Mathematics for Computer Science
Discrete Maths and Linear Algebra

Lecture 1: Mathematical Induction

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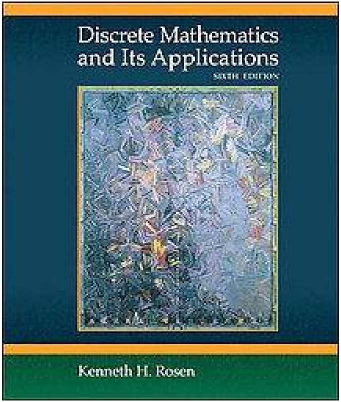
Timetable and organisation

- Lectures:
  - One lecture each week in Terms 1 and 2; revision lectures in Term 3
  - Always in this room at this time
- Practicals:
  - Demonstrators: Chris Lindop, Siani Smith, Giacomo Paesani, and Sam Hunt
  - Even-numbered teaching weeks
  - Practical group allocation is on DUO

Timetable and organisation

- Slides and exercises:
  - All obtainable from DUO: http://duo.dur.ac.uk;
  - The lecture slides will be put on DUO 1-2 days before the lectures;
  - The exercises will be put on DUO prior to the practical classes;
  - The answers will appear after the classes.
- Assessment:
  - One combined summative assignment (aka coursework) for both submodules, worth 34% of the final mark;
  - The hand-in deadline is Wednesday 12th December, check the time on DUO;
  - The assignment will be put on DUO (at least) two weeks before that;
  - Exam at the end of the year.

Recommended textbook (for the first 12 lectures)



Kenneth H. Rosen, Discrete Mathematics and Its Applications (6th Edition), McGraw-Hill, 2007.

Feedback and Advice

If you have any problem with the module or the way it is delivered (e.g. too difficult, too easy, too fast, too slow), ask for an explanation or give me feedback:

- During lectures;
- Before or after lectures;
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Important things:

- To know what's on the slides is not enough - attend lectures!
- This course has a lot of material - engage/revise during the year!
  - Impossible to revise all of it just before the exam!
- Pay attention even you think you already know the material!
- Use the recommended books for additional explanation, examples, etc!

Overview of this submodule

Term 1

- Combinatorics: Counting Principles
- Probability Theory
- Graph Theory
- Proof techniques

Term 2

- Number Theory
- Linear Algebra

Examples relevant to CS will appear throughout

Contents for today's lecture

- Proofs by induction
- Examples
- Exercises

Exercises and introduction

Suppose we want to prove that the following statements are valid for every positive integer n:

- $n < 2^n$
- $1 + 2 + \ldots + n = n(n+1)/2$
- $n^3 - n$ is divisible by 3

Although these three cases look very different, there is a general approach to prove such statements, called a proof by induction.

See Rosen, Chapter 4.

The general principle of a proof by induction

Suppose we want to prove that a given statement $S(n)$ holds for all integers $n \geq j$ for a fixed integer j.

The simplest proof by induction works as follows (if it works):

- Step 1 (Basis Step): Check that $S(n)$ is true for $n = j$; If this is not the case, then the statement cannot be true. If $S(j)$ is true, then proceed to Step 2.
- Step 2 (Induction Step): Prove the following conditional statement.
  If $S(n)$ holds for a fixed value $n = k \geq j$ (Ind. Hypothesis, or Assumption) then it also holds for $n = k + 1$.

The two steps together then imply that $S(n)$ holds for $n = j$ (by Step 1), for $n = j + 1$ (by Step 1 and Step 2 applied for $k = j$), for $n = j + 2$ (by Step 2 applied for $k = j + 1$), and so on, so it holds for all $n \geq j$.

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  - Need to use that to derive the statement for $n = k + 1$, that is, $k + 1 < 2^{k+1}$.
  - We have

$$k + 1 < 2^k + 1 < 2^k + 2^k = 2^{k+1}.$$

    The first inequality above is by the inductive assumption.
- Since the statement is valid for $n = 1$, and that it is valid for $n = k + 1$ if it is valid for $n = k$, we conclude that it is **valid for all positive integers** n.

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- Induction Step: Let $k \geq 1$ be an integer.
  - Assume that the statement holds for $n = k$, that is, $k^3 - k = 3m$ for some integer $m \geq 0$.
  - Need to use that to derive the statement for $n = k + 1$, that is, $(k+1)^3 - (k+1) = 3m'$ for some integer $m' \geq 0$.

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  - We have

    $$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - (k+1) = k^3 - k + 3k^2 + 3k = 3\cdot(m + k^2 + k)$$

    The last equality above is by the inductive assumption.
- Since both the basis and the induction step are completed, we conclude that the statement is valid for all positive integers n.

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$$1 + 2 + \ldots + k + (k+1) = (k+1)k/2 + (k+1) = (k+2)(k+1)/2$$

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- Since we know that the statement is valid for $n = 1$, and that it is valid for $n = k+1$ if it is valid for $n = k$, we conclude that it is valid for all positive integers n.

