Linear Systems

1 Systems of linear equations

- A **linear equation** in n variables $x_1, ..., x_n$ is an equation of the form

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$$

  where the a_i's and b are co constant and not all a_i's are equal to 0

- A finite set of linear equations is called a **system of linear equations**, or simply a **linear system**

- A general linear system can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

- A **solution** to such a system is a sequence of $s_1, .., s_n$ of numbers such that the assignment $x_1 = s_1, ..., x_n = s_n$ satisfies every equation

- A linear system is called consistent if it has at least one solution and it is inconsistent otherwise

2 A linear system with one solution

Solve linear system

$$x - y = 1$$

$$2x + y = 6$$

Eliminate x from the 2nd equation by adding -2 times the 1st equation to the 2nd

$$x - y = 1$$

$$2y = 4$$

We have $y = 4/3$, and from the 1st equation $x = 7/3$
This system has **one solution**

3 A linear system with no solutions

Solve linear system

$$x + y = 4$$

$$3x + 3y = 6$$

Eliminate x from the 2nd equation by adding -3 times the 1st equation to the 2nd

$$x + y = 4$$

$$0 = -6$$

The 2nd equation is contradictory. This system has **no solutions**

4 A linear system with infinitely many solutions

Solve linear system

$$2x + 2y = 1$$
$$8x - 4y = 2$$

Eliminate x from the 2nd equation by adding -3 times the 1st equation to the 2nd

$$2x - 2y = 1$$
$$0 = 0$$

The 2nd equation implies no restrictions on x and y, can be omitted
Any pair of values for x and y that satisfies $4x - 2y = 1$ is a solution
Solving it for x, we get $x = \frac{1}{4} + \frac{1}{2}y$
The solution set can be described as the set of all pairs of numbers of the form $x = \frac{1}{4} + \frac{1}{2}y$,y (y is a free variable here)
This system has **infinitely many solutions**

5 Matrix form of a linear system

A linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

can be written in a matrix form as $Ax = b$ where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

The matrix A is called the coefficient matrix of the system
If A is (square and) invertible then the solution can be found as $x = A^{-1}b$

6 The augmented matrix and elementary row operations

The augmented matrix of a linear system is the matrix

$$(A|\mathbf{b}) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

The basic method for solving a linear system is to perform algebraic operations on the system that:

(a) Do not alter the equation set

(b) Produce increasingly simpler systems

Typically the operations are

- Multiply an equation through by a non zero constant
- Interchange two equations
- Add a constant times one equation to another

This corresponds to the **elementary row operations** on the augmented matrix

- Multiply a row through by a non zero constant
- Interchange two rows
- Add a constant times one row to another

7 Row echelon form

Assume that we transform the augmented matrix of a linear system in variables x,y,z to the form

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Then we know the solution: it's x=1, y=2, z=3
A matrix is in **row echelon form** if it has the following properties:

- If a row is not all 0s then the first non zero number in it is 1 (the leading 1)
- The rows that are all 0s (if any) are grouped together at the bottom
- If two successive rows are not all 0s then the leading 1 of the higher row occurs further to the left than the leading 1 of the lower row

A matrix is in **reduced row echelon form** if it has the above properties, plus

- Each column that contains a leading 1 has 0s everywhere else

Strategy for solving linear systems: use elementary row operations to transform the augmented matrix to (reduced) row echelon form

8 Extracting solutions from row echelon form

Assume that we have transformed the augmented matrix of a linear system to a (reduced) row echelon form
Examples:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)\left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We have the following possibilities:

- Some row has a leading 1 in the last column.
  Then the system includes equation $0 \cdot x_1 + ... + 0 \cdot x_n = 1$
  Then we know the system has no solutions.
- The number of leading 1s is equal to the number of variables
  (and there is no leading 1 in the last column)
  Then the system has a unique solution
- The number of leading 1s is smaller than the number of variables
  (and there is no leading 1 in the last column)
  Then the system has infinitely many solutions

9 General solution (and an example)

Assume the matrix in reduced row echelon form is as follows:

$$\left(\begin{array}{cccr|c} 1 & -1 & 0 & 2 & 2 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

