First order Logic - Logical Equivalence

1 Logical Equivalence

Two formulae ϕ and ψ are logically equivalent of they are true for the same set of models, in which case we write $\phi \equiv \psi$
All logical equivalences from propositional logic give rise to equivalences in first-order logic: for example, as

$$p \Rightarrow q \equiv \neg p \vee q \text{ for any propositional variables p and q}$$

We must have that

$$\phi \Rightarrow \psi \equiv \neg\phi \vee \psi, \text{ for any first-order formulae } \phi \text{ and } \psi$$

Note, however, that care must be taken as to exactly what an interpretation is when we "plug in" formulae as in the previous example: if

- ϕ is over the signature consisting of the binary relation symbol E and the constant symbol C
- ψ is over the signature consisting of the binary relation symbol E and the ternary relation symbol M

Then an interpretation for $\neg\phi \vee \psi$ is over the signature consisting of the symbols E, C and M

2 Some tricks

2.1 Renaming variables

Consider some first-order formula of the form $\forall x\phi(x)$ where y does not appear in $\phi(x)$
If we replace every occurrence of the variable x in ϕ with the variable y, we claim that $\forall x\phi(x) \equiv \forall y\phi(y)$:

- Let I be some interpretation for $\forall x\phi(x)$ in which $\forall x\phi(x)$ is true
- For every value u in the domain of I, we have that $(I, x = u) \models \phi(x)$
- So, for every value u in the domain of I, we have that $(I, y = u) \models \phi(y)$
- Hence, I is an interpretation in which $\forall y\phi(y)$ is true.
  Similarly, if I is an interpretation in which $\forall y\phi(y)$ is true then I is an interpretation in which $\forall x\phi(x)$ is true

In general, and by the same reasoning, if we ever have some formula ϕ in which there is a quantification, $\forall x$, say, then we can replace

- Every occurrence of x in the scope of this quantification with the variable y
- The quantification $\forall x$ by $\forall y$

So long as y does not appear in ϕ, without changing the semantics
Of course, the same can be said of $\exists x\phi(x)$ and, more generally, any formula containing a quantification $\exists x$

But, consider the formula $\exists xE(x, y)$
If we simply replace x with y and $\exists x$ with $\exists y$ then we get $\exists yE(y, y)$ which is semantically very different from $\exists xE(x, y)$

2.2 Substitution

Consider the formula ϕ in which there is contained a sub formula ψ
Suppose further that ψ has free variables $x_1, x_2, ..., x_k$
If ψ is logically equivalent to a formula $\chi(x_1, x_2, ..., x_k)$ then we can replace ψ in ϕ with the formula χ and not change the semantics

3 Some common equivalences

More interesting re the interactions between the quantifiers $\forall$ and $\exists$ and the logical connectives $\neg$, $\vee$ and $\wedge$
Consider the formula $\neg\forall x\phi$, where $\phi(x)$ is a first-order formula with free variable x
Let I be some interpretation fr $\neg\forall x\phi$. We have that:

- $I \models \neg\forall x\phi$
  Iff it is not the case that $I \models \forall x\phi$
  Iff it is not the case that for every value u in the domain of I, we have that $\phi(u)$ holds in I
  Iff there exists some value u in the domain of I such that $\neg\phi(u)$ holds in I
  Iff $I \models \exists x\neg\phi$

($\phi(u)$ is shorthand for saying that x is to be interpreted as u).
So for evert first-order formula $\phi(x)$

$$\neg\forall x\phi \equiv \exists x\neg\phi$$

Consider the formula $\neg\exists x\phi$, where $\phi(x)$ is a first-order formula with free variable x
Let I be some interpretation for $\neg\exists x\phi$. We have that:

- $I \models \neg\exists x\phi$
  Iff it is not the case that $I \models \exists x\phi$
  Iff it is not the case that there exists some value u in the domain of I such that $\phi(u)$ holds in I
  Iff for every value u in the domain of I, we have that $\neg\phi(u)$ holds in I
  Iff $I \models \forall x\neg\phi$

So, for every first order formula $\phi(x)$:

$$\neg\exists x\phi \equiv \forall x\neg\phi$$

General rule: negations can be "pushed through" universal quantifiers if we change the universal quantifier to an existential quantifier
Another general rule: negations can be "pushed through" existential quantifiers if we change the existential quantifier to a universal quantifier

3.1 Example

Consider the formula $\neg\exists x\forall y(\neg E(x,y) \vee M(y,y,z,x))$. We have

$$\begin{aligned} &\neg\exists x\forall y(\neg E(x,y) \vee M(y,y,z,x)) \\ \equiv\ &\forall x\neg\forall y(\neg E(x,y) \vee M(y,y,z,x)) \\ \equiv\ &\forall x\exists y\neg(\neg E(x,y) \vee M(y,y,z,x)) \end{aligned}$$

