SIMPLE MODEL FOR POTENTIAL OF STARS IN GALAXIES

- We try to address the following questions: what kinds of orbits are possible in the following potentials:

  a). a spherical symmetric potential

  b). an axisymmetric potential (axially symmetric)

  c). non-rotating bar-shaped potential

  d). a rotating bar

  e). A triaxial elliptical galaxy.

- First obtain analytic solutions in simpler potentials, then develop intuitive understanding of orbits in more general potentials.

- In the orbital study, a fundamental approximation: we "forget" that galaxies are made of stars, and consider only the large-scale forces from overall mass distribution. i.e., we assume the gravitational fields of galaxies are smooth, this approximation is valid for $t < t_{relax} = \frac{0.1N}{lnN} t_{\rm cross}$

- Since the trajectory of a star in a gravitational field does not depend on its mass, we examine the dynamics of a particle of unit mass. The following quantities are all written per unit mass:

  - momentum
  - angular momentum
  - force
  - energy
  - Lagrangian and Hamiltonian.

ORBITS IN STATIC SPHERICAL POTENTIAL, THE SIMPLEST CASE!

- Centrally directed gravitational field $\vec{g} = g(r)\hat{e_r}$
- The equation of motion:

$$\frac{d^2\overrightarrow{r}}{dt^2} = \vec{F} = g(r)\hat{e_r}$$

$$\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = \underbrace{\frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt}}_{=0} + \vec{r} \times \frac{d^2\vec{r}}{dt^2} = \vec{r} \times g(r)\hat{e_r} = 0$$

$$\Rightarrow \boxed{\vec{r} \times \frac{d\vec{r}}{dt} = \vec{r} \times \vec{v} = \vec{L} = \text{constant vector}} \quad \text{or} \quad \frac{d\vec{L}}{dt} = \vec{T} = \vec{r} \times \vec{F} = 0$$

- By definition, $\vec{r}(\vec{r} \times \vec{v}) = \vec{r}\vec{L} = 0$, $\vec{r}$ is always $\perp$ which is a constant vector. A star moves on a plane, the orbital plane, which $\perp \vec{L}$
- Use planar polar coordinates (r, ψ) with centre of attraction at O:

$$\ddot{\vec{r}} = \frac{d^2\vec{r}}{dt^2} = \vec{F} = -\nabla\Phi = \frac{d\Phi}{dr}\hat{e_r}$$

- Since $\vec{r} = r\hat{e_r}$, we obtain $\ddot{r} = \dfrac{d\Phi}{dr}$, correct?
- Notice here, $\hat{e_r} \neq 0$ and $\hat{e_\psi} \neq 0$:

$$\frac{d\hat{e_r}}{d\psi} = \hat{e_\psi}; \frac{d\hat{e_\psi}}{dr} = -\hat{e_r} \quad \text{(Eq.B.21)}, \quad \begin{cases} \hat{e}_r & = cos\psi\hat{e}_x + sin\psi\hat{e}_y \\ \hat{e}_\theta & = -sin\psi\hat{e}_x + cos\psi\hat{e}_y \end{cases}$$

$$\dot{\hat{e_r}} = \frac{d\hat{e}_r}{d\psi}\dot{\psi} = \dot{\psi}\hat{e}_\psi \quad ; \quad \dot{\hat{e_\psi}} = \frac{d\hat{e}_\psi}{d\psi}\dot{\psi} = -\dot{\psi}\hat{e}_r$$

- As $\dot{\vec{r}} = \dfrac{d(r\hat{e}_r)}{dt} = \dot{r}\hat{e}_r + r\dot{\psi}\hat{e}_\psi$, so $\ddot{\vec{r}} = (\ddot{r} - r\dot{\psi}^2)\hat{e}_r + (2\dot{r}\dot{\psi} + r\ddot{\psi})\hat{e}_\psi$,

  $\hat{e_r}$: $\ddot{r} - r\dot{\psi}^2 = -\dfrac{d\Phi(r)}{dr}$;

  $\hat{e_\psi}$: $2\dot{r}\dot{\psi} + r\ddot{\psi} = \dfrac{1}{r}\dfrac{d(r^2\dot{\psi})}{dt} = 0 \Leftrightarrow L = r^2\dot{\psi} = \text{constant}$

  (Generalised Kepler's 2rd Law: $L = 2\dfrac{\Delta A}{\Delta t}$, ΔA is the area swept out by $\vec{r}$ in Δt.)

A SIMPLER WAY

- Is there any simpler approach?
- Simpler way:

  Lagrangian: $L = K - V = \frac{1}{2}\dot{\vec{r}}^2 - \Phi$, $\dot{\vec{r}} = \dot{r}\hat{e_r} + r\dot{\psi}\hat{e_\psi}$

$$L = \frac{1}{2}[\dot{r}^2 + (r\dot{\psi}^2)] - \Phi(r)$$

- Equation of motion: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{q}}}\right) - \frac{d\Phi}{d\vec{q}} = 0$

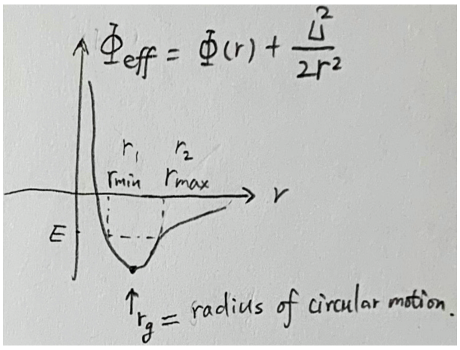
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \boxed{\ddot{r} - r\dot{\psi}^2 + \frac{d\Phi}{dr} = 0} \qquad (*)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) - \frac{\partial L}{\partial \psi} = \frac{d}{dt}\left(r^2\dot{\psi}^2\right) = 0$$

- Define: $E = \underbrace{\frac{1}{2}\dot{r}^2}_{\text{radial E}} + \underbrace{\frac{1}{2}\left(r\dot{\psi}\right)^2}_{\text{azimuthel E}} + \Phi(r) = \frac{1}{2}\dot{r}^2 + \overbrace{\frac{L^2}{2r^2} + \Phi(r)}^{\Phi_{eff} = \Phi_{eff}(r, L)} = \text{constant}$
  - unbound: $E \geq 0$, only $r_{\min}$, uninteresting
  - bound: $E < 0$

$$\dot{r}^2 = 2\left[E - \Phi(r)\right] - \frac{L^2}{r^2}$$

$$\boxed{\dot{r} = \pm\sqrt{2\left[E - \Phi(r)\right] - \frac{L^2}{r^2}}}$$



  - if $\dot{r} = 0 \Rightarrow$ two roots $\underset{\substack{\downarrow \\ \text{peri-} \\ \text{center}}}{r_1}$, $\underset{\substack{\downarrow \\ \text{ape-} \\ \text{center}}}{r_2}$

$$r_1 < r < r_2$$

- Another version of the Equation of motion, replacing t with ψ.
- Since $L = r^2 \frac{d\psi}{dt} \to \frac{d}{dt} = \frac{L}{r^2}\frac{d}{d\psi}$

Eq.* $\to$ $\frac{L^2}{r^2}\frac{d}{d\psi}\left(\frac{1}{r^2}\frac{dr}{d\psi}\right) - \frac{L^2}{r^3} = -\frac{d\Phi}{dr}$

$\to$ $\boxed{\frac{d^2u}{d\psi^2} + u = \frac{1}{L^2u^2}\frac{d\Phi\left(\frac{1}{u}\right)}{dr}}$ Binet Equation

- Generally no analytical solution, can be solved numerically.
- $\frac{du}{dx} * (\quad)$ then integrate over ψ to obtain

$$\left(\frac{du}{d\psi}\right)^2 + \frac{2\Phi}{L^2} + u^2 = \text{const} = \frac{E}{L^2} \quad \left(\text{note } : \frac{d\Phi}{dr} = -u^2\frac{d\Phi}{du}\right)$$

- For bound orbit, $\frac{du}{d\psi} = 0 \Rightarrow u^2 + \frac{2\left[\Phi\left(\frac{1}{u}\right) - E\right]}{L^2} = 0$

2 roots : $r_1 = r_{\text{peri}}$

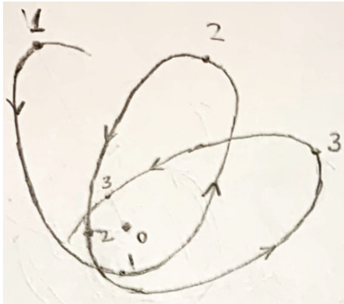
$r_2 = r_{\text{apo}}$

GENERAL SHAPE OF BOUND ORBITS: ROSETTE

- $\sim$ ellipse, "eccentricity"$= \dfrac{r_{apo} - r_{peri}}{r_{apo} + r_{peri}}$

$$= \begin{cases} 0 & : \text{circular} \\ 1 & : \text{radial orbit} \end{cases}, \text{ generally not closed.}$$

r_{apo}, r_{peri} do not change in every circle.



- radial period: T_r $\quad (r_p \to r_a \to r_p)$

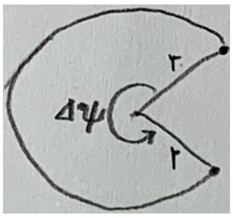
$$\dot{r} = \pm\sqrt{2(E-\Phi) - \frac{L^2}{r^2}} = \frac{dr}{dt}$$

$$T_r = 2\int_{r_p}^{r_a} \frac{dr}{\sqrt{2(E-\phi) - L^2/r^2}}$$

- $\Delta\Phi$ = azimuthal angle increased during T_r, $\dfrac{d\psi}{dr} = \dfrac{d\psi}{dt}\cdot\dfrac{1}{\dot{r}} = \dfrac{L}{r^2}\dfrac{1}{\dot{r}}$

$$\Delta\psi = 2\int_{r_p}^{r_a} \frac{d\psi}{dr} dr = 2\int_{r_p}^{r_a} \frac{L}{r^2}\cdot\frac{1}{\dot{r}} dr$$

$$= 2L\int_{rp}^{r_a} \frac{1}{r^2\sqrt{2(E-\Phi) - L^2/r^2}}$$



$$T_\psi = \frac{2\pi}{\Delta\psi/T_r} = \frac{2\pi}{\Delta\psi} T_r$$

- In general $\dfrac{\Delta\psi}{2\pi}$ is not a rational number, orbit is not closed!
- The orbit eventually passes close to every point in the annulus $(r_p < r < r_a)$

Two special cases: $\dfrac{\Delta\psi}{2\pi}$ is rational $\Rightarrow$ closed orbits

$\longrightarrow$ Two and only two

1. SPHERICAL HARMONIC OSCILLATOR

- Generated by a homogeneous sphere of matter

$$\Phi = \frac{1}{2}\Omega^2 r^2 + const. \quad \Omega^2 = \frac{U_c^2}{r^2} = \frac{GM}{r^3} = \frac{4}{3}\pi G\rho_0$$

- Easier to use Cartesian coordinates

$$\ddot{x} = -\Omega^2 x \quad \ddot{y} = -\Omega^2 y$$ (harmonic oscillator's Eq.)

$$x = X\cos(\Omega t + \epsilon_x) \qquad X, Y, \epsilon_x, \epsilon_y = \text{constant}$$
$$y = Y\cos(\Omega t + \epsilon_y)$$

Tilted ellipse! How is it different from the Keperian motion?

- We reorient the ellipse so SMA = x-axis

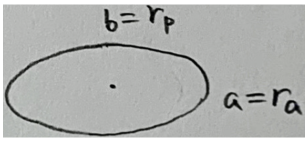
$$x(t) = a\cos(\Omega t)$$
$$y(t) = b\sin(\Omega t)$$

$$E = \frac{(a^2+b^2)\,\Omega^2}{2} = a^2\left(1 - \frac{e^2}{2}\right)\Omega^2$$
$$L = ab\Omega = a^2\sqrt{1-e^2}\Omega$$
$$E = \frac{\Omega^2}{2}a^2 + \frac{L^2}{2a^2} = \frac{1}{2}\Omega^2 b^2 + \frac{L^2}{2b^2}$$

- $b = 0, e = 1 \Rightarrow \begin{cases} E &= \frac{1}{2}a^2\Omega^2 \\ L &= 0 \end{cases}$ ← familiar?

