_4 where it is more likely to traverse towards the opposite level (up/down) than to the opposite side (left/right). The unfair weighted tail-arc incidence matrix is encoded in Figure 1.

From now on, we will only consider quantum walks defined by real weighted tail-arc incidence matrices N. The following results are special cases of characterizations of perfect and pretty good state transfer in [2, 8].

Theorem

[1, Theorem 2.2] Let X be a graph with real weighted tail-arc incidence matrix N. Let H be the real weighted adjacency matrix of X associated with N. Let a and b be two vertices of X. The quantum walk on X with respect to N admits perfect state transfer from a to b at time t if and only if the following hold:

(i) *a and b are strongly cospectral;*

(ii) *If $\lambda \in \Lambda_{ab}^{+}$, then $\lambda = \cos(j\pi/t)$ for some even integer j;*

(iii) *If $\lambda \in \Lambda_{ab}^{-}$ then $\lambda = \cos(j\pi/t)$ for some odd integer j.*

[8, Theorem 3.6]

Theorem

[8, Theorem 3.6] Let X be a graph with real weighted tail-arc incidence matrix N. Let H be the real weighted adjacency matrix of X associated with N. Let a and b be two vertices of X. The quantum walk on X with respect to N admits pretty good state transfer from a to b if and only if the following hold:

(i) *a and b are strongly cospectral;*

(ii) *For any set $\{\ell_\lambda : \lambda \in \Lambda_a\}$ of integers such that*

$$\sum_{\lambda \in \Lambda_a} \ell_\lambda \arccos \lambda \equiv 0 \pmod{2\pi},$$

we have

$$\sum_{\lambda \in \Lambda_{ab}^-} \ell_\lambda \equiv 0 \pmod{2}.$$

Unfair (Weighted) Result C_4

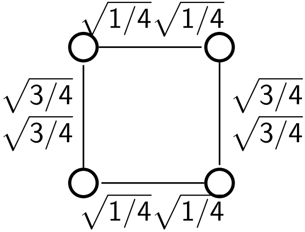

Figure: A weighted quantum walk on C_4

The associated Hermitian adjacency matrix H is

$$H = \begin{bmatrix} 0 & 1/4 & 0 & 3/4 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 3/4 & 0 & 1/4 \\ 3/4 & 0 & 1/4 & 0 \end{bmatrix}.$$

Unfair (Weighted) Result C_4

Theorem

Let X be the graph from Figure 1. Let

$$H = \begin{bmatrix} 0 & 1/4 & 0 & 3/4 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 3/4 & 0 & 1/4 \\ 3/4 & 0 & 1/4 & 0 \end{bmatrix}.$$

PST occurs at time 3 between adjacent vertices a and b for which $H_{ab} = 1/4$. From the definition of H these are the vertex pairs $(0, 1)$ and $(2, 3)$.

Proof. Here we use Theorem [8][Theorem 3.6]. The spectral decomposition of the matrix H is the following.

$$H = -1 \cdot E_{-1} + -0.5 \cdot E_{-0.5} + 0.5 \cdot E_{0.5} + 1 \cdot E_1 \ ,$$

$$E_{-1} = \frac{1}{4} \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix} ,$$

Unfair (Weighted) Result C_4

$$E_{-0.5} = \frac{1}{4}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix},$$

$$E_{0.5} = \frac{1}{4}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix},$$

$$E_{1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Unfair (Weighted) Result C_4

As we can see, all pairs of vertices are strongly cospectral. Therefore the pairs of vertices $(0, 1)$ and $(2, 3)$ are strongly cospectral, satisfying condition (i) in Theorem [1, Theorem 2.2]. Additionally we can see that for both pairs of vertices, $(0, 1)$ and $(2, 3)$, the eigenvalue supports are partitioned into $\Lambda^+_{ab} = \{1, -0.5\}$ and $\Lambda^-_{ab} = \{-1, 0.5\}$. For instance, the columns for the vertices 0 and 1 are of equal, or opposite sign:

$$E_{-1} e_0 = -E_{-1} e_1 \ ,$$

$$E_1 e_0 = +E_1 e_1 \ .$$

??

We found that PST occurs at time $t = 3$, (see Section 4.3). We then see that to satisfy Theorem 1, j will be the even integers 0 and 4 for Λ_{ab}^{+} and the odd integers 3 and 1 for Λ_{ab}^{+}. Thus we satisfy condition (ii) in Theorem 1, so vertices $(0, 1)$ and $(2, 3)$ have PST at time 3. $\square$

Can we find perfect state transfer on any graph?

