# Basic Counting Principles

## 1 The Product Rule

### 1.1 Definition

Suppose that a procedure can be broken down into a sequence of two tasks. If there are $n_1$ ways to do the first task and for each of these ways of doing the first task, there are $n_2$ ways to do the second task. Then there are $n_1 \times n_2$ ways to do the procedure.

### 1.2 Example

*How many different passwords can be constructed using two (or k) symbols from a set of N distinct symbols*

For each of the N choices of the first symbol there are again N choices for the second symbol, so the answer is $N \times N$. For passwords consisting of k symbols the answer is $N \times N \times ... \times N = N^k$

If we do not allow repetitions of symbols, the answer is $N \times (N-1)$, respectively $N \times (N-1) \times ... \times (N-k+1)$

## 2 The sum rule

### 2.1 Definition

If a task can be done either in one of $n_1$ ways or in one of $n_2$ ways, where none of the set of $n_1$ ways is the same as any of the set of $n_2$ ways, then there are $n_1 + n_2$ ways to do the task

### 2.2 Example

*Suppose we have a set of N characters and a set of M integers. In how many ways can we choose one symbol which is either a character or an integer?*

It is clear that we can choose the symbol in $N + M$ ways

In how many ways can we construct a sequence of three symbols where the first one is a character, the second one an integer, and the third one either of them?

A combination of the product and sum rules gives the answer: $N \times M \times (N + M)$

## 3 Example: Counting IP addresses

- The internet is made up of interconnected physical networks of computers
- Each computer (actually, each network connection of a computer) is assigned to an Internet address
- Version 4 of the Internet Protocol is still in use
  - An address in IPv4 is a string of 32 bits (looks like 172,16.254.1 in decimal)
  - It consists of **netid** (network number) and **hostid** (host number)
  - There are three classes of addresses: class A,B and C
  - Class A is for large networks, B for medium sized and C for small
  - Actually, there are also classes E and D, but for a separate purpose
- Class A address 0 7-bit netid 24-bit hostid (Short netid as small number of large networks, but lots of hosts on those networks)
  - Technical restriction: Class A netid cannot be 111111
- Class B address 10 14-bit netid 16-bit hostid
- Class C address 110 21-bit netid 8-bit hostid
  - Technical restriction: hostid in any class cannot be all 0 or 1

*How many different IPv4 addresses are available for a computer on the internet*
Let x be the number we want to compute

- Let $x_a$ be the number of class A addresses,
- Let $n_a$ be the number of class A netids, and
- Let $h_a$ be the number of class A hostids;
- define $x_b, n_b, h_b$ and $x_c, n_c, h_c$ similarly

- By the sum rule $x = x_a + x_b + x_c$
- By the product rule, $x_a = n_a \times h_a$
  - By the product rule, $n_a = 2^7 - 1 = 127$ (since 111111 is not available)
  - By the product rule, $h_a = 2^{24} - 2 = 16,777,124$ (since can't have all 0s or all 1s)
  - Hence $x_a = 127 \times 16,777,124 = 2,130,706,178$
- Similarly $x_b = n_b \cdot h_b = 2^{14} \cdot (2^16 - 2) = 1,073,709,056$
- Also, $x_c = n_c \cdot h_c = 2^{21} \cdot (2^8 - 2) = 532,676,608$
- All in all, $x = x_a + x_b + x_c = 3,737,091,842$ - this is a small number

IPv4 addresses are exhausted now and 128-bit IPv6 addresses are now in use

# 4 Factorial Function

## 4.1 Definition

The factorial of an integer $n \geqslant 0$, denoted $n!$, is defined by:

$$0! = 1$$

$$n! = 1 \cdot 2 \cdot ... \cdot (n-1) \cdot n \quad n \geqslant 1$$

## 4.2 Example

*How many different passwords of length 8 can we construct with the letters A,b,c,D,E,f,g,h if each symbol occurs exactly once?*

The number is 8! (for the first symbol we have 8 possibilities; for each of these choices there are 7 possibilities for the second symbol, etc)

## 4.3 Example 2

The factorial function $n!$ grows extremely fast with increasing $n$

*If $n!$ > the age of the universe in seconds, how large should n be?*
n=20 is enough, $20! = 2.43 \cdot 10^{18} > age \approx 4.32 \cdot 10^{17}$ sec

# 5 Permutations

## 5.1 Definition

The **permutation** of a set of distinct objects is an **ordered** arrangement of these objects

## 5.2 r-Permutations

### 5.2.1 Definition

An **ordered** arrangement of r elements from a set of at least r distinct objects is called an **r-permutation**.
This is different from a standard permutations as some elements from the set will be unused

### 5.2.2 Theorem

If n and r are integers with $1 \leqslant r \leqslant n$ then there are

$$P(n,r) = n \cdot (n-1) \cdot \ldots \cdot (n-r+1)$$

r-permutations of a set with n distinct elements
Easy to prove using the product rule:

- n different choices for the first position
- for each of these choices, n-1 choices for the second position
- and so on, until the final position, for which there are n-r+1 different choices given any of the choices for the first r-1 position
- The product rule yields the given formula

### 5.2.3 Corollary

*If n and r are integers with* $1 \leqslant r \leqslant n$ *then*

$$P(n,r) = \frac{n!}{(n-r)!}$$

This is also easy to prove by using the definition of the factorial function and writing out the expansions of $n!$ and the product of $P(n,r)$ and $(n-r)!$.

For any set of n distinct elements, there are n! permutations of the set

### 5.2.4 Examples

*Suppose there are 8 runners in the final race, there can be no ties. How many ways are there to award the three medals if all outcomes are possible?*
Answer: $P(8,3) = 8 \cdot 7 \cdot 6 = 336$

*How many permutations of the letters ABCDEFGH contain the string ABC?*
Answer: Since ABC must occur as a block, treat this string as one symbol. Then we need to count the number of permutations of symbols (ABC),D,E,F,G,H. It's 6!=720

# 6 R combinations

## 6.1 Definition

An r-combination of elements of a set of at least r elements is an **unordered** selection of r elements from the set

The difference between a permutation and combination is that a permutation is ordered, whereas a combination is not.

## 6.2 Theorem

The number of r-combinations of a set with n elements, where n and r are integers with $0 \leqslant r \leqslant n$, equals

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

We can prove this as follows, using what we know about r-permutations:

- Each r-combination can be ordered in r! different ways to obtain an r-permutation
- So $P(n,r) = r! \cdot C(n,r)$ Writing this out and dividing by $r!$ gives the above formula

## 6.3 Example

*The department needs to form a committee by selecting 3 of 9 post-doctoral researchers and 4 of 11 PhD students. How many ways can this be done?*

Answer: the ordering of the selecting committee members does not matter, so:

- $C(9,3) = \frac{9!}{3!6!} = 84$ ways to select postdocs and
- $C(11,4) = \frac{11!}{4!7!} = 330$ ways to select PhD students
- By the product rule, there are $84 \cdot 330 = 27,720$ ways to select the committee