Basic Counting and Binomial Coefficients

1 Binomial Coefficients

1.1 Definition

The number of r-combinations of a set with n distinct elements (with $r \leqslant n$) is denoted by C(n,r). It is also denoted by:

$$\binom{n}{r}$$

and is called a **binomial coefficient**

"n choose r" refers to counting the number of different subsets of r elements from a set of n elements, so by literally choosing r elements from n elements in all possible ways

1.2 Example

How many National Lottery combinations are there?

$$\binom{49}{6} = \frac{49!}{6!43!} = \frac{49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = 13,938,816$$

1.3 A simple property of binomial coefficients

1.3.1 Proposition

For any integers $0 \leqslant k \leqslant n$

$$\binom{n}{k} = \binom{n}{n-k}$$

1.3.2 Proof

- Selecting k objects out of n is the same as leaving out k-n elements
- LHS counts the number of ways to select k
- RHS counts the number of ways to leave n-k out

2 Pascal's identity

2.1 Theorem

Let n and k be positive integers with $n \geqslant k$. Then:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

2.2 Proof

2.2.1 Direct Proof

$$
\begin{aligned}
\binom{n}{k-1}+\binom{n}{k} &= \frac{n!}{(k-1)!(n-(k-1))!}+\frac{n!}{k!(n-k)!}= \\
&= \frac{n!}{(k-1)!(n+1-k))!}+\frac{n!}{k!(n-k)!}= \\
&= \frac{k\cdot n!}{k\cdot(k-1)!(n+1-k))!}+\frac{(n+1-k)\cdot n!}{(n+1-k)\cdot k!(n-k)!}= \\
&= \frac{k\cdot n!}{k!(n+1-k))!}+\frac{(n+1-k)\cdot n!}{k!(n+1-k)!}= \\
&= \frac{k\cdot n!+(n+1-k)\cdot n!}{k!(n+1-k)!}=\frac{k\cdot n!+(n+1)\cdot n!-k\cdot n!}{k!(n+1-k)!}= \\
&= \frac{(n+1)!}{k!(n+1-k)!}=\binom{n+1}{k}.
\end{aligned}
$$

2.2.2 Combinatorial Proof

Pascal's Identity:

$$\binom{n+1}{k}=\binom{n}{k-1}+\binom{n}{k}$$

Show that both sides of the identity count the same things in a different way:

- The left hand side counts all the possible subsets of k elements from a set of n+1 elements
- Fix one element x. The right hand side counts:
  - All the possible k subsets containing x (we have to choose another k-1 elements from the other n elements) plus,
  - All the possible k subsets not containing x (we still have to choose k elements from the other n elements)

3 Pascal's Triangle and Binomial Coefficients

$$
\begin{array}{lccccccccccccc}
n=0: & & & & & & & \binom{0}{0} & & & & & & \\
n=1: & & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
n=2: & & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
n=3: & & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
n=4: & & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
n=5: & & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
n=6: & \binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}
$$

$$
\begin{array}{lccccccccccccc}
n=0: & & & & & & & 1 & & & & & & \\
n=1: & & & & & & 1 & & 1 & & & & & \\
n=2: & & & & & 1 & & 2 & & 1 & & & & \\
n=3: & & & & 1 & & 3 & & 3 & & 1 & & & \\
n=4: & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
n=5: & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
n=6: & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

4 Binomial Theorem

4.1 Theorem

Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + ... + \binom{n}{n-2}x^2y^{n-2} + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

4.2 Examples

$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

The coefficients are rows in Pascal's triangle

4.3 Idea of the proof

- In the expansion of $(x+y)^n$ we obtain the value of "n choose k" as the coefficient of the term $x^{n-k}y^k$
- This is because one has to choose a y from k of the n brackets x+y in the expansion; the other n-k are all x

4.4 Examples

Prove that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + ... + \binom{n}{n} = 2^n$$

Solution: Use the binomial formula with $x = y = 1$

5 Permutations with indistinguishable objects

5.1 Theorem

The number of different permutations of n objects, of which there are
n_1 indistinguishable objects of type 1
n_2 indistinguishable objects of type 2
...,
n_k indistinguishable objects of type k, is

$$\frac{n!}{n_1!n_2!...n_k!}$$

As with anagrams

- There are $C(n, n_1)$ choices to put n_1 type 1 objects in n positions
- Then $C(n-n_1, n_2)$ choices t put n_2 type 2 objects in the remaining $n - n_1$ positions, and so on

The total number is:

$$C(n, n_1) \cdot C(n-n_1, n_2), ..., C(n - n_1 - n_2 - ...n_{k-1} - n_k) = \frac{n!}{n_1!n_2!...n_k!}$$

6 Combinations with Indistinguishable Objects

Multiset - A set in which each element a has a multiplicity (times they appear) n_a
Example: [0,0,1,2,2,2]=[1,0,2,2,0,2] is a multiset of size 6, with $n_0 = 2$, $n_1 = 1$, $n_2 = 3$ An r-combination of n objects with repetition is a multiset of size r whose elements come from these n objects

Using the idea with bars and stars, one can show the following:

6.1 Theorem

The number of different r combinations of n objects with repetitions is $C(n+r-1, n-1) = C(n+r-1, r)$

7 Additional Slides

8 Distributing Objects into Boxes

How many ways are there to distribute n objects into k boxes?
The answer depends of whether the objects/boxes are distinguishable

- D objects into D boxes
  - Ex. Dealing 52 cards to 4 people
- inD objects into D Boxes
  - Identical balls into numbered boxes
- D objects into inD Boxes
  - Numbered balls into identical boxes
- inD objects into inD Boxes
  - Identical balls into identical boxes

8.1 Objects into D-Boxes

D-Objects into D-Boxes: numbered balls into numbered boxes

8.1.1 Theorem

The number of ways to distribute n D-Objects into k D-boxes so that box i receives n_i objects is:

$$\frac{n!}{n_1!n_2!,...,n_k!}$$

inD objects into D boxes: Identical balls into numbered boxes

8.1.2 Theorem

The number of ways to distribute n inD objects into k D-Boxes is

$$C(n+k-1,k)$$

8.2 Objects into inD-Boxes

D-Objects into inD-Boxes: n numbered balls into k identical boxes
Same as counting partitions of the balls into $\leqslant$ k parts

8.2.1 Fact

A simple closed formula for the number of ways to distribute n D-Objects into k inD-Boxes is
NOT KNOWN **D-Objects into inD Boxes: n identical balls into k identical boxes**

8.2.2 Fact

A simple closed formula for the number of ways to distribute n inD-Objects into k inD boxes is:
NOT KNOWN