The Basics of Graph Theory

1 What is a Graph?

- A mathematical model
- A representation of objects and relations between them
- The objects can be 'anything'
- The relations between pairs of anything

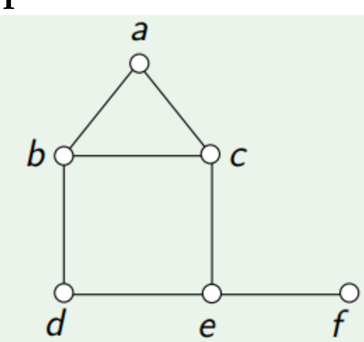
2 Formal Definitions

2.1 Definitions

A **graph** G is a pair $(V(G), E(G))$, where $V(G)$ is a **nonempty** set of **verticies**(or nodes) and $E(G)$ is a set of **unordered pairs** {u,v} with $u, v \in V(G)$ and $u \neq v$ called the **edges** of G.

- V(G) can be infinite, but all our graphs will be finite
- If no confustion can arise we write uv instead of $\{u, v\}$
- If the graph G is clear from the context, we write V and E instead of V(G) and E(G)
- It often helps to draw graphs
  - represent each vertex by a point
  - each edge by a line or curve connecting the corresponding points
  - only endpoints of lines/curves matter, not the exact shape

3 A drawing of a graph



This is a drawing of the graph $G = (V, E)$ with $V = \{a, b, c, d, e, f\}$ and $E = \{ab, ac, bc, bd, ce, de, ef\}$.

4 Types of graphs

- **directed** graphs or **digraphs** - edges can have directions
  - The web graph: vertices are webpages and edges are hyperlinks
  - the precedence graph: vertices are program statements, edges reflect execution order
  - the influence graph: vertices are people in the group, edges mean "influences"
- multigraphs - multiple edges are allowed between two vertices
  - the air link graph - several different airlines can fly between two towns
- pseudographs - edges of the form uu, called loops are allowed

  - region pseudograph in computer graphics: Vertices are connected regions edges mean ”can get from one to the other by crossing a fence”
- vertex or edge weighted graphs - vertices and/or edges can have weights
  - the road map graph: weights on edges

By default, all our graphs are **simple undirected** graphs, that is, the above things are not allowed

5 More examples of graph models

Graphs can be useful to express **conflicting** situations between objects

- vertices - base stations for mobile phones, Edges: overlapping service areas
- vertices - traffic flows at a junctions, Edges: conflicting flows

Graphs can be useful for **analysing strategies** and **solutions**

- vertices: states in a game, edges: transitions between states
- vertices: steps in a solution, Edges: transitions between steps

6 Terminology

6.1 Definitions

Let G be a graph and uv cn edge in it. Then

- u and v care called endpoints of the edge uv
- u and v are called neighbours or adjacent vertices
- uv is said to be incident to u (and to v)
- if vw is also an edge and $w \neq u$ then uv and vw are called adjacent

6.2 Definitions

Let $G = (V, E)$ be a graph. The **neighbourhood** of a vertex $v \in V$, notation $N(v)$, is the set of neighbours of v i.e., $N(v) = \{u \in V | uv \in E\}$.
The **degree** of a vertex $v \in V$ notation deg(v), is the number of neighbours of v i.e. $deg(v) = |N(v)|$
With $\delta(G)$ or δ we denote the **smallest degree** in G, and with $\Delta(G)$ or Δ or the **largest degree**
A vertex with degree 0 will be called an **isolated vertex**
A vertex with a degree 1 an **end vertex** or a **pendant vertex**

6.3 Definition

A subgraph $G' = (V', E')$ of $G = (V, E)$ is a graph with $V' \subseteq V$ and $E' \subseteq E$; this subgraph is called **proper** if $G' \neq G$ and spanning if $V' = V$

7 First theorem in Graph theory

Can you guess the relationship between the sum of the degrees of the vertices of a graph G and the number of edges of G

7.1 Theorem(Handshaking Lemma)

$$\text{Let } G = (V, E) \text{ be a graph. Then } \sum_{v \in V} \deg(v) = 2|E|$$

This is useful for proving that a graph cannot exist

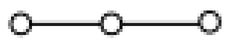
7.2 Proof

Every edge has two endpoints and contributes one to each of their degrees, so contributes two to the sum of the degrees of all the vertices of V

