

SIR models

Longmei Shu

Department of Mathematics,
Emory University

April 2, 2020

Problem Introduction

Problem Introduction



coronavirus



[All](#)

[News](#)

[Videos](#)

[Images](#)

[Books](#)

[More](#)

[Settings](#)

[Tools](#)

About 8,370,000 results (0.68 seconds)

Top stories

COVID-19 alert

Coronavirus disease

Overview

[Symptoms](#)

[Prevention](#)

[Treatments](#)

[Statistics](#)



Share



[Live Coronavirus News and Updates](#)

The New York Times
1 hour ago



Live updates:
Coronavirus death toll
passes 50,000
worldwide; U.S....

Washington Post
5 mins ago



Israel's Defense And
Spy Agencies Step Up
Anti-Coronavirus Efforts

NPR
6 mins ago

Help and information

COVID-19 (Novel Coronavirus) - Georgia

Georgia Department of Public Health



dph.georgia.gov

Information about COVID-19 in the United States

Centers for Disease Control and Prevention



cdc.gov/coronavirus/2019-ncov

Coronavirus Self-Checker

Worldwide cases



[View full map](#)

Source: [Wikipedia](#) · [About this data](#)

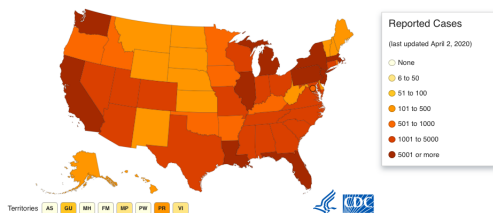
Worldwide cases

Location	Confirmed	Recovered	Death
Worldwide	996,047	208,630	51,331
United States	235,787	10,324	5,761

Problem Introduction

Problem Introduction

States Reporting Cases of COVID-19 to CDC*

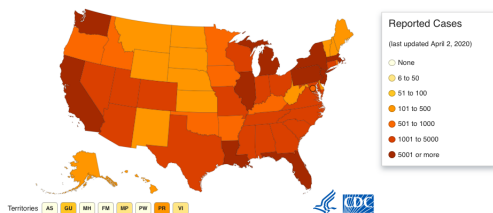


left: CDC

right: <https://coronavirus.1point3acres.com/#stat>

Problem Introduction

States Reporting Cases of COVID-19 to CDC*



left: CDC right: <https://coronavirus.1point3acres.com/#stat>

This is bad, what can we do?

SIR Models

number of **S**usceptible people

$S(t)$

number of people **I**nfected

$I(t)$

number of people who have **R**ecovered

$R(t)$

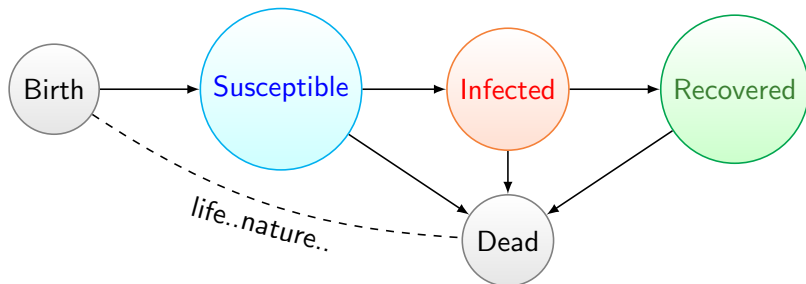


Figure: The cycle of life

Diseases with permanent immunity

Assumptions

- Once you recover, you will never be susceptible or infected again.
- The total population K is fixed. Death rate = birth rate = μ .
- The death rate is uniform among different groups.
- The infected recover at rate γI .
- The susceptible become infected at rate βSI .

Diseases with permanent immunity

Assumptions

- Once you recover, you will never be susceptible or infected again.
- The total population K is fixed. Death rate = birth rate = μ .
- The death rate is uniform among different groups.
- The infected recover at rate γI .
- The susceptible become infected at rate βSI .

Model

$$\frac{dS}{dt} = \mu K - \beta SI - \mu S \quad (1)$$

$$\frac{dI}{dt} = \beta SI - \gamma I - \mu I \quad (2)$$

$$\frac{dR}{dt} = \gamma I - \mu R \quad (3)$$

Model Analysis

Model Analysis

Our wish

No more new cases!

Model Analysis

Our wish

No more new cases!

Math translation

$$\frac{dI}{dt} \leq 0.$$

Model Analysis

Our wish

No more new cases!

Math translation

$$\frac{dI}{dt} \leq 0.$$

$$\frac{dI}{dt} = \beta SI - \gamma I - \mu I = I(\beta S - \gamma - \mu) = 0$$

$$I = 0, \text{ or } S = \frac{\gamma + \mu}{\beta} = S_*.$$

If $K \leq S_*$, then $S \leq S_*$, $dI/dt \leq 0$. No worries.

Model Analysis

What if $K > S_*$? Will we approach some stable equilibrium?

Model Analysis

What if $K > S_*$? Will we approach some stable equilibrium?

$$\begin{aligned}\frac{dS}{dt} = 0 &\Rightarrow S = \frac{\mu K}{\mu + \beta I} \\ \frac{dI}{dt} = 0 &\Rightarrow I = 0, \text{ or } S = S_* \\ \frac{dR}{dt} = 0 &\Rightarrow R = \frac{\gamma}{\mu} I\end{aligned}$$

Model Analysis

What if $K > S_*$? Will we approach some stable equilibrium?

$$\begin{aligned}\frac{dS}{dt} = 0 &\Rightarrow S = \frac{\mu K}{\mu + \beta I} \\ \frac{dI}{dt} = 0 &\Rightarrow I = 0, \text{ or } S = S_* \\ \frac{dR}{dt} = 0 &\Rightarrow R = \frac{\gamma}{\mu} I\end{aligned}$$

Equilibrium points

$(K, 0, 0)$ - disease free;

$(S_*, I_* = \frac{\mu(\beta K - \gamma - \mu)}{\beta(\gamma + \mu)}, R_* = \frac{\gamma}{\mu} I_*)$ - stable coexistence?

