Divide-and-Conquer Colouring for Restricted Inputs

1 Divide-and-conquer algorithms

Many useful algorithms are recursive, i.e. call themselves recursively to deal with subproblems

At each level of recursion:

- **Divide** the problem into a number of subproblems
- **Conquer** the subproblems by solving them: recursively, or straightforward if the subproblem sizes are small enough
- **Combine** the solutions for the subproblems into a solution for the original problem

2 Problem Definition

Definition: Colouring

An assignment of colours to the vertices of a graph such that no two adjacent vertices get the same colour

Definition: k-colouring

A colouring using at most k colours

Definition: Chromatic number χ_G

The smallest integer k for which G has a k-colouring

The task of given a graph G determining χ_G is an NP hard problem

3 How to deal with NP-hardness

This can be done in several ways. For example:

- Heuristics
- Approximation algorithms
- Exact algorithms
- Parametrized algorithms
- Algorithms with restricted inputs

We focus on the last approach. That is, we exploit the structure of the input with an aim to develop faster algorithms. Justification

1. In practice problem inputs may exhibit structure
2. By focussing on the input structure we understand better what structure makes a problem computationally hard

4 Definition

Let G_1 and G_2 be two vertex-disjoint graphs

The **disjoint union** $G_1 + G_2$ of G_2 and G_2 is the graph with the vertex set and edge set of the union of them. There is no overlap between G_1 and G_2

The **join** $G_1 \times G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup E^*$, where E* is the set of all edges between vertices of G_1 and vertices of G_2. Adding all possible edges between the graphs
A graph G is a cograph iff G can be created by the following rules

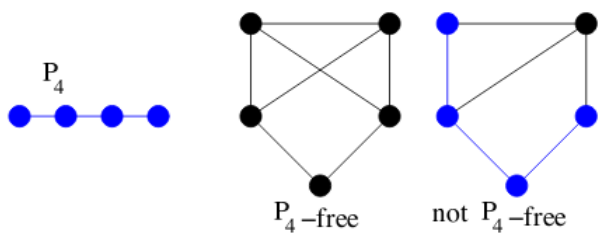
- The 1-vertex graph K_1 is a cograph
- If G_1 and G_2 are cographs, then so is $G_1 + G_2$
- If G_1 and G_2 are cographs, then so is $G_1 \times G_2$

Important: Cographs

P_4 is the smallest graph that is not a cograph

5 Alternate definition

Let P_4 denote the path on 4 vertices
A graph G contains P_4 as an induced subgraph if G can be modified to P_4 via vertex deletions
If not, then G is said to be P_4-free



It is known that a graph is a cograph iff it is P_4-free

6 Definition of a cotree

A cotree T_G of a cograph G is the unique decomposition tree satisfying

1. Its root r corresponds to the graph $G_r = G$
2. Every left x of T corresponds to exactly one vertex of G, and vice versa
3. Every internal node x of T has at least two children, is either labelled + of ×, and corresponds to an induced subgraph G_x of G defined as follows:
   - If x is a +- node, then G_x is the disjoint union of all graphs G_y, where y is a child of x
   - If x is a ×-node, then G_x is the join of all graphs G_y where y is a child of x
4. Labels of internal nodes on the (unique) path from any leaf to r alternate between + and ×

This tree shows the composition of a cograph, each internal node is an operation and each leaf is a vertex of the graph

7 Solving colouring for cographs

Let G be a cograph on n vertices and m edges
We construct its cotree T_G. This can be done in $O(n+m)$ time

We follow a bottum up approach: only process node x after first processing its children.

Leaves: Each leaf corresponds to a single vertex of G. The chromatic number of a single-vertex graph is 1

Union-nodes: Let x be a +-node. Then χ_{G_x} is the maximum number of the chromatic numbers of the graphs G_y, where y is a child of x

Join-nodes. Let y be a $\times$-node. Then χ_{G_x} is the sum of the chromatic numbers of the graphs G_y, where y is a child of x

In this way the last node we process is node r, which corresponds to the graph $G_r = G$

As the number of nodes of T is $O(n)$ and we used linear time per node, the total running time is polynomial