Mathematical Induction

1 Proof By Induction

Suppose we want to prove that the following statements are valid for every **positive integer** m:

- $n < 2^n$
- $1 + 2 + .. + n = n(n + 1)/2$
- $n^3 - n$ is divisible by 3

Although these 3 cases look very different there is a **general approach** to prove such statements, called **proof by induction**

1.1 The general principle of proof by induction

Suppose we want to prove that a given statement $S(n)$ holds for all integers $n \geqslant j$ for a fixed integer j.
The simplest proof by induction works as follows:

- Step 1 (**Basis Step**): Check that S(n) is true for n=j (smallest possible value of n); If this is not the case, then the statement cannot be true. If S(j) is true, then proceed to Step 2
- Step 2: (**Induction Step**): Prove the following **conditional** statement. If S(n) holds for a fixed value $n = k \geqslant j$ (**Induction Hypothesis or Assumption**) then it also holds for $n = k + 1$

The two steps together then imply that $S(n)$ holds for $n = j$ (by Step 1), for $n = j + 1$ (by Step 1 and Step 2 applied for k=j), for $n = j + 2$ (by Step 2 applied for $k = j + 1$, and so on, so it holds for all $n \geqslant j$

1.2 Example 1: Proving $n < 2^n$ by induction

1. Basis Step: Check that the statement is valid for n=1. It is, as $1 < 2$
2. Induction Step: Let $k \geqslant 1$
    - Assume that the statement holds for $n = k$, that is, $k < s^k$
    - Need to use that to derive the statement for $n = k + 1$, that is $k + 1 < 2^{k+1}$
    - We have
      $k + 1 < 2^k + 1 < 2^k + 2^k = 2^k + 1$
      The first inequality above is by the inductive assumption
3. Since the statement is **valid for n=1**, and that is **valid for n=k+1 if it is valid for n=k**, we conclude that it is **valid for all positive integers n**

1.3 Example 2: Proving that $n^3 - n$ is divisible by 3

1. Basis Step: Check that the statement is valid for $n = 1$. It is as $1^3 - 1 = 0$ is divisible by 3
2. Induction Step: Let $k \geqslant 1$ be an integer
    - Assume that the statement holds for $n = k$, that is $k^3 - k = 3m$ for some integer $m \geqslant 0$
    - Need to use that to derive the statement for $n = k + 1$, that is, $(k + 1)^3 - (k + 1) = 3m'$ for some integer $m' \geqslant 0$
    - We have
      $(k + 1)^3 - (k + 1) = k^3 + 3k^2 + 3k + 1 - (k + 1) = k^3 - k + 3k^2 + 3k = 3(m + k^2 + k)$
      The m appears due to replacing some of the terms in k with the assumption $k^3 - k = 3m$
3. Since both the basis and inductive step are completed, we conclude that the statement is valid for all positive integers n

1.4 Example 3: Proving that $1 + 2 + ... + n = (n + 1)n/2$

1. Basis Step: Check that the statement is valid for $n = 1$. It is, as $1 = 2 \cdot 1/2$
2. Induction Step: Let $k \geqslant 1$ be an integer
   - Assume that the statement holds for $n = k$, that is
     $1 + 2 + ... + k = (k + 1)k/2$
   - Need to use that to derive the statement for $n = k + 1$, that is
     $1 + 2 + ... + k + (k + 1) = (k + 2)(k + 1)/2$
   - We have
     $1 + 2 + ... + k + (k + 1) = (k + 1)k/2) + (k + 1) = (k + 2)(k + 1)/2$
3. Since we know that the statement is valid for $n = 1$, and that it is valid for $n = k + 1$, if it is valid for $n = k$, we conclude that it is valid for all positive integers n

1.5 Variations

- Sometimes a statement is valid only for $n \geqslant j$. Then the Basis Step is about $S(j)$, rather than just 1
- Sometimes in the basis step we have to check a number of small cases, not just $n = j$
- Sometimes in Induction Step we have to assume that $S(n)$ holds for all $n \leqslant k$, not just for $n = k$ (Strong Induction in Textbook)

2 Geometry

2.1 Some Geometry

- A **polygon** is a closed geometric figure formed by a series of line segments $s_1 ..., s_n$ called sides
- Two consecutive sides share an endpoint, as do s_1 and s_n. The endpoints are called **vertices**
- A polygon is **simple** if no two non-consecutive sides intersect
- Each polygon divides the plane into two regions: **interior** and **exterior**
- A polygon is **convex** if each line segment between two interior points is within the interior. Line segment joining points doesn't go outside shape
- A **diagonal** is a line segment connecting two non-consecutive vertices
- An **interior diagonal** is one that lies entirely inside the polygon
  - Lemma: Every simple polygon has an interior diagonal

2.2 Triangulation in Computational Geometry

- **Triangulation** is the process of dividing a simple polygon into triangles by adding non-intersecting diagonals
- In order to computationally process a complicated surface, it is divided into simple polygons, which are then triangulated
- Delaunay triangulations are especially popular

2.3 Triangulation Proof By Strong Induction

Theorem: Each simple polygon with $n \geqslant 3$ sides can be triangulated into $n-2$ triangles

1. Basis Step: The theorem trivially holds for $n = 3$
2. Induction step: let $k \geqslant 3$ be any integer
   - Assume that the theorem is true for all $n \leqslant k$
   - Take a simple polygon P with $n = k + 1$ sides
   - By Lemma above, it has an interior diagonal
   - The diagonal splits P into two simple polygons. Q with s sides and R with t sides
   - We have $3 \leqslant s \leqslant k$ and $3 \leqslant t \leqslant k$. Moreover $k + 1 = s + t - 2$. Using base assumption Q and R can also be split using a diagonal
   - By assumption Q and R can be triangulated into $s-2$ and $t-2$ triangles, respectively. Assume the theorem is true
   - This gives triangulation of P into $(s-2) + (t-2) = k-1 = n-2$ triangles

2.4 Eye proof by induction

Doesn't work for n=2