Homework Set 3
"Galactic Dynamics" course at SJTU

Due Date: April 30, 2020

Warning: please start working on these problems as early as possible.

1. (5 points) Prove that at any point in an axisymmetric system at which the local density is negligible, the epicycle, vertical, and circular frequencies κ, ν, and Ω are related by $\kappa^2 + \nu^2 = 2\Omega^2$.

2. (5 points) Prove that circular orbits in a given potential are unstable if the angular momentum per unit mass on a circular orbit decreases outward. Hint: evaluate the epicycle frequency.

3. (10 points) [BT08, Problem 3.10 (b) and 3.16] $\Delta\psi$ denotes the increment in azimuthal angle during one complete radial cycle of an orbit. (a) Prove in the epicycle approximation that along orbits in a potential with circular frequency $\Omega(R)$,

$$\Delta\psi = 2\pi(4 + \frac{\mathrm{d}\ln\Omega^2}{\mathrm{d}\ln R})^{-1/2}.$$

(b) Using the epicycle approximation, prove that the azimuthal angle $\Delta\psi$ between successive pericenters lies in the range $\pi \leq \Delta\psi \leq 2\pi$ in the gravitational field arising from any spherical mass distribution in which the density decreases outwards.

4. (10 points) [BT08, Problem 3.25] Consider two point masses m_1 and $m_2 > m_1$ that travel in a circular orbit about their center of mass under their mutual attraction. (a) Show that the Lagrange point L_4 of this system forms an equilateral triangle with the two masses. (b) Show that motion near L_4 is stable if $m_1/(m_1 + m_2) < 0.03852$. (c) Are the Lagrange points L_1, L_2, L_3 stable?

5. (10 points) Prove that the orbits in Figure 1(a) and Figure 1(b) are impossible in a spherical potential (they are the cover page of the

internationally renowned textbook *Classical Mechanics* (Goldstein et al. 2001), and Figure 3.13 in their book). You may use the curvature of a plane curve $R(\phi)$ in polar coordinates:

$$\kappa = \frac{R^2 + 2(R')^2 - RR''}{[R^2 + (R')^2]^{3/2}},$$

where$R' = \mathrm{d}R/\mathrm{d}\phi$. The local radius of curvature is κ^{-1}. Prove that the curvature of an orbit with energy E and angular momentum L in the spherical potential $\Phi(r)$ is

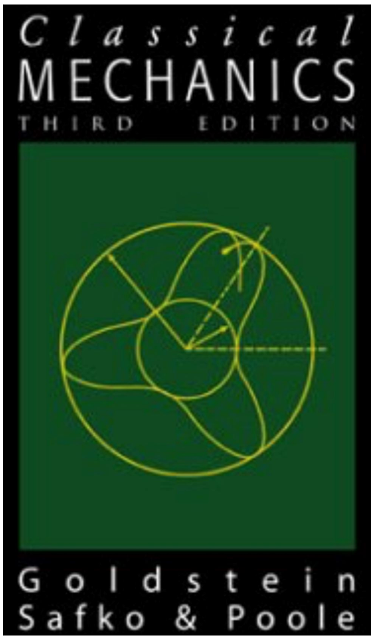
$$\kappa = \frac{L\mathrm{d}\Phi/\mathrm{d}r}{r[2(E - \Phi(r))]^{3/2}}.$$

Hint: use Equation 3.10 in BT08.

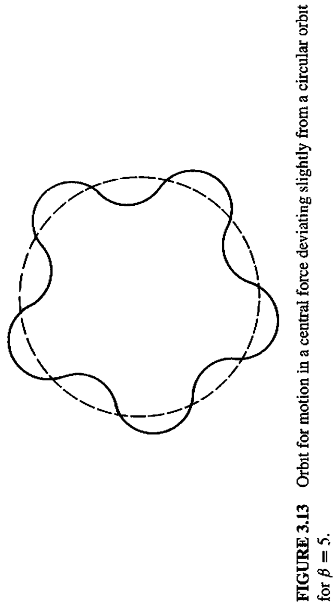
6. (10 points) The Hipparcos satellite has enabled us to determine accurate velocities for nearby stars, so that we can plot the density of nearby stars in velocity space. Let us set up a rotating coordinate system (x, y, z) centered on the present location of the Sun, such that the positive x-axis points radially inward, positive y-axis points towards the rotation direction of the Sun, and the positive z-axis points towards the North Galactic Pole. Let $U = \dot{x}$ and $V = \dot{y}$, and plot the distribution on the (U, V) plane of all stars whose distance from the Sun is $< d_{max}$.

