Basis and dimension of a vector space

1 Basis

1.1 Definition

If V is a vector space and $S = \{v_1, ..., v_r\}$ is a set of vectors in V then S is a basis for V if

1. S is linearly independent
2. S spans V

1.2 Description

- The standard unit vectors form a basis for $\mathbb{R}^n$, called the standard basis
- The $m \times n$ matrices M_{ij} whose entries are all 0 except $a_{ij} = 1$ form the standard basis for the space $\mathbb{M}_{mn}$ of all $m \times n$ matrices
  Consider the case $m = n = 2$ (other cases are similar) Then:

$$M_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, M_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, M_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, M_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

  It is clear that $aM_{11} + bM_{12} + cM_{21} + dM_{22} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
  Any 2×2 matrix is a linear combination of our matrices (i.e. they span $\mathbb{M}_{22}$). If $aM_{11} + bM_{12} + cM_{21} + dM_{22}$ is the zero matrix then $a = b = c = d = 0$ (i.e. they are linearly independent)

1.3 Example

Show that the vectors $\mathbf{v}_1 = (1, 2, 1), \mathbf{v}_2 = (2, 9, 0), \mathbf{v}_3 = (3, 3, 4)$ form a basis in $\mathbb{R}^3$
For a vector $b = (b_1, b_2, b_3) \in \mathbb{R}^3$ consider the equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{b}$
The equation can be re-written as a linear system and in matrix form

$$\begin{array}{rrl} k_1 + 2k_2 & +3k_3 & = b_1 \\ 2k_1 + 9k_2 & +3k_3 & = b_2 \\ k_1 & +4k_3 & = b_3 \end{array} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

We need to check

1. linear independence, i.e. if b=0 then the system only has the trivial solution $k_1 = k_2 = k_3 = 0$, and
2. the vector span $\mathbb{R}^3$ i.e. the system has a solution for every b

- Condition (1) holds iff the matrix of the system has a non zero determinant
- If the determinant is non-zero then condition (2) holds too
- The determinant is -1, so we are done

2 Basis representation is unique

2.1 Theorem

If $S = \{v_1, ..., v_n\}$ is a basis for a vector space V then each vector $v \in V$ can be expressed as $\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n$ in exactly one way

2.2 Proof

S spans V, hence each vector can be represented as in at least one way. Assume some vector v has two different representations:

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n$$
$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

Subtracting one from the other, one gets

$$\mathbf{0} = (k_1 - c_1)\,\mathbf{v}_1 + (k_2 - c_2)\,\mathbf{v}_2 + \ldots + (k_n - c_n)\,\mathbf{v}_n$$

Since the two representations of v are different, we have $k_1 \neq c_i$ for some i. Then the last equality contradicts the fact that $S = \{v_1, ..., v_n\}$ is linearly independent

3 Coordinates

3.1 Definition

If $S = \{v_1, ..., v_n\}$ is a basis for a vector space V then the coordinates of a vector $v \in V$ relative to the basis S are the (unique) numbers $k_1, k_2, ..., k_n$ such that $\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n$ The vector $(\mathbf{v})_S = (k_1, k_2, \ldots, k_n) \in \mathbb{R}^n$ is the coordinate vector of v relative to S

3.2 Example

When $V = \mathbb{R}^n$ and S is the standard basis then v and $(v)_s$ are the same
In general $v \leftrightarrow (v)_s$ is a one to one correspondence between V and $\mathbb{R}^n$
How do we find the coordinates of a given vector relative to a given basis?

We checked that $\mathbf{v}_1 = (1, 2, 1), \mathbf{v}_2 = (2, 9, 0), \mathbf{v}_3 = (3, 3, 4)$ form a basis in $\mathbb{R}^3$. Find the coordinates of $v = (5, -1, 9)$ in this basis
We need to find numbers k_1, k_2, k_3 such that $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{v}$. This equation can be rewritten as a linear system

$$\begin{aligned} k_1 + 2k_2 \quad &+3k_3 = 5 \\ 2k_1 + 9k_2 \quad &+3k_3 = -1 \\ k_1 \qquad\quad &+4k_3 = 9 \end{aligned}$$

Solving the system, we get $k_1 = 1, k_2 = -1, k_3 = 2$, these are the coordinates of $\mathbf{v}$

4 Dimension

A vector space V is finite-dimensional if it can be spanned by a finite set of vectors. Otherwise, V is infinite-dimensional