8 Some graph classes

Some graphs appear so often they have special names

8.1 P_3

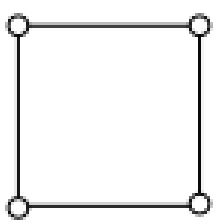


This is denotes an P_3 and in general we define P_n as a path on n vertices i.e. a graph with vertex set $\{v_1, v_2, ..., v\}$ and edge set $\{v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$ So P_n has n-1 edges

8.1.1 Definition

A path in a graph G is a subgraph of G which is P_k for some integer $k \geqslant 1$. This notion is called a **simple path**

8.2 C_4

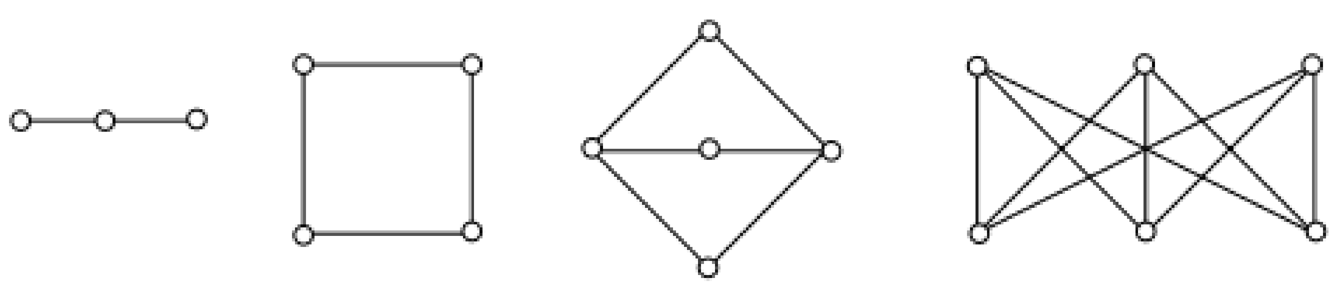


In general a cycle C_n on n verticies is defined similarly as a P_n, but with an additional edge between v_n and v_n. So C_n has n edges

8.2.1 Definition

A cycle in a graph G is a subgraph of G which is a C_k for some integer $k \geqslant 3$. This notion is called a **simple circuit**

8.3 $K_{p,q}$



All four of these graphs can be described as $K_{p,q}$: a graph consisting of two disjoint vertex sets on p and q vertices and all the edges between the two vertex sets. So $K_{p,q}$ has $p \cdot q$ edges

8.3.1 Definitions

$K_{p,q}$ is called a **complete bipartite** graph. So a graph is bipartite iff we can partition its vertex set into two sets such that every edge has endpoints in each set

8.4 Definition

A **complete** graph on n vertices, denote by K_n contains all the possible edges between pairs of vertices.
A K_n graph has $\binom{n}{2} = \frac{1}{2}n(n-1)$

8.5 Definition

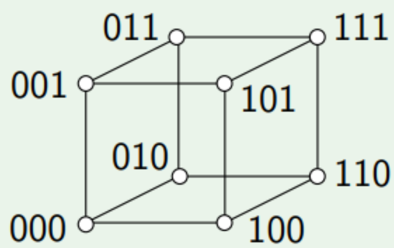
The (n dimensional) hypercube or n cube $Q_n (n \geqslant 1)$ is the graph with

$$V = \{(e_1, ..., e_n) | e_i \in \{0, 1\} (i = 1, ..., n)\}$$

in which two vertices are neighbours iff the corresponding rows differ in exactly one entry

8.5.1 Example

$Q_1 = P_2 = K_2$; $Q_2 = C_4$. For $n = 3$ the set V consists of $2^3 = 8$ elements, namely all rows (in short hand notation) 000, 001, 010, 011, 100, 101, 110, 111.



Q_3

9 More on n cubes

9.1 Theorem

All n cubes are bipartite

9.2 Proof

- We give a bipartition of the vertex set of the n cube
- Let V_1 contain all the vertices with an odd number of 1s
- Let V_2 contain all vertices with an even number of 1s
- This is clearly a partition of V into two disjoint sets
- It is easy to see that each edge has one endpoint in each of the sets
- So it proves that all n-cubes are bipartite

10 Questions

p_n is only k regular for p_2
c_n is k regular
$K_{p,q}$ is only k regylar for p=q
Q_n is k regular

p_n is bipartite
C_n is bipartite only for even n

In Q^n the number of verticies is 2^n and each has n connections. The number of edges is $n \times 2^{n-1}$