Variations

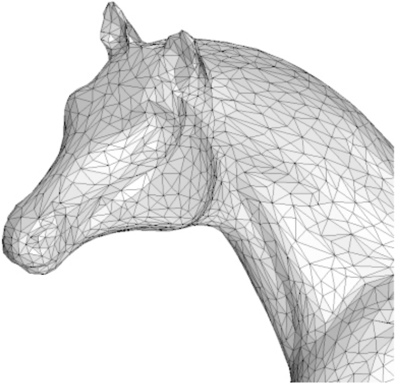
- Sometimes a statement is valid only for $n \geq j$. Then the Basis Step is about $S(j)$.
- Sometimes in Basis Step we have to check a number of small cases, not just $n = j$.
- Sometimes in Induction Step we have to assume that $S(n)$ holds for all $n \leq k$, not just for $n = k$.

The last item is called *strong induction* in Rosen.

Some Geometry

- A polygon is a closed geometric figure formed by a sequence of line segments $s_1 \ldots, s_n$ called sides.
- Two consecutive sides share an endpoint, as do s_1 and s_n. The endpoints are called vertices.
- A polygon is simple if no two non-consecutive sides intersect.
- Each polygon divides the plane into two regions: interior and exterior
- A polygon is convex if each line segment between two interior points is within the interior.
- A diagonal is a line segment connecting two non-consecutive vertices
- An interior diagonal is one that lies entirely inside the polygon.
    - Lemma: every simple polygon has an interior diagonal.

Triangulation in Computational Geometry



- Triangulation is the process of dividing a simple polygon into triangles by adding non-intersecting diagonals.
- In order to computationally process a complicated surface, it is divided into simple polygons, which are then triangulated.
- Delaunay triangulations are especially popular (click `here` for more info)

Triangulation Proof by Induction

Theorem

Each simple polygon with $n \geq 3$ sides can be triangulated into $n - 2$ triangles.

Proof.

- Basis Step: the theorem trivially holds for $n = 3$.
- Induction Step: let $k \geq 3$ be any integer.
  - Assume that the theorem is true for all $n \leq k$
  - Take a simple polygon P with $n = k + 1$ sides.

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  - By Lemma above, it has an interior diagonal.
  - The diagonal splits P into two simple polygons: Q with s sides and R with t sides.
  - We have $3 \leq s \leq k$ and $3 \leq t \leq k$. Moreover, $k + 1 = s + t - 2$.

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  - Assume that the theorem is true for all $n \leq k$
  - Take a simple polygon P with $n = k+1$ sides.
  - By Lemma above, it has an interior diagonal.
  - The diagonal splits P into two simple polygons: Q with s sides and R with t sides.
  - We have $3 \leq s \leq k$ and $3 \leq t \leq k$. Moreover, $k+1 = s+t-2$.
  - By assumption, Q and R can be triangulated into $s-2$ and $t-2$ triangles, respectively.
  - This gives a triangulation of P into $(s-2)+(t-2) = k-1 = n-2$ triangles.

$\square$

(Erroneous?) Proof by Induction

Theorem

All people are brown-eyed.

Proof.

We prove the following statement $S(n)$ by induction: If a collection of n people includes one brown-eyed person then they are all brown-eyed.

- Basis Step: $S(1)$ is true, if 1 person is brown-eyed then (s)he is so.
- Induction Step: Let $k \geq 1$ be any integer.
  - Assume $S(n)$ holds for $n = k$.
  - Take any group of $k + 1$ people including one brown-eyed person.
  - Remove a different (not that) person from the group.
  - We now have a group of k. By assumption they are all brown-eyed.
  - Bring the different person back and remove someone else.
  - Again, we have a group of k people including someone brown-eyed.
  - Conclusion: $S(n)$ holds for $n = k + 1$, and everyone is brown-eyed.

□

Exercises

Prove the following statements by induction on n.

- $1 + 2 + 2^2 + \ldots + 2^n = 2^{(n+1)} - 1$ for every positive integer n;
- $1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for all positive integers $n \geq 2$.

Short answers to the exercises

We prove by induction on n that $1 + 2 + 2^2 + \ldots + 2^n = 2^{(n+1)} - 1$ for every positive integer n. We first check the case $n = 1$: the left hand side results in $1 + 2 = 3$; the right hand side results in $2^2 - 1 = 3$; so the statement holds for $n = 1$. Next we assume it holds for $n = k \geq 1$, and we consider the case $n = k + 1$: $1 + 2 + 2^2 + \ldots + 2^k + 2^{k+1} =$ (by the assumption on the sum of the first k terms) $= 2^{k+1} - 1 + 2^{k+1} = 2^{k+2} - 1$. So assuming the statement holds for $n = k$ we have shown it holds for $n = k + 1$. This proves the first result.

Next we prove by induction on n that $1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for all positive integers $n \geq 2$. We first check the case $n = 2$: left hand side is $1 + \frac{1}{4} = \frac{5}{4}$; right hand side is $2 - \frac{1}{2} = \frac{3}{2} = \frac{6}{4}$, so the statement holds for $n = 2$. Next we assume it holds for $n = k \geq 2$, and we consider the case $n = k + 1$:
$1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{k^2} + \frac{1}{(k+1)^2} =$ (by the assumption on the sum of the first k terms) $< 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$. If we use that $\frac{1}{(k+1)^2} < \frac{1}{k(k+1)}$, we see that $2 - \frac{1}{k} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k} + \frac{1}{k(k+1)} = 2 - \frac{1}{k+1}$. So assuming the statement holds for $n = k$ we have shown it holds for $n = k + 1$. This proves the second result.