Machine Architecture
Number Systems

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Binary

So far a real CPU looks very much like the LMC, but with a few more instructions.

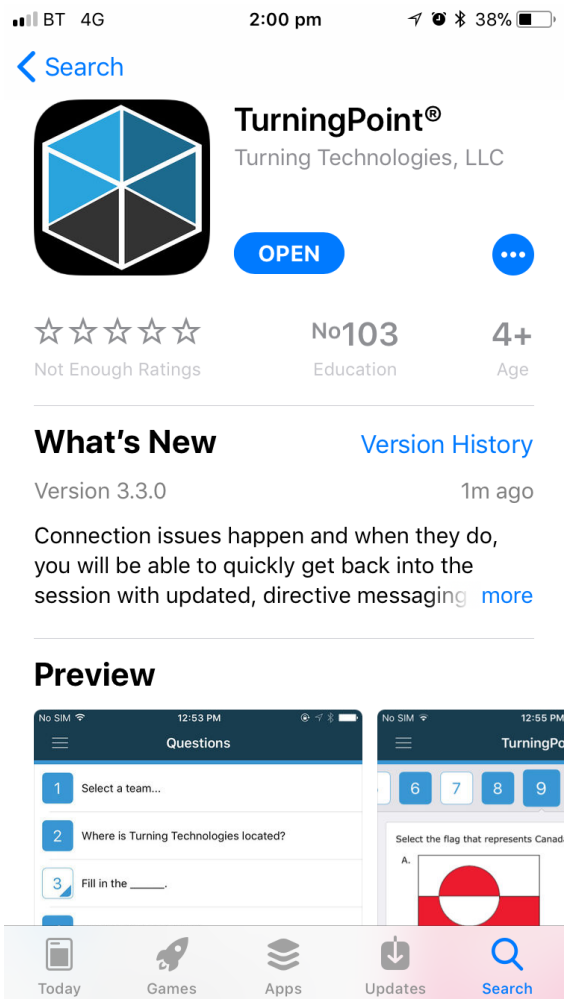
Of course, in a real CPU, everything is in **binary**…

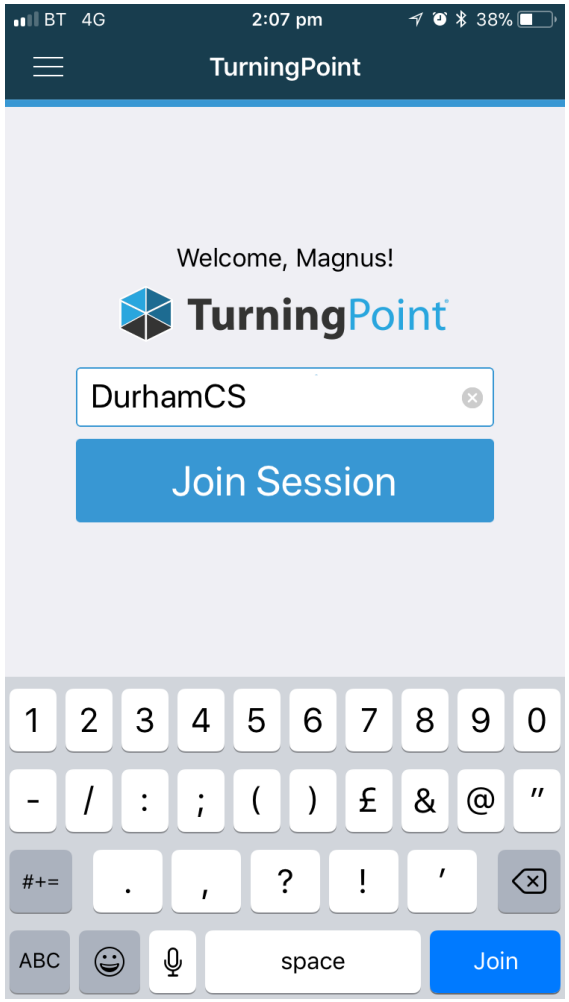
So we need to know how to:

- represent numbers in binary
- represent negatives
- add, subtract, multiply and divide binary numbers

See the CPU animation in machine code.

