SIR models

Longmei Shu

Department of Mathematics,
Emory University

April 2, 2020

Problem Introduction

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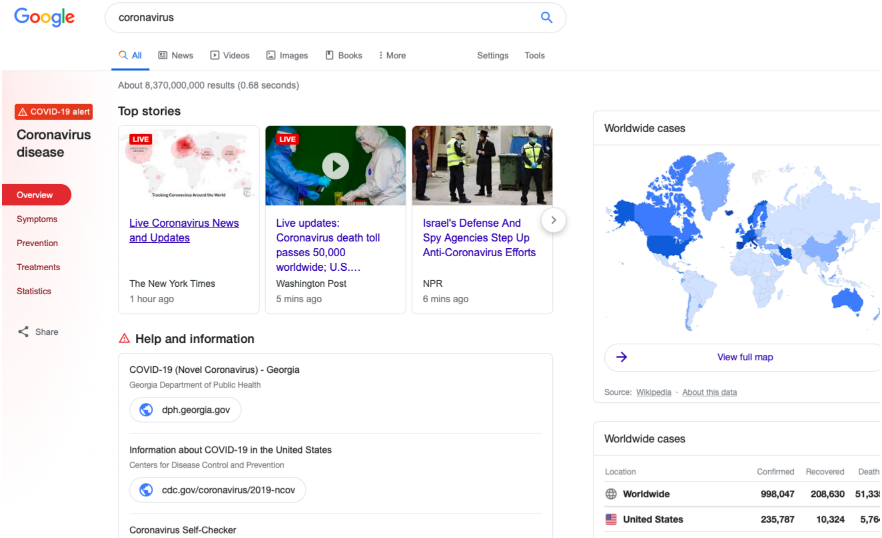

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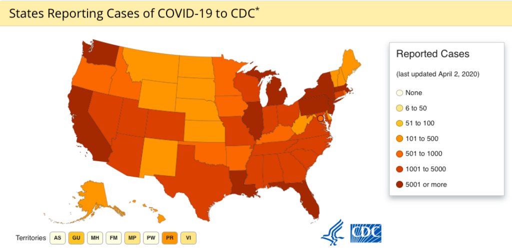

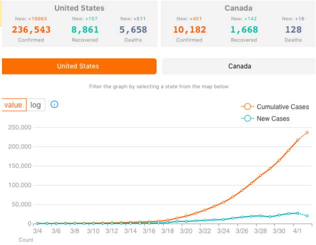

left: CDC right: https://coronavirus.1point3acres.com/#stat

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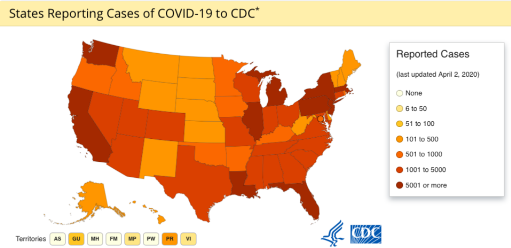

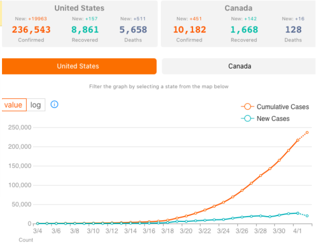

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This is bad, what can we do?

SIR Models

number of Susceptible people	$S(t)$
number of people Infected	$I(t)$
number of people who have Recovered	$R(t)$

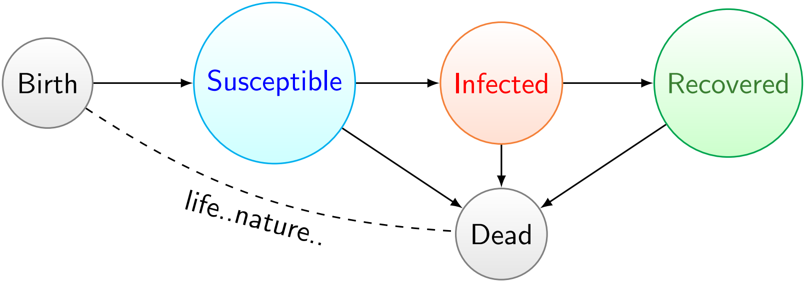

Figure: The cycle of life

Diseases with permanent immunity

Assumptions

- Once you recover, you will never be susceptible or infected again.
- The total population K is fixed. Death rate$=$ birth rate$= \mu$.
- The death rate is uniform among different groups.
- The infected recover at rate γI.
- The susceptible become infected at rate βSI.

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Model

$$\frac{dS}{dt} = \mu K - \beta SI - \mu S \tag{1}$$

$$\frac{dI}{dt} = \beta SI - \gamma I - \mu I \tag{2}$$

$$\frac{dR}{dt} = \gamma I - \mu R \tag{3}$$

Model Analysis

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Our wish

No more new cases!

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Math translation

$$\frac{dI}{dt} \leq 0.$$

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Math translation

$$\frac{dI}{dt} \leq 0.$$

$$\frac{dI}{dt} = \beta SI - \gamma I - \mu I = I(\beta S - \gamma - \mu) = 0$$

$$I = 0, \text{ or } S = \frac{\gamma + \mu}{\beta} = S_*.$$

If $K \leq S_*$, then $S \leq S_*$, $dI/dt \leq 0$. No worries.