- $r_a = a$, $r_p = b$

$$T_\psi = \frac{2\pi}{\Omega} \quad T_r = \frac{1}{2}T_\psi = \frac{\pi}{\Omega} \quad \boxed{\Delta\psi = \pi}$$



2. KEPLER POTENTIAL

- Point mass $\Phi = -\frac{GM}{r}$
- Binet Eq. $\frac{d^2u}{d\phi^2} + u = \frac{1}{L^2u^2}\frac{d\Phi}{dr} = \frac{GM}{L^2}$

$$\rightarrow \quad r(\psi) = \frac{1}{u(\psi)} = \frac{1}{c \cdot \cos(\psi - \psi_0) + \frac{GM}{L^2}} \qquad c > 0, \psi_0 = constant$$

- This is the equation of ellipse with the origin at one focus, define

$$\boxed{e \equiv \frac{cL^2}{GM} \propto L^2, \quad a = \frac{L^2}{GM(1-e^2)}}$$

① First expression $\boxed{r(\psi) = \frac{a(1-e^2)}{1 + e\cos(\psi - \gamma_0)}}$ $\quad \psi - \psi_0$ = true anomaly

- bound: $r_p = a(1-e) \qquad r_a = a(1+e) \xrightarrow{motivation}$ orbital "eccentricity" = $\frac{r_a - r_p}{r_a + r_p}$
- unbound $e > 1$: hyperbola $\qquad e = 1$: parabola

$$E/L^2 = \left(\frac{du}{d\psi}\right)^2 + \frac{2\Phi}{L^2} + u^2 \Rightarrow \boxed{E = -\frac{GM}{2a}} \quad \boxed{a = a(E) = \frac{GM}{-2E}}$$

like circular case! E depends only on a, $\boxed{L^2 = GMa(1-e^2)}$

② Second expression

$$r = a(1 - e\cos\eta) \qquad \eta \text{ :eccentric anomaly}$$

$$\sqrt{1-e}\tan\frac{1}{2}(\psi - \psi_0) = \sqrt{1+e}\tan\frac{\eta}{2}$$

$$\Omega t = \frac{2\pi}{T_r}t = \eta - e\sin\eta \text{ : mean anomaly}$$

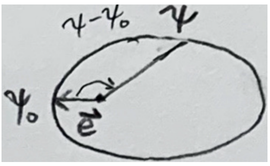
$$\Omega = \sqrt{GM/a^3}$$

$$t = \int_{\psi_0}^{\psi}\frac{d\psi}{\dot\psi} = \int d\psi\frac{r^2}{L} = \frac{a^2}{L}\int_0^{\eta} d\eta\frac{d\psi}{d\eta}(1-e\cos\eta)^2 = \frac{a^2}{L}\sqrt{1-e^2}(\eta - e\sin\eta)$$

$$T_r = T_\psi = \frac{a^2}{L}\sqrt{1-e^2} = \boxed{2\pi\sqrt{\frac{a^3}{GM}}} = 2\pi\frac{GM}{(-2E)^{3/2}}$$ depends only on a or E

$\longrightarrow$ Kepler's 3rd Law: $\boxed{\frac{a^3}{T^2} = \frac{GM}{4\pi^2}}$

- $\boxed{\Delta\psi = 2\pi}$



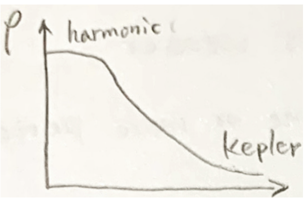
Laplace, Runge-Lenz, eccentricity } vector $\boxed{\vec{e} = \frac{\vec{v}\times\vec{u}}{GM} - \hat{e}_r}$

- $\frac{d\vec{e}}{dt} = 0 \quad |\vec{e}| = e, \quad \vec{e}$ $\underbrace{\text{always points from central mass towards pericenter}}_{\text{"line of apsides"}}$

- Other central fields $\approx$ "elliptical" orbits, but they do not have vector integral of motion like $\vec{e}$, because orbits are not closed.

$$T_\psi = \frac{2\pi}{\Delta\psi}T_r$$

- General form of galatix potential is between harmonic and Keplerian.



$$\underset{\text{harmonic}}{\pi} \le \Delta\psi \le \underset{\text{Kepler}}{2\pi}$$

- In non-Keplerian potentia, orbit $\approx$ ellipse

- The "ellipse" precesses in the $\underbrace{\text{opposite}}_{\text{retrograde}}$ sense to the rotation of the star

- Precession rate $\Omega_p = \frac{\psi_p}{T_r} = \frac{\Delta\psi - 2\pi}{T_r} < 0 \quad T_r$: time for one radial oscillation

---

- Timing the Local Group (Box.3.1) "Kahn-Woltjer timing argument" for M_{MW}

- Local Group $\approx$ MW + M31, relatively isolated. $\approx$ two point masses
- Andromeda = M31: $D = (740 \pm 40)\ kpc$, $V_{GC} = -125\ kms^{-1}$, relative to the G.C.
- As $V_{GC} < 0$, approaching, tangential motion negligible
- Gravity has halted the original <u>motion</u> of M31 away from the MW.
  "Hubble recession"
- Assume $L = 0$, $e = 1$, Kepler orbit.

$$r = a(1 - \mathrm{e}\cos\eta) \to \frac{d\ln r}{d\ln t} = \frac{t}{r}\frac{dr}{dt} = \frac{e\sin\eta(\eta - e\sin\eta)}{(1 - e\cos\eta)^2}$$

- At $\eta = 0, r = t = 0$ (big bang).
  Set $e = 1$, $r = 740\ kpc$, $\dfrac{dy}{dt} = -125\mathrm{km/s}$, $t = 13.7\mathrm{Gyr}$, solve the nonlinear equation for η numerically, $\eta = 4.29$

$$\to a = \frac{r}{1 - e\cos\eta} = 524\mathrm{kpc}, \quad T_r = 16.6\ \mathrm{Gyr}$$

- $\dfrac{GM}{4\pi^2} = \dfrac{a^3}{T^2} \to M = M_{MW} + M_{M31} = 4.6 \times 10^{12} M_\odot$, collode in $\sim$ 3 Gyr. "Milkomeda"

- M is larger than conventional estimate. What are the possible factors that could change M?
  If $M_{M31} = M_{MW} \times 1.5 \Rightarrow M_{MW} = 1.84 \times 10^{12} M_\odot$
  $M_* = 5 - 6 \times 10^{10} M_\odot$, mostly DM!

3. ISOCHRONE POTENTIAL

- Invented by Hénon(1959), more realistic, all orbits are analytic

$$\Phi = -\frac{Gm}{b+\sqrt{b^2+r^2}} = -\frac{GM}{b \cdot s} \qquad S = 1+\sqrt{1+\frac{r^2}{b^2}} = -\frac{GM}{b\Phi}$$

$\begin{cases} r \ll b, \text{ harmonic} & \Phi \propto r^2 \\ r \gg b, \text{ Keplerian} & \Phi \propto \frac{1}{r} \end{cases}$. $\frac{r^2}{b^2} = s^2\left(1-\frac{2}{s}\right)$ $s \gg 2$, so one-to-one relation between r and s.

- $\boxed{T_r = \frac{2\pi Gm}{(-2E)^{3/2}}} = T_r(E)$, does NOT depend on L, same as Keplerian + Circular.

$$|\Delta\psi| = \pi\left(1+\frac{|L|}{\sqrt{L^2+4GMb}}\right)$$

$$\pi < |\Delta\psi| < 2\pi$$

$\uparrow$ $\uparrow$

$L^2 \ll 4GMb$, fly through the core $\qquad$ $L^2 \gg 4GMb$, never approaches core($r<b$)

$$T_\psi = \frac{2\pi}{|\Delta\psi|}T_r = \frac{4\pi GM}{(-2E)^{3/2}}\frac{\sqrt{L^2+4GMb}}{|4|+\sqrt{L^2+4GMb}}$$

INTEGRAL OF MOTION

- $I\left[\vec{x}\left(t_1\right), \vec{v}\left(t_1\right)\right] = I\left[\left(\vec{x}\left(t_2\right), \vec{v}\left(t_2\right)\right]\right.$ for any t_1 and t_2, $\frac{dI(\vec{x}, \vec{v})}{dt} = 0$ along an orbit, i.e. orbital invariant.
- Not to be confused with constants of motion $C(\vec{x}, \vec{v}, t)$, $\frac{dc}{dt} = 0$. C is not very meaningful.
- Any orbit in any force field always have 6 constants

$$(\vec{x}_0, \vec{v}_0) = [\vec{x}(t-0), \vec{v}(t=0)]$$

$$\vec{x}_0 = \vec{x} - \int_{t_0}^{t} \vec{v} dt, \quad \vec{v}_0 = \vec{v} - \int_{t_0}^{t} \vec{a} \cdot dt$$

- Example: circular orbit $\psi = \Omega t + \psi_0$: $\psi_0 = \Omega t - \psi$ = constant of motion but not integral.
  $\mathrm{L} = rV_\phi$= Integral
- Any integral of motion is a constant of motion, but not vice versa.
- Orbits can have $0-5$ integrals of motion.
- Only isolating integrals are of great importance; non-isolating integrals do not affect the phase-space distribution of an orbit $\Rightarrow$ no value.
- BT P157: $\psi = \psi_0 + G(E, L, r)$, ψ_0: the fifth integral of motion. ψ_0: isolating only if $G(E, L, r)$ increases by 2π every azimuthal period.

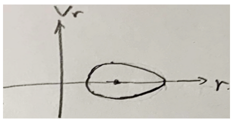
  $\rightarrow$ e.g. Keplerian

  otherwise, ψ_0 is non-isolating integral
- Motion is not further restricted to a low-dimensional sub-space of the hyper-surface defined by the other integrals, because if we wait long enough, the star will pass orbitrarily close to any point on the 2-D torus.

- The trajectory of a star in phase space lies on a hypersurface of one lower dimensions for each <u>isolating integral of motion</u>.

  spherical: 4 integrals$(E+\vec{L})$, constrains an orbit on a 2-D torus

$$V_r = \pm\sqrt{2[t-\Phi(r)] - L^2/r^2} \qquad V_\phi = \frac{L}{r}$$



- Given E, $\vec{L}$, the star's position and velocity can be specified by 2 quantities (r, ψ).

Any static (time-independent) potential		# of Integral
Any Φ_r	$E = \frac{1}{2}v^2 + \Phi(\vec{r})$	1
Axisymmetric $\Phi(R, z)$	$E, L_z = R^2\dot{\phi}$	2
Sperical $\Phi(r)$	$E, \vec{L}$	4 isolating and 1 non-isolating
Harmonic Oscillator	E_x, E_y, L_x, L_y, L_z	5
Kepler	$E, \vec{u}, \vec{e}$ $\vec{e} = \frac{\vec{v}\times\vec{\imath}}{GM} - \hat{e}_r$	

- $\boxed{\text{Noether Theorem}}$: When a potential presents a symmetry(i.e. its functional form is unchanges by a spacial and or temporal transformation), there is a corresponding physical quantity that is conserved.

Invariant of L under time translation	$\rightarrow E$ conserved
spacial	$\rightarrow \vec{P}$ conserved
rotational	$\rightarrow L$ conserved
Gauge Invariance of electric potential	$\rightarrow$ charge conservation

More symmetries $\Rightarrow$ more integrals of motion

EQUATION OF MOTION IN MERIDIONAL PLANE

- Spherical sustems are too idealistic!
  What kinds of orbits are possible in axisymmetric potentials?
  ↑
  possible for many red dgalaxies

- Naturally, a cylindrical coords (R, ϕ, z) is convenient with the z-axis aligned with the symmetry axis.

⋆ If orbits are confined to $z = 0$ (equatorial plane), a particle cannot "perceive" that potential is not spherically symmetric
$\rightarrow$ Orbits are identical to those in spherical case!
i.e. rosette figure between $r_{\rm a}$ and $r_{\rm p}$.