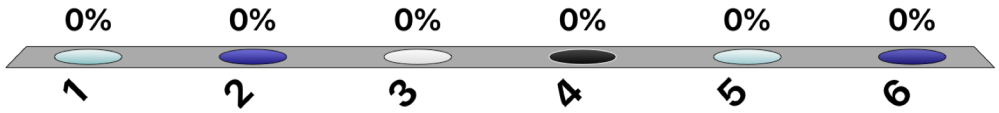
TurningPoint





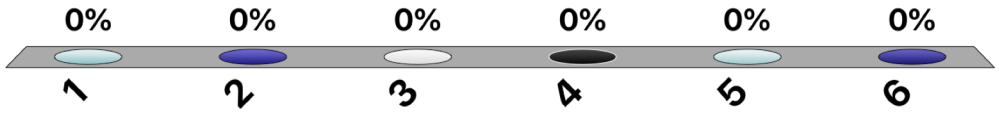
What is decimal 10 in binary?

Rank	Responses
1	
2	
3	
4	
5	
6	OTHER



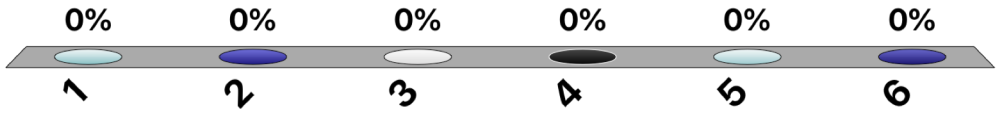
What is binary 111 in decimal?

Rank	Responses
1	
2	
3	
4	
5	
6	OTHER



What is hex 1B in decimal?

Rank	Responses
1	
2	
3	
4	
5	
6	OTHER



Number systems

This lecture's objective: develop an understanding of the different number systems used when discussing computers including:

- **Decimal** – used by humans
- **Binary** – used when discussing issues close to the machine
- **Hexadecimal** – used when humans are try to interpret what is happening in the machine

Decimal: 232

Binary: 11101000

Hex: E8

Roman: CCXXXII

All represent the **same number of objects.**

Decimal

Ten unique symbols 0 1 2 3 4 5 6 7 8 9

What does the decimal notation 432.75 **mean**?

The “decimal point”, in general called **the radix point**, indicates the position of the ‘units’, immediately to the left of the radix point.

So we have 2 ‘units’

What is the 3?	10s
What is the 4?	100s
The 7?	1/10ths
The 5?	1/100ths

Positional number systems

We start with a particular ordered set of symbols. E.g. a,b,c or 0,1,2

The **base** (or **radix**) of the number system is the number of symbols (including 0).

E.g. 10 for decimal (0,1,2,3,4,5,6,7,8,9), 2 for binary (0,1).

We use **positional number systems** to represent values

$$cab.bc_3 \quad \text{or} \quad 201.12_3$$

Note: subscript after a number gives the base

The contribution of a symbol *x,* which is the i^{th} symbol in the order, is $(i-1)*\text{base}^{\text{position}}$, where position is number of places to the **left** of the units.

E.g $cab.bc_3$ is $1*3^0+0*3^1+2*3^2 + 1*3^{-1} + 2*3^{-2}$

$= 1+0+18+1/3+2/9 = 19\ 5/9$

Binary

2-symbol positional number system: symbols 0,1

Position	2	1	0	.	-1	-2
Base^{position}	2^2	2^1	2^0	.	2^{-1}	2^{-2}
Decimal value	4	2	1	.	.5	.25
Example	1	1	0	.	1	1

110.11_2 =

$1 * 2^2 = 1 * 4 = 4.$

$1 * 2^1 = 1 * 2 = 2.$

$0 * 2^0 = 0 * 1 = 0.$

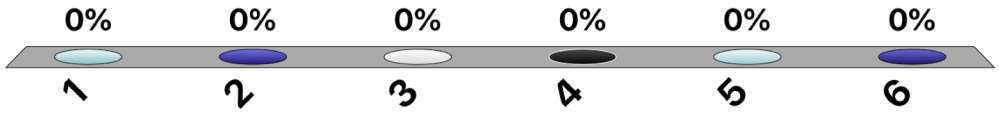
$1 * 2^{-1} = 1 * .5 = 0.5$

$1 * 2^{-2} = 1 * .25 + = 0.25$

6.75_{10}

What is 101000.01_2 in decimal?