Model Analysis

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Equilibrium points

$(K, 0, 0)$ - disease free;

$(S_*, I_* = \frac{\mu(\beta K - \gamma - \mu)}{\beta(\gamma + \mu)}, R_* = \frac{\gamma}{\mu} I_*)$ - stable coexistence?

Model Analysis

Jacobian

$$J = \begin{pmatrix} -\beta I - \mu & -\beta S & 0 \\ \beta I & \beta S - \gamma - \mu & 0 \\ 0 & \gamma & -\mu \end{pmatrix}$$

At $(K, 0, 0)$

$$J = \begin{pmatrix} -\mu & -\beta K & 0 \\ 0 & \beta K - \gamma - \mu & 0 \\ 0 & \gamma & -\mu \end{pmatrix},$$

$$\begin{aligned} |J - \lambda I| &= \begin{vmatrix} -\mu - \lambda & -\beta K & 0 \\ 0 & \beta K - \gamma - \mu - \lambda & 0 \\ 0 & \gamma & -\mu - \lambda \end{vmatrix} \\ &= (\beta K - \gamma - \mu - \lambda)(\lambda + \mu)^2. \end{aligned}$$

eigenvalue $\lambda = \beta K - \gamma - \mu > 0$, unstable.

Model Analysis

At (S_*, I_*, R_*)

$$J = \begin{pmatrix} -\beta I_* - \mu & -\gamma - \mu & 0 \\ \beta I_* & 0 & 0 \\ 0 & \gamma & -\mu \end{pmatrix},$$

$$|J - \lambda I| = \begin{vmatrix} -\beta I_* - \mu - \lambda & -\gamma - \mu & 0 \\ \beta I_* & -\lambda & 0 \\ 0 & \gamma & -\mu - \lambda \end{vmatrix}$$

$$= -\beta I_*(\lambda + \mu)(\gamma + \mu) - \lambda(\lambda + \mu)(\lambda + \mu + \beta I_*) = p(\lambda)$$

$p(\lambda)$ is a degree 3 polynomial with all negative coefficients. It can't have positive real roots. So it either has 3 negative real roots (stable), or a negative real root with a complex pair of roots. The real part of the complex roots could be negative (stable) or positive (unstable).

Model Analysis

Consider any degree 3 polynomial with a negative real root $-c$ and a pair of complex roots $a \pm bi$ where $a > 0$. It is a constant multiplied by

$$(\lambda + c)(\lambda - a - bi)(\lambda - a + bi) = (\lambda + c)[(\lambda - a)^2 + b^2]$$

$$= (\lambda + c)(\lambda^2 - 2a\lambda + a^2 + b^2)$$

$$= \lambda^3 - 2a\lambda^2 + (a^2 + b^2 - 2ac)\lambda + (a^2 + b^2)c.$$

The coefficients won't have the same sign as $1 > 0, -2a < 0$. So $p(\lambda)$ can't have a negative real root and a complex pair of roots with positive real part.

Conclusion

(S_*, I_*, R_*) is a stable fixed point. (endemic disease)

Discussion

Results

The long term behavior of the disease depends on the basic reproduction number

$$R_0 = \frac{\beta K}{\gamma + \mu}.$$

If $R_0 \leq 1$, then $dI/dt \leq 0$, the disease will go away on its own or not become a problem.

If $R_0 > 1$, then the disease becomes endemic. The number of infected people will approach a stable limit $I_* = \frac{\mu}{\beta}(R_0 - 1)$.

Model Analysis under vaccination

Vaccination

Assume a fixed fraction ρ of all new borns is vaccinated against the disease, so they are born into the "Recovered" compartment.

$$\frac{dS}{dt} = (1-\rho)\mu K - \beta SI - \mu S \tag{4}$$

$$\frac{dI}{dt} = \beta SI - \gamma I - \mu I \tag{5}$$

$$\frac{dR}{dt} = \gamma I - \mu R + \rho\mu K \tag{6}$$

fixed points

$((1-\rho)K, 0, \rho K)$- no disease
stable when $(1-\rho)K \leq \frac{\gamma+\mu}{\beta}$; unstable when $(1-\rho)K > \frac{\gamma+\mu}{\beta}$

Model Analysis under vaccination

fixed points

$(S_*, \frac{(1-\rho)\mu K}{\gamma+\mu} - \frac{\mu}{\beta}, \frac{\gamma(1-\rho)K}{\gamma+\mu} - \frac{\gamma}{\beta} + \rho K)$
only exist when $(1-\rho)K > \frac{\gamma+\mu}{\beta}$, is stable as long as it exists. (endemic disease)

basic reproduction number

$$\tilde{R}_0 = \frac{(1-\rho)\beta K}{\gamma+\mu} = (1-\rho)R_0$$

To make $\tilde{R}_0 < 1$, we can try to make $\rho > 1 - \frac{1}{R_0}$.

Discussion

Quarantine, Social Distancing

What happens when we implement quarantine and social distancing? How do we modify the model to describe the effects?

Social distancing means β becomes smaller? Then R_0 becomes smaller, the disease should go away on its own if $R_0 \leq 1$.

Other modifications

Some SIR models don't include the death/birth rates or the assumption that the total population is fixed.
There are other features we could look at in more details, for example, we can further divide the population into finer groups by age/gender, and use different β, γ for different groups..

Thank You!