⋆ In this section, we are generally interested in orbits with $z \neq 0$(not confined to the equatorial plane)

  – we generally assume potential is symmetric about $z = 0$ plane as in real galaxy. = reflection-symmetric
  – First we can reduce the 3D to 2D problem by taking advantage of $L_{\rm Z} = \mathrm{constant}$
    $\rightarrow$motion is in the (R, z) plane = "meridinal"

$$\Phi = \Phi(R, z)$$

  – Cylindrical coord. $\vec{q} = (R, \phi, z)$
  – Lagrangian $\mathcal{L} = \frac{1}{2}\left[\dot{R}^2 + (R\dot{\phi})^2 + \dot{z}^2\right] - \Phi(R, z) = k - \Phi$

$$\vec{p} = \left(\frac{\partial L}{\partial \dot{\vec{q}}}\right)_{\vec{q},t}, \mathrm{i\cdot e\cdot} \quad p_i = \left(\frac{\partial l}{\partial \dot{q}_i}\right)_{q_i,t}$$

$$\rightarrow P_R = \dot{R}, \quad p_\phi = R^2\dot{\phi}, \quad P_z = \dot{z}$$

– Hamiltonian: $\mathcal{H} = \vec{p}\cdot\vec{q} - \mathcal{L} = \frac{1}{2}\left[p_R^2 + \frac{p_\phi^2}{R^2} + p_z^2\right] + \Phi(R,z)$, eliminating $\dot{\vec{q}}$, $\mathcal{H}(= H(\vec{q},\vec{p},t)$

– Hamilton's Equation $\dot{p} = -\frac{\partial H}{\partial \vec{q}}$

$$\begin{cases} \dot{p_R} = \ddot{R} = \frac{p_\phi^2}{R^3} - \frac{\partial \Phi}{\partial R} \\ \dot{P_\phi} = \frac{d}{dt}\left(R^2\dot{\phi}\right) = 0 \quad \Leftrightarrow P_\phi = L_z = Rv_\phi = R^2\phi = c_0nSt \\ P_z = \ddot{z} = -\frac{\partial \Phi}{\partial z} \end{cases}$$

– Define $\boxed{\Phi_{\text{eff}} \equiv \Phi(R,z) + \frac{L_z^2}{2R^2}} = \Phi_{\text{eff}}\,(R,z,L_z)$

– Eq. of motion in (R,z) $\rightarrow$ $\begin{cases} \ddot{R} = -\frac{\partial \Phi_{eff}}{\partial R} \\ \ddot{z} = -\frac{\partial \Phi_{eff}}{\partial z} \end{cases}$ 3D motion $\rightarrow$ 2D motion in (R,z)

– Generally can only be solved numerically.

$$H = \frac{1}{2}\left[P_R^2 + P_\phi^2/R^2 + P_z^2\right] + \Phi(R,z) = \underbrace{\frac{1}{2}\left(P_R^2+P_z^2\right)}_{\text{K.E. in (R,z)}} + \underbrace{\Phi_{eff}}_{\substack{\text{K.E. azimuthal } +\Phi(R,z) \\ \text{conservation of } L_z}} = E$$

– Allowed region: $E \geq \Phi_{\text{eff}}(R,z)$

– Zero velocity curve $E = \Phi_{\text{eff}}\,(R,z))$. Φ_{eff} rises steeply near z-axis, centrifugal barrier.

– Φ_{eff} has a minimum at the point

$\boxed{\frac{\partial \Phi_{eff}}{\partial R} = \frac{\partial \Phi}{\partial R} - \frac{Lz^2}{k^3} = 0}$ and $\boxed{-\frac{\partial \Phi}{\partial z} = 0}$ $\rightarrow$ Satisfied anywhere in the $z=0$ plane since $\Phi(R,z)$ is symmetric about $z=0$ plane.

$$\left(\frac{\partial \Phi}{\partial R}\right)_{(R_g,0)} = \frac{L_z^2}{R_g^3} = R_g\dot{\phi}^2 = \frac{v^2}{R}$$

(circular orbit condition)

$\rightarrow$ satisfied at $R = R_g$, <u>guiding-center radius</u>

– The minimum of Φ_{eff} occurs at $R = R_g$ at which a circular orbit with L_z

- Since $E = \frac{1}{2}\left(\dot{R}^2 + \dot{z}^2\right) + \Phi_{\rm eff}$, a circular orbite at $R = R_{\rm g}$ has the least energy with the given Lz. Note, $\Phi_{\textsf{eff}} = \Phi_{\textsf{eff}}\ (R, z, L_z)$
- Eq. of motion can only be solved numerically.

THE EPICYCLE APPROXIMATION

- Next we derive approximate solutions to Eq. of motion for nearly circular orbits $\left(= \boxed{\text{epicycle approximation}}\right)$
- Define $x = R - R_g$. then minimum of Φ_{eff} is at $(x, z) = (0, 0)$ or $(R, z) = (R_g, 0)$
- Expand Φ_{eff} in a Taylor series expansion around minimum.

$$\Phi_{\text{eff}} = \Phi_{\text{eff}}(R_{\text{g}}, 0) + \frac{1}{2}\frac{\partial^2 \Phi_{\text{eff}}}{\partial R^2}\bigg|_{(R_{\text{q}},0)} x^2 + \frac{1}{2}\frac{\partial^2 \Phi_{\text{eff}}}{\partial z^2}\bigg|_{(R_{\text{g}},0)} z^2 + O\left(xz^2\right)$$

Note: $\frac{\partial^2 \Phi_{\text{eff}}}{\partial x \partial z}\bigg|_{z=0} = 0$. Since Φ_{eff} eff is an even function of z, $\frac{\partial}{\partial z}\bigg|_{z=0} = 0$.

- Physically: fit Φ_{eff} near $(R_{\text{g}}, 0)$ with a 2D parabola bowl.
- Eq. of motion: $\boxed{\ddot{x} = -k^2 x \quad \ddot{z} = -\nu^2 z}$

where epicycle frequency $\kappa^2 = \frac{\partial^2 \Phi_{\text{eff}}}{\partial R^2}\bigg|_{(R_g,0)} = \frac{\partial^2 \Phi}{\partial R^2}\bigg|_{(R_g,O)} + \frac{3L_z^2}{R_g^4} = 3\Omega^2\big|_{R_{\text{g}}}$,
(or radial frequency)

vertical frequency $\nu^2 = \frac{\partial^2 \Phi_{eff}}{\partial z^2}\bigg|_{(R_g,0)}$

- Can write $\boxed{\kappa^2 = \left(R\frac{d\Omega^2}{dR} + 4\Omega^2\right)_{R_{\text{g}}}}$, since $\Omega^2(R) = \frac{1}{R}\left(\frac{\partial \Phi}{\partial R}\right)_{(R,0)} = \frac{L_z^2}{R^4}$
- What is the condition for stability of these 2 oscillations?
- $x = R - R_g$, radial freq. $\kappa \approx \Omega_{\text{r}}$
- Note: $T_r = \frac{2\pi}{\kappa} \quad T_\psi = \frac{2\pi}{\Omega}$

  $\Omega \lesssim \kappa \lesssim 2\Omega \qquad \Omega(R), \kappa(R)$: generally decreasing function

  $\downarrow$ Kepler $\quad \downarrow$ harmonic

  1) Uniform rotation $\Omega = \text{constant}$, $\rho = \text{constant}$. Harmonic oscillator, $v_c \propto R\,(\text{rigid rotation})$

  $$\rightarrow K = 2\Omega = \text{constant}$$

  2) Kepler $\Omega = \sqrt{\frac{GM}{R^3}} \rightarrow K(R) = \Omega(R)$

3) Constant circular velocity (Flat $v_c = constant$)

$$\Omega = \frac{V_c}{R} \propto R^{-1} \longrightarrow \kappa(R) = \sqrt{2}\Omega(R)$$

$\Delta\psi = \frac{2\pi}{T_\psi} \cdot T_r \approx 2\pi \cdot \frac{\Omega}{\kappa} \approx \frac{2\pi}{\sqrt{2}}$, if $v_c = \mathrm{constant}$ and epicycle approximation valid. i.e. angular separation between 2 consecutive apocenters

$$\Delta\Phi = 2\pi - \Delta\psi = 2\pi - \frac{2\pi}{\sqrt{2}} \approx \boxed{105.48^\circ}$$

– Oort "constant" of galactic differential rotation (dimension: frequency)

$$A \equiv \frac{1}{2}\left(\frac{V_c}{R} - \frac{dV_c}{dR}\right)_{R_0} = \frac{1}{2}\left(R\frac{d\Omega}{dR}\right)_{R_0}$$

"shear rate" $= 2A$

$$B \equiv -\frac{1}{2}\left(\frac{V_c}{R} + \frac{dV_c}{dR}\right)_{R_0}$$

"vorticity" of star and gas motions near Sun

$$\left|\nabla \times \left(v_c e_\phi^\wedge\right)\right| = -2B$$

– $A = A(R)$; $B = B(R)$, constants only near the Sun $(R = R_0)$!

$$\boxed{\Omega_0 = A - B \quad \kappa_0^2 = -4B(A - B) = -4B\Omega_0}$$

$$A = 14.8 \pm 0.8 \ \mathrm{km/s/kpc} \quad \mathrm{B} = -12.4 \pm 0.6 \ \mathrm{km/s/kpc} \Rightarrow \begin{array}{l} \kappa_0 = 37 \pm 3 \ \mathrm{km/s/kpc} \\ \frac{\kappa_0}{\Omega_0} = 2\sqrt{\frac{-\mathrm{B}}{A - B}} = 1.35 \pm 0.6 \end{array}$$

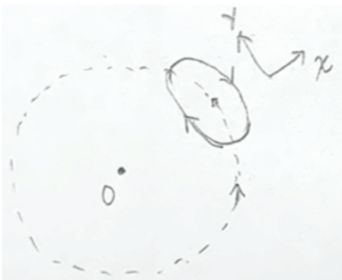
BT. Table 1-2

RADIAL AND AZIMUTHAL MOTIONS

- $\ddot{x} = -x^2 x \quad x = R - R_g$
- Radial: $\boxed{x(t) = X\cos(kt + \alpha)}$, $\quad \bar{x} \geqslant 0, \quad \alpha = \text{constant}$

$$\dot{\phi} = \frac{Lz}{R^2} \underset{\uparrow}{=} \frac{L_z}{R_g^2}\left(1 + \frac{x}{R_g}\right)^{-2} \approx \Omega_g\left(1 - \frac{2x}{R_g}\right)$$
$$R = R_g + x$$

- Azimuthal: $\phi = \underline{\Omega gt + \phi_0} - \gamma \frac{X}{Rg} \sin\kappa t + \alpha$, here $\boxed{y = \frac{2\Omega q}{\kappa} = -\frac{\kappa}{2B} \geqslant 1}$ moving guiding center $(R, \phi) = (R_g, \Omega_g t + \phi_0)$
- The azimuthal mothon also harmonic. The azimuthal motion also harmonic.
- Erecting Pseudo-Cartesian coordinates (x, y, z) centered on this moving guiding center, y is in the direction of guiding center rotation, $y = R_g\left[\phi - (\Omega gt + \phi_0)\right]$



$$\boxed{\begin{aligned} x &= \cos(\kappa t + \alpha) \\ y &= -Y\sin(\kappa t + \alpha) \\ z &= Z\cos(\nu t) + z_0 \end{aligned}}$$

$$\frac{X}{Y} = \frac{1}{\gamma} = \frac{\kappa}{2\Omega} \leq 1$$
$$\boxed{Y = \frac{2\Omega}{\kappa} X = \gamma X \geqslant X}$$

Harmonic oscillations in (x, y, z)! $\begin{cases} \dot{x} = -\kappa X \sin(\kappa t + \alpha) \\ \dot{y} = -(\kappa x) \cdot \gamma \end{cases}$

- Retrograde epicycle at a rate $\frac{2\pi}{x}$
- Elliptical epicycle is elongated in the azimuthal direction!
- Closed ellipse in the frame of the moving guiding center still a rosette figure in the inertial frame
- Ancient Greek astronomer Hipparchus $(190 - 120BC)$ invented round epicycles to explain the retrograde motions of planets on the sky. - not very successful with only I epicycle, need multiple epicycles (first known perturbation

expansion!) If he used elliptical epicycles $\left(\frac{x}{r} = \frac{1}{z}\right)$ $\rightarrow$ correct! $\rightarrow$ keperian ellipse $\left(k = \Omega, \frac{x}{r} = \frac{1}{z}\right)$

MEASURING RADIAL AND VELOCITIES

- Every star has its own guiding center, in general we do not know the location of the guiding center of any given star?
- But we can measure v_R and $v_\phi(R_0) - v_c(R_0)$ for a group of stars, each of which has its own $R_{\rm g}$.

$$V_\phi(R_0) - V_c(R_0) = R_0\left(\dot\phi - \Omega_0\right) = R_0\left(\dot\phi - \Omega_g + \Omega_g - \Omega_0\right)$$
$$\simeq R_0\left[\left(\dot\phi - \Omega_g\right) - \left(\frac{d\Omega}{dR}\right)_{R_g} x\right]$$

Note difference between Ω_0, $\dot\phi$, Ω_g $\quad R = R_0 = R_g + x$

$$\dot\phi = \frac{L_z}{R_0} = \frac{L_z}{R_g}\left(1 + \frac{x}{R_g}\right)^{-2} \approx \Omega_g\left(1 - \frac{2x}{R_g}\right)$$
$$\to v_{\phi(R_0) - V_c(R_0)} \approx -R_0 x\left(\frac{2\Omega}{R} + \frac{d\Omega}{dR}\right)_{R_{\rm g}} \qquad x = R_0 - R_g$$
$$\approx -R_0 x\left(\frac{2\Omega}{R} + \frac{d\Omega}{dR}\right)_{R_0} = 2Bx$$
$$V_\phi(R_0) - V_c(R_0) \approx 2Bx = -\frac{\kappa}{\gamma}x = -\frac{\kappa}{\gamma}X\cos(\kappa t + \alpha)$$

- Averaging phases α of stars near the Sun

$$\overline{[V_\phi - V_c(R_0)]}^2 = \frac{x^2\bar{x}^2}{2\gamma^2} = 2B^2\bar{x}^2$$

For an ensemble of stars seen by an comoving observer. at R_0 with $v_c(R_0)$.