Rank	Responses
1	
2	
3	
4	
5	
6	OTHER



Binary

Exercise:

What is 11101000.101_2 in decimal?

It is $0+0*2+0*2^2+1*2^3+0*2^4+1*2^5+1*2^6+1*2^7 \; + \; 1*2^{-1}+0*2^{-1}+1*2^{-3}$

i.e. 8 + 32 + 64 + 128 + 1/2 + 1/8 = 232 5/8 in decimal + fraction

Binary

Each digit in a binary number system is known as a bit

- **B**inary dig**IT**

A bit can have only one of two possible values

- 0 or 1 (sometimes referred to as false & true or off & on).

Groups of bits are known as:

- **Nibble** – 4 bits, 2^4 = 16 possible values.
- **Bytes** – 8 bits, 2^8 = 256 possible values.
- **Half word** – 16 bits, 2^{16} = 65,536 possible values.
- **Word** – 32 bits, 2^{32} = 4,294,967,296 possible values.
- **Double word** – 64 bits, 2^{64} = 18,446,744,073,709,551,616 values.

Note: “word” is CPU dependent – e.g. sometimes refers to 16 or 64 bits

Hexadecimal

16 distinct symbols: **0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F**

Why do we need Hexadecimal?

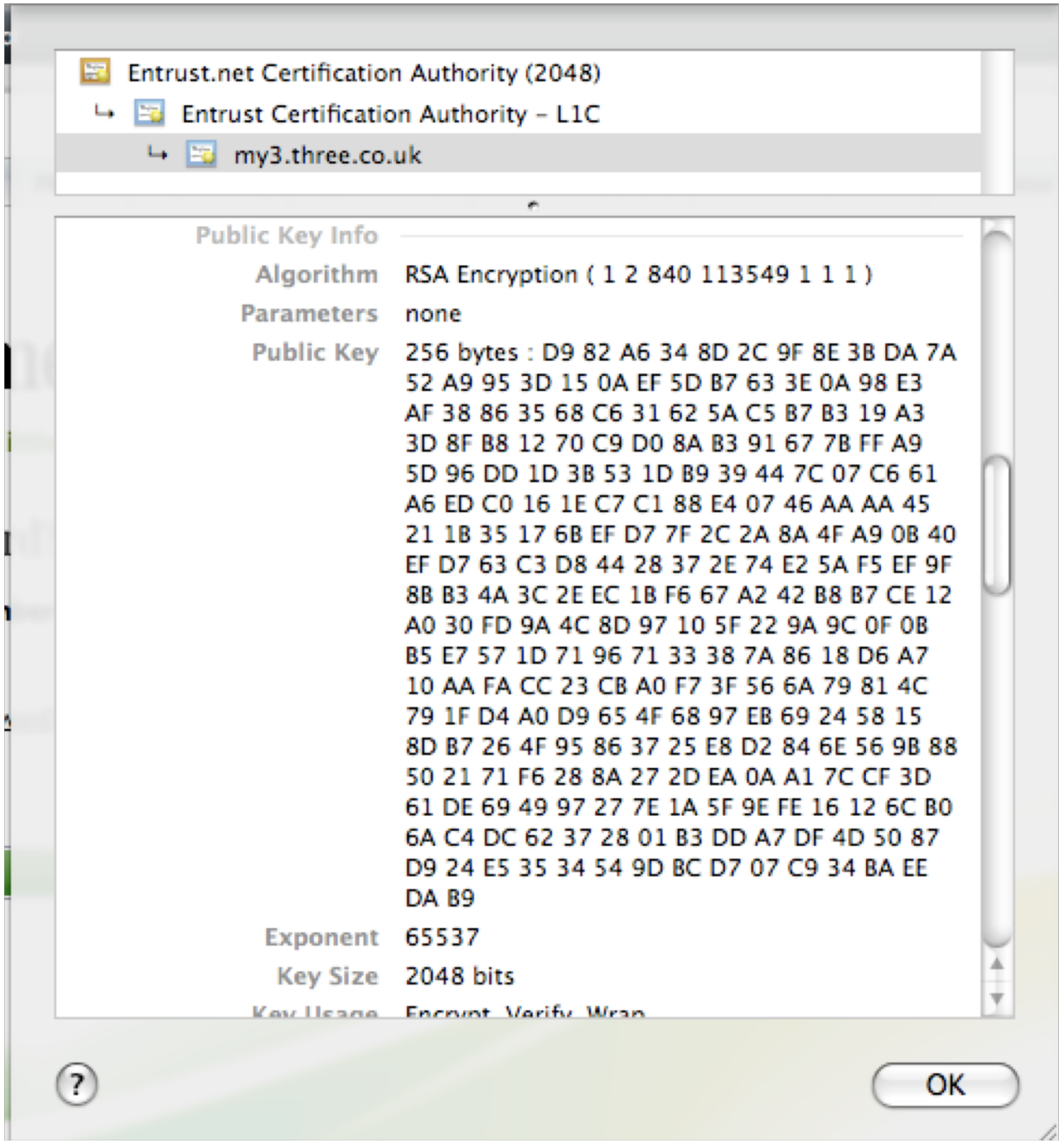
- Reading and writing binary values is difficult for humans