Now we introduce a diagonal matrix M that changes the weight of each arc in X, and we form a new unfair, weighted arc-tail incidence matrix N_t'.

$$N_t' = N_t M,$$

where

$$M = diag(w_{a_1}'/w_{a_1}, w_{a_2}'/w_{a_2}, \cdots).$$

The numerators of each entry on the diagonal are the new weights for each arc, that is, w_{a_i} is the current weight of arc a_i, and w_{a_i}' is the new desired weight of arc a_i.

We have defined the orthonormal eigenbases V and V' for H and H' respectively:

$$H = VDV^*$$

$$H' = V'D'V'^*$$

Then we can find a matrix P that represents a change of basis matrix between the eigenbases of H and H', so,

$$PV = V'$$

and forall

$$i, Pv_i = v_i'$$

Where v_i is the i^{th} column of V and v_i' is the i^{th} column of V'.

At this point we assume that the matrix P preserves the grouping of eigenvectors. This simplifies the problem, and the proofs below. Further research is required to understand what happens without this assumtion.

Theorem

Assume H and H' have the same eigenvector grouping. λ_i is any eigenvalue for H and λ_i' is the corresponding eigenvalue for H'. Then the eigenprojections of H and H' have the following relationship:

$$E'_{\lambda_i'} = PE_{\lambda_i}P^T$$

Theorem 10.1

Theorem

Vertex states x_a and x_b are strongly co-spectral relative to matrix H' if and only if $P^T x_a$ and $P^T x_b$ are strongly co-spectral relative to matrix H.

Theorem 10.2

Theorem

States x_c and x_d, defined as complex unit vectors on the vertices of X are strongly co-spectral relative to H if and only if

$$x_c = a_1 v_1 + \cdots + a_n v_n$$
$$x_d = \alpha_1 a_1 v_1 + \cdots + \alpha_n a_n v_n$$

where for all $i, |\alpha_i| = 1$, v_i are eigenvectors of H, and v_i, v_j have the same eigenvalue $\implies \alpha_i = \alpha_j$.

Theorem

The relative spins (α_i) in each eigenprojection, as defined for the strongly co-spectral states x_c and x_d, are the same relative spins for the states x_a and x_b when $P^T x_a = x_c$ and $P^T x_b = x_d$. That is,

$$x_c = a_1 v_1 + \cdots + a_n v_n$$
$$x_d = \alpha_1 a_1 v_1 + \cdots + \alpha_n a_n v_n$$

where for all $i, |\alpha_i| = 1$, v_i are eigenvectors of H, and v_i, v_j have the same eigenvalue $\implies \alpha_i = \alpha_j$.

$$P^T x_a = x_c$$
$$P^T x_b = x_d$$

$\implies$

Theorem 10.3 (cont.)

$$x_a = b_1 v_1' + \cdots + b_n v_n'$$
$$x_b = \alpha_1 b_1 v_1' + \cdots + \alpha_n b_n v_n'$$

Lemma

The second condition for PST in Theorem 2.3, is satisfied,

$$e^{it\theta_k}\mu_k = e^{it\theta_s}\mu_s$$

, if and only if

$$t \arccos \lambda_j + \theta_{\mu_k} = t \arccos \lambda_l + \theta_{\mu_s} \pmod{2\pi}$$

Theorem

States x_a and x_b have perfect state transfer relative to the Hermitian adjacency matrix H which is from the quantum walk U if and only if:

1. *Vertex states x_a and x_b are strongly co-spectral, that is, for all λ_j there exists $|\mu_j| = 1$, such that*

$$E_{\lambda_j} x_a = \mu_j E_{\lambda_j} x_b$$

2. *There exists $t \in \mathbb{Z}$ such that for all*

$$\lambda_j, \lambda_l$$

$$t \arccos \lambda_j + \theta_{\mu_k} = t \arccos \lambda_l + \theta_{\mu_s} \pmod{2\pi}$$

Theorem

States x_a and x_b have perfect state transfer with respect to the matrix H' defined as

$$H' = PVD'V^*P^{-1}.$$

Where V is the eigenbasis of H, and D' is a diagonal matrix of eigenvalues maintaining the grouping of eigenvectors (Theorem 7.1),
if and only if:

(i)

$$P^T x_a = x_c = a_1 v_1 + \cdots + a_n v_n$$
$$P^T x_b = x_d = \mu_1 a_1 v_1 + \cdots + \mu_n a_n v_n$$

where for all $i, |\mu_i| = 1$, v_i are eigenvectors of H, and v_i, v_j have the same eigenvalue $\implies \mu_i = \mu_j$, and $D' = diag(\cdots, \lambda_i, \cdots)$,