- Known $\begin{cases} \cos^2\alpha = \frac{1}{2}(1 + \cos 2\alpha) \\ \sin^2\alpha = \frac{1}{2}(1 - \cos 2\alpha) \end{cases}$, $v_R = \dot{x} = -\kappa X \sin(\kappa t + \alpha)$, similarly

$$v_R^2 = \frac{1}{2}\kappa^2 X^2 = -2B(A - B)X^2$$

$$\frac{\overline{[v_\phi - v_c(R_0)]}^2}{\overline{v_R}^2} \approx \frac{-B}{A - B} = \frac{\kappa_0^2}{4\Omega_0^2} = \frac{1}{\gamma^2} \simeq 0,46$$

This is counter-intuitive!

- Note mean-square azimuthal and radial velocities relative to the guiding center of a single star

$$\frac{\overline{\dot{y}^2}}{\overline{\dot{x}^2}} = \frac{\frac{1}{2}(\kappa Y)^2}{\frac{1}{2}(\kappa Z)^2} = \gamma^2 \geqslant 1, \text{ completely opposite!}$$

$$\begin{cases} \ddot{x} = -\kappa^2 x \\ \dot{z} = -\gamma^2 z \end{cases}$$, for harmonic oscillators with spring constant = $$\begin{cases} \kappa^2 \\ \gamma^2 \end{cases}$$

INTEGRALS OF MOTION IN EPICYCLE APPROXIMATION

– Consider integrals of motion in epicycle approximation

$$H_R = \frac{1}{2}\left(\dot{x}^2+\kappa^2 x^2\right) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\gamma^2\left[v_\phi\left(R_0\right) - V_c\left(R_0\right)\right]^2 = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\frac{1}{\gamma^2}\dot{y}^2$$

$$= \frac{1}{2}\gamma^2 X^2 = \frac{1}{2}\kappa^2\left(\frac{Y}{\gamma^2}\right)^2 = \frac{1}{2}\gamma^2\tilde{v}^2_{\phi,\mathrm{max}}$$

$$H_z = \frac{1}{2}\left(\dot{z}^2+\nu^2\nu^2\right)$$

$$= \frac{1}{2}\nu^2 Z^2 = \frac{1}{2}v^2_{Z,\mathrm{max}}$$

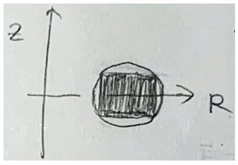
$$H = H_R + H_z + \Phi_{eff}\left(R_g, 0\right)$$

$$\kappa^2 x^2 = \left(\gamma\tilde{V}_\phi\right)^2 = \frac{1}{\gamma^2}\dot{y}^2$$

$$\begin{cases} \gamma = \dfrac{2\Omega_g}{x} \\ \dot{y} = -2\Omega g x = -\partial X x \\ \tilde{v}_\phi = V_\phi - V_c\left(R_0\right) = -\dfrac{K}{\partial}x \end{cases}$$

– 3 integrals of motion: (H_R, H_z, L_z) or (H, L_z, H_z), harmonic motion is all 3 directions (x, y, z).

– Epicyclic motion fills up a rectangle in the $(R-Z)$ plane: Lissajous Figure. (4 corners are limited by $\Phi_{eff} \leqslant E$)



– When is the epicycle approximation valid?

1. Nearly circular orbits. Taytor expansion is only accurate near $R = R_g$. Epicycle results generally give us great insights even for orbits that deviate greatly from circular orbits.

2. Z-direction $\Phi \propto z^2$ is only true for z small enough that $\rho_{\mathrm{disk}}(z) \simeq \mathrm{constant}$, i.e, for $z \ll 300\mathrm{pc}$: $z = Z\cos\nu t + z_0$. In general, epicycle approximation is poor in z-direction, as the orbits of most disk stars have $z > 300\mathrm{pc}$!

– Is epicycle approximation valid for nearly circular orbits in a spherical potential?

INVESTIGATION OF FREQUENCY RATIOS

- Vertical oscillation frequency, ν
- Poisson Eq.

$$4\pi G\rho = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Phi}{\partial R}\right) + \frac{\partial^2 \Phi}{\partial z^2}$$
$$\simeq \frac{1}{R}\frac{\mathrm{d}v_c^2}{\mathrm{d}R} + \nu^2$$

- Disk with flat $v_c = \mathrm{constant} \rightarrow \boxed{\nu^2 = 4\pi G\rho}$
- Flat $v_c \Rightarrow \kappa^2 = 2\Omega^2$, also $\Omega^2 \approx \frac{GM}{R^3} = \frac{4}{3}\pi G\bar{\rho}$. Ronghly true even for a disk $\Rightarrow \frac{\nu^2}{\kappa^2} = \frac{\nu^2}{2\Omega^2} = \frac{4\pi G\rho}{\frac{8}{3}\pi G\bar{\rho}} = \frac{3}{2}\frac{\rho}{\bar{\rho}}$ measures the degree to which mass is concentrated towards the plane.
- Table 1.1 $\quad \rho \approx 0.1\mathrm{M_\odot pc^{-3}}$ near the Sun
- Vertical oscillation period $\frac{2\pi}{\nu} = \frac{2\pi}{\sqrt{4\pi G\rho}} = \boxed{87\mathrm{Myr}}$
- $T = \frac{2\pi}{\Omega} = \frac{2\pi R_0}{v_0} = \frac{8.2\mathrm{kpc}}{240\mathrm{km/s}} \cdot 2\pi = \frac{8200\mathrm{pc}}{240\mathrm{pc/Myr}} \cdot 2\pi = \boxed{210\mathrm{Myr}}$
  $1\mathrm{km} = 1.023\mathrm{pc/Myr}$
- $\frac{2\pi}{k} \approx \boxed{155\mathrm{Myr}}$ if $\frac{\kappa_0}{\Omega_0} = 2\sqrt{\frac{-13}{A-B}} = 1.35$
- $\frac{\nu}{k} \approx 1.8$ for the Sun $\Rightarrow \bar{\rho} = \frac{3}{2}\rho \cdot \left(\frac{x}{\nu}\right)^2 = 0.046\mathrm{M_\odot pc^{-3}}$
- Harvard professor Lisa Randall wrote a book "How dark matter killed dinosaurs". Massive extinction period $\sim 30\mathrm{Myr}$? Dinosaurs, $\sim 66\mathrm{Myr}$ ago. dissipative DM particle $\rightarrow$ DM disk, $\Sigma = 10\mathrm{M_\odot/pc^2}, \quad z_d = 10\mathrm{pc}$
  $\nu = \sqrt{4\pi G\rho} \quad \rho_{DM} \sim 1M_\odot/pc^3$

$\rightarrow \nu$ is increased by a factor 3 ?

$$\rightarrow \frac{2\pi}{\nu} \rightarrow 30\mathrm{Myr?}$$

THE THIRD INTEGRAL OF MOTION

- General orbits in an axisymmetric potential E and Lz are integrals of motion, but are there more? The two orbits with the same E and Lo, but they cook very different difference does not diminish, no matter how long they are integrated. $\rightarrow$ third integral?
- Eq. (3.70) $\Phi_{\rm eff} = \frac{1}{2}v_0^2 \ln\left(R^2 + \frac{Z^2}{q^2}\right) + \frac{L_z^2}{2R^2}$

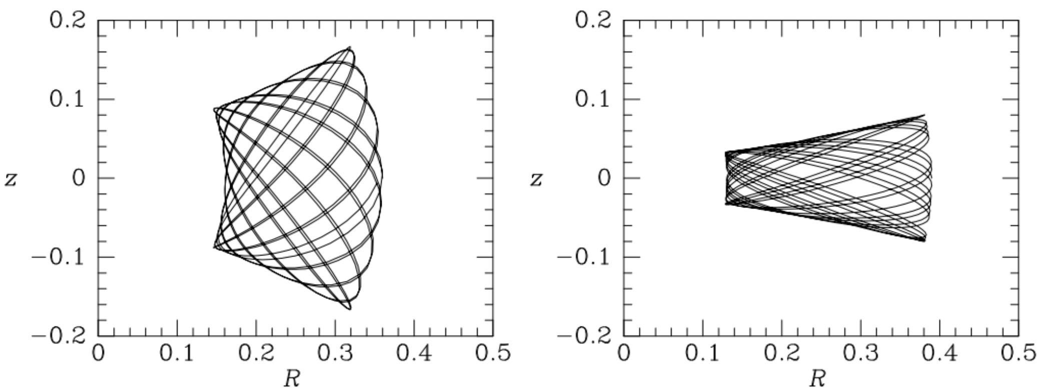


Figure 3.4 Two orbits in the potential of equation (3.70) with $q = 0.9$. Both orbits are at energy $E = -0.8$ and angular momentum $L_z = 0.2$, and we assume $v_0 = 1$.

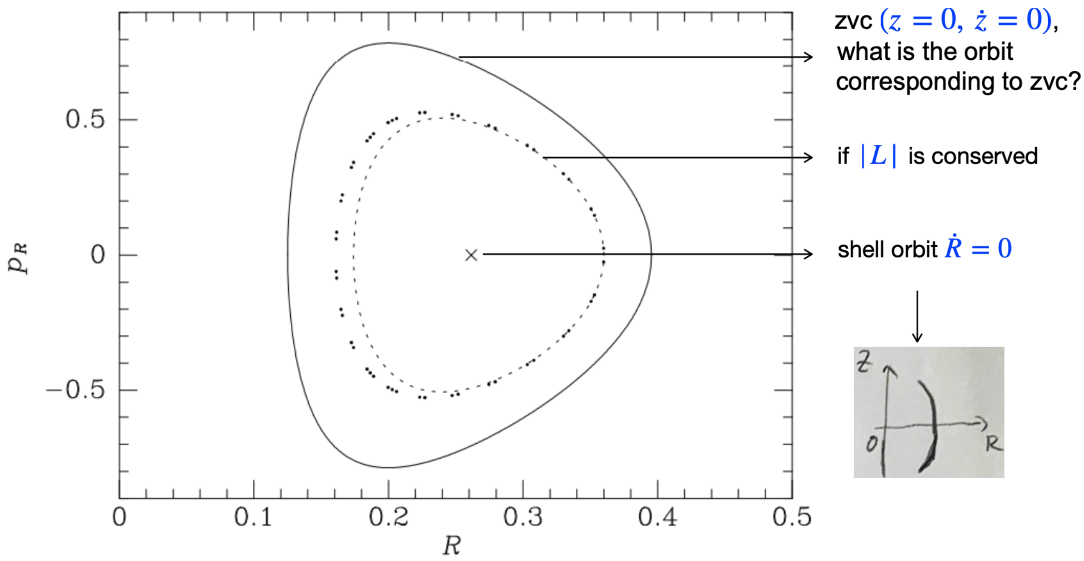


Figure 3.5 Points generated by the orbit of the left panel of Figure 3.4 in the (R, p_R) surface of section. If the total angular momentum L of the orbit were conserved, the points would fall on the dashed curve. The full curve is the zero-velocity curve at the energy of this orbit. The $\times$ marks the consequent of the shell orbit.

