Intro to Discrete Probability

1 Probability Theory

Some applications in CS:

- Monte Carlo algorithms - Faster algorithms that give correct answers with high probability
- Information Theory - Good models for data compression and error correction
- Computer Graphics - Automated random generation of objects
- Statistical analysis - empirical analysis of computer system performance
- Machine learning - Probabilistic classification methods
- Bioinformatics - Finding genes associated with particular diseases
- ...

2 Sample space, events and probability

- An **experiment** is a procedure that yields one of a given set of possible **outcomes**
- The **sample space** of the experiment is the set of possible outcomes
- An **event** is any subset of the sample space

For a finite sample space S and event E in it, let $|S|$ and $|E|$ denote the number of possible outcomes in S and E respectively.
$|S|$ represents the number of elements in S

2.1 Definition

If S is finite sample space of equally likely outcomes, and E is an event in it, then the **probability** of E is

$$p(E) = \frac{|E|}{|S|}$$

All of our sample spaces are finite, so we can use counting principles

3 Examples

3.1 Example 1

An urn contains four blue balls and five red balls. What is the probability that a ball chosen from the urn is blue?

- Sample Space: 9 balls. We are interested in the event: blue ball
- The number of successful outcomes (blue ball) is 4
- The total number of possible outcomes is 9
- Hence, the probability is $\frac{4}{9}$

3.2 Example 2

When two (fair) dice are rolled, which sum of the numbers on the dice is more probable: 7 or 8?

- Sample space: 36 combinations (x,y), x is on the 1st die and y is on the 2nd
- The successful outcomes for 7 are (1,6),(2,5),(3,4),(4,3),(5,2),(6,1)
- The number of successful outcomes for 7 is 6
- The total number of possible outcomes is 36
- Hence, the probability of sum 7 is 6/36=1/6
- What is the probability that the sum is 8? 5/36, so sum 7 is more probable

3.3 Example 3

What is the probability to win the big prize in the National Lottery, i.e. to correctly guess 6 numbers out of 49

- Sample Space: 6 combinations out of the set of 49 numbers $\binom{49}{6} = 13,938,816$
- The number of successful outcomes is 1
- Hence, the probability is 1/13,938,816

3.4 Example 4

What is the probability that a hand of (5) cards in poker contains 4 cards of the same kind?

- Sample space: 5 combinations of the set of 52 numbers,
  C(52,5)=$\frac{52!}{47!5!} = 2,598,960$ in total
- The number of successful outcomes is

$$C(13,1) \cdot C(4,4) \cdot C(48,1) = 13 \cdot 1 \cdot 48 = 624$$

  We choose
  - 1 kind out of 13
  - then all 4 cards of that kind, and
  - then 1 card from the remaining 48
- Hence, the probability os 624/2,598,960≈0.00024

3.5 Example 5

What is the probability that a hand of (5) cards in poker contains 3 cards of 1 kind and 2 of another?

- Sample space: 5 combinations of the set of 52 numbers,
  C(52,5)=$\frac{52!}{47!5!} = 2,598,960$ in total
- The number of successful outcomes is

$$P(13,2) \cdot C(4,3) \cdot C(4,2) = 13 \cdot 12 \cdot 4 \cdot 6 = 3774$$

  We choose
  - One of a kind for three cards and a different one for 2,
  - Then 3 out of 4 cards for the 1st kind, and
  - 2 out of 4 for the other
- Hence, the probability is 3,744/2,598,960≈0.0014

3.6 Example 6

What is the probability that number 11,4,17,39 and 23 are drawn in that order from a bin containing 50 balls labelled 1,2,...,50 if:
(a) the ball selected is not returned to the bin
(b) the ball selected is returned to the bin before the next one is drawn

(a) "Sampling without replacement"

- Sample space S: 5 permutations from the set of 50 number P(50,5)=254,251,20
- The number of successful outcomes is 1
- Hence, the prob is...

(b) "Sampling with replacement

4 The probability of the complementary event

Let E be an event in sample space S. The probability of $\overline{E}$ is given by

$$P(\overline{E}) = 1 - P(E)$$

4.1 Example 1

A sequence of 10 bits is randomly generated. What is the probability that at least one of these is zero

For this the complimentary value, which is that the sequence is all 1s is much easier to calculate

$$1 - \frac{1}{1024} = \frac{1023}{1024}$$

5 The probability of combinations of events

Let E_1,E_2 be events in a sample space S then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - p(E_1 \cap E_2)$$

5.1 Example 1

What is the probability that a number selected at random from the numbers from 1 to 100 is divisible by at least one of 2 and 5?

- Sample space is 100
- Prob divisible by 2 is 1/2
- Prob divisible by 5 is 1/5
- Prob divisible by 10 is 1/10
- Therefore the probability is calculated by

$$\frac{1}{2} + \frac{1}{5} - \frac{1}{10} = \frac{3}{5}$$

6 The 3 door puzzle

- Asked to open 1 of 3 doors
- Prize behind only 1
- First select a door
- One of other doors discounted
- Should you change your choice or not?

7 Assigning probabilities

- There can be some sample space where the probabilities $s_1 ... s_n$ are not equally likely
- This makes a probability distribution

7.1 Example

- Biased die such that 3 appears twice as often as any other number
- Probability of 3 is 2/7, the probability of a different number is 1/7
- The probability of odd numbers is 4/7