Homework Set 2
"Galactic Dynamics" course at SJTU

Due Date: Wed., April 1

Warning: please start working on these problems as early as possible.

1. (10 points) Stellar collisions. For most luminous ellipticals stellar densities are low and stellar collisions are extremely rare. The possible exceptions are low-luminosity ellipticals like M32. M32 can be approximated as a spherical mass distribution with the density $\rho(r) = 3 \times 10^5 (r/1\ \mathrm{pc})^{-2.4}\,\mathrm{M}_\odot/\mathrm{pc}^3$ (e.g. Lauer et al. 1998, AJ, 116, 2263). Assume that most of this mass is in stars similar to the Sun. Assume that a typical red giant star has radius $10\,\mathrm{R}_\odot$ and lives for 10^8 years. Estimate inside what distance from the center of M32 do most red giants collide with other stars (not necessarily another red giant) during its life time?

2. (10 points) For a sphere of uniform density (Mass = M, radius = R_0).
(a) what is its total potential energy?
(b) Prove that this sphere, if composed of a pressureless fluid with density ρ, will collapse to a point in time $t_{\mathrm{ff}} = \frac{1}{4}\sqrt{3\pi/(2G\rho)}$ when it is released from test. The time t_{ff} is called the **free-fall time** of a system of density ρ.? How does it compare to the time that a test particle takes from any radius to $r = 0$ when released with initial velocity of zero? (Slightly revised from BT08, Problem 3.4)

3. (10 points) [BT08, Problem 2.7] Astronauts orbiting an unexplored planet find that (i) the surface of the planet is precisely spherical and centered on r = 0; and (ii) the potential exterior to the planetary surface is $\Phi = -GM/r$ exactly, that is, there are no non-zero multipole moments other than the monopole. Can they conclude from these observations that the mass distribution in the interior of the planet is spherically symmetric? If not, give a simple example of a non-spherical mass distribution that would reproduce the observations.

4. (15 points) [BT08, Problem 1.2] The **luminosity density** $j(\boldsymbol{r})$ of a stellar system is the luminosity per unit volume at position $\boldsymbol{r}$.

(a) For a transparent spherical galaxy, show that the surface brightness $I(R)$ (Box 2.1) and luminosity density $j(r)$ are related by the formula

$$I(R) = 2\int_R^\infty \mathrm{d}r \frac{r j(r)}{\sqrt{r^2 - R^2}}. \tag{1}$$

(b) What is the surface brightness of a transparent spherical galaxy with luminosity density $j(r) = j_0(1 + r^2/b^2)^{-5/2}$ (this is the Plummer model of §2.2.2c)?
(c) Invert the above equation using Abel's formula (eq. B.72) to obtain

$$j(r) = -\frac{1}{\pi}\int_r^\infty \frac{\mathrm{d}R}{\sqrt{R^2 - r^2}} \frac{\mathrm{d}I}{\mathrm{d}R}. \tag{2}$$

(d) Determine numerically the luminosity density in a spherical galaxy that follows the $R^{1/4}$ surface-brightness law. Plot $\log_{10} j(r)$ versus $\log_{10} r/R_e$, where R_e is the effective radius.

5. (10 points) [BT08, Problem 2.16] Prove that the potential $\Phi(r)$ is a non-decreasing function of r in any spherical system. Does the same conclusion hold in an axisymmetric razor-thin disk? If so, prove it; if not, find a counter-example.

6. (10 points) [BT08, Problem 2.10] Consider an axisymmetric body whose density distribution is $\rho(R, z)$ and total mass is $M = \int \mathrm{d}^3\boldsymbol{r}\rho(R, z)$. Assume that the body has finite extent, $\rho(R, z) = 0$ for $r^2 = R^2 + z^2 > r_{\mathrm{max}}^2$, and is symmetric about its equator, that is, $\rho(R, -z) = \rho(R, z)$.

   (a) Show that at distances large compared to r_{max}, the potential arising from this body can be written in the form

$$\Phi(R, z) \simeq -\frac{GM}{r} - \frac{G}{4}\frac{R^2 - 2z^2}{r^5}\int \mathrm{d}^3\boldsymbol{r'}\rho(R', z')(R'^2 - 2z'^2), \tag{3}$$

   where the error is of order $(r_{\mathrm{max}}/r)^2$ smaller than the second term.
   (b) Show that at large distances from an exponential disk with surface density $\Sigma(R) = \Sigma_0 \exp(-R/R_d)$, the potential has the form

$$\Phi(R, z) \simeq -\frac{GM}{r}[1 + \frac{3R_d^2(R^2 - 2z^2)}{2r^4} + \mathrm{O}(R_d^4/r^4)], \tag{4}$$

   where M is the mass of the disk.

7. (10 points) [BT08, Problem 2.11] Show that the potential energy of an exponential disk is $W \simeq -11.627\ G\Sigma_0^2 R_d^3$. Show further that if all stars move on circular orbits, the disk's angular momentum is

$J \simeq 17.462\ G^{1/2}\Sigma_0^{3/2}R_d^{7/2}$ and its kinetic energy is $K \simeq 5.813\ G\Sigma_0^2 R_d^3$. Hence show that for this disk the dimensionless spin parameter $\lambda \equiv J|E|^{1/2}/GM^{5/2} \simeq 0.4255$, where $E = K + W$ is the total energy.

8. (10 points) The Gaussian disk. A razor-thin disk has surface density $\Sigma(R) = \Sigma_0 \exp\bigl(-R^2/2a^2\bigr)$. Compute its potential $\Phi(R)$ in the disk plane.