- $\ddot{R} = -\frac{\partial \Phi_{fff}}{\partial R} \quad z = -\frac{\partial \Phi_{\text{eff}}}{\partial z}$ used $L_z = \text{constant}$ to reduce motion in meridional plane, (R, z)
- How to visualize. 4-D phase space $(R, \dot{R}, z, z)$?
  $H_{\text{eff}}(R, z, \dot{R}, \dot{z}) = \text{constant} = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 + \Phi_{\text{eff}} \to 3D$
- Poincoré surface of section (SOS): cut 3D ellipsoidal volume in $(R, z, \dot{R})$, construct a SOS diagram to show the phase space in 2D subspace $(R, \dot{R})$
- $z = 0$, and $z > 0$ (moving upwards), record $(R, \dot{R})$ consequences to remove sign ambiguity. $\to$ no distinct orbits at the same E can occupy the same point.
- Zero velocity curve (zvc) in sos. ($\dot{z} = 0$)

$$H_{eff} \geqslant \frac{1}{2}\dot{R}^2 + \Phi_{eff}(R, z = 0)$$

- If I_3 exists, orbits lie on a smooth curve: Invariant curve (1-D curve in 2-D space), otherwise can fill up the area inside zvc.
- $I_3 =$ non-classical integral, because I_3 has no analytical expression in $(\vec{x}, \vec{v})$
- We may get an intuitive picture of the nature of $I_3 =$ by considering two special cases:

  ① since $|L| =$ integral for a spherical potential so for a nearly spherical potential. $I_3 \approx |L|$ see the dashed line in Fig. 3.5.
  $|L(t)|$ oscillates rapidly, but its mean value does not change. So $|L|$ is an approximately conserved quantity, even in a flattened potential. Orbits are approximately planar with r_{peri} and r_{apo}. The approximate orbital plane has a fixed inclination to the z-axis but precesses about z. Precession rate $\to$ 0, for a increasingly spherical.

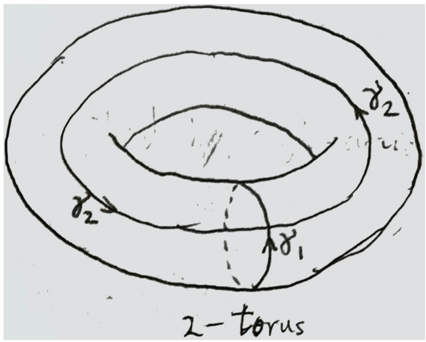
  ② Potential separable in R and z

$$\Phi(R, z) = \Phi_R(R) + \Phi_z(z)$$

  Then I_3 can be $H_Z = \frac{1}{2}P_z^2 + \Phi_z(z)$ In the case of epicycle approximation, what is the shape of the invariant curves?

DESCRIBING OSCILLATORY MOTION USING ANGLE-ACTION VARIABLES

- The motion of a star on a regular orbit is over the surface of a hyper-torus, see Fig. 3.24 for an example of a 2-torus. It therefore oscillates in three independent directions (triply periodic motion)



- Oscillatory motion is most elegantly described by angle-action variables

- Orbits are the building blocks of galaxies. Describing a star as $(\vec{x}, \vec{v})$ is good for Poisson Eq. but sometimes not helpful for understanding for orbits. $(\vec{x}, \vec{v})$ change!

- Better: describe a star as on orbit labelled $\vec{J}$ at $\vec{\theta}$, $\vec{J}$ stays $\sim$ fixed

$$\vec{J} = (J_1, J_2, J_3) \quad \vec{\theta} = (\theta_1, \theta_2, \theta_3)$$

- Angle-action variables are a set of canonical coordinates.
  3 momenta are integrals of motion the $\vec{J} =$ “actions“ (dimension: angular mom.)
  3 conjugate coordinates = “angles” $\vec{\theta}$
  canonical $\to$ can utilize all the apparatus of Hamiltonian theory.

- Regular orbits: orbits that have the angle-action variables.
  $\to$ have the some $\#$ of isolating integrals as the spatial dimension
  $\to$ have three well-defined. frequencies.

- Angle-action variables cannot be defined for many potentials of practical importance. But the conceptual framework is extremely useful for understanding complex phenomenon in real potentials that do not admit angle-action variables.

- Angle-action variables $(\vec{\theta}, \vec{J})$ $\quad \vec{J} = (J_1, J_2, J_3)$ are integrals of motion.Hamilton's Equation for the motion of generalized momentum, J_i:

$$\dot{J}_i = -\frac{\partial H}{\partial \theta_i} = 0$$

- so $\boxed{H = H(\vec{J})}$, independent of coordinates $\vec{\theta}$.

$$\dot{\theta}_i = \frac{\partial H}{\partial J_i} = \Omega_i(\vec{J}) = \text{constant} \Rightarrow \theta_i(t) = \theta_i(0) + \Omega_i t$$

  $\vec{\theta}$ is a linear function of time.

- Advantages of actions (or angle-action variables)
  - A regular orbit has a label $\vec{J}$ at point $\vec{\theta}$, $\vec{J}$, stays $\sim$ fixed
    * $\vec{\theta}$ increases linearly with time, the dynamics is relatively easy, motion is "harmonic". $\{\Omega_i\} = \text{constant}$ for an orbit. $(\vec{x}, \vec{v})$ space, motion complicated and anharmonic.
    * Jeans' theorem = a steady-state distribution function $f(\vec{x}, \vec{v})$ is $f(\vec{J})$. 6D structure $\rightarrow$ 3D.
  - To describe a galaxy/model describe structure in $\vec{J}$
    * actions are adiabatical invariants
    * reasonable intuitive (J_x, J_ϕ, J_z, ranges from 0 to ∞)
    * A Hamiltonian that admits angle-action variables is <u>integrable</u>
    * natural coordinates of perturbation theory.
    * approximate analytic solution of more complex real potentials.
    * useful for stability analysiss of stellar systems.
    * give insights into dynamics of orbits

$$H = H_0 + \epsilon\Phi_1$$
$$\dot{\vec{I}} = -\epsilon\frac{\partial\Phi_1}{\partial\vec{\theta}}$$
$$\dot{\vec{\theta}} = \vec{\Omega}(\vec{I}) + \epsilon\frac{\partial\Phi_1}{\partial\vec{J}}$$

Equation of motion can be solved in Φ_0.

SIMPLIFIED INTRODUCTION TO ACTION-ANGLE VARIABLES

- An actibn is defined as the area of an oscillation in phase space, divided by 2π

$$J_i = \frac{1}{2\pi}\oint_{\gamma_i} \vec{p}\cdot d\vec{q}$$

 where path γ_i on which θ_z increases from 0 to 2π while every this else = constant.

- Example: a spherical potential.

$$\dot{r} = \left[2(E-\Phi(r)) - \frac{L^2}{r^2}\right]^{\frac{1}{2}}$$

- The period of its radial oscillation is:

$$T_r = \int \frac{dr}{\dot{r}} = \frac{2\pi}{\Omega_r}$$

where $\Omega_r \equiv \dfrac{2\pi}{T_r}$ is defined as a constant for the star and define angle θ_r to be the phase of this oscillation. $\dot{\theta}_r = \Omega_r = \mathrm{constant}$, $\theta_r = \Omega_r t + \mathrm{constant}$ increases linearly with time.

  * Note phase angle ϕ of a star relative to GC is not exactly θ_r, because radial oscillation is anharmonic.
  * The action conjugate to this angle, "radial action" is the area of the oscillation in phase space.

$$\boxed{\text{radial action, } J_r} = \frac{1}{2\pi}\oint \dot{r}dr$$

   Integral is over one radial period.

  * Actions have the dimension of angular momentum.
  * J_z: measures the amplitude of a star's radial oscillation - amount of "in and out" motion. In general, J_z has no simple form.

$$J_r = 0 \text{ for circular arbits,} \quad J_r = \frac{1}{2}\kappa X^2 \text{ for epicycle motion}$$

– Since $L = r^2\dot{\phi} = \mathrm{constant}$, $\dot{\phi} = \dfrac{L}{r^2} \neq \mathrm{constant}$, except for circular orbits in which guiding center $\equiv$ circular motion with L. But we introduce an angle θ_a so $\dot{\theta}_a = \mathrm{constant}$, θ_a= phase angle of GC.

$$\dot{\theta}_a \equiv \Omega_a = \frac{\Delta\theta}{T_r}$$

where $\Delta\theta$ is the azimuthal angle change in one radial period.

∗ The azimuthal action conjugate to θ_a is:

$$J_a = \frac{1}{2\pi}\int r^2\dot{\phi}\, d\phi, \qquad \text{if } \Phi(\vec{r}) \text{ is axisymmetric } \boxed{J_a = L_z}$$

J_a characterize the amount of "round and round" motion.

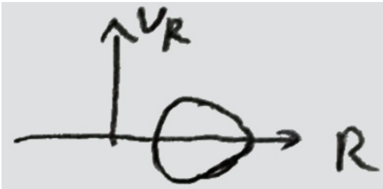
THE THIRD ACTION

- In 3D, there are generally 3 actions, but the third action, which characterizes the "up and down" motion is zero for all orbits in a spherical potential, because all orbits are planar.
- Orbits that do not stay in the symmetry plane of a spherical potential have a third non-zero action.

J_e (or J_θ) : latitudinal action or sometimes vertical action

⋆ How could actions be estimated for the surface of section (SOS)?

Actibn $\equiv$ area of osillation in phase space
$=$ area inside the inslariant curves(IC)
in the appropriate SOS



Orbits have larger ICs have larger actibus.

⋆ Disadvantages of angle-action variables

1. Cannot be used in irregular orbit. Fortunately, most orbits are regular in realistic potentials.
2. Transformation between $(\vec{J}, \vec{\theta}) \leftrightarrow (\vec{x}, \vec{v})$ are not all easy.

⋆ Even for spherical potentials. can have different sets of angle-action variables. See Table 3.1 . (P224)

⋆ Actions constitute an alternative set of integrals, $H = H(\vec{J})$, $I_i \leftrightarrow J_i$. P222 and Table 3.1. Any spherical potential has 4 isolating integrals of motion $J_1 = L_z$, $J_2 = L$, $J_3 = J_r$, and θ_1; since $H = H(J_2, J_3)$, $\Omega_1 = \dfrac{\partial H}{\partial J_1} = 0$.

HOW TO COMPUTE/FIND ACTIONS?

- We can only find them analytically for the $\boxed{\text{isochrone potential}}$.

$$\Phi_{\rm ISO} = -\frac{GM}{b + \sqrt{b^2 + r^2}}$$

  This compasses Kepler ($(r \gg b)$) and spherical harmonic $(r \ll b)$.

- $J_r = J_r(E, L)$

$$J_r = \frac{(GM)^2}{\sqrt{-2E}} - \frac{1}{2}\left(L + \sqrt{L^2 + 4GMb}\right) \qquad (3.225)$$

typo in BT08

$$H = H(\vec{J}) = -\frac{(GM)^2}{2\left[Jr + \frac{1}{2}\left(L(+\sqrt{L^2 + 4GMB}\right]^2} \qquad L = J_\theta + |J_\phi| \qquad (3.226)$$

$$\Omega_r = \Omega_r(\vec{J}) = \frac{(GM)^2}{\left[J_r + \frac{1}{2}(L + \sqrt{L^2 + 4GMb})\right]^3}$$

$$\Omega_\theta = \frac{1}{2}\left(1 + \frac{L}{\sqrt{L^2 + 4GMb}}\right)\Omega_r$$

$$\Omega_\phi = \mathrm{sgn}\,(J_\phi)\,\Omega_\theta$$

- Use 1D numerical integrals for any spherical potential.

$$J_r = \frac{1}{\pi}\int_{r_{min}}^{r_{\rm max}} dr\sqrt{2E - 2\Phi(r) - \frac{L^2}{r^2}}$$

  b $\rightarrow$ 0, Kepler Φ. $J_r = \frac{GM}{\sqrt{2|E|}} - L = L\left[\frac{1}{\sqrt{1-e^2}} - 1\right] = \sqrt{GMa}(1 - \sqrt{1-e^2})$.
  See Table (E.1) in Appendix.

