Maths for CS - Discrete Maths and Linear Algebra
Practical #1 (week 2)

This practical is devoted to exercises on mathematical induction. See the lecture slides for the basic definitions and examples.

1. Prove that, for every positive integer n,

$$1 \cdot 2 + 2 \cdot 3 + \ldots + n(n+1) = n(n+1)(n+2)/3.$$

2. Prove that 3 divides $n^3 + 2n$ for every positive integer n.

3. For which non-negative integers n is $2n + 3 \leq 2^n$? Prove your answer.

4. What is wrong with this proof?

   "Theorem". For every positive integer n, if x and y are positive integers with $\max(x, y) = n$ then $x = y$.

   Basis Step. If $n = 1$ and $\max(x, y) = n$ then $x = y = 1$.

   Induction Step. Let $k \geq 1$ be any integer.

   - Assume that the claim holds for $n = k$, i.e. if $\max(x, y) = k$ then $x = y$.
   - Take any positive integers x and y with $\max(x, y) = k + 1$.
   - We have $\max(x - 1, y - 1) = k$ and then, by inductive assumption, $x - 1 = y - 1$.
   - Hence $x = y$ and the induction step is complete.

5. Recall definitions of a simple polygon and a triangulation from the lecture.

   Let $E(n)$ be the statement that, in a triangulation of a simple polygon with n sides, at least one of the triangles in the triangulation has two sides bordering the exterior of the polygon.

   (a) Explain where a proof using strong induction that $E(n)$ is true for all $n \geq 4$ runs into difficulties.

   (b) Prove that $E(n)$ is true for all $n \geq 4$ by proving, by strong induction, a stronger claim $T(n)$ that, in a triangulation of a simple polygon with n sides, at least two of the triangles in the triangulation have two sides bordering the exterior of the polygon.

6. Suppose there are n people in a group, each aware of a scandal no one else in the group knows about. These people communicate by phone; when two people talk, they share information about all scandals they know about. For example, on the first call, two people share information, so by the end of the call each of them knows about two scandals. The *gossip problem* asks for $G(n)$, the minimum number of telephone calls that are needed for all n people to learn about all scandals.

   (a) Find $G(n)$ for $n = 1, 2, 3, 4$.

   (b) Prove that $G(n) \leq 2n - 4$.

   (c) Prove that $G(n) = 2n - 4$ for $n \geq 4$. (Quite some challenge!).

   (d) What happens if they send texts instead of calling? (Bonus question)