Discrete Structures - Functions

1 Functions

- To make associations between elements of sets we use functions
- A function f from A to B, written $f : A \to B$ is an assignment of an element of B to every element of A. If $b \in B$ is the element assigned to $a \in A$ then we write $f(a) = b$
- Functions can be defined in a number of ways
- In the case of a function $f : A \to B$
  - The set A is known as the domain (or source) of f
  - The set B is the **codomain** (or **target**) of f
- If $f(a) = b$ then b is the **image** of a (under f)
- The **pre-image** of $b \in B$ (under f) is the subset $\{a : f(a) = b\}$ of A
- The image (or range) of f is the set of images of elements of A

2 Illustrations of function concepts

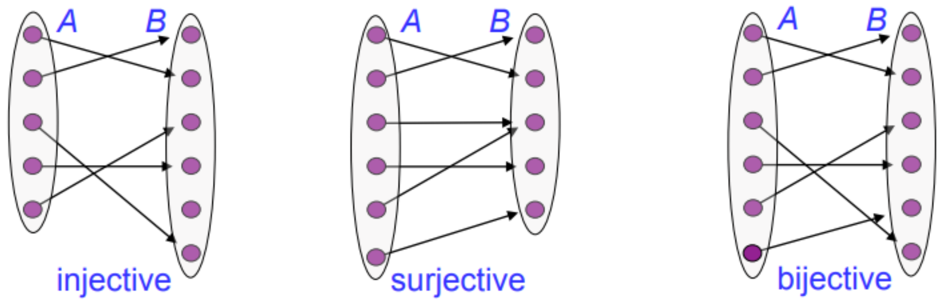
- Let $f : N \to Q$ be defined by the formula $f(x) = x/2 + 3$
  - The domain of f is N and the codomain is Q
  - The image of 5 under f is 5.5
  - The pre-image of 8 is $\{10\} \subseteq N$
  - The image of f is $3, 3.5, 4, 4.5, ...$
- Let $f : P(N) \to N \cup \{\perp\}$ be defined by the property: $f(x)$ is the minimal element of the set x if $x \neq \emptyset$ and $\perp$ if $x = \emptyset$
  - The domain of f is P(N) and the codomain is $N \cup \{\perp\}$
  - The pre image of 5 is the set $\{X : X \subseteq N, 5 \in X \wedge 0, 1, 2, 3, 4 \notin X\} \subseteq P(N)$

3 Partial functions

- Partial functions are variations of functions where the function may not be defined for every element in the domain
- A partial function $f : A \to B$ is either $f(a) \in B$ or $f(a)$ is undefined
- Partial functions are particularly relevant in CS, as when finding the input output correspondence of a particular program, the program might not provide an output for every input

4 Special types of function

- A function $f : A \to B$ is injective or one-to-one (with f being an injection) if for every (written $\forall$) $a \in A$ and $a' \in A$ if $f(a) = f(a')$ then $a = a'$
- A function $f : A \to B$ is surjective or onto (with f being a surjection) if every $b \in B$ is such that there exists (written $\exists$) some $a \in A$ such that $f(a) = b$
- If a function $f : A \to B$ is both injective and surjective then it is bijective or a one-to-one correspondence (with f being a bijection)



5 More on bijections

- Suppose that $f : A \to B$ is a bijection
- We can build a set of ordered pairs
$$P = \{(a, f(a)) : a \in A\} \subseteq A \times B$$
- As f is onto every $b \in B$ must appear as the second component in some pair (a,b)
- As f is one-to-one every $b \in B$ must appear as the second component in at most one pair (a,b)
- So, each element of A appears in exactly one pair in P, as does each element of B
- The set P is a "pairing" of the elements of A and B so that every element of A is associated with a unique element of B, and vice versa

6 Compositions of functions

Suppose that $f : A \to B$ and $g : B \Rightarrow C$ are functions. We can define the composition of g and f as the function $g \circ f : A \to C$ defined as $(g \circ f)(x) = g(f(x))$

