Sets

1 Some notation

- We write $x \in X$ to denote that x is an element or member of the set X, or that X contains x, with $x \notin X$ denoting that x is not an element of X
- We can describe a set by listing it's elements
  - the set of pairs of prime numbers less than 6 is $\{\{2,3\}\{2,5\},\{3,5\}\}$
- It is always possible if a set is finite
- However if a set is infinite, then it is not possible, unless we cheat by adding dots
- We often describe a set by it's defining property, e.g.:
  - The set of natural numbers $\mathbb{N} = \{x : x$ x is a natural number$\} = \{0,1,2,...\}$
  - The set of integers $\mathbb{Z} = \{x : x \in N$ or $x = -y$ for $y \in \mathbf{N}\} = \{0, 1, -1, 2, -2, \ldots\}$
  - The set of rational numbers $\mathbb{Q} = \{x : x = y/z$ for $y, z \in \mathbf{Z}$ with $z \neq 0\}$
  - The set of real numbers $\mathbb{R} = \{x : x$ is a real number $\}$
- We regard 0 as a natural number

2 Cardinality

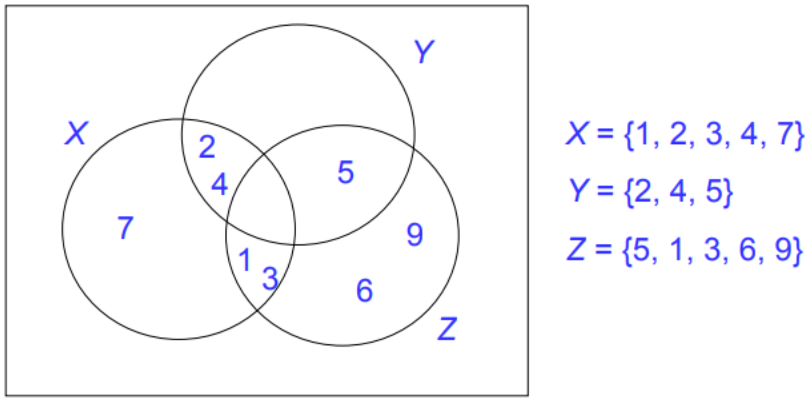
- If there are exactly n distinct elements in the set S, where $n \in \mathbf{N}$ then S is finite and has size or cardinality n and we write $|S| = n$
  - As we remarked earlier, if S is not finite then it is infinite
- Of course, the empty set $\emptyset$ has size 0
- We can also define the size of an infinite set
- One might be tempted to think that all infinite sets have the same size, however there are different sizes of infinity

3 Set Equality

- Two sets X and Y are equal when we write $X = Y$ iff they contain exactly the same elements
- Equivalently X and Y are equal iff:
  - for every object x, $x \in X$ implies that $x \in Y$
  - for every object x $x \in Y$ implies that $x \in X$
  - For example:
    - $\{1, 2, 3, 4, 5\} = 3, 2, 4, 5, 1$
- **Singleton Set** - A set containing exactly one element
- Note that strictly speaking $\{1, 3, 3, 5\}$ is not a set but a multiset, but we regard it as a description of the set $\{1, 3, 5\}$
- Recall also that our sets are objects and so we can have sets containing sets as elements, indeed, we can have sets containing sets as elements, e.g.
  - $\{\emptyset\} \neq \emptyset$
  - $\{\{\emptyset\}\} \neq \{\emptyset\}$

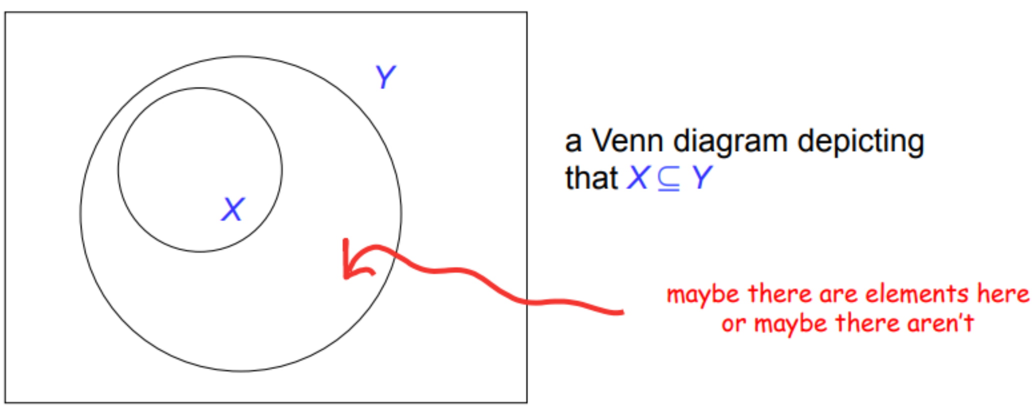
4 Venn Diagrams

- Sometimes it is useful to have a pictoral representation of a set or sets
- Any venn diagram is contained within (usually) a rectangle, depicting the set of all objects
- Sets are represented by circles and elements by points or items