Advantages to using Hexadecimal

- **More compact** that other number systems
- **Easy to convert** between binary and hexadecimal

Programmers must be aware of what they are writing

- BEEF and $BEEF_{16}$ have very different meanings
- In Java use a prefix to denote a hexadecimal value: **0x**BEEF = $BEEF_{16}$


Entrust.net Certification Authority (2048)
Entrust Certification Authority – L1C
my3.three.co.uk
Public Key Info
Algorithm
RSA Encryption (1 2 840 113549 1 1 1)
Parameters
none
Public Key
256 bytes : D9 82 A6 34 8D 2C 9F 8E 3B DA 7A
52 A9 95 3D 15 0A EF 5D B7 63 3E 0A 98 E3
AF 38 86 35 68 C6 31 62 5A C5 B7 B3 19 A3
3D 8F B8 12 70 C9 D0 8A B3 91 67 7B FF A9
5D 96 DD 1D 3B 53 1D B9 39 44 7C 07 C6 61
A6 ED C0 16 1E C7 C1 88 E4 07 46 AA AA 45
21 1B 35 17 6B EF D7 7F 2C 2A 8A 4F A9 0B 40
EF D7 63 C3 D8 44 28 37 2E 74 E2 5A F5 EF 9F
8B B3 4A 3C 2E EC 1B F6 67 A2 42 B8 B7 CE 12
A0 30 FD 9A 4C 8D 97 10 5F 22 9A 9C 0F 0B
B5 E7 57 1D 71 96 71 33 38 7A 86 18 D6 A7
10 AA FA CC 23 CB A0 F7 3F 56 6A 79 81 4C
79 1F D4 A0 D9 65 4F 68 97 EB 69 24 58 15
8D B7 26 4F 95 86 37 25 E8 D2 84 6E 56 9B 88
50 21 71 F6 28 8A 27 2D EA 0A A1 7C CF 3D
61 DE 69 49 97 27 7E 1A 5F 9E FE 16 12 6C B0
6A C4 DC 62 37 28 01 B3 DD A7 DF 4D 50 87
D9 24 E5 35 34 54 9D BC D7 07 C9 34 BA EE
DA B9
Exponent
65537
Key Size
2048 bits
Key Usage
Encrypt, Verify, Wrap
?
OK