(ii) and there exists

$$t \in \mathbb{Z}$$

such that for all λ_j, λ_l

$$t \arccos \lambda_j + \theta_{\mu_k} = t \arccos \lambda_l + \theta_{\mu_s} \pmod{2\pi}$$

Theorem 11.2 (cont.)

where

$$\mu_k = e^{i\theta_{\mu_k}}$$
$$\mu_s = e^{i\theta_{\mu_s}}$$

We know the matrix P is invertible, and will be used in the following way to construct H':

$$H' = PVD'V^*P^{-1}$$

And we have requirements for the matrix H', representing the quantum walk. These are:

Theorem

H' represents a quantum walk on graph X if and only if

1. *H' is Hermitian*
2. *H' respects the structure of graph X, i.e. is a weighted adjacency matrix.*
3. *$H' = N_t M R M^* N_t^*$ where M is such that $\sum_{a_i \in A_v}(|w_{a_i}|^2) = 1$, A_v is the set of all arcs that are outgoing from vertex v.*

Results

At this point we need inverse eigenvalue problem results which I didn't have time to finish during this MQP.

We say there is *pretty good state transfer* (PGST) from a to b if there is a unimodular $\gamma \in \mathbb{C}$ such that for any $\epsilon > 0$, there is a time $t \in \mathbb{Z}$ such that

$$\left| U^t x_a - \gamma x_y \right| < \epsilon.$$

[8]

Folded d-Cube - Model

Given an abelian group G and an inverse-closed subset C of $G\backslash\{0\}$, the *Cayley graph* over G with connection set C, denoted $X(G, C)$, is the graph with vertex set G, where two vertices u and v are adjacent if they differ by an element in C. The folded d-cube graph, denoted FQ_d, can be constructed from the d-cube by connecting each pair of antipodal vertices. Equivalently, it is the Cayley graph over $\mathbb{Z}_2^d$ with connection set $\{e_1, e_2, \ldots, e_d, \mathbf{1}\}$.

Folded d-Cube - Model

Let A_i be the adjacency matrix for $X(\mathbb{Z}_2^d, \{e_i\})$, and $A_{\mathbf{1}}$ the adjacency matrix for $X(\mathbb{Z}_2^d, \{\mathbf{1}\})$. Then the adjacency matrix for FQ_d is

$$A = A_1 + A_2 + \cdots + A_d + A_{\mathbf{1}} \ .$$

Consider a real-weighted adjacency matrix H of X:

$$H = w_1 A_1 + w_2 A_2 + \cdots + w_d A_d + w_{\mathbf{1}} A_{\mathbf{1}} \ .$$

We will find weights $w_1, w_2, \ldots, w_d, w_{\mathbf{1}}$ so that any quantum walk on FQ_d with H as associated real weighted adjacency matrix admits pretty good state transfer between the antipodal vertices. Recall that $H = NRN^*$. To this end, we cite a sufficient condition for pretty good state transfer below.

Theorem

[8, Theorem 5.1]. Let X be a connected graph. Let H be a symmetric non-negative adjacency matrix of X, with spectral radius p for some prime p. Let N be any real weighted tail-arc incidence matrix of X such that $H = NRN^T$. Suppose a and b are strongly cospectral relative to H, and Λ_a is a subset of

$$\{p - 2r : r = 0, 1, \cdots, p\} .$$

Then the quantum walk with transition matrix $U = R(2N^TN - I)$ admits pretty good state transfer between a and b if one of the following holds:

(i) *for any pair $\lambda, -\lambda \in \Lambda_a$,*

$$\lambda \in \Lambda_{ab}^{\pm} \iff -\lambda \in \Lambda_{ab}^{\pm} ;$$

(ii) *for any pair $\lambda, -\lambda \in \Lambda_a$,* Theorem

Let $X = X(\mathbb{Z}_2^d, \{c_1, c_2, \cdots, c_k\})$*. For every element* $g \in \mathbb{Z}_2^d$*, the character* ψ_g *is an eigenvector for the weighted adjacency matrix*

$$H = w_1 A(X(\mathbb{Z}_2^d, \{c_1\})) + w_2 A(X(\mathbb{Z}_2^d, \{c_2\})) + \cdots + w_k A(X(\mathbb{Z}_2^d, \{c_k\}))$$

and $\sum_{i=1}^{k} w_i \psi_g(c_i)$ *is the associated eigenvalue. Equivalently, for every* $g \in \mathbb{Z}_2^d$*,*

