Discrete Structures - Relations

1 Binary relations

Let A and B be sets. A **binary relation R from A to B** is a subset of the cartesian product $A \times B$
(Or, equivalently, a binary relation R is a set of ordered pairs where the first element comes from A and the second element from B.)

- We write $(a, b) \in R$ or say that $R(a, b)$ **holds** if the ordered pair (a,b) is in the binary relation R
- We write $(a, b) \notin R$ or say that $R(a, b)$ **does not hold** or say that $\neg R(a, b)$ **holds** if the ordered pair (a, b) is not in R

2 Functions as binary relations

Functions can be viewed as binary relations.
If $f : A \to B$ then the graph of the function f is the binary relation $\{(a, f(a)) : a \in A\} \subseteq A \times B$
Conversely, not every binary relation R from A to B can be considered to be the graph of a function:

- we need R to have the property that every element a of A appears in (as the first component) **exactly** 1 element (a,b) of R
- if this is so then the corresponding function f is defined as $f(a)$ = the unique element $b \in B$ for which $(a, b) \in R$

In general, binary relations are generalizations of functions and can be used to describe a much wider class of relationships

3 Relations on a set

We can have **relations** involving **more** than two sets; that is, relations that are subsets of the Cartesian product of any number of sets.
If a relation R is a subset of $A_1 \times A_2 \times \cdots \times A_n$, for some $n \geq 1$ then we say that R has an arity n or is n-ary relation
Henceforth, we simply refer to all relations as simply 'relations' and only add terms such as 'binary' or 'n-ary' when required.
The **binary relation R on a set A** is a relation from A to A; that is, a subset of $A \times A$

We can also have n-ary relations on sets; that is, subsets of $A \times A \times \cdots \times A$ (repeated n times)

4 Properties of relations

4.1 Reflexive

A binary relation R on A is **reflexive** if $(a, a) \in R, \forall a \in A$

A relation R on A of arity greater than 2 can also be reflexive, we insist that $(a, a, \ldots, a) \in R, \forall a \in A$

4.2 Irreflexive

A binary relation R on A is irreflexive if $(a, a) \notin R, \forall a \in A$

A relation R on A of arity greater than 2 can also be irreflexive: we insist that $(a, a, \ldots, a) \notin R, \forall a \in A$

4.3 Symmetry and anti-symmetry

A binary relation R on A is symmetric if:

$$(a, b) \in R, \text{ then } (b, a) \in R \forall a, b \in A$$

A binary relation R on A is anti-symmetric if:

$$(a, b), (b, a) \in R, \text{ then } a = b \forall a, b \in A$$

(or we do not have $(a, b), (b, a) \in R$ if $a \neq b$)

4.4 Transitivity

A binary relation R on A is **transitive** if:

$$(a, b), (b, c) \in R \text{ then } (a, c) \in R, \forall a, b, c \in A$$

5 Combining Relations

As relations are just sets (of tuples), all operations that can be applied to sets can be applied to relations.
Also, just as functions can be **composed**, so can relations
Let $R \subseteq A \times B$ and $S \subseteq B \times C$ be relations. The **composite relation** $S \circ R \subseteq A \times C$ is defined as:

$$\{(a, c) : a \in A, c \in C, \exists b \in B \text{ s.t. } (a, b) \in R \text{ and } (b, c) \in S\}$$

6 Composing a relation with itself

Suppose that $R \subseteq A \times A$ We can **compose R with itself** so that

$$R \circ R = \{(a, c) : a, c \in A, \exists b \in A \text{ s.t. } (a, b), (b, c) \in R\}$$

We denote $R \circ R$ by R^2
In general, we denote $R \circ R \circ \cdots \circ R$ (repeated n times) by R^n (with R^0 and $\emptyset$ and R^1 defined as R)
Note that it does not matter the order we choose to build R^n. For example, $R \circ (R \circ R)$ is the same relation as $(R \circ R) \circ R$

- As an illustration, suppose that we have a **directed graph** G with vertex set V and edge set E. The edge set is a relation $E \subseteq V \times V$

7 Projections

Suppose that we have some n-ary relation R such that

$$R \subseteq A_1 \times A_2 \times \cdots \times A_n$$

We can build a new m-cry relation from R, where $m < n$, by projecting

- Choose $i_1, i_2, ..., i_m \in \{1, 2, ..., n\}$, where $i_1 < i_2 < .. < i_m$
- Build the m-ary relation $S \subseteq A_{i_1} \times A_{i_2} \times \cdots \times A_{i_m}$ by taking all those m-tuples that are obtained from some n-tuple of R bu only including the elements in components $i_1, i_2, ..., i_m$
- The relation S is the **projection** of R in components $i_1, i_2, ..., i_m$

