Matrices and Determinants

1 Matrices

1.1 Definition

A matrix is a rectangular array of (real) numbers. The numbers in the array are called the **entries** of the matrix. The entry in row i and column j is denoted by aij

1.2 Dimensions

- A matrix with m rows and n columns is said to have size $m \times n$
- A general $m \times n$ matrix can be written as

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- A matrix of size $n \times n$ is called a square matrix of order n
- Two matrices are **equal** when they have the same size and the corresponding entries are equal

2 Matrix operations

Let $A = a_{ij}$ and $B = (b_{ij})$ be $m \times n$ matrices

- The **sum** A+B is defined as the $m \times n$ matrix $C = (c_{ij})$ such that $c_{ij} = a_{ij} + b_{ij}$
- The difference A-B is defined similarly
- If α is a number (scalar) then the product (of a matrix by a scalar) αA is the $m \times n$ matrix $C = (c_{ij})$ such that $c_{ij} = \alpha \cdot a_{ij}$

Example: Let

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{pmatrix}$$

Then:

$$2A - B = \begin{pmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 1 \\ 3 & 3 & 7 \end{pmatrix}$$

3 Matrix Multiplication

- If A is an $m \times r$ matrix and B an $r' \times n$ matrix and the product matrix AB is defined only if $r = r'$
- If $A = (a_{ij})$ is an $m \times r$ matrix and $B = (b_{ij})$ an $r \times n$ matrix then the **product** AB is the $m \times n$ matrix $C = (c_{ij})$ such that:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ir}b_{rj}$$

4 Properties of matrix arithmetic

Assuming that the sizes of the matrices are such that the operations can be preformed, the following rules are valid:

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $A(BC) = (AB)C$
4. $A(B \pm C) = AB \pm AC$

5. $(B \pm C)A = BA + CA$
6. $\alpha(B \pm C) = \alpha B \pm \alpha C$
7. $(\alpha \pm \beta)A = \alpha A \pm \beta A$
8. $\alpha(\beta A) = (\alpha\beta)A$
9. $\alpha(BC) = (\alpha B)C = B(\alpha C)$

5 Special matrices

- A matrix whose entities are all 0 is called a zero matrix and denoted by 0.
  We have A+0=0+A=A and 0A=0
- A square matrix (a_{ij}) such that $a_{ii} = 1$ and $a_{ij} = 0$. If $i \neq j$ is called the identity matrix. denotes I_n

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

- It is easy to check that, for any $m \times n$ matrix A $AI_n = A = I_m A$

6 AB vs BA

In general, even for square matrices, it is possible that

- $AB \neq BA$
- AB=0, but $A \neq 0$ and $B \neq 0$
- AC=BC, but $A \neq B$

Example:

$$\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3 & 7 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 3 & 7 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 17 \\ 0 & 0 \end{pmatrix}$$

7 Matrix Transpose

If A is an $m \times n$ matrix then the **transpose** of A is the $n \times m$ matrix A^T such that the ith row of A is the ith column of A^T
Example:

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{pmatrix} \text{ then } A^T = \begin{pmatrix} 2 & 1 \\ 3 & 3 \\ 4 & 1 \end{pmatrix}$$

7.1 Theorem

If the sizes of the matrices are such that the operations can be performed then

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(A - B)^T = A^T - B^T$
4. $(\alpha A)^T = \alpha A^T$
5. $(AB)^T = B^T A^T$

8 Matrix inverse and its properties

- If A is a square matrix of order n and if a matrix B of the same size can be found such that $AB = BA = I_n$, then A is said to be **invertable** (or **non singular**), and B is called an **inverse** of A
- In this case A and B are inverse of each other
- If no such B can be found then A is singular
- If B and CD are both inverses of A then $B = C$. Indeed, we have

$$B = BI = B(AC) = (BA)C = IC = C$$

- So, we can speak of the inverse of A, it is usually denoted by A^{-1}
- If A and B are invertible matrices of the same size then AB is invertable and $(AB)^{-1} = B^{-1}A^{-1}$. Indeed,

$$(AB)\left(B^{-1}A^{-1}\right) = A\left(BB^{-1}\right)A^{-1} = AIA^{-1} = AA^{-1} = I$$

- If A is invertible then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$. Indeed,

$$A^T\left(A^{-1}\right)^T = \left(A^{-1}A\right)^T = I^T = I$$

9 Finding the inverse of a 2×2 matrix

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The **determinant** of A is the number $det(A) = ad - bc$. This number is also denoted by $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

9.1 Theorem

The matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible iff $det(A) \neq 0$, in which case

$$A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

10 Minors and cofactors

- We defined the determinants of 2×2 matrices, so will now define them for general square matrices
- Assume that we can compute determinants of square matrices of order n-1
- If A is a square matrix of order n, then the **minor of the entry** a_{ij} denoted my M_{ij}, is the determinant of the matrix (of order n-1) obtained from A by removing its ith row and jth column
- The number $C_{ij} = (-1)^{i+j}M_{ij}$ is called the **cofactor of** a_{ij}

Example: Let

$$A = \begin{pmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{pmatrix}$$

The minor of a_{32} is

$$M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$$

The cofactor of a_{32} is

$$C_{32} = (-1)^{3+2} \cdot 26 = -26$$

11 Determinants

If A is an $n \times n$ matrix then the **determinant** of A can be computed by any of the following **cofactor expressions** along the ith row and along the jth column, respectiveley:

$$\det(A) = a_{i1}C_{i1} + a_2C_{i2} + \ldots + a_{in}C_{in}$$

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$