$$H\psi_g = \left(\sum_{i=1}^{k} w_i \psi_g(c_i) \right) \psi_g \ .$$

Lemma

*Let H be the weighted adjacency matrix of the folded d-cube as defined in Equation (**??**). For each element $g \in \mathbb{Z}_2^d$, the character ψ_g defined by $\psi_g(a) = (-1)^{\langle g,a \rangle}$ for each $a \in \mathbb{Z}_2^d$, is an eigenvector for H with eigenvalue λ_g, where*

$$\lambda_g = \begin{cases} 2d - 4\,\mathrm{wt}(g) + m, & \text{if } \mathrm{wt}(g) \equiv 0 \pmod 2 \\ 2d - 4\,\mathrm{wt}(g) - m, & \text{if } \mathrm{wt}(g) \equiv 1 \pmod 2 \end{cases}.$$

Lemma

The spectral radius of the weighted adjacency matrix H of the folded d-cube is $2d + m$.

Lemma

Let

$$H = 2A_1 + 2A_2 + \cdots + 2A_d + mA_{\mathbf{1}}$$

be a weighted adjacency matrix of the folded d-cube such that $\rho(H)$ is a prime p. Then the eigenvalues of H form a subset of

$$\{p - 2r : r = 0, 1, \cdots, p\} \ .$$

Lemma

Let g and h be two elements of $\mathbb{Z}_2^d$. Let λ_g and λ_h be the eigenvalues of H associated with the eigenvectors ψ_g and ψ_h. Then $\lambda_g = \lambda_h$ if and only if $\mathrm{wt}(g) = \mathrm{wt}(h)$.

Lemma

Let H be a real weighted adjacency matrix of X with order n. Let a and b be two vertices in X. Let $\{x_1, x_2, \ldots, x_n\}$ be a set of orthogonal eigenvectors of H. Then a and b are strongly cospectral relative to H if and only if the following hold.

(i) *For each i, we have $\langle e_a, x_i \rangle = \pm \langle e_b, x_i \rangle$.*

(ii) *If x_i and x_j are eigenvectors of H for the same eigenvalue, then either*

$$\langle e_a, x_i \rangle = \langle e_b, x_i \rangle, \quad \langle e_a, x_j \rangle = \langle e_b, x_j \rangle \ ,$$

or

$$\langle e_a, x_i \rangle = -\langle e_b, x_i \rangle, \quad \langle e_a, x_j \rangle = -\langle e_b, x_j \rangle .$$

Folded d-Cube Lemma 3.8, Corollary 3.9

Lemma

Let H be the adjacency matrix of the folded d-cube. The eigenvectors found by Theorem 13 are orthogonal.

Corollary

Let d be a positive integer. Let m be an odd integer such that $2d + m$ is a prime. Let

$$H = 2A_1 + 2A_2 + \cdots + 2A_d + mA_{\mathbf{1}}$$

be the weighted adjacency matrix of the folded d-cube FQ_d. Then the vertices $\mathbf{0}$ and $\mathbf{1}$ are strongly cospectral relative to H.

Lemma

The eigenvalue supports of $a = \mathbf{0}$ and $b = \mathbf{1}$ satisfy condition (ii) from Theorem 12 ([8, Theorem 5.1]).

Theorem

Let

$$H = 2A_1 + 2A_2 + \cdots + 2A_d + mA_{\mathbf{1}}$$

be a weighted adjacency matrix for the folded d-cube

$$FQ_d = X(\mathbb{Z}_2^d, \{e_1, e_2, \cdots, e_k, \mathbf{1}\})$$

such that $2d + m$ *is prime. Let* N *be any real weighted arc-tail incidence matrix of* FQ_d *which satisfies the equation* $H = NRN^*$*. Then the quantum walk on* FQ_d *with transition matrix* $U = R(2N^T N - I)$ *has PGST between vertices* $a = \mathbf{0}$ *and* $b = \mathbf{1}$*.*

[4, Chapter 6]

Proof. The weighted folded d-cube has a non-negative adjacency matrix and by Lemma 15, has a spectral radius of $2d + m$ which is prime. Let $p = 2d + m$. By Lemma 16 we have shown that the eigenvalues, and therefore the eigenvalue support Λ_a for vertex a is a subset of

$$\{p - 2r : r = 0, 1, \cdots, p\} .$$

By Lemma 20 we have shown that **0** and **1** are strongly cospectral vertices, thus $\Lambda_a = \Lambda_{ab}$. By Lemma 21 we have shown that the eigenvalue support of a, $\Lambda_a = \Lambda_{ab}$ satisfies condition (ii) in Theorem 12. Thus this weighting of the folded d-cube enables PGST between vertices **0** and **1**. □

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Thank you!

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