   This distribution turns out to be lumpy rather than smooth (Figure 2, also see Figure 10.15 in the textbook *Galactic Astronomy*, Binney & Merrifield, 1998). These lumps are called moving groups and are commonly believed to consist of a set of stars that were born in a small region with very similar velocities, perhaps as part of a cluster that later dissolved. (This interpretation is not necessarily correct, by the way).

   In the epicycle approximation, a simple model of a moving group is a set of stars with the same guiding center radius R_g and the same radial epicycle amplitude X, but with a range of epicycle and azimuthal phases (ϕ_0 and α in Equation 3.93a of BT08); this model is intended to reflect the fact that the group spreads out in phase much more rapidly than it spreads out in guiding-center radius or epicycle amplitude. We shall neglect motions perpendicular to the galactic plane and assume that $X \gg d_{max}$. What would be the signature of such a group in the (U, V) plane?



(a) The cover of *Classical Mechanics*



(b) Figure 3.13 in *Classical Mechanics*

Figure 1: The impossible orbits

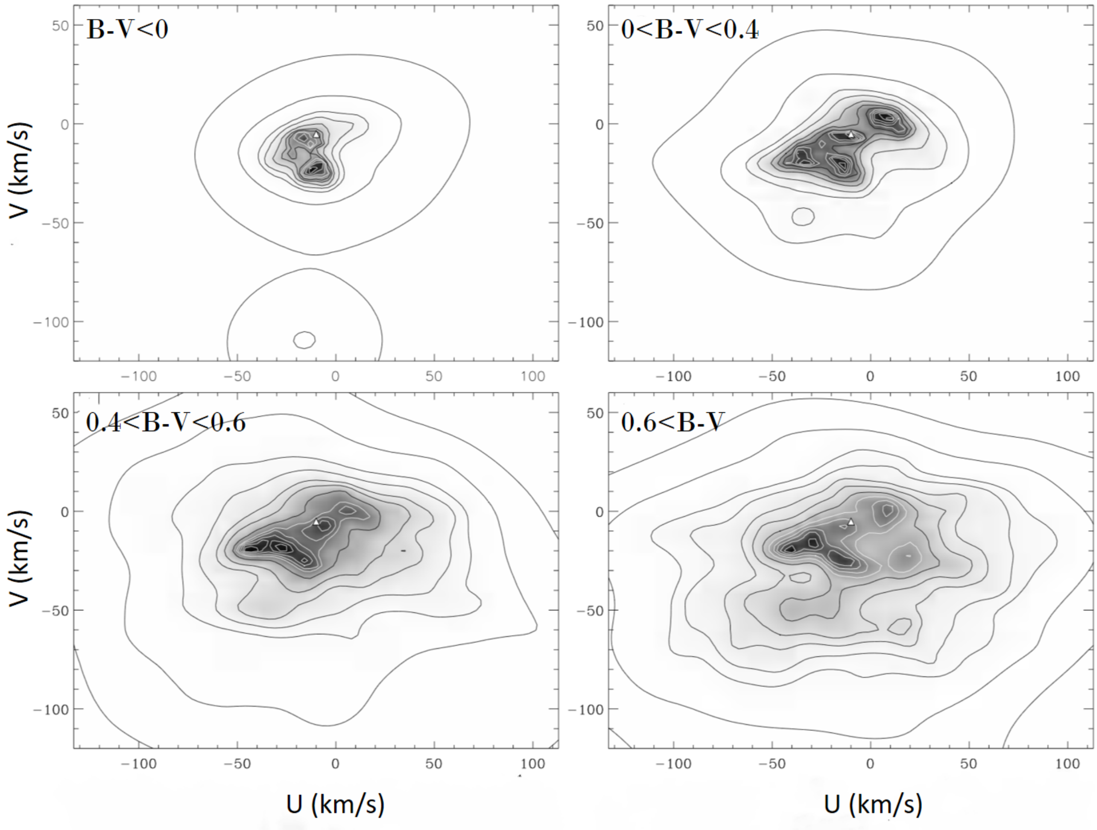


Figure 2: Figure of Problem 6. The density of stars near the Sun in velocity space. Each panel shows the density of the main sequence stars projected onto the (U, V) plane for a different range of $B-V$ color. The Sun's velocity is at $U = V = 0$ and the velocity of LSR is marked by a triangle.