More on Natural Deduction for Propositional Logic

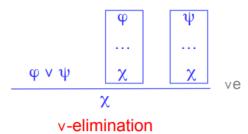
1 More rules

• Rules for introducing disjunction



v-introduction

• Rules for eliminating disjunction



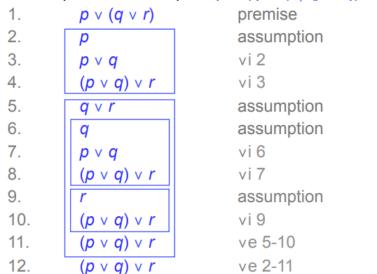
- In order to apply the rule $\vee e$, we use boxes as previously
 - But now there is a box starting with each disjunct φ and ψ
 - Each box needs to end with the same intended formula, χ

2 A proof using ∨ elimination

Here is a proof of the sequent $q \Rightarrow r \mid (p \lor q) \Rightarrow (p \lor r)$

- 1. $q \Rightarrow r$ premise 2. $p \lor q$ assumption 3. p assumption 4. $p \lor r$ vi 3 5. q assumption 6. r \Rightarrow e 1 5 7. $p \lor r$ vi 6 8. $p \lor r$ $p \lor r$
- Assume that p is true, so the RHS is true
- $p \lor r$ is true
- Open box assuming q is true
- Eliminate the implies symbol from the LHS
- Shown that no matter if p or q is true, the statement $p \lor r$ will be true
- It has been shown that if $p \vee q$ is true, that $p \vee r$ is true, so $(p \vee q) \Rightarrow (p \vee r)$ will always be true

Here is a proof of the sequent $p \vee (q \vee r) \vdash (p \vee q) \vee r$



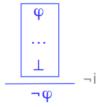
• Basically trying to build the RHS from the LHS

3 More rules

• Rules for negation



- The symbol ⊥, known as bottom, represents a contradiction, in natural deduction if one has a contradiction then one can infer **any** formula
- Rules for introducing negation



¬-introduction

4 A proof using rules for negation

Here is a proof of the sequent $x \lor \neg y \vdash y \Rightarrow x$.

- 1. $x \lor \neg y$ 2. x3. y4. x5. $y \Rightarrow x$
- 4. x5. $y \Rightarrow x$ 6. $\neg y$ 7. y8. \bot 9. x
- 10. $y \Rightarrow x$ 11. $y \Rightarrow x$

- premise
 - assumption
 - assumption copy 2
 - ⇒i 3-4
 - assumption
 - assumption
 - ¬e67
 - ⊥e 8
 - ⇒i 7-9
 - ve 1 2-5 6-10
- $p \land \neg p \Rightarrow \varphi$ holds as $p \land \neg p$ will always be false, and if the LHS of an implication is false, then the whole statement will be true
- Lines $2 \to 5$ say $x \Rightarrow (y \Rightarrow x)$

Here is a proof of the sequent $x \Rightarrow (y \Rightarrow z)$, x, $\neg z \mid \neg y$.

- 1. $X \Rightarrow (y \Rightarrow z)$
- 2. x
- 3. ¬*z*
- 3. ¬Z
 4. *y*
- 5. $y \Rightarrow z$
- 6. **z** 7. ⊥
- 8. ¬*y*

- z) premise
 - premise
 - premise
 - assumption
 - ⇒e 1 2
 - ⇒e 4 5
 - ¬e36
 - ¬i 4-7

5 A derived rule

- We can derive other rules in natural deduction
- Consider modus tollens $\varphi \Rightarrow \psi, \neg \psi \vdash \neg \varphi$
 - 1. $\varphi \Rightarrow \psi$
 - 2. ¬ψ
 - 3. 4.

5.

6.

φ ψ <u>⊥</u>

- premise
- premise
- assumption
- ⇒e 1 3
- ¬e 2 4
- ¬i 3-5
- Note that we can use derived rules just as if they were rules of natural deduction
 - e.g., in a proof with
 - * a line reading $\varphi \Rightarrow \psi$

- * and another line reading $\neg \psi$
- we could immediately infer $\neg \psi$ and write modus tollens as an explaining remark

--e 4

6 More derived rules

- Proof by contradiction is the principle "if from $\neg \varphi$ I can prove \bot then I can deduce φ
- Here is a proof that this principle can be applied in natural deduction
 - 1. $\neg \phi \Rightarrow \bot$ premise 2. $\neg \phi$ assumption 3. \bot \Rightarrow e 1 2 4. $\neg \neg \phi$ \neg i 2-3
- We denote reductio ad absurdum by RAA

7 More derived rules

- The law of excluded middle states that either φ is true or $\neg \varphi$ is true
- Here is a proof of it

5.

- 1. assumption 2. assumption 3. vi 2 4. ¬e 13 5. $\neg i 2-4$ $\neg \varphi$ 6. 7. ¬e 1 6 8. ¬i 1-7 9. --e8
- We denote the law of excluded middle by LEM

8 Some facts about Natural Deduction

- Natural deduction is sound and complete
- Let $\varphi_1, \varphi_2, ..., \varphi_m$ and ψ be formulae
- Soundness
 - If the sequent $\varphi_1, \varphi_2, ... \varphi_m \vdash \psi$ is provable then the formula $\varphi_1 \land \varphi_2 \land ... \land \varphi_m \Rightarrow \psi$ is a tautology
- Completeness
 - If $\varphi_1 \wedge \varphi_2 \wedge ... \wedge \varphi_m \Rightarrow \psi$ is a tautology then the sequent $\varphi_1, \varphi_2, ... \varphi_m \vdash \psi$ is provable
- A **theorem** is a formula ψ for which the sequent $\vdash \psi$ is provable, thus, the soundness and completeness of natural deduction tells us that every theorem is a tautology and every tautology is a theorem

9 Proving Theorems

• Here is a proof that the sequent $(p \Rightarrow (\neg p \lor q)) \lor (p \Rightarrow \neg q)$ is a theorem