8 Closures of relations

Let $R \subseteq A \times A$
The **reflexive closure** of R is the smallest reflexive relation that contains R. It is obtained by adding to R all the pair (x,x) that do not already lie in R
The **symmetric closure** of R is the smallest symmetric relation that contains R. It is obtained by adding to R all the pairs(x,y) for which (y,x) (but not (x,y)) lies in R
The **transitive closure** of R is the smallest transitive relation that contains R. It is the relation defined as

$$\{(a, b) : a, b \in A, (a, b) \in R^n, \text{ for some } n \geq 1\} = \bigcup_{n=1}^{\infty} R^n$$

9 Equivalence Relations

A relation $R \subseteq A \times A$ is called an **equivalence relation** if it is **reflexive, symmetric and transitive**
If R is an equivalence relation and $(a, b) \in R$ then a and b are **equivalent** and we sometimes write $a \equiv b$ or $a \sim b$ (note that $b \equiv a$ also)
Let $a \in A$. The **equivalence class** containing a, written as $[a]_R$ is the set of all elements z that are equivalent to a.
Note that all elements in $[a]_R$ are equivalent to each other
So, if $x \in [a]_R$ then $a \in [x]_R$ and $[a]_R = [x]_R$
In particular, no element of a can be in two different equivalence classes; that is, if $[a]_R \neq [b]_R$ then $[a]_R \cap [b]_R = \emptyset$
So the distinct equivalence classes of R partition A; that is, A can be written as the disjoint union of equivalence classes.

10 Partial orders

A binary relation that is reflexive, anti symmetric and transitive is called a partial order.
A set S together with a partial order R on S is called a partially ordered set (or **poset**) and written (S,R).

We often denote the partial order relation in a poset by $\leqslant$ even though we way not be referring to the usual ordering on numbers, and write $a \leqslant b$ rather than $\leqslant (a, b)$
If $(S, \leqslant)$ is some poset then two elements of S are comparable if either $a \leqslant b$ or $b \leqslant a$, and incomparable otherwise

11 Using posets in communication protocols

Suppose that we have two devices, A and B, sending messages to one another according to some protocol.
A **trace** of this system is a sequence of messages where each message comes with:

- the time it was sent and the time it was received

However, the two device clocks are not synchronised and might run at different speeds:

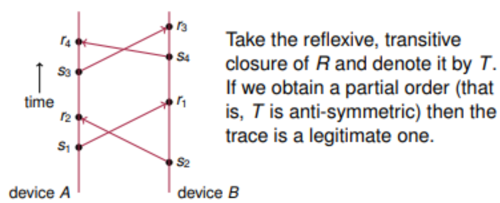
- so a time on device A cannot be compared with a time on device B

The delivery of a message takes a non-zero amount of time. A typical trace might be (with events in increasing time order)

- device A: send m_1; receive m_2; send m_3, receive m_4
- device B: send m_2; receive m_1; send m_4; receive m_3

We can think of our example trace as a relation R on a set E

$$E = \{s_1, r_1, s_2, r_2, s_3, r_3, s_4, r_4\}$$
$$R = \{(s_1, r_1), (s_2, r_2), (s_3, r_3), (s_4, r_4), (s_1, r_2)$$
$$(r_2, s_3), (s_3, r_4), (s_2, r_1), (r_1, s_4), (s_4, r_3)\}$$



12 Total and well orders

If $(S, \leqslant)$ is a poset, and further, every two elements in S are comparable then S is a **totally ordered set** or **linearly ordered set**, with $\leqslant$ a **total ordering** or **linear ordering**

- The poset $(\mathbb{Z}, \leqslant)$ is totally ordered (as $a \leqslant b$ or $b \leqslant a$ $\forall a, b \in \mathbb{Z}$)

If $(S, \leqslant)$ is a poset and, further, $\leqslant$ is a total ordering and every non empty subset of S has a least element (under $\leqslant$) then $(S, \leqslant)$ is a **well-ordered set**.

13 Lexicographic orders

If $(A, \leqslant_A)$ and $(B, \leqslant_B)$ are two posets then define the lexicographic ordering $\leqslant$ on $A \times B$ by $(a, b) \leqslant (a', b')$ if and only f

- $a \leqslant_A a'$ and $a \neq a'$, or
- $a = a'$ and $b \leqslant_B b'$

With $(a, b) = (a', b')$ if and only if $a = a'$ and $b = b'$
$(A \times B, \leqslant)$ is a poset.

Lexicographic orders can be extended to more than two posets.
Let $(A_j, \leqslant_i)$ be a poset, for i=1,2,...,n. Consider the cartesian product $A_1 \times A_2 \times ... \times A_n$, where $n \geqslant 2$. We say that $(a_1, a_2, ..., a_n) \leqslant (b_1, b_2, ..., b_n)$ in the lexicographic order $\leqslant$ on $A_1 \times A_2 \times ... \times A_n$ if and only if

- $a_1 = b_1, a_2 = b_2, ..., a_n = b_n$ or
- $\exists j \in \{1, 2, ..., n\}$ such that (A , \{}leq A) $a_1 = b_1, a_2 = b_2, \ldots, a_{j-1} = b_{j-1}, a_j \leq_j b_j, a_j \neq b_j$