Hexadecimal

Position	2	1	0	.	-1	-2
$Base^{position}$	16^2	16^1	16^0	.	16^{-1}	16^{-2}
Decimal Value	256	16	1	.	.0625	.00390625
Example	C	2	D	.	1	0

$C * 16^2 = 12 * 256 = 3072.$

$2 * 16^1 = 2 * 16 = 32.$

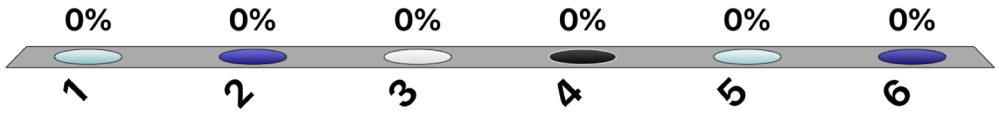
$D * 16^0 = 13 * 1 = 13.$

$1 * 16^{-1} = 1 * .0625 = .0625$

3117.0625_{10}

What is hex 10A.8 in decimal?

Rank	Responses
1	
2	
3	
4	
5	
6	OTHER



Hexadecimal

Exercise:

What is $1F8C.C_{16}$ in decimal?

It is $12+8*16+15*16^2+1*16^3 + 12*16^{-1}$

i.e. 12 + 8*16 + 15*256 + 1*4096 + 12/16

= 12 + 128 + 3840 + 4096 + 3/4 = 8076 3/4 in decimal + fraction

GCHQ challenge

Can you crack it?

```
eb 04 af c2 bf a3 81 ec   00 01 00 00 31 c9 88 0c
0c fe c1 75 f9 31 c0 ba   ef be ad de 02 04 0c 00
d0 c1 ca 08 8a 1c 0c 8a   3c 04 88 1c 04 88 3c 0c
fe c1 75 e8 e9 5c 00 00   00 89 e3 81 c3 04 00 00
00 5c 58 3d 41 41 41 41   75 43 58 3d 42 42 42 42
75 3b 5a 89 d1 89 e6 89   df 29 cf f3 a4 89 de 89
d1 89 df 29 cf 31 c0 31   db 31 d2 fe c0 02 1c 06
8a 14 06 8a 34 1e 88 34   06 88 14 1e 00 f2 30 f6
8a 1c 16 8a 17 30 da 88   17 47 49 75 de 31 db 89
d8 fe c0 cd 80 90 90 e8   9d ff ff ff 41 41 41 41
```

This is machine code written in hexadecimal.

GCHQ challenge

```
00171D4F   EB 04              JMP SHORT 00171D55
00171D51   AF                 SCAS DWORD PTR ES:[EDI]
00171D52   C2 BFA3            RETN 0A3BF
00171D55   81EC 00010000      SUB ESP,100
00171D5B   31C9               XOR ECX,ECX
00171D5D   880C0C             MOV BYTE PTR SS:[ESP+ECX],CL
00171D60   FEC1               INC CL
00171D62  ^75 F9              JNZ SHORT 00171D5D
00171D64   31C0               XOR EAX,EAX
00171D66   BA EFBEADDE        MOV EDX,DEADBEEF
00171D6B   031C0C             ADD EBX,DWORD PTR SS:[ESP+ECX]
00171D6E   00D0               ADD AL,DL
00171D70   C1CA 08            ROR EDX,8
00171D73   8A1C0C             MOV BL,BYTE PTR SS:[ESP+ECX]
```

This code implements the Rivest Cipher 4 algorithm

Translation from Binary to Hex

1. Starting from the **radix point**, separate the binary number into groups of **four** binary digits (nibbles)
2. Then **translate each group** (nibble) into it hexadecimal equivalent, group by group, maintaining right to left order