5 Subsets

- A set X is a subset of set Y when we write $X \subseteq Y$ iff every element that is in X is also in Y
- So, X is not a subset of Y, when we write $X \subsetneq Y$ iff there is some element that is in X that is not in Y
- A subset X of Y is a proper subset when we write $X \subset Y$, if $X \subseteq Y$ and there may be at least one element of Y that is not in X



6 Some facts about subsets

- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
- For any set S $S \subseteq S$
- For any set S $\varnothing \subseteq S$
- For any sets A and B
  - $A = B$ iff $A \subseteq B$ and $B \subseteq A$
    - Trivially if $A = B$ then $A \subseteq B$ and $B \subseteq A$
    - Conversely suppose that $A \subseteq B$ and $B \subseteq A$
      - If $x \in A$ then $x \in B$
      - If $x \notin A$ then $x \notin B$

    * So A=B

- For every set A if $A \subseteq \emptyset$ then $A = \emptyset$
  - Suppose that $A \subseteq \emptyset$ and let $x \in A$, so $x \in \emptyset$, a contradiction

7 The Power Set

- There are a number of common operations upon sets which enable us to create new sets out of old ones
- Let S be a set. The **power set** P(S) (or P(S) or 2^s) is the set of all subsets of S
- We already have seen that every non empty set S has at least 2 subsets, $\emptyset$ and S
- However, in general, there are many more subsets, e.g:
  - If $S = \{0, 1, 2, 3\}$ then P(S) is all combinations of 0,1,2,3 and the empty set
  - If $S = \mathbb{N}$ then P(S) is any set of natural numbers
  - If $S = \emptyset$ then:
    * $P(S) = \{\emptyset\}$
    * $P(P(S)) = \{\emptyset, \{\emptyset\}\}$
    * $P(P(P(S))) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
- In general, if S is finite of size n, then $P(S)$ is finite of size 2^n

8 The Cartesian Product

- Often, the order of a collection of elements matters, though the order of the elements in a set is of no importance
- An ordered n-touple $(a_1, a_2, ..., a_n)$ is an ordered collection of elements
- If $n = 2(resp.n = 3)$ then we call the touple an ordered pair (resp touple)
- Two ordered n touples are equal iff $a_i = b_i$ for all $i = 1, 2, ..., n$
- Cartesian products allow us to talk about "order"
- For any two sets X and Y, the cartesian product $X \times Y$ is the set

$$\{(x, y) : x \in X \text{ and } y \in Y\}$$

- For example:
  - The Cartesian product of $\{0, 1, 2\}$ and $\{a, b\}$ is

$$\{(0, a), (1, a), (2, a), (0, b), (1, b), (2, b)\}$$

  - The Cartesian product of $\{a, b\}$ and $\{0, 1, 2\}$ is:

$$\{(a, 0), (a, 1), (a, 2), (b, 0), (b, 1), (b, 2)\}$$

- We can also define the Cartesian product of more than two sets
- Let $A_1, A_2, ..., A_n$ be sets. The Cartesian product $A_1 \times A_2 \times ... \times A_n$ is the set:

$$\{(a_1, a_2, \ldots, a_n) : a_i \in A_i, \text{ for all } i = 1, 2, \ldots, n\}$$

- If $A_1, A_2, ..., A_n$ are all finite sets with $|A_i| = m_i$ for $i = 1, 2, ..., n$ then:

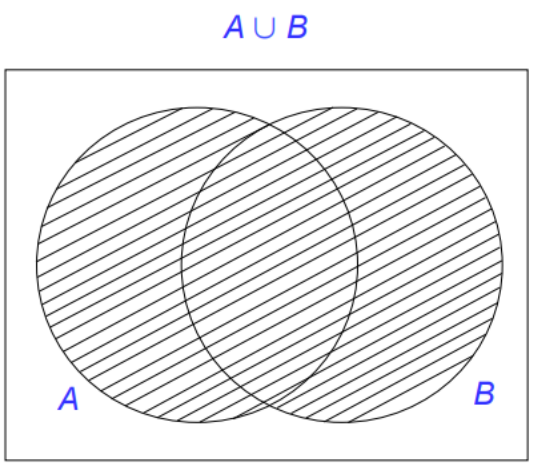
$$|A_1 \times A_2 \times \ldots \times A_n| = m_1 \times m_2 \times \ldots \times m_n$$

