An overview of first order logic

1 Predicates and atomic formulae

Whereas the fundamental building block in propositional logic is the propositional variable, with first order logic it is the **predicate** (we have already been introduced to predicates when we studied relations)
A **predicate symbol** (or **relation symbol**) is just a symbol with an associated arity e.g., P might be defined as a predicate symbol with arity r
Given a predicate symbol P of arity r and some variables $x_1, x_2, ..., x_r$ (where it might be the case that some of these variables are the same), the formula

$$P(x_1, x_2, ..., x_r)$$

is an **atomic formula** of first order logic
In order to know whether this atomic formula is **true** or **false**, we need to be given an r-ary relation P′ over some domain D, say, and values $v_1, v_2, ..., v_r$ from D for $x_1, x_2, ..., x_r$

2 Atomic formula: an example

Suppose T is the ternary relation symbol. Then

$$T(x, y, x)$$

is an **atomic formula**
Now let T′ be the following ternary relation on ℕ

$$\{(u, v, w) : u, v, w \in \mathbb{N}, u = 2v \text{ and w is even}\}$$

and consider the interpretation (or model) of $T(x, y, x)$ in T′ with x=6 and y=3. This is true

In this case, we write $(T', x = 6, y = 3) \models T(x, y, x)$ or sometimes $(\mathbb{N}, T', x = 6, y = 3) = T(x, y, x)$

2.1 Lecture example

$$(u, v, w) = (6, 3, 6)$$

This is in T′ as the last digit is even and the first digit is twice the second digit

3 Building formula

Given some atomic formulae, we can build more complicated formulae from these atomic formulae by using the usual connectives of propositional logic, namely $\neg, \wedge, \vee, \Rightarrow$ and $\Leftrightarrow$. For example,

$$E(x_1, x_2) \vee (T(x_1, x_1, x_3) \Rightarrow \neg E(x_2, x_3))$$

is a formula of first-order logic, where E is a predicate symbol of arity 3, and x_1, x_2 and x_3 are variables.
In order to interpret this formula, we need a binary relation for E, a ternary relation for T and values for x_1, x_2 and x_3. The domains of the relations for E and T must be the same.

Is the following interpretation true?
$E = \left\{(u_1, u_2) \in \mathbb{N}^2 : u_1 \leq u_2\right\}, T = \left\{(u_1, u_2, u_3) \in \mathbb{N}^3 : u_1 \cdot u_2 = u_3\right\}$
and $x_1 = 3, x_2 = 2$ and $x_3 = 9$

This is not true as $F \vee T \Rightarrow F$, which is false.

Not only do we allow formulae such at $P(x_1, x_2, ..., x_r)$ as atomic formulae, but we are also allowed formulae of the form $x = y$, where x and y are variables (this constitutes all atomic formulae)
The semantics of x=y is that this atomic formula is true only if the value of x is equal to the value of y (in an interpretation)

3.1 Example

Let E be a binary predicate symbol. Consider the formula

$$(E(x, y) \wedge E(y, z)) \Rightarrow \neg(x = z)$$

(We sometimes abbreviate $\neg(x = z)$ by $x \neq z$)
If E is interpreted as:

$$E = \left\{(x, y) \in \mathbb{N}^2 : x < y\right\}$$

and x=5, y=7 and z=11 then is the formula true in this interpretation?

$E(x, y)$ is true as $5 < 7$
$E(y, z)$ is true as $7 < 11$
$\neg(x = z)$ is true as $5 \neq 11$
So the whole formula is true

3.2 More on building formulae

Formulae built from atomic formulae are called quantifier-free formula and the free variables are those variables appearing in a formula

4 Quantifiers

Given a formula with **free** variables, we can now "quantify" over these variables using the universal quantifier (or the for-all quantifier) $\forall$ and the existential quantifier (or the exists quantifier) $\exists$
Suppose that $\phi(x)$ is a quantifier-free formula with one free variable x. Then $\forall x \phi(x)$ is a formula of first order logic and has no free variables. The variable x is a **bound** variable in $\forall x \phi(x)$

4.1 Example

Suppose that Q is a unary relation symbol. Consider the formula $\forall x Q(x)$. Is it true for the following interpretations?