4 More complicated equivalences

Consider the formula $\forall x\phi \wedge \exists y\psi$ where $\phi(x)$ and $\psi(y)$ are first order formulae with free variables x and y, respectively.
By renaming bound variables (if necessary), we may assume that x does not appear in ψ and y does not appear in ϕ
Let I be some interpretation for $\forall x\phi \wedge \exists y\psi$
We have $I \models \forall x\phi \wedge \exists y\psi$ iff $I \models \forall x\phi$ and $I \models \exists y\psi$:

- $I \models \forall x\phi$ iff no matter which value from the domain of I we give to the variable x, we have that $\phi(x)$ holds in I
- $I \models \exists y\psi$ iff there exists some value from the domain of I for the variables y such that $\psi(y)$ holds in I

Thus: $I \models \forall x\phi \wedge \exists y\psi$ iff:
No matter which value we give to x, we have that $\phi(x)$ holds in I, and there exists some value for y such that $\psi(y)$ holds in I

Consider $\forall x\exists y(\phi \wedge \psi)$
Suppose that $I \models \forall x\exists y(\phi \wedge \psi)$
Choose any u for x. There exists a v for y such that $\phi(u) \wedge \psi(v)$ holds.
So, $I \models \forall x\phi \wedge \exists y\psi$
Hence, $\forall x\phi \wedge \exists y\psi \equiv \forall x\exists y(\phi \wedge \psi)$.
Indeed, by the same token, $I \models \forall x\phi \wedge \exists y\psi$ iff $I \models \exists y\forall x(\phi \wedge \psi)$.
General rule: Quantifications can be "pulled out" from inside logical connectives and the order of quantifiers doesn't matter, so long as the names of the quantified variables are not used elsewhere

5 Some more complicated equivalences

5.1 Example 1

If we assume that

- x does not appear in ψ and χ
- y does not appear in ϕ and χ
- z does not appear in ϕ and ψ

Applying this general rule yields:

$$\begin{aligned}(\forall x\phi \wedge \exists y\psi) \vee \forall z\chi &\equiv \forall x\exists y(\phi \wedge \psi) \vee \forall Z\chi \\ &\equiv \forall x\exists y\forall z((\phi \wedge \psi) \vee \chi)\end{aligned}$$

5.2 Example 2

Consider the formula $(\forall x\phi \vee \forall x\psi) \wedge \exists x\chi$
We can rename two of the bound occurrences of x to get

$$(\forall x\phi(x) \vee \forall y\psi(y)) \wedge \exists z\chi(z)$$

(assuming y and z do not appear in ϕ and χ, respectively).
Now we get the equivalent formulae

$$\begin{aligned}&(\forall x\phi(x) \vee \forall y\psi(y)) \wedge \exists\chi(z) \\ &\qquad\equiv \forall x\forall y(\phi(x) \vee \psi(y)) \wedge \exists\chi\chi(z) \\ &\qquad\equiv \forall x\forall y\exists z(\phi(x) \vee \psi(y) \wedge \chi(z))\end{aligned}$$

6 Be careful when applying general rules

Great care has to be taken when manipulating quantifiers:

- The order of the quantification matters
- Consider other occurrences of a quantified variable **outside the scope**

6.1 Example

Consider the first-order sentence $\forall x\exists yE(x,y)$
Let I be the interpretation with domain $\{1,2,3,4\}$ where $E = \{(1,2),(2,3),(3,4),(4,1)\}$
Clearly, $I \models \forall x\exists yE(x,y)$ but $I \not\models \exists x\forall yE(x,y)$
Consider the first order sentence $\forall x\exists yE(x,y) \wedge \forall z\neg E(z,z)$
Whilst $I \models \forall x\exists yE(x,y) \wedge \forall z\neg E(z,z)$
$I = \forall z\forall x\exists y(E(x,y) \wedge \neg E(z,z))$
$I = \forall x\forall z\exists y(E(x,y) \wedge \neg E(z,z))$
It is not the case that $I = \forall z\exists y\forall x(E(x,y) \wedge \neg E(z,z))$

7 More on bound occurrences

Consider the first order formula $\forall x\exists yE(x,y) \wedge \exists xU(x)$
It does not make sense to pull the quantifiers out, as we could get $\forall x\exists y\exists x(E(x,y) \wedge U(x))$
Actually, semantically this second sentence is logically equivalent to

$$\exists y\exists x(E(x,y) \wedge U(x))$$

(as existentially quantified x "overwrites" the universally quantified x) which is certainly not equivalent to the sentence we started with. To see this, consider the interpretation where the domain is $\{1,2\}$, $E = \{(1,2)\}$ and $U = \{1\}$
We need to ensure that the two original bound occurrences of x have "nothing to do with each other". In order to ensure this, we need to rename one of them:

$$\begin{aligned}\forall x\exists yE(x,y) \wedge \exists xU(x) &\equiv \forall x\exists yE(x,y) \wedge \exists z\mathrm{U}(z) \\ &\equiv \forall x\exists y\exists z(E(x,y) \wedge U(z))\end{aligned}$$