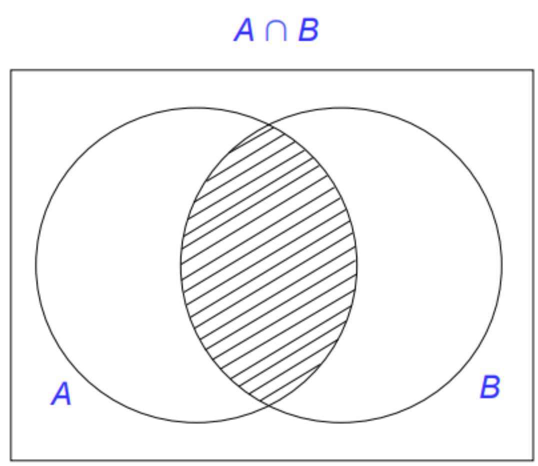
9 Union and Intersection

- Let A and B be sets, the union of A and B, written as $A \cup B$ is the set that contains all elements that are in A, in B or both
  - That is $A \cup B = \{x : x \in A \lor x \in B\}$
- Let A and B be sets. The intersection of A and B, written $A \cap B$ is the set of elements that are in A and B
  - That is, $A \cap B = \{x : x \in A \land x \in B\}$
- Two sets are called disjoint if their intersection is the empty set
- Principle of inclusion-exclusion: if A and B are finite sets then:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

9.1 Union and Intersection Venn Diagrams

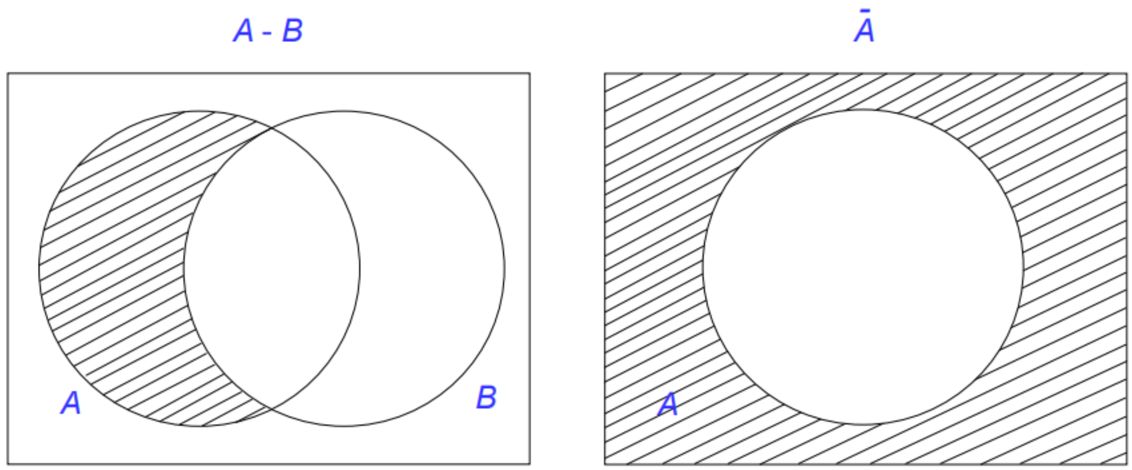




10 Difference and Compliment

- Let A and B be sets. The difference of A and B, written $A - B(or\ A \backslash B)$ is the set that contains all elements that are in A but not in B
  - That is, $A - B = \{x : x \in A \land x \notin B\}$
- Let A be a set. The compliment of A, written $\overline{A}$ is the set that contains all elements that are not in A
  - That is, $A = \{x : x \notin A\}$
- The difference A-B is sometimes called the **complement of B with respect to A**

10.1 Venn Diagram of Difference and Complement



11 A Different Semantics

- Note that we can define different semantics for propositional logic
- Consider some formula ϕ of propositional logic such as:

$$(X \wedge (Y \wedge Z)) \vee \neg(\neg X \vee (Y \wedge Z))$$

- Previously we interpreted ϕ using truth assignments, with a truth assignment making ϕ either true or false
- We can interpret ϕ by assigning sets to each of the propositional variables:
  - We get that ϕ denotes a set of elements via:
    - Interpret $\wedge$ as intersection
    - Interpret $\vee$ as union
    - Interpret $\neg$ as complement
- Write $\phi \equiv_s \psi$ iff ϕ and ψ always denote the same set of elements
- We get the identities from propositional logic

$$\phi \equiv_S \psi \text{ if, and only if, } \phi \equiv \psi$$

- So $(X \wedge (Y \wedge Z)) \vee \neg(\neg X \vee (Y \wedge Z))$ denotes the set of elements shown, i.e. X