- Interpret Q as the relation $Q = \{u \in \mathbb{N} : \text{u is even}\}$
  This is not true as there are odd natural numbers
- Interpret Q as the relation $Q\{u \in \mathbb{N} : \text{u has a square root}\}$
  This is true as every natural number is a square root

Then thinking about the formula $\exists x Q(x)$ and the relation $Q = \{u \in \mathbb{N} : \text{u is even}\}$, it is then true as 2 is even.

A formula e.g. $\neg(\forall x Q(x))$ is the same as $\exists x \neg Q(x)$

5 More complicated formulae

We can apply quantifiers to quantifier-free formula even when there is more than one free variable in the formula.
Let $\phi(x_1, x_2, ..., x_r)$ be a quantifier free formula with free variables $x_1, x_2, ..., x_r$. Then the following are two examples of formulae of first order logic.

$$\forall x_1 \phi\left(x_1, x_2, \ldots, x_r\right) \quad \exists x_3 \phi\left(x_1, x_2, \ldots, x_r\right)$$

The first has free variables $x_2, ..., x_r$ and bound variable x_1 (as it is outside the ϕ); and the second has free variables $x_1, x_2, x_4, ..., x_r$ and bound variable x_3
An **interpretation** of such formulae are as before except that relations and values for the free variables have to be supplied in order for any interpretation to make sense.

5.1 Examples

- If $\phi(x)$ is the formula $\forall y(x = y \vee E(x, y))$ and $E = \left\{(u, v) \in \mathbb{N}^2 : u < v\right\}$ then

  $$(E, x = 0) \models \phi(x)$$

  $$\forall y(0 = y \vee E(0, y))$$

$$\forall y(0 = y \vee 0 < y)\text{True as only natural numbers is either 0 } or\ > 0$$

but

$$(E, x = v) \models \neg\phi(x) \text{ wherever } v \neq 0$$

- If $\phi(x)$ is the formula $\exists y E(y, x)$ and $E = \left\{(u, v) \in \mathbb{N}^2 : u < v\right\}$ then we have

$$(E, x = 0) \models \neg\phi(x)$$

but

$$(E, x = v) \models \phi(x) \text{ wherever } v \neq 0$$

6 More complicated formulae

We can also apply quantifiers to formulae already involving quantifiers.
Consider the formula $\forall y(x = y \vee E(x, y))$. There is one free variable and we can quantify over this free variable; like this

$$\exists x \forall y(x = y \vee E(x, y))$$

Let the binary relation $E = \left\{(u, v) \in \mathbb{N}^2 : u < v\right\}$
For formula above to be **true** in this interpretation, we need that there exists some value $u \in \mathbb{N}$ for x such that for any value $v \in \mathbb{N}$ for y, we have that $u = v \vee E(u, v)$; that is, either $u = v$ or $u < v$
There clearly does exist such a value u, namely $u = 0$. However, if $E = \left\{(u, v) \in \mathbb{Z}^2 : u < v\right\}$ then the formula is **false** as given any value for x, there is always some integer that is strictly less than this value for x

We can also build new formula, using the usual **propositional connectives**, from existing formulae that involve **quantifiers**. Consider the formula

$$\exists x \forall y(x = y \vee E(x, y)), \text{ and } \exists x \forall w(x = w \vee E(w, x))$$

If we **interpret** E as $\left\{(u, v) \in \mathbb{N}^2 : u < v\right\}$ then is the following formula true?

$$\exists x \forall y(x = y \vee E(x, y)) \wedge \exists x \forall w(x = w \vee E(w, x))$$

What if we interpret E as

$$\{(u, v) \in \{0, 1, \ldots, 9\} \times \{0, 1, \ldots, 9\} : u < v\}$$

Notice how the same variable, x, is quantified twice in the same formula yet the two quantifications are entirely separate!