Network flows

1 Flow networks, flow, maximum flow

- Material is transferred in a network from a ”source” to a ”sink”
- Source produces material at a steady rate, sink consumes at same rate

Edges have a given capacity

Vertices (other than source/sink) are junctions

- Material flows through them without collecting in them
- Entering rate = exiting rate

2 Definitions

Definition: Maximum-flow problem

We wish to compute the greatest possible rate of transportation from source to sink

Definition: Flow network

- G=(V,E)
- Two distinguished vertices: source s and sink t
- Each edge $(u,v) \in E$ as non-negative capacity $c(u,v) \geqslant 0$
- If$(u,v) \notin E$, we assume $c(u,v) = 0$
- For each $v \in V$, there is a path $s \rightarrow v \rightarrow t$

2.1 Flow constraints

Definition: Capacity constraint

For all $u,v \in V$, we require $f(u,v) \leqslant c(u,v)$
Flow from one vertex to another must not exceed given capacity

Definition: Skew symmetry

For all $u,v \in V$, we require $f(u,v) = -f(v,u)$
Flow from vertex u to vertex v is negative of flow in reverse direction

Definition: Flow conservation

For all $u \in V - \{s,t\}$ we require

$$\sum_{c \in V} f(u,v) = 0$$

Total flow out of a vertex is 0, likewise for total flow into a vertex (just saying what goes in, comes out), this doesn't apply to the source or drain

2.2 Total flows

Definition: Total positive flow

The total positive flow entering vertex v is

$$\sum_{u \in V : f(u,v) > 0} f(u,v)$$

The total positive flow leaving vertex u is

$$\sum_{v \in V : f(u,v) > 0} f(u,v)$$

Definition: Total net flow

The total net flow at a vertex v is
total positive flow leaving v - total positive flow entering v

Definition: Flow value

The value of flow f is defined as the total flow leaving the source (and thus entering the sink)

$$|f| = \sum_{v \in V} f(s,v)$$

Note that $|\cdot|$ does not mean absolute value

If there is an arrow only in one direction on the graph, then the capacity in the other direction is 0. There is no assumption of symmetric capacities.

3 Technical tools

Implicit summation

Let $X, T \subseteq V$. Then

$$f(X,Y) = \sum_{x \in X} \sum_{y \in Y} f(x,y)$$

Commonly occurring identities

1. For all $X \subseteq V$, we have $f(X,X) = 0$. Because each $f(u,v)$ and $f(v,u) = -f(u,v)$ cancel each other
2. For all $X, Y \subseteq V$, we have $f(X,Y) = -f(Y,X)$. Generalisation of $f(X,X) = 0$, with the same reasoning

3. For all $X, Y, Z \subseteq V$ with $X \cap Y = \varnothing$, we have
$$f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$$
$$f(Z, X \cup Y) = f(Z, X) + f(Z, Y)$$
Split summation into two: one over X, one over Y

Three important ideas:

1. Residual networks
2. Augmenting paths
3. Cuts

Method is iterative:

1. Start with $f(u, v) = 0$ for all $u, v \in V$
2. At each iteration, increase flow value by finding an augmenting path (a path from source to sink along which we can increase flow) and then augment flow along this path
3. Repeat until no augmenting paths can be found

4 Residual networks

Idea: Residual network consists of edges that can admit more flow

Formally: Consider vertices u and v. Amount of additional flow we can push from u to v before exceeding capacity $c(u, v)$ is residual capacity of (u, v)
$$c_f(u, v) = c(u, v) - f(u, v)$$
Note that when flow $f(u, v)$ is negative, then residual capacity $c_f(u, v)$ is greater than $c(u, v)$
Interpretation:

- Flow of $-x$ from u to v (i.e. $f(u, v) = -x < 0$)
- Implies flow of x from v to u (i.e. $f(v, u) = x > 0$)
- Can be cancelled by pushing x units from u back to v
- Can then push another $c(u, v)$ from u to v
- We can push in total $c_f(u, v) = c(u, v) + x > c(u, v)$ from u to v

Given flow network $G = (V, E)$ and flow f, the residual network of G induced by f is $G_f = (V, E_f)$ with
$$E_f = \Big\{(u, v) \in V \times V : c_f(u, v) > 0\Big\}$$
i.e. each residual edge that can admit flow that is strictly positive
$$G : \; u \underset{1}{\overset{3/5}{\rightleftharpoons}} v \;\Rightarrow\; G_f : \; u \underset{4}{\overset{2}{\rightleftharpoons}} v$$
Note that $|E_f| \leqslant 2|E|$

Lemma:
Let $G = (V, E)$ be a flow network, f be a flow in G, G_f be the residual network of G induced by f, and let f' be a flow in G_f. Then the flow sum $f + f'$ with
$$(f + f')(u, v) = f(u, v) + f'(u, v)$$

Is a flow in G with value $|f+f'|=|f|+|f'|$ **Proof**:
Must verify that skew symmetry, capacity constraints and conservation are obeyed
Skew symmetry:

$$\begin{aligned}(f+f')(u,v) &= f(u,v)+f'(u,v)\\ &= -f(v,u)-f'(v,u)\\ &= -\left(f(v,u)+f'(v,u)\right)\\ &= -\left(f+f'\right)(v,u)\end{aligned}$$

Capacity constraint:
Note f' is flow in $G_f r$, so $f'(u,v)\leqslant c_f(u,v)$. Since by def. $c_f(u,v)=c(u,v)-f(u,v)$

$$\begin{aligned}(f+f')(u,v) &= f(u,v)+f'(u,v)\\ &\leq f(u,v)+c_f(u,v)\\ &= f(u,v)+(c(u,v)-f(u,v))\\ &= c(u,v)\end{aligned}$$

