Paths, Cycles and Connectivity

1 Walks, paths, cycles and distances

- A walk in a graph G is a sequence $v_0v_1, v_1, v_2, ..., v_{n-1}v_n$. In this case we also say that $v_0, v_1, ...v_n$ is a walk of G
- A walk $v_0, v_1, ..., v_n$ in G is a path is all v_i's are distinct. In this case we also say that $v_0, v_1, ..., v_n$ is a graph in G. A **path** is a walk that never crosses itself
- A walk $v_0, v_1, ..., v_n$ with $v_0 = v_n$ is called a **circuit** or **closed walk**
- A closed walk is a **cycle** if all v_i's in it are distinct except $v_0 = v_n$
- If G is a directed graph then the **directed paths** and **directed cycles** are defined in a natural way, with each edge being directed from v_i to v_{i+1}
- The **length** of a path or a cycle is the number of edges in it
- The distance between vertices u and v in a graph, denotes $dist(u, v)$, is the length of a shortest path from u to v if such a path exists, and ∞ otherwise
- The **diameter** of a graph is the largest distance between two vertices in it

2 The acquaintance graph and six degrees of separation

- All vertices are people
- There is an edge between two of them if they are acquainted

3 The collaboration graph and the Erdos number

- The vertices are all people
- There is an edge between two of then if they have written a joint paper

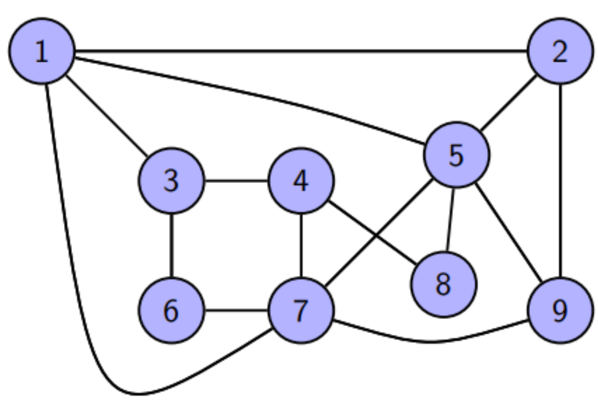
4 Shortest Path Problems

In an edge weighted graph, the problem of finding the shortest distance from u to v is the shortest path problem

5 Connectivity

5.1 Definition

A graph is called **connected** if any two distinct vertices are connected by a path. A **connected component** of a graph G is a maximal connected subgraph of G



- Is this graph connected?
- What about this graph?
- How many connected components does this graph have?

- Is connected - In one piece
- Removing 6-7, 5-8 and 4-7 makes two connected components, making it not connected

6 Exercise

δ - the smallest degree of a vertex in a graph

Prove that if G is a graph on n vertices are $\delta(G) \geqslant (n-1)/2$ then G is connected

- Proof **by contradiction:** Assume G is not connected, derive a contradiction
- Take two vertices u and v in different connected components
- We have that $deg(u) \geqslant (n-1)/2$ and $deg(v) \geqslant (n-1)/2$
- The set N(u) contains none of the neighbours of v nor v itself
- Similarly, N(v) contains none of the neighbours of u nor u itself
- Hence G contains at least $\frac{n-1}{2} + 1 + \frac{n-1}{2} + 1 = n + 1$ vertices, a contradiction
- In fact, we even proved more: any two vertices in G are at distance at most 2 (so the diameter of G is at most 2)

7 Strong connectivity

7.1 Definition

- A directed graph G is called **(weakly) connected** if the graph obtained from G by forgetting directions is connected
- A directed graph is called **strongly connected** if any two distinct vertices are connected by directed paths in both directions (can go from u to v and v to u)
- A **strongly connected component** (or simply **strong component**) of a digraph G is a maximal strongly connected subgraph of G

Graph

- 3,4,6,7 is a strongly connected component as a loop is formed, the only way out is to go to 8, which you cannot leave
- 2,1,5,9 is another strongly connected component
- 8 is a strong component on its own

8 Special circuits/cycles in graphs

- Can we travel along the edges of a given graph G so that we start and finish at the same verted and traverse **each edge exactly once**?
  - Such a circuit in G is called a Eulerian circuit
- Can we travel along the edges of a given graph so that we start and finish at the same vertex and visit **each vertex exactly once**
  - Such a called a **Hamiltonian** cycle