- $\boxed{\text{Hamilton - Jacobi Eq.}}$ use a generating function for the transformation from (g_i, p_i) to (θ_i, J_i), both are sets of canonical coordinates $\rightarrow$ $d\vec{q} \cdot d\vec{p} = d\vec{\theta} \cdot d\vec{J}$

- $S(\vec{q}, \vec{J})$ is the (unknown) generating function of the transformation.

$$q = \vec{x}, \qquad p = \vec{v}$$

$$\vec{\theta} = \frac{\partial S}{\partial \vec{J}}, \quad \vec{p} = \frac{\partial S}{\partial \vec{q}}$$

- Use $S(\vec{I}, \vec{J})$ to eliminate $\vec{p}$ from Hamiltonian $H(\hat{\Omega}, \vec{p}) \to \frac{\partial S}{\partial \dot{q}}(\vec{q}; \vec{J})$
- By moving along an orbit, we can vary the q_i while holding constant the J_i. As we vary q_i this way, H must remain constant at energy E of the orbits.
- solve for $S(\vec{q}, \vec{J})$, $\boxed{H\left(\vec{q}, \frac{\partial S}{\partial \vec{q}}\right) = E}$ at fixed $\vec{J}$, κ_i is some arbitrary constant in the solution.

$$J_i = \frac{1}{2\pi} \oint_{\cdot\gamma_i} \vec{p} \cdot d\vec{q} = \frac{1}{2\pi} \oint_{\gamma_i} \frac{\partial S}{\partial \vec{q}} \cdot d\vec{q} = \frac{\Delta S(\vec{\kappa})}{2\pi} \to J_i = J_i(\kappa_i)$$

COORDINATE TRANSFORMATION IN ACTION SPACE

- In general. $(\vec{J}, \vec{\theta}) \leftrightarrow (\vec{x}, \vec{v})$ are not all easy!
- How this process works for 2D harmonic oscillator?

$$H(\vec{x}, \vec{p}) = \frac{1}{2}\left(p_x^2 + p_x^2 + \omega_x^2 x^2 + \omega_y^2 y^2\right)$$

Plug $p_x = \frac{\partial S}{\partial x} \quad p_y = \frac{\partial S}{\partial y} \quad S = S(x, y, \vec{J})$ into H to eliminate $\vec{p}$.

- Hamilton-Jacobi Eq.

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \omega_x^2 x^2 + \omega_y^2 y^2 = 2E$$

need to be solved for S.

- Solve it by method of separation of variables, $S(x, y, \vec{J}) = S_x(x, \vec{J}) + S_y(x, \vec{J})$

$$\left(\frac{\partial S_x}{\partial x}\right)^2 + \omega_x^2 x^2 = 2E - \left(\frac{\partial S_y}{\partial y}\right)^2 - \omega_y^2 y^2 = \kappa^2(\vec{J}) \geqslant 0$$

$$\kappa^2 = \left(\frac{\partial S_x}{\partial x}\right)^2 + \omega_x^2 x^2 = p_x^2 + \omega_x^2 x^2$$

$$S_x(x, \vec{J}) = \kappa \int_0^x dx \epsilon \sqrt{1 - \frac{\omega_x^2 x^2}{\kappa^2}}$$

where $\epsilon = \pm 1$ so $S_x(x, \vec{J})$ increases monotonically along a path over the orbital torus and $x = -\frac{\kappa}{\omega_x}\cos\psi$.

- This gives $S_x(x, \vec{J}) = \frac{\kappa^2}{\omega_x^2}\int d\psi \sin^2\psi$

$$\Rightarrow \boxed{S_x(x, \vec{J}) = \frac{\kappa^2}{2\omega_x}\left(y - \frac{1}{2}\sin 2\psi\right)}$$

- $p_x = \frac{\partial S}{\partial x} = \epsilon\kappa\sqrt{1 - \frac{\omega_x^2 x^2}{\kappa^2}} = \kappa \sin\psi$, so both x and p_x return to their old values when ψ increase by 2π incrementing ψ by 2π carries us around the path with γ_x that is associated with J_x.

$$J_x = \frac{\Delta S}{2\pi} = \frac{\Delta S_x}{2\pi} = \frac{\kappa^2}{2u_x} = \frac{p_x^2 + \omega_x^2 x^2}{2\omega_x}$$

Similarly,

$$J_y\left(y, p_y\right)=\frac{2E-\kappa^2}{2\omega_y}=\frac{p_x^2+\omega_y^2 y^2}{2\omega_y}$$

$$H=H(\vec{J})=\omega_x J_x+\omega_y J_y$$
$$\rightarrow \Omega_x=\frac{\partial H}{\partial J_x}=\omega_x, \quad \Omega_y=\frac{\partial H}{\partial J_y}=\omega_y$$

- As $J_x=\frac{1}{2}\omega_x X^2$ and $\psi=-\sqrt{\frac{\omega_x}{2J_x}}x$,

$$\theta_x=\frac{\partial S}{\partial T_x}=\psi-\frac{1}{2}\sin 2\psi+J_x(1-\cos 2\psi)\frac{\partial \psi}{\partial T_x}=\psi \tag{3.217}$$

- For 1-D harmonic Oscillator, $H=J\omega$

$$\boxed{J=\frac{1}{2\omega}\left(p^2+\omega^2 x^2\right)=\frac{H}{\omega}=\frac{E}{\omega}}$$

where $H=J\omega$

- $x=X\cos\left(\omega t+\theta_0\right) \quad P=\dot{x}=-X\omega\sin(\,)\rightarrow\frac{p}{\omega x}=-\frac{\omega X\sin(\,)}{\omega X\cos(\,)}=-\tan\theta$

  where angle variable $\dot{\theta}=\frac{\partial H}{\partial J}=\omega$, $\theta(x,p)=-\tan^{-1}\left(\frac{P}{\omega x}\right)=\omega t+\theta_0$ (See problem 3.33, P272)

STÄCKEL POTENTIALS

- Stäckel Potential is very important and special in galactic dynamics.
  - All integrals of motion are analytical function of $(\vec{x}, \vec{v}), (E, I_i, I_3) = I_2(\vec{x}, \vec{v})$
  - All orbits are regular, no chaotic orbits (or irregular)
  - J_r can be computed in a ID numerical integral.
- Example: non-rotating 2D bar potential
  - constant u: ellipses
  - constant v: hyperbola

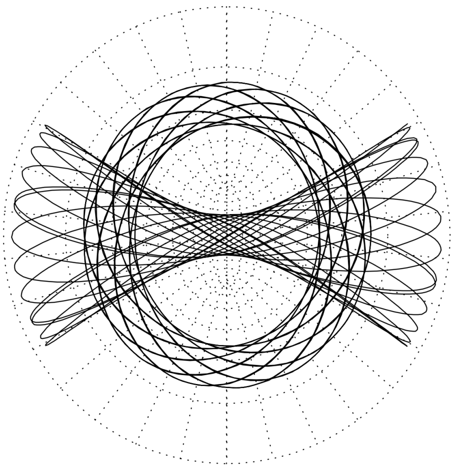
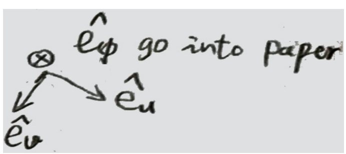


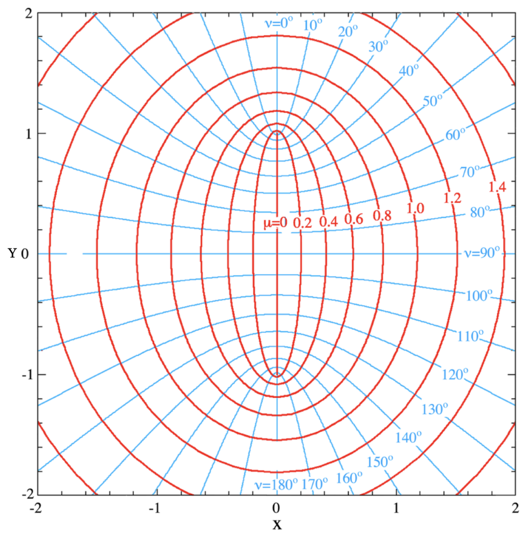
Figure 3.30 The boundaries of loop and box orbits in barred potentials approximately coincide with the curves of a system of spheroidal coordinates. The figure shows two orbits in the potential Φ_L of equation (3.103), and a number of curves on which the coordinates u and v defined by equations (3.267) are constant.

- Oblate potential $\leftrightarrow$ prolate spheroidal coordinate system.
- prolate spheroidal coordinates

  $x(R) = \Delta \sinh u \sin v$

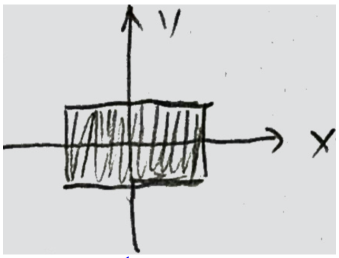
  $y(z) = \Delta \cosh u \cos v$
- $v = 0$, $x = 0$, +y-axis is minor axis of Φ





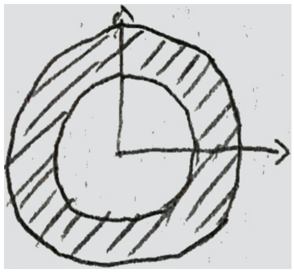
- What is Stäckel potential? First consider two very special cases.

1.2D harmonic oscillator (Lissajours figure)



$$H_x = \frac{1}{2}\left(p_x^2 + \omega_x^2 x^2\right)$$
$$H_y = \frac{1}{2}\left(p_y^2 + \omega_y^2 y^2\right)$$

2. planar orbits in an axisymmetric potential Φ



– rectangle bounded by (constant x, y)

– annulus bounded by (constant R)

- Common points
  - Boundaries of orbits are $v_x = 0$ or $v_y = 0$ (constant x, y), $v_r = 0$; Boundary reflects the symmetry of the coordinate system.
  - Momenta in the coordinate system can be written as a function of one variable only

$$\begin{aligned} p_x &= p_x(x) \\ &= \pm\sqrt{2E_x - \omega_x^2 x^2} \\ p_y &= p_y(y) \\ &= \pm\sqrt{2E_y - \omega_y^2 y^2} \end{aligned}$$

$$\begin{aligned} p_r &= p_r(r) \\ &= \pm\sqrt{2(E - \Phi(r)) - \frac{L_z^2}{r^2} p_\phi} \\ p_\phi &= L_z \end{aligned}$$

  - These expressions arise by slitting up a function of coordinates $\times H$ into two parts, each of which is a function of only one coord.

$$\begin{aligned} H &= \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2\right) \\ &= E_x\left(x, V_x\right) + E_y\left(x, v_y\right) \end{aligned}$$

$$r^2 H = r^2\left[\frac{1}{2}p_r^2 + \Phi(r)\right] + \frac{1}{2}p_\phi^2$$

- For a non-rotating planar bar, we look for a coordinate system whose coordinate curves run parallel to the edge of box and/or loop orbits
$\Rightarrow$ spheruidal coordinate system (u, v, ϕ)

- Stäckel potential has the form:

$$\boxed{\Phi(u,v) = \frac{U(u) - V(v)}{\sinh^2 u + \sin^2 v}}$$

$$x = \Delta \sinh u \sin v \qquad (x,y) \leftrightarrow (u,v)$$
$$y = \Delta \cosh u \cos v$$

- First write H in terms of (u, v, p_u, p_v)

$$|\dot{\vec{x}}|^2 = |h_u \dot{u} \hat{e}_u + h_v \dot{v} \hat{e}_v|^2$$
$$= \Delta^2 \left(\sinh^2 u + \sin^2 v\right)\left(\dot{u}^2 + \dot{v}^2\right)$$

(h_u, h_v, h_ϕ) metric tensor. See. Appendix. B2: $ds^2 = h_i^2 dq_i^2$

$$L = \frac{1}{2}|\dot{r}|^2 - \Phi = \frac{1}{2}\Delta^2 \left(\sinh^2 u + \sin^2 v\right)\left(\dot{u}^2 + \dot{v}^2\right) - \Phi(u,v)$$

$$p_u \equiv \frac{\partial L}{\partial \dot{u}} = \Delta^2 \left(\sinh^2 u + \sin^2 v\right) \dot{u}$$

$$p_v \equiv \frac{\partial L}{\partial \dot{v}} = \Delta^2 \left(\sinh^2 u + \sin^2 v\right) \dot{v}$$

$$H\left(u, v, p_u, p_v\right) = p_u \dot{u} + p_v \dot{v} - L = \frac{1}{2}\Delta^2 \left(\sinh^2 u + \sin^2 v\right)\left(\dot{u}^2 + \dot{u}^2\right) + \Phi$$

$$\boxed{H\left(u, v, p_u, p_v\right) = \frac{p_u^2 + p_v^2}{2\Delta^2 \left(\sinh^2 u + \sin^2 v\right)} + \Phi(u,v)}$$

