Matrix Inversion

1 Row equivalence of matrices

Recall that the three **elementary row operations** on a matrix are:

- Multiply a row my a non zero constant c
- Interchange two rows
- Add a constant c times one row r_1 to another row r_2

Observation: If B is obtained from A by using an elementary row operation then A can be obtained from B by using the **inverse elementary row operation**:

- Multiply the same row by a non zero constant $1/c$
- Interchange the same two rows
- Add -c times row r_1 to row r_2

Matrices A and B are called **row equivalent** if either (hence each) can be obtained from the other by a sequence of elementary row operations

2 Elementary matrices

An $n \times n$ matrix is called an **elementary matrix** if it is obtained from the identity matrix I_n by performing a single elementary row operation
Examples of elementary matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.1 Lemma

Suppose that an elementary matrix E is obtained from I_m by performing an elementary row operation. If A is an $m \times n$ matrix then the product EA is the matrix obtained from A by performing the same row operation

Thus, performing an elementary row operation has the same effect as multiplying by the corresponding elementary matrix (from the left)

3 Elementary matrices are invertible

3.1 Lemma

Every elementary matrix E is invertible, and the inverse is also elementary

3.2 Proof

By definition, E can be obtained from I by using some elementary row operation. Then I can be obtained from E by using the inverse elementary row operation. By the above, there is a matrix E_0 such that $E_0E = I$, hence E is invertible. We have $E_0 = E^{-1}$, so $EE_0 = I$, which implies that E_0 is also elementary

4 Theorem about invertible matrices

4.1 Theorem

If A is an $n \times n$ matrix, then the following are equivalent, i.e. all true or all false

1. A is invertible
2. The linear system $Ax = 0$ has only the trivial solution $x = 0$
3. The reduced row echelon form of A is I_n
4. A can be expressed as a product of elementary matrices
5. $det(A) \neq 0$

4.2 Proof

We will prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) and 1 $\Leftrightarrow$ (5) later today
(1) $\Rightarrow$ (2). Assume that A is invertible. If $Ax = 0$ then $x = A^{-1}Ax = A^{-1}0 = 0$

(2) $\Rightarrow$ (3). Assume that the system $Ax = 0$ has only the trivial solution $x = 0$. The augmented matrix of the system is $[A|0]$. If the reduced echelon form of this matrix is not $[I_n|0]$ then the system has a non trivial solution, which can't exist. Hence the reduced echelon form of $[A|0]$ is $[I_n|0]$, which immediately implies (3).

(3) $\Rightarrow$ (4) If I_n is obtained from A by a sequence of elementary row operations then there are elementary matrices $E_1, ...E_k$ such that

$$E_k \cdots E_2 E_1 A = I_n$$

We proved today that each E_i is invertible and each E_i^{-1} is elementary. Hence

$$A = E_1^{-1}E_2^{-1} \cdots E_k^{-1} I_n = E_1^{-1}E_2^{-1} \cdots E_k^{-1}$$

(4) $\Rightarrow$ (1). The product of invertible matrices is also invertible

5 Inversion algorithm

As an application of the above theorem, we give an algorithm which

1. decides whether a given matrix A is invertible
2. and, if so, finds the inverse A^{-1}

Assume that $E_k \cdots E_2 E_1 A = I_n$. Multiplying by A^{-1}, get $E_k \cdots E_2 E_1 I_n = A^{-1}$
Therefore, if a sequence of elementary row operations transforms A to I_n then the same sequence transforms I_n to A^{-1}
Inversion algorithm:

1. Write the matrix $[A|I_n]$
2. Apply elementary row operations to the whole matrix to transform its left half (i.e. A) to reduced row echelon form
3. If this form (of the left half) is not I_n then the matrix is not invertible
4. Otherwise, the obtained matrix is $[I_n|A^{-1}]$

5.1 Example

Find the inverse (if it exists) of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

We have $A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$

6 Determinants reminder

- The determinant of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the number

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- If A is a square matrix of order n, then the minor of the entry a_{ij} denoted by M_{ij}, is the determinant of the matrix (of order n-1) obtained from A by removing its ith row and jth column
- The number $C_{ij} = (-1)^{i+j} M_{ij}$ is called the cofactor of a_{ij}
- If A is an $n \times n$ matrix then the determinant of A can be computed by any of the following **cofactor expansions** along the ith row and along the jth column, respectively

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

- Easy to see: if A has a row of 0s or a column of 0s then det(A)=0
- Easy to see: it holds that $det(A) = det(A^T)$

7 Determinants and elementary row operations

How do elementary row operations affect the determinant of a square matrix?