In equations, this is

$$\begin{aligned} x_1 - x_2 + 2x_4 &= 2 \\ x_3 - x_4 &= 5 \end{aligned}$$

- The variables corresponding to the leading 1s (x_1 and x_3 in this example) are the **leading variables**
- The other variables are **free variables**
- **General solution**: the leading variables expressed via free variables
- For the above system $x_1 = x_2 - 2x_4 + 2$, $x_3 = x_4 + 5$ (where x_2 and x_4 are arbitrary numbers)

10 Gaussian elimination procedure

Goal: Transform a matrix to row echelon form by using row operations

Step 1: Locate the leftmost column that contains a non zero

$$\begin{pmatrix} \mathbf{0} & 0 & -2 & 0 & 7 & 12 \\ \mathbf{2} & 4 & -10 & 6 & 12 & 28 \\ \mathbf{2} & 4 & -5 & 6 & -5 & -1 \end{pmatrix}$$

Step 2: Interchange the first row with another row (if necessary) to move a non zero to the top in this column

$$\begin{pmatrix} \mathbf{2} & 4 & -10 & 6 & 12 & 28 \\ \mathbf{0} & 0 & -2 & 0 & 7 & 12 \\ \mathbf{2} & 4 & -5 & 6 & -5 & -1 \end{pmatrix}$$

Step 3: If a is at the top, multiply the first row by 1/a (to get a leading 1)

$$\begin{pmatrix} \mathbf{1} & 2 & -5 & 3 & 6 & 14 \\ \mathbf{0} & 0 & -2 & 0 & 7 & 12 \\ \mathbf{2} & 4 & -5 & 6 & -5 & -1 \end{pmatrix}$$

Step 4: Add suitable multiples of the first to the rows below so that all numbers below the leading 1 are 0s

$$\begin{pmatrix} \mathbf{1} & 2 & -5 & 3 & 6 & 14 \\ \mathbf{0} & 0 & -2 & 0 & 7 & 12 \\ \mathbf{0} & 0 & 5 & 0 & -17 & -29 \end{pmatrix}$$

Step 5: now separate the top row from the rest ("draw a line below it") and repeat steps 1-5 for the matrix below the line

$$\left(\begin{array}{cccccc} \mathbf{1} & 2 & -5 & 3 & 6 & 14 \\ \hline \mathbf{0} & 0 & -2 & 0 & 7 & 12 \\ \mathbf{0} & 0 & 5 & 0 & -17 & -29 \end{array}\right)$$

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{pmatrix}$$

$$\left(\begin{array}{cccccc} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ \hline 0 & 0 & 0 & 0 & 1/2 & 1 \end{array}\right)$$

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

11 Gauss-Jordan elimination

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

To find the reduced row echelon form, we need one step on top of Gaussian.

Step 6: Beginning from the last non 0 row and working upward, add suitable multiples of each row to create 0s before the leading 1s

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \quad \text{added } (7/2) \times \text{3rd row to 2nd row}$$

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \quad \text{added } (-6) \times \text{3rd row to 1st roe}$$

$$\begin{pmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \quad \text{added } 5 \times 2nd \text{ row to 1st row}$$

12 Example

Solve linear system by Gauss-Jordan elimination:

$$\begin{aligned} -2x_3 \quad\quad + 7x_5 &= 12 \\ 2x_1 + 4x_2 - 10x_3 \quad +6x_4 + 12x_5 &= 28 \\ 2x_1 + 4x_2 - 5x_3 \quad +6x_4 - 5x_5 &= -1 \end{aligned}$$

The augmented matrix of system is

$$\left(\begin{array}{ccccc|c} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{array}\right)$$

We have already transformed the above matrix to reduced row echelon form (see four previous slides)

$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

The general solution of the system is $x_1 = -2x_2 - 3x_4 + 7, x_3 = 1, x_2$

13 Homogeneous linear systems

- A linear system $Ax = b$ is **homogeneous** if b is all 0s
- Such a system has a **trivial solution**: x is all 0s. Any other solution is called **non-trivial**

13.1 Theorem

If a homogeneous linear system has n variables and the reduced row echelon form of its augmented matrix has r non-0 rows then the system has $n - r$ free variables.

The above theorem follows immediately from the shape of the reduced row echelon form.
Being consistent and having free variables implies having infinitely many solutions.

13.2 Corollary

A homogeneous linear system with more variables than equations has infinitely many solutions