4.1 Theorem

Let V be a finite-dimensional vector space and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be any basis in V

1. Any subset of V with more than n vectors is linearly independent
2. Any subset of V with fewer than n vectors does not span V

4.2 Corollary

All bases of a finite-dimensional vector space have the same number of vectors

4.3 Definition

The dimension of a finite-dimensional vector space V, denoted by $\dim(V)$, is the number of vectors in any of its bases. By convention, $\dim(\{0\}) = 0$

4.4 Examples

- $\dim(\mathbb{R}^n) = n$
- $\dim(\mathbb{M}_{mn}) = mn$

5 Plus/Minus Theorem

5.1 Theorem

Let S be a non empty set of vectors in a vector space V

1. If S is linearly independent and $\mathbf{v} \in V$ is not in $\text{span}(S)$ then $S \cup \{\mathbf{v}\}$ is also linearly independent
2. If some $\mathbf{v} \in S$ can be expressed as a linear combination of other vectors in S then $\text{span}(S) = \text{span}(S \backslash \{\mathbf{v}\})$

5.2 Corollary

Let V be an n-dimensional vector space and let S be a subset of V with exactly n vectors. If S is linearly independent or S spans V then S is a basis for V

6 Dimension of a subspace

6.1 Theorem

Let W be a subspace of a finite-dimensional vector space V. Then

1. W is finite-dimensional and $\dim(W) \leqslant \dim(V)$
2. $W = V$ iff $\dim(W) = \dim(V)$

7 Row space, column space and null space

7.1 Definition

Let A be an $m \times n$ matrix
The **row space** of A is the subspace of $\mathbb{R}^n$ spanned by the row vectors of A
The **column space** of A is the subspace of $\mathbb{R}^m$ spanned by the column vectors of A
The **null space** of A is the solution set of the linear system $ax = 0$

7.2 Example

Let

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{pmatrix}$$

The row vectors of A are

$$\mathbf{r}_1 = (2, 1, 0) \text{ and } \mathbf{r}_2 = (3, -1, 4)$$

The column vectors of A are

$$\mathbf{c}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{c}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \text{ and } \mathbf{c}_3 = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

8 Elementary row operations and the column space

Elementary row operations change neither the row space nor the null space of a matrix, but they can change the column space
However, elementary row operations do not change dependencies between column vectors of a matrix: if $\mathbf{w}_1, \ldots, \mathbf{w}_r$ are some column vectors of a matrix such that $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \ldots + c_r\mathbf{w}_r = \mathbf{0}$ holds before an operation then it also holds after

8.1 Theorem

Let A and B be row equivalent matrices. Then a given set of column vectors in A is linearly independent iff the corresponding set of column vectors in B is such.Similarly, a given set of column vectors in A is a basis for the column space of A iff the corresponding set of column vectors in B is a basis for the column space of B.

8.2 Theorem

If a matrix R is in row echelon form then the column (row) vectors with the leading 1s form a basis for the column (row, respectively) space of R.

9 Finding basis for the row, column and null spaces

In order to find a basis for the column space of a matrix A do

- Transform A (by elementary row operations) to row echelon form R
- Select all columns in R that have leading 1s
- The corresponding columns in A form a basis

In order to find a **basis for the row space** of a matrix A, do

- Transform A (by elementary) row operations to row echelon form R
- The rows in R with the leading 1s form a basis for the row space of A

In order to find the **basis for the null space** of A, do

- Find the general solution to the system $Ax = 0$
- For each free variable x, take the solution (vector $\mathbf{v}_x$) om which x=1 and the other free variables are set to 0
- These vectors $\mathbf{v}_x$ together form a basis for the null space

In order to find a basis for Span(S), where S is a set of vectors, do

- form a matrix whose row vectors are the vectors in S and then do as above

10 Rank and nullity

10.1 Theorem

The row space and column space of a matrix have the same dimension

10.2 Definition

The **rank** of a matrix A, denoted by rank(A), is the dimension of its row space.
The nullity of A, denoted by nullity(A), is the dimension of the null space of A

10.3 Lemma

For any $m \times n$ matrix A, rank(A) and nullity(A) are the numbers of leading and free variables,. respectively, in the general solution to $Ax = 0$

10.4 Theorem (Dimension Theorem for Matrices)

For any matrix A with n columns, rank(A)+nullity(A)=n

10.5 Proof

The system $Ax = 0$ has n variables. Now use the previous lemma