Example:

$$11\ 0101\ 1101\ 1000.001_2 = 35D8.2_{16}$$

0011 0101 1101 1000. 0010
3 5 D 8 . 2

Translation from Hex to Binary

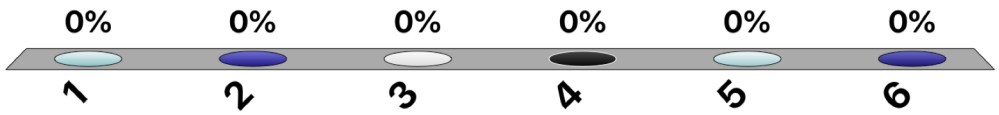
1. Starting from the **radix point**, separate the hexadecimal number into digits.
2. Then translate each digit into a **4-digit binary nibble**, maintaining right to left order

Example:

$EF02A.B4_{16} = 1110\ 1111\ 0000\ 0010\ 1010\ .\ 1011\ 0100_2$

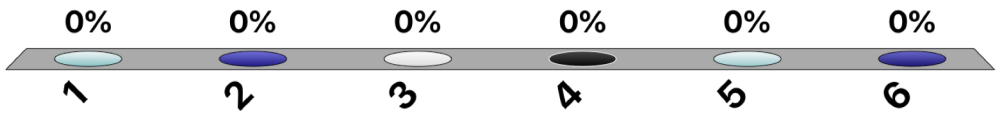
What is binary 111100.01 in hex?

Rank	Responses
1	
2	
3	
4	
5	
6	OTHER



What is hex BB.8 in binary?

Rank	Responses
1	
2	
3	
4	
5	
6	OTHER



Converting Decimal to Binary

- Repeatedly divide the number by 2, until you reach 0 / 2
- Put the remainders down **right to left** from radix point

Example: Convert 13_{10} to its binary representation

		remainder	
13/2	= 6	1	digit closest to radix point
6/2	= 3	0	
3/2	= 1	1	
1/2	= 0	1	left most digit

Result is 1101_2

Decimal Fractions to Binary

- Repeatedly multiply the number by 2, until fractional part is 0.
- If the ith result is greater than or equal to 1, place 1 in the ith position to the right of the radix, retain only fractional part.
- Else, place a 0 in the ith position to the right of the radix.

Example: 0.40625_{10}

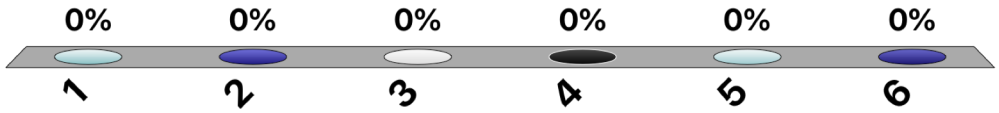
0.40625*2	= 0.8125	0
0.8125*2	= 1.625	1
0.625*2	= 1.25	1
0.25*2	= 0.5	0
0.5*2	= 1.0	1

Answer 0.01101_2

Other bases:

What is 404_7 in decimal?

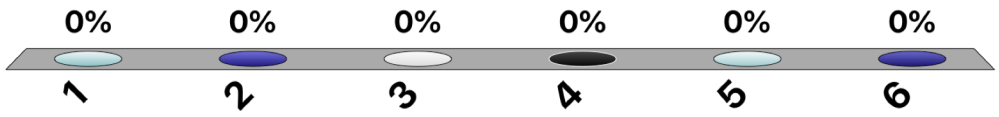
Rank	Responses
1	
2	
3	
4	
5	
6	Other



Other bases:

What is 119_{10} in base 6?

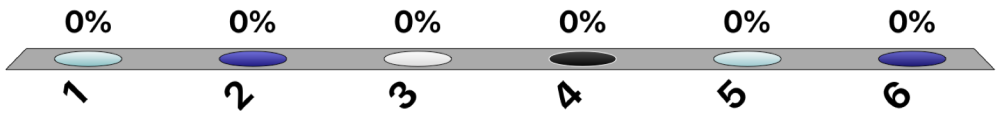
Rank	Responses
1	
2	
3	
4	
5	
6	Other



Other bases:

What is 232_4 in base 5?

Rank	Responses
1	
2	
3	
4	
5	
6	Other



Animated examples

http://courses.cs.vt.edu/~csonline/NumberSystems/Lessons/index.html