7.1 Theorem

Let A be an $n \times n$ matrix

- If B is obtained from A by multiplying a row by a constant k then $det(B) = k \cdot det(A)$
- If B is obtained from A by interchanging two rows then $det(B) = -det(A)$
- If B is obtained from A by adding a multiple of one row to another row then $det(B) = det(A)$

7.2 Lemma

If $A = (a_{ij})$ is an $n \times n$ upper triangular matrix, i.e. $a_{ij} = 0$ whenever $i > j$ (all 0s under the diagonal), then $\det(A) = a_{11} \cdot a_{22} \cdots a_{(n-1)(n-1)} \cdot a_{nn}$

8 Computing determinants

The previous section suggests a strategy for computing the determinant of a matrix:

- Use elementary row operations to transform the matrix into row echelon form
- Record how determinant changes during the transformation
- The row echelon form is upper triangular, its determinant is easy to find

Example:

$$\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} = -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} = -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} = -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix} = (-3)\cdot(-55) = 165$$

We now have two ways of computing determinants:
by cofactor expansion (i.e. by definition) and by row reduction (as above)
They can be mixed: create many 0s by row reduction and use cofactor expansion

9 Determinants of elementary matrices

We have $det(I_n) = 1$. The following is a special case of the previous theorem

9.1 Corollary

Let E be an $n \times n$ elementary matrix

- If E is obtained from I_n by multiplying a row by a constant k then $det(E) = k$
- If E is obtained from I_n by interchanging two rows then $det(E) = -1$
- If E is obtained from I_n by adding a multiple of one row to another row then $det(E) = 1$

9.2 Lemma

If E and B are $n \times n$ matrices and E is elementary then $det(EB) = det(E)det(B)$

9.3 Proof

We consider only the 1st case from the above corollary, the other two are similar. If E is obtained from I_n by multiplying a row by k, then EB is obtained from B by the same operation, so $det(EB) = k \cdot det(B) = det(E)det(B)$

10 Inconvertibility criterion

The following theorem is (1) $\Leftrightarrow$ (5) from the theorem about invertible matrices

10.1 Theorem

A square matrix A is invertible iff $det(A) \neq 0$

10.2 Proof

Let R be the reduced row echelon form of A. We have the following facts:

- Either R=I (and $det(R) = 1$) of R contains a row of 0s (and $det(R) = 0$)
- A is invertible iff R=I, by the theorem about invertible matrices (1) $\Leftrightarrow$ (3)
- We know that $R = E_r \cdots E_2E_1A$ for some elementary matrices E_i
- $\det(R) = \det(E_r)\cdots\det(E_2)\det(E_1)\det(A)$ by the previous lemma
- $det(E_i) \neq 0$ for all i, so $det(R)$ and $det(A)$ are either both 0 or both non 0
- Finally, A is invertible $\Leftrightarrow R = I \Leftrightarrow \det(R) \neq 0 \Leftrightarrow \det(A) \neq 0$

11 Properties of determinants

11.1 Theorem

If A and B are square matrices of the same size then $det(AB) = det(A)det(B)$

11.2 Proof

- It can be shown that if A is invertible then neither is AB. In this case $det(A) = det(AB) = 0$
- Assume that A is invertible, then $A = E_1E_2 \cdots E_r$ for some elementary E_i
- Then $AB = E_1E_2 \cdots E_rB$ and $\det(AB) = \det(E_1)\det(E_2)\cdots\det(E_r)\det(B)$
- Since $\det(A) = \det(E_1)\det(E_2)\cdots\det(E_r)$, we have the required equality.

Applying the above theorem to the case when A is invertible and $B = A^{-1}$, we get

11.3 Corollary

If A is invertible then $det(A^{-1}) = 1/det(A)$

Note that $det(A + B) \neq det(A) + det(B)$ in general. Try $A = I_2$ and $B = -I_2$

12 Inverting a matrix via cofactors/adjoint

- If A is a square matrix of order n, then the **minor of the entry** a_{ij} denoted by M_{ij}, is the determinant of the matrix (of order n-1) obtained from A by removing its ith row and jth column
- The number $C_{ij} = (-1)^{i+j}M_{ij}$ is called the **cofactor of** a_{ij}
- The matrix
$$\text{cof}(A) = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & C_{n2} & \cdots & c_{nn} \end{pmatrix}$$
is called the **matrix of cofactors** of A
- The transpose of cof(A) is the **adjoint** of A, denoted by $adj(A)$

12.1 Theorem

If A is an invertible matrix then $A^{-1} = \dfrac{1}{det(A)} \cdot adj(A)$

12.2 Example

Find the inverse (if it exists) of the following matrix $A = \begin{pmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{pmatrix}$

We have computed $det(A) = 165$ earlier, so the inverse exists
We have:

$$\text{cof}(A) = \begin{pmatrix} -60 & 15 & 30 \\ 29 & -10 & 2 \\ 39 & 15 & -3 \end{pmatrix}, \text{ so adj } (A) = \begin{pmatrix} -60 & 29 & 39 \\ 15 & -10 & 15 \\ 30 & 2 & -3 \end{pmatrix}$$

Therefore,

$$A^{-1} = \frac{1}{165}\begin{pmatrix} -60 & 29 & 39 \\ 15 & -10 & 15 \\ 30 & 2 & -3 \end{pmatrix} = \begin{pmatrix} -60/165 & 29/165 & 39/165 \\ 15/165 & -10/165 & 15/165 \\ 30/165 & 2/165 & -3/165 \end{pmatrix}$$