- H does not explicitly depend on time, $\rightarrow H = \text{constant} = E$

- If Stäckel potential $\Phi(u,v) = \dfrac{U(u) - V(v)}{\sinh^2 u + \sin^2 v}$, then wey can rewrite $H = E$ as

$$\underbrace{2\Delta^2 \left[E \sinh^2 u - U(u)\right] - p_u^2}_{\text{left does not depend on v}} = \underbrace{p_v^2 - 2\Delta^2 \left[E^2 \sin^2 v + V(u)\right]}_{\text{right does not depends on u}} = I_2 = \text{constant}$$

$$p_u = \pm \Delta \sqrt{2\left[E \sinh^2 u - I_2 - U(u)\right]} = \pm \Delta \sinh u \sqrt{2\left[E - U_{eff}(u)\right]}$$
$$p_u = \pm \Delta \sqrt{2\left[E \sin^2 v + I_2 + V(v)\right]} = \pm \Delta \sin v \sqrt{2\left[E - V_{eff}(u)\right]}$$

$$U_{\text{eff}}(u) \equiv \frac{I_2 + U(u)}{\sinh^2 u} \qquad V_{\text{eff}}(v) \equiv -\frac{I_2 + V(v)}{\sin^2 u}$$

- Eliminating E between the above equations, we obtain the integral I_2

$$\boxed{I_2\left(u, v, p_u, p_v\right)=\frac{\sinh^2 u\left(\frac{p_v^2}{2\Delta^2}-V\right)-\sin^2 v\left(\frac{p_u^2}{2\Delta^2}+U\right)}{\sinh^2 u+\sin^2 v}=\text{constant}}$$

- In the limit foci, $\Delta \to 0$ and $u \to w$ (rounder ellipse, more axisymmetric), $\Delta \sinh u \to \Delta \cosh u \to R$ and $v \to \frac{\pi}{2}-\phi$ in (R, ϕ) polar coordinates

$$2\Delta^2 I_2 \to L_z^2$$

The generalized angular momentum $2\Delta^2 I_2 = L_z^2$

- An orbit of specified E and I_2 is confined to values of u and v at which both $U_{\rm eff} \le E$ and $V_{\rm eff} \le E$

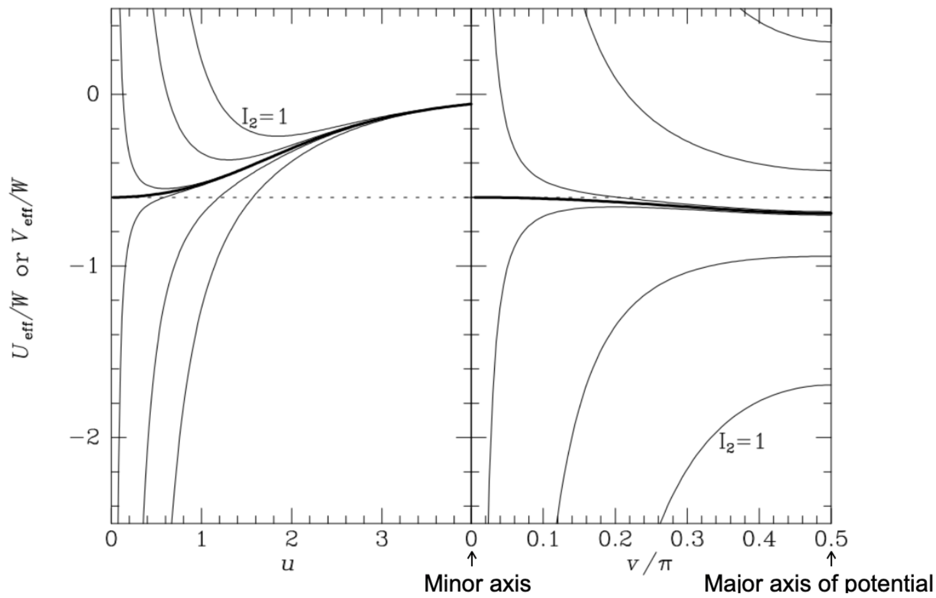


Figure 3.31 The effective potentials defined by equations (3.268b) when U and V are given by equations (3.252). The curves are for $I_2 = 1, 0.25, 0.01, 0, -0.01, -0.25$ and -1, with the largest values coming on top in the left panel and on the bottom in the right panel. The thick curves are for $I_2 = 0$.

- If $U(u)$ and $V(v)$ has the form of the potentail of the perfect prolate spheroid, $\rho(m) = \dfrac{1}{\left(1+\frac{m^2}{a_1^2}\right)^2}$ (Problem 2.14):

$$U(u) = -W \sinh u \tan^{-1}\left(\frac{\Delta \sinh u}{a_3}\right) \qquad V(v) = \omega \sin v \tanh^{-1}\left(\frac{\Delta \sin u}{a_3}\right)$$

  where Δ, W, a_3 are all constants,

$$U_{\rm eff}(u) = \frac{I_2 + U(u)}{\sinh^2 u} \qquad V_{\rm eff}(v) = -\frac{I_2 + V(v)}{\sin^2 v}$$

- $\boxed{I_2 > 0 \quad \text{loop orbits}}$
  - $U_{\rm eff}(u)$ has a minimum, $\rightarrow$ u coordinate has a lower limit $u_{\rm min} < |u| < u_{\rm max}$, elliptical ring.
  - v can cover all the azmuthal range. $0 < v < \pi$

- $\boxed{I_2 < 0 \quad \text{box orbits}}$

  - u coordinate can reach center $0 < |u| < u_{\max}$
  - v can only cover the major axis, but not minor axis. $v > v_{\min}$

- I_2 is the generalized angular momentum

$$2\Delta^2 I_2 \to L_z^2$$

- (u, p_u) surface of section: contour plit of $I_2\,(u, p_u, E)$ with a given E.

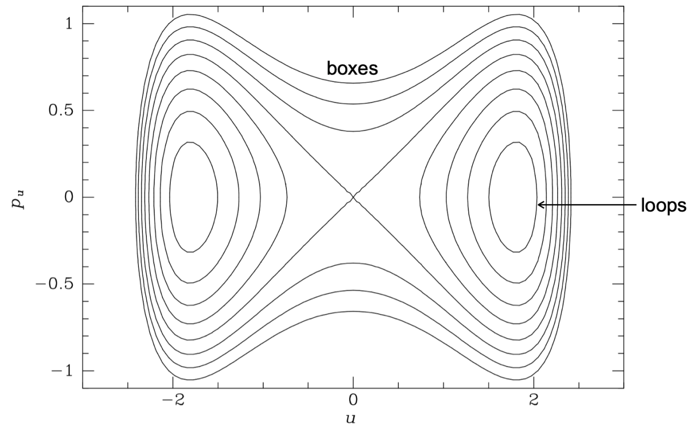


Figure 3.32 The (u, p_u), $v = 0$ surface of section for motion at $E = -0.25$ in the Stäckel potential defined by equations (3.247) and (3.252) with $\Delta = 0.6$ and $a_3 = 1$. Each curve is a contour of constant I_2 (eqs. 3.268). The invariant curves of box orbits ($I_2 = -0.6, -0.4, \ldots$) run round the outside of the figure, while the bull's-eyes at right are the invariant curves of anti-clockwise loop orbits. Temporarily suspending the convention that loops always have $u > 0$, we show the invariant curves of clockwise loops as the bull's-eyes at left.

- $p_u = p_u(u) = \pm\Delta \sinh u \sqrt{2\left[E - U_{eff}(u)\right]}$, redefine $u < 0$ as retrograde loops.

- The resemblance to $x - v_x$ S.O.S. is very clear!

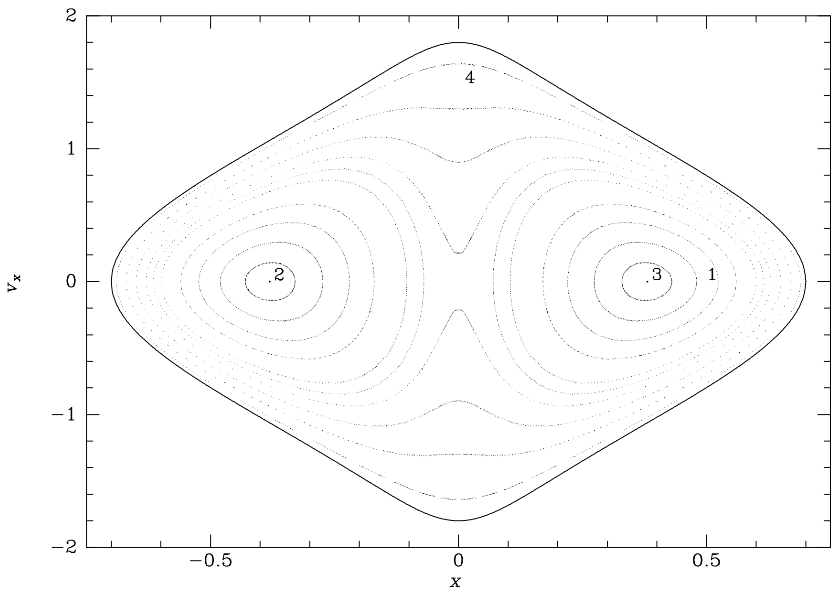


Figure 3.9 The $(x, \dot{x})$ surface of section formed by orbits in Φ_L of the same energy as the orbits depicted in Figure 3.8. The isopotential surface of this energy cuts the long axis at $x = 0.7$. The curves marked 4 and 1 correspond to the box and loop orbits shown in the top and bottom panels of Figure 3.8.

$$J_r = \frac{1}{2\pi}\oint p_u(u)du = \frac{1}{\pi}\int_{u_{min}}^{U_{max}} |p_u(u)|\, du$$

$$J_a = \frac{1}{2\pi}\oint P_v(v)dv = \frac{1}{\pi}\int_{V_{\min}}^{V_{\max}} |p_v(v)|\, dv$$

- Actions = the areas inside the invariant curves in which the tori puncture the surfaces of section $v = \mathrm{constant}$ and $u = \mathrm{constant}$.

STÄCKEL POTENTIAL FOR A FLATTENED AXISYMMETRIC POTENTIAL

- (E, L_z) conserved.
- Loop orbits that have vertical extension in the meridional (R, z) plane.
- prolate spheroidal coordinates in the (R, z) plane

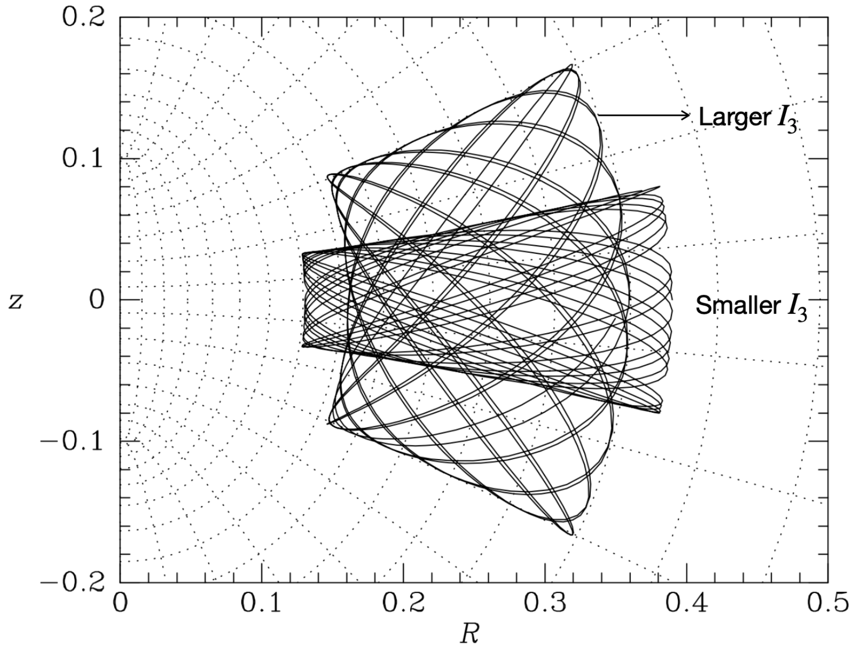


Figure 3.27 The boundaries of orbits in the meridional plane approximately coincide with the coordinate curves of a system of spheroidal coordinates. The dotted lines are the coordinate curves of the system defined by (3.242) and the full curves show the same orbits as Figure 3.4.