Flow conservation:
For all $u\in V-\{s,t\}$

$$\begin{aligned}\sum_{v\in V}(f+f')(u,v) &= \sum_{v\in V}\left(f(u,v)+f'(u,v)\right)\\ &= \sum_{v\in V}f(u,v)+\sum_{v\in V}f'(u,v)\\ &= 0+0\\ &= 0\end{aligned}$$

Value

$$\begin{aligned}\left|f+f'\right| &= \sum_{v\in V}(f+f')(s,v)\\ &= \sum_{v\in V}\left(f(s,v)+f'(s,v)\right)\\ &= \sum_{v\in V}f(s,v)+\sum_{v\in V}f'(s,v)\\ &= |f|+\left|f'\right|\end{aligned}$$

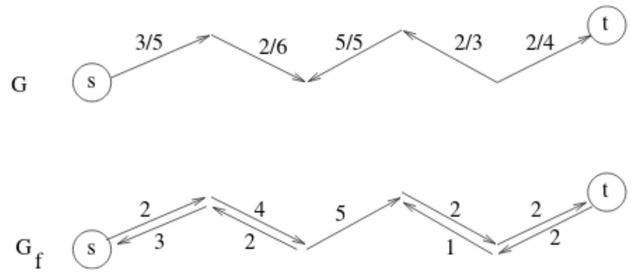
5 Augmenting Paths

Given a flow network $G(V,E)$ and flow f an augmenting path P is a simple path in residual network G_f

Recall that each edge (u,v) in G_f admits some additional positive flow, obeying capacity constraint

Flow value can be increased by

$$c_f(P)=\min_{(u,v)\in P}c_f(u,v)$$



Lemma:

Let $G = (V, E)$ to be a flow network, f be a flow in G, and let P be an augmenting path in G_f. Define f_p by

$$f_P(u,v) = \begin{cases} c_f(P) & \text{if } (u,v) \text{ is on } P \\ -c_f(P) & \text{if } (v,u) \text{ is on } P \\ 0 & \text{otherwise} \end{cases}$$

Then f_p is a flow in G_f with value $|f_p| = c_f(P) > 0$ **Corollary (Improve)**
Let G, f, P, f_p be as above, Define $f' = f + f_p$. Then f' is a flow in G with value $|f'| = |f| + |f_p| > |f|$

6 Basic Ford-Fulkerson algorithm

Listing 1 Ford-Fulkerson(G,s,t)

```
1  for each edge (u,v) ∈ E do
2      f(u,v)=0
3      f(v,u)=0
4  end for
5  while there exists a path P = $s \rightsquigarrow t$ in the residual network $G_f$ do
6      $c_f$(P)=min\{$c_f$(u,v):(u,v)\in P\}
7      for each edge $(u,v) \in P$ do
8          f(u,v)=f(u,v)+$c_f$(P)
9          f(v,u)=-f(u,v)
10     end for
11 end while
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6.1 Running Time

The running strongly depends on how the augmenting paths in line 4 are determined

If chosen poorly:

- Value of flow increases with each iteration
- But possibly too slowly
- In extreme cases it can never terminate and it can even not converge to the value of maximum flow (can only happen if capacities are irrational numbers)

In practice

- Capacities are integers so always terminates
- If capacities are small then it is an efficient algorithm (polynomial time)

6.1.1 Running time analysis

Assume integral capacities. Simple bound $O(E \cdot |f^*|)$ for running time when choosing paths arbitrarily, $|f^*|$ being the value of maximum flow

- Initialisation in lines 1-4 take $O(E)$
- While loop in lines 5-11 is executed at most $|f^*|$ times (value of flow increases by at least one unit in each interaction)
- In lines 6-10 (within the while loop) we need $O(V + E) = O(E)$ time.

6.1.2 Problematic case

Algorithm needs $\Omega(E|f^*|)$ time in worst case, because (u, v) is always chosen to be part of augmenting path.

Note that the value of the maximum flow f* can be arbitrarily large (might even be exponential on the size of the network)

This algorithm has not polynomial running time in the worst case

7 Edmonds-Karp algorithm

- A special implementation of Ford-Fulkerson algorithm
- The augmenting path P is always chosen to be a shortest path from s to t (e.g. using BFS) in the residual network G_f
- Regardless of the flow that fits in this path P

This has running time $O(V \cdot E^2)$

8 Cuts of flow networks

A cut of a flow network $G = (V, E)$ is a partition of V into the sets S and T=V-S such that $s \in S$ and $t \in T$

If f is a flow in G, then $f(S, T)$ is the net flow across cut (S,T); its capacity is c(S,T)

A minimum cut is a cut with minimum capacity over all cuts

Lemma: Net Flow
Let f be a flow in a flow network G with source s and sink t, let (S,T) be a cut of G. Then the flow across (S,T) is f(S,T)=$|f|$

Corollary (Upper Cut)
The value of a flow f in a network G is upper bounded by the capacity of any cut (S,T) in G

Proof

$$|f| = f(S,T) = \sum_{u\in S}\sum_{v\in T} f(u,v) \leq \sum_{u\in S}\sum_{v\in T} c(u,v) = c(S,T)$$

Therefore max-flow $\leqslant$ min-cut
Theorem: Max-flow min-cut
If f is a flow in a flow network $G = (V, E)$ with source s and sink t, then the following conditions are equivalent

1. f is a maximum flow in G
2. The residual network G_f contains no augmenting path
3. $|f| = c(S, T)$ for some cut (S, T) of G