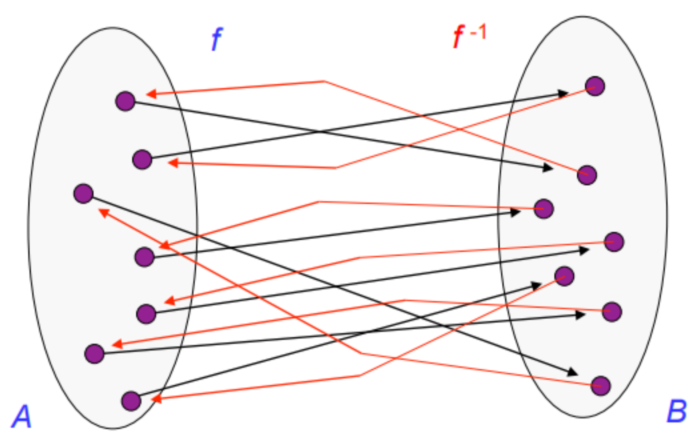
Note that even if the function $g \circ f$ exists, the function $f \circ g$ might not exist

Also, even if both $g \circ f$ and $f \circ g$ exist, it could well be that they are different

7 Inverses

Often we want the inverse of a function, where the inverse of the function $f : A \to B$ is the function $f^{-1} : B \to A$ where:

- $f^{-1}(f(a)) = a, \forall a \in A$
- $f\left(f^{-1}(b)\right) = b, \forall b \in B$



- Note that it may not always be the case that the inverse function exists

- Let $f : A \to B$
  - Suppose that f is not one to one, i.e. $\exists$ distinct $a, a' \in A$ s.t. $f(a) = f(a')$
  - If f^{-1} exists then $a = f^{-1}(f(a)) = f^{-1}(f(a')) = a'$ which yields a contradiction
  - So, if an inverse of f exists then f must be one to one
  - Suppose that f is not onto, i.e. $\exists b \in B$ s.t. there is not $a \in A$ s.t. $f(a) = b$
  - If f^{-1} exists then $f^{-1}(b) = a'$ for some $a' \in A$ with $b = f\left(f^{-1}(b)\right) = f(a')$, which yields a contradiction
  - So, if an inverse of f exists then f must be onto
  - So, if an inverse of f exists then f must be a bijection
  - Conversely, if f is one-to-one and onto then the inverse exists. We simply define $f^{-1}(b)$ as the unique element $ainA$ for which $f(a) = b$. Since f is a bijection, we can "pair" elements of A and B so that each element of A is associates with a unique element of B, and vice versa
- This, we have proven that f has an inverse if, and only if, f is a bijection

8 Cardinality Revisited

- Two sets A and B (which may be finite or infinite) have the same cardinality iff there is a bijection from A to B
- A set is countable if it is finite or has the same cardinality as N when we say it has cardinality $\aleph_0$
- A set is uncountable if it does not have cardinality $\aleph_0$

9 Uncountable sets

- Up until not, we have not even shown that there exist uncountable sets, however these sets do exist and $\mathbb{R}$ is one of them
- Suppose that R is countable
  - This the set I of real number strictly between 0 and 1 is countable, that is, there is a bijection $f : \mathbb{N} \to I$
  - List all the elements of $\mathbb{R}$
  - "pull out" those between 0 and 1 and put them in a sub-list
- Form a new decimal number x between 0 and 1 by building the number whose ith digit behind the decimal point is 5 ith the ith digit of f(i-1) is 4 and 4 otherwise
- By definition x is not equal to any number on the list
  - its ith digit of x behind the decimal point is different from the ith digit behind the decimal point of the ith number in the list
  - So f is not onto, which yields a contradiction
- Thus, $\mathbb{R}$ is uncountable and has cardinality "bigger" that $\aleph_0$
- The generic technique employed is called **diagonalization**