Resolution for Propositional Logic

1 Resolution

Recall the rule of inference known as resolution:

$$\frac{p \vee q \qquad \neg p \vee r}{q \vee r}$$

- Forms the basis of the proof system for propositional logic known as **resolution**

However, the basic rule of resolution is a more general one than that above

$$\frac{p_1 \vee \dots \vee p_{i\text{-}1} \vee \boxed{x} \vee p_{i+1} \vee \dots \vee p_m \qquad q_1 \vee \dots \vee q_{i\text{-}1} \vee \boxed{\neg x} \vee q_{i+1} \vee \dots \vee q_n}{p_1 \vee \dots \vee p_{i\text{-}1} \vee p_{i+1} \vee \dots \vee p_m \vee q_1 \vee \dots \vee q_{i\text{-}1} \vee q_{i+1} \vee \dots \vee q_n}$$

- The ps and qs are literals - that is, variables or negated variables (not necessarily distinct)
- This is the **only** rule of resolution

This idea can also be implied with just x and $\neg x$ the empty set can be inferred as there is a contradiction.

2 The proof system Resolution

- Natural deduction proves theorems starting from scratch, whereas resolution takes a given formula and works with it in order to decide whether it is a theorem or not
- In the proof system resolution, we proceed as follows
  - We are given a propositional formula φ
  - We take $\neg\varphi$ and write it in cnf as $C_1 \wedge C_2 \wedge ... \wedge C_m$
  - We start with the clauses $C_1, C_2, ..., C_m$
  - We continually apply the resolution rule of inference to infer new clauses
    * If we infer the empty clause $\varnothing$, then we halt and output that φ is a theorem
    * If we get to the point where we have not inferred the empty clause and we cannot infer any new clauses then we halt and output that φ is not a theorem
- We have one minor remark
  - When resolving, we are also allowed to delete the repeated literals in any clause
- Resolution is both sound and complete
  - if Resolution announces that φ is a theorem then φ is a tautology
  - if φ is a tautology then Resolution announces that φ is a theorem

3 Resolution in action

Consider the propositional formula φ

$$\begin{array}{l}(((A \wedge W) \Rightarrow I) \wedge (\neg A \Rightarrow P) \wedge (\neg W \Rightarrow S) \wedge \neg I \wedge (D \Rightarrow (\neg P \wedge \neg S))) \Rightarrow \neg D \\ (\quad \varphi_1 \quad \wedge \quad \varphi_2 \quad \wedge \quad \varphi_3 \quad \wedge \quad \varphi_4 \quad \wedge \quad \varphi_5 \quad) \Rightarrow \quad \psi \\ \neg(\quad \varphi_1 \quad \wedge \quad \varphi_2 \quad \wedge \quad \varphi_3 \quad \wedge \quad \varphi_4 \quad \wedge \quad \varphi_5 \quad) \vee \quad \psi\end{array}$$

So $\neg\varphi$ is

$$((A \wedge W) \Rightarrow I) \wedge (\neg A \Rightarrow P) \wedge (\neg W \Rightarrow S) \wedge \neg I \wedge (D \Rightarrow (\neg P \wedge \neg S))) \wedge D$$
$$\equiv (\neg(A \wedge W) \vee I) \wedge (A \vee P) \wedge (W \vee S) \wedge \neg I \wedge (\neg D \vee (\neg P \wedge \neg S) \wedge D$$
$$\equiv (\neg A \vee \neg W \vee I) \wedge (A \vee P) \wedge (W \vee S) \wedge \neg I \wedge (\neg D \vee \neg P) \wedge (\neg D \vee \neg S) \wedge D$$

So, the set of clauses to which we apply resolution is:

$$\neg A \vee \neg W \vee I \qquad A \vee P \qquad W \vee S \qquad \neg I \qquad \neg D \vee \neg P \qquad \neg D \vee \neg S \qquad D$$

4 Resolution in action

So, we have our set of clauses

$$\neg A \vee \neg W \vee I \qquad A \vee P \qquad W \vee S \qquad \neg I \qquad \neg D \vee \neg P \qquad \neg D \vee \neg S \qquad D$$

Now we start resolving

- $\neg A \vee \neg W$ ($\neg A \vee \neg W \vee I$ and $\neg I$)
- $P \vee \neg W$ ($\neg A \vee \neg W$ and $A \vee P$)
- $P \vee S$ ($P \vee \neg W$ and $W \vee S$)
- $\neg D \vee S$ ($P \vee S$ and $\neg D \vee \neg P$)
- $\neg D \vee \neg D$ ($\neg D \vee S$ and $\neg D \vee \neg S$)
- $\neg D$
- $\emptyset$

So φ is a theorem, and so a tautology

5 Resolution in action

Let φ be the formula $((p \vee q) \wedge (\neg p \vee \neg q) \vee (r \Rightarrow (p \wedge q))) \Rightarrow r$

So, $\neg\varphi$ is $\neg(((p \vee q) \wedge (\neg p \vee \neg q) \wedge (r \Rightarrow (p \wedge q))) \Rightarrow r)$
$$\equiv ((p \vee q) \wedge (\neg p \vee \neg q) \wedge (r \Rightarrow (p \wedge q))) \wedge \neg r$$
$$\equiv (p \vee q) \wedge (\neg p \vee \neg q) \wedge (\neg r \vee (p \wedge q)) \wedge \neg r$$
$$\equiv (p \vee q) \wedge (\neg p \vee \neg q) \wedge (\neg r \vee p) \wedge (\neg r \vee q) \wedge \neg r$$

Hence, the set of clauses to which we apply resolution is

$$p \vee q \qquad \neg p \vee \neg q \qquad \neg r \vee p \qquad \neg r \vee q \qquad \neg r$$

Now we start resolving

- $\cancel{q \vee \neg q}$ - we can ignore this as it will never yield a new clause
- $\cancel{p \vee \neg p}$ - we can ignore this as it will never yield a new clause
- $\neg q \vee \neg r$
- $\neg p \vee \neg r$
- $\cancel{p \vee \neg r}$ - we have this phrase already
- $\cancel{\neg r \vee \neg r}$ - i.e. $\neg r$ and we have this clause already
- $\cancel{\neg q \vee \neg r}$ - we have this clause already
- $\cancel{\neg r \vee \neg r}$ - i.e. $\neg r$ and we have this clause already
- No new clauses can be inferred

If you have n non negated clauses and l negated clauses, then the number of new clauses is $n \times l$

6 Is Resolution the silver bullet

- Resolution works by taking the negation of a formula φ we wish to prove true and showing that this negation $\neg\varphi$ is unsatisfiable (in essence)
- One might be inclined to think (from our examples) that resolution will always give a ”quick” answer as to whether a formula is a tautology or not
- However this is not the case, for the worst case resolution involves an exponential number of applications

7 Satisfiability vs tautologies

- SAT-solvers check whether or not a given formula of propositional logic is satisfiable, whereas proof systems, such as resolution, aim to prove theorems
- To some extent, these two tasks are different sides of the same coin
- Let φ be some propositional formula
  - If φ is satisfiable
    - Then there exists a truth assignment making φ true
    - Therefore $\neg\varphi$ is not a tautology
  - Conversely, if $\neg\varphi$ is not a tautology
    - Then there exists some truth assignment making $\neg\varphi$ false
    - So φ is satisfiable
- So, φ is satisfiable if, and only if $\neg\varphi$ is not a tautology
  - This leads to strong links between SAT-solving and automated theorem proving