Model Analysis

Jacobian

$$J = \begin{pmatrix} -\beta I - \mu & -\beta S & 0 \\ \beta I & \beta S - \gamma - \mu & 0 \\ 0 & \gamma & -\mu \end{pmatrix}$$

At $(K, 0, 0)$

$$J = \begin{pmatrix} -\mu & -\beta K & 0 \\ 0 & \beta K - \gamma - \mu & 0 \\ 0 & \gamma & -\mu \end{pmatrix},$$

$$\begin{aligned} |J - \lambda I| &= \begin{vmatrix} -\mu - \lambda & -\beta K & 0 \\ 0 & \beta K - \gamma - \mu - \lambda & 0 \\ 0 & \gamma & -\mu - \lambda \end{vmatrix} \\ &= (\beta K - \gamma - \mu - \lambda)(\lambda + \mu)^2. \end{aligned}$$

eigenvalue $\lambda = \beta K - \gamma - \mu > 0$, unstable.

Model Analysis

At (S_*, I_*, R_*)

$$J = \begin{pmatrix} -\beta I_* - \mu & -\gamma - \mu & 0 \\ \beta I_* & 0 & 0 \\ 0 & \gamma & -\mu \end{pmatrix},$$

$$|J - \lambda I| = \begin{vmatrix} -\beta I_* - \mu - \lambda & -\gamma - \mu & 0 \\ \beta I_* & -\lambda & 0 \\ 0 & \gamma & -\mu - \lambda \end{vmatrix}$$

$$= -\beta I_*(\lambda + \mu)(\gamma + \mu) - \lambda(\lambda + \mu)(\lambda + \mu + \beta I_*) = p(\lambda)$$

$p(\lambda)$ is a degree 3 polynomial with all negative coefficients. It can't have positive real roots. So it either has 3 negative real roots (**stable**), or a negative real root with a complex pair of roots. The real part of the complex roots could be negative (**stable**) or positive (**unstable**).

Model Analysis

Consider any degree 3 polynomial with a negative real root $-c$ and a pair of complex roots $a \pm bi$ where $a > 0$. It is a constant multiplied by

$$\begin{aligned}(\lambda + c)(\lambda - a - bi)(\lambda - a + bi) &= (\lambda + c)[(\lambda - a)^2 + b^2] \\&= (\lambda + c)(\lambda^2 - 2a\lambda + a^2 + b^2) \\&= \lambda^3 - 2a\lambda^2 + (a^2 + b^2 - 2ac)\lambda + (a^2 + b^2)c.\end{aligned}$$

The coefficients won't have the same sign as $1 > 0$, $-2a < 0$. So $p(\lambda)$ can't have a negative real root and a complex pair of roots with positive real part.

Conclusion

(S_*, I_*, R_*) is a stable fixed point. (endemic disease)

Results

The long term behavior of the disease depends on the **basic reproduction number**

$$R_0 = \frac{\beta K}{\gamma + \mu}.$$

If $R_0 \leq 1$, then $dl/dt \leq 0$, the disease will go away on its own or not become a problem.

If $R_0 > 1$, then the disease becomes endemic. The number of infected people will approach a stable limit $I_* = \frac{\mu}{\beta}(R_0 - 1)$.

Model Analysis under vaccination

Vaccination

Assume a fixed fraction ρ of all new borns is vaccinated against the disease, so they are born into the "Recovered" compartment.

$$\frac{dS}{dt} = (1 - \rho)\mu K - \beta SI - \mu S \quad (4)$$

$$\frac{dI}{dt} = \beta SI - \gamma I - \mu I \quad (5)$$

$$\frac{dR}{dt} = \gamma I - \mu R + \rho\mu K \quad (6)$$

fixed points

$((1 - \rho)K, 0, \rho K)$ - no disease

stable when $(1 - \rho)K \leq \frac{\gamma + \mu}{\beta}$; unstable when $(1 - \rho)K > \frac{\gamma + \mu}{\beta}$

Model Analysis under vaccination

fixed points

$$(S_*, \frac{(1-\rho)\mu K}{\gamma+\mu} - \frac{\mu}{\beta}, \frac{\gamma(1-\rho)K}{\gamma+\mu} - \frac{\gamma}{\beta} + \rho K)$$

only exist when $(1 - \rho)K > \frac{\gamma+\mu}{\beta}$, is stable as long as it exists. (endemic disease)

basic reproduction number

$$\tilde{R}_0 = \frac{(1 - \rho)\beta K}{\gamma + \mu} = (1 - \rho)R_0$$

To make $\tilde{R}_0 < 1$, we can try to make $\rho > 1 - \frac{1}{R_0}$.

Quarantine, Social Distancing

What happens when we implement quarantine and social distancing? How do we modify the model to describe the effects?

Social distancing means β becomes smaller? Then R_0 becomes smaller, the disease should go away on its own if $R_0 \leq 1$.

Other modifications

Some SIR models don't include the death/birth rates or the assumption that the total population is fixed.

There are other features we could look at in more details, for example, we can further divide the population into finer groups by age/gender, and use different β, γ for different groups..

Thank You!