- the (u, v) coordinate system defined by

$$R = \Delta \sinh u \sin v \qquad z = \Delta \cosh u \cos v$$

- H does not depend on t and $\phi \to H = \mathrm{constant}$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$$

$$p_\phi = L_z = \Delta^2 \sinh^2 u \sin^2 v \dot{\phi}$$
$$= \mathrm{constant}$$

$$H\left(u,v,p_u,p_v,p_\phi\right)=\frac{p_u^2+p_u^2}{2\Delta^2\left(\sinh^2 u+\sin^2 v\right)}+\frac{p_\phi^2}{2\Delta^2 \sin h^2 u \sin^2 u}+\Phi(u,v)$$
(Eq. 3.246)

$$\Phi(u,v)=\frac{U(u)-V(v)}{\sinh^2 u+\sin^2 v}=\frac{u'(\lambda)-v'(\mu)}{\lambda-\mu}$$

where $\lambda=\Delta^2\sinh^2 u$, $\mu=-\lambda^2\sin^2 v$ for prolate or $\mu=-\Delta^2\cos^2 v$ for oblate,

- Stäckel potential logarithmic Φ_L admit I_3, but it is not of Stäckel form.

$$p_u=\pm\sqrt{2\Delta^2\left[E\sinh^2 u-I_3-U(u)\right]-\frac{L_{z^2}}{\sinh^2 u}}$$

$$p_v=\pm\sqrt{2\Delta^2\left[E\sin^2 u+I_3+V(v)\right]-\frac{L_z^2}{\sin^2 v}}$$

∗ "Stäckel fudge"

we can find a system of spheroidal coordinates that approximately bounds any given orbit, but in general different orbits require different spheroidal coordinate systems.

- Eliminating E.

$$I_3\left(u,v,p_u,p_v,p_\phi\right)=\frac{1}{\sinh^2 u+\sin^2 v}\times$$
$$\left[\sinh^2 u\left(\frac{p_v^2}{2\Delta^2}-V\right)-\sin^2 v\left(\frac{p_u^2}{2\Delta^2}+U\right)\right]+\frac{p_\phi^2}{2\Delta^2}\left(\frac{1}{\sin^2 v}-\frac{1}{\sinh^2 u}\right)$$

Problem 3.39

- $I_3\approx$ vertical/radial energy, $I_3\downarrow$, more vertical energy is transferred into the energy of radial oscillation.

$$J_u=\frac{1}{\pi}\int_{u_{min}}^{U_{\max}}du p_u(u)\qquad J_v=\frac{1}{\pi}\int_{v_{\min}}^{v_{\max}}p_v(v)dv\qquad J_\phi=L_z$$

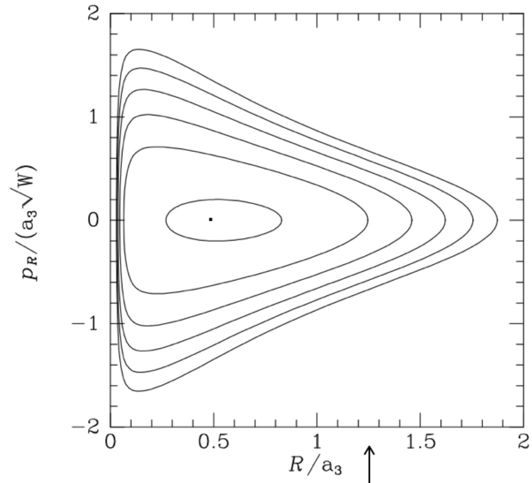
Still need numerical ID integral! even though I_3 or I_2 is known analytically.

- If $U(v)$ and $V(v)$ has the same form as the perfect prolate ellipsoid.

$$H\left(R, z, p_R, p_z\right) = \text{constant}$$

- Each conserved integral confine the motion in phase space to one dimension lower

- Addition integral $I_2 \leftrightarrow$ torus



center: shell orbit

Figure 3.29 Surface of section at $E = -0.5W$ and $L_z = 0.05a_3\sqrt{W}$ constructed from equations (3.249) and (3.252) with $\Delta = 0.6a_3$.

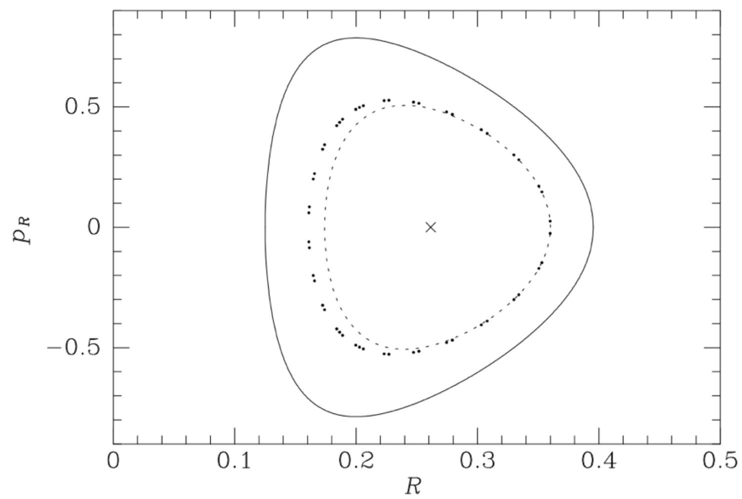


Figure 3.5 Points generated by the orbit of the left panel of Figure 3.4 in the (R, p_R) surface of section. If the total angular momentum L of the orbit were conserved, the points would fall on the dashed curve. The full curve is the zero-velocity curve at the energy of this orbit. The $\times$ marks the consequent of the shell orbit.

- A Stäckel potential: is specified by for two or three arbitrary functions of one variable.
- Few galactic potentials are of Stäckel's form, since ρ, therefore Φ, are specified by arbitrary function of three variables.
- Why have we devoted so much space to Stäckel potentials?
  Because orbits in these potentials turn out to have the same structure as the much more general class of regular orbits, which admit as many isolating integrals as they have spatial dimensions.
- Any 3-D orbit admitting 3 integrals (H, I_2, I_3), forms a 3- torus in phase space (Arnold 1978)

LABEL THE TORI

- We have seen that the basic structures in phase space are orbital tori. How should we label the tori?
- The obvious labels are E and I_2. For $2D$ spatial dimension (cuntour levels in S.O.S.). Why actions, which are the areas inside the invariant curves in S.O.S. are better labels than E and I_2 ? they are harder to compute!

  (1) actions alone enjoy the properties of adiabatic invariants.

  (2) more importantly, we need coordinates to tell us where we are on any given torus. The actions, unlike H and I_2, generate beautiful coordinates for the tori, the so-called angle variables.

  - $\theta_i = \Omega_i t + \theta_i(\theta)$ $\quad \theta_i$ can $\rightarrow \omega$, but x_i cannot
  - We can scale θ_i so that $\vec{x}$ returns to its original value after θ_i has increase by 2π. Our paths form a near grid on the surface of the torus, and θ_i can be used as coordinates on the torus.
  - If we try to repeat this construction with H and J_2 replacing J_i the paths/curves we generate would not close on themselves, and we would not obtain a coordinate system valid everywhere on the torus.

SLOWLY VARYING POTENTIALS

- Adiabatic: potential variations are slow compared to a typical orbital frequency.
- Actions are adiabatic invariants (Proof is in §3.6.1)

  Exception: J_i. with $\Omega_i = 0$ is not an adiabatic invariant.
- BUT How slow is slow?

  Potential does not have to change very slowly for the actions to be well conserved.
- Applications:

$$\frac{\Delta J}{J} \propto e^{-\omega_0 T} \quad \omega_0 T = \frac{T}{T_0} \cdot 2\pi \gg 1$$

T is typical timescale of potential change. If the at the timescale off T_0 it is slow enough to be considered adiabatic.

(1) Hamonic oscillatior. $\Phi = \frac{1}{2}\omega^2 x^2$

$$J = \frac{1}{2\omega}\left(p^2 + \omega^2 x^2\right) = \frac{H}{\omega} = \frac{E}{\omega}$$

– $H(x,p) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2 = \frac{1}{2}\omega^2 x^2$, if $x(t) = X\cos(\omega t + \phi)$

– $J = \frac{H}{\omega} = \frac{1}{2}\omega X^2$, if ω is increased to $\omega' = s\omega$ adiabatically, $X' = \frac{X}{\sqrt{s}}$ decreases.

– Note $J = \mathrm{constant}$, but $E' = \omega' J = s\omega J = sE$ not conserved.

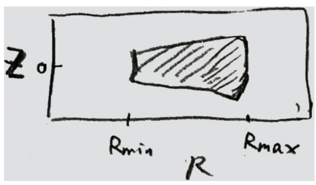
(2) Eceentric orbit $\ddot{z} = -\nu^2 z \quad \nu(t) = \left(\frac{\partial^2\Phi}{\partial z^2}\right)^{1/2}_{(R(t),0)} = \sqrt{\Phi_{zz}(R,0)}$

– $\frac{\nu}{\kappa} \sim 2$ so change in R can be considered as much slower (“adiabatic”) than vertical.

$$z(R) = z\left(R_0\right)\left(\frac{\Phi_{xx}\left(R_0,0\right)}{\Phi_{tz}(R,0)}\right)^{1/4}$$

where R_0 is the guiding center’s radius.

– At $R_{\max}/R_{\min}$, the vertical restoring force is weaker/stronger, $\to$ $|\Delta z|$ is larger/smaller



(3) Transient perturbations

- slowly distorting potential in some arbitrary fashion. (P241)
- slowly varying external perturbation of a stellar system, even strong perturbations.

EPICYCLE APPROXIMATION

- $\Phi_{\text{eff}} = \Phi + \frac{L_z^2}{2R^2} = \underbrace{\Phi\left(R_g\right) + \frac{L_z^2}{2R_g^2} + \frac{1}{2}\kappa x^2}_{\Phi(R)=E_c\left(J_\phi\right)+\frac{1}{2}\kappa x^2} + \frac{1}{2}\nu z^2$

- $H_R = \frac{1}{2}\left(\dot{x}^2 + \kappa^2 \cdot X^2\right) = E_R \quad H_z = \frac{1}{2}\left(z^2 + U^2 z^2\right) = E_z$

  $E_c\left(J_\phi\right) = \Phi\left(R_g\right) + \frac{J_\phi^2}{2R_S^2}$

- Epicycle approximation

$$\boxed{\begin{aligned} J_R &= \frac{E_R}{\kappa} = \frac{1}{2}\kappa X^2 \\ J_z &= \frac{E_z}{\nu} = \frac{1}{2}\nu Z^2 \\ J_\phi &= L_Z = R_g^2\Omega_g \end{aligned}}$$

$$x = R - R_g = X\cos\theta_R = -\overbrace{\sqrt{\frac{2J_R}{\kappa}}}^{=X}\cos\theta_R$$

$$\theta_R = \kappa_t + \alpha + \pi$$

$$z = Z_1\cos\theta_z = -\sqrt{\frac{2Jz}{\nu}}\cos\theta_z$$

$$\theta_\phi = \phi - \frac{\gamma}{R_g}\sqrt{\frac{2J_R}{\kappa}}\sin\theta_R - \underbrace{\frac{J_R}{2}\frac{d\ln x}{dJ_\phi}\sin 2\theta_R}_{} \tag{2.65}$$

A correction to $\dot{\phi} = \frac{L_z}{R^2} \approx \Omega_g\left(1 - \frac{2x}{R_g}\right)$

to make $\left(\theta_R, \theta_\phi, J_R, J_\phi\right)$ canonical.

- Compared to: $\phi = \Omega_g t + \phi_0 - \gamma\frac{X}{R_g}\sin(\kappa t + \alpha)$

- $\boxed{\Omega_\phi = \dot{\theta}_\phi \neq \Omega_g! \quad \text{when } J_R \neq 0}$

- $H = E_R + E_c + E_z, \quad E_R = J_R\kappa$

$$E_c\left(J_\phi\right) = \Phi(R_g) + \frac{J_\phi^2}{2R_\delta^2}, \quad \frac{dE_c}{dJ_\phi} = \frac{2J_\phi}{2R_g^2} = \frac{2R_g^2\Omega_g}{2R_g^2} = \Omega_g$$

$$\Omega_\phi = \frac{\partial H}{\partial J_\phi} = \frac{d\kappa}{dJ_\phi}J_R + \Omega_g \neq \Omega_g$$

- Note $J_\kappa = \text{constant} = \frac{1}{2}\kappa X^2 = \frac{1}{2}X^2 \cdot \Omega \cdot \left(\frac{\kappa}{\Omega}\right) \xlongequal{\Omega=\frac{L}{R^2}} \frac{1}{2}L^2\left(\frac{X}{R_g}\right)^2\left(\frac{\kappa}{\Omega}\right)$, so if $\frac{\Omega}{\kappa}$ associated with a nearly circular orbit is unchanged by a slow variation of the potential.

  $\rightarrow$ $\boxed{\text{oritial eccentricity}} = \frac{R_{\max} - R_{\min}}{R_{\max} + R_{\min}} \approx \boxed{\frac{X}{R_g} \text{ is adiabatic invariant}}$