Formal Syntax and Semantics

1 Syntax of first-order logic

Every (well formed) formula of first order logic is constructed from **atoms** (or **atomic formula**). We completely define the **syntax** of first-order logic by defining what we mean by atoms and the constructions we are allowed to use.

1.1 Atoms

- If P is a relation symbol of arity r and $y_1, .., y_r$ are (not necessarily distinct) variables or constant symbols, then $P(y_1, ..., y_r)$ is an **atom** with free variables from $y_1, ..., y_r$ (this sequence can also contain constants and repeated items)
- If C and D are constant symbols and x and y are variables then C=D, C=x and x=y are all **atoms** with, respectively, set of free variables $\emptyset, \{x\}, \{x, y\}$

The **signature** of formula is its finite set of predicate (relation) and constant symbols

1.2 Constructions

- If ϕ and ψ are formulae, with free variables free(ϕ) and free(ψ), then

$$\phi \vee \psi, \phi \wedge \psi, \neg\phi$$

  are formulae, with, respectively, free variables $free(\phi) \cup free(\psi), free(\phi) \cup free(\psi)\ and\ free(\phi)$
- If ϕ is a formula with free variables $free(\phi)$ then

$$\exists x(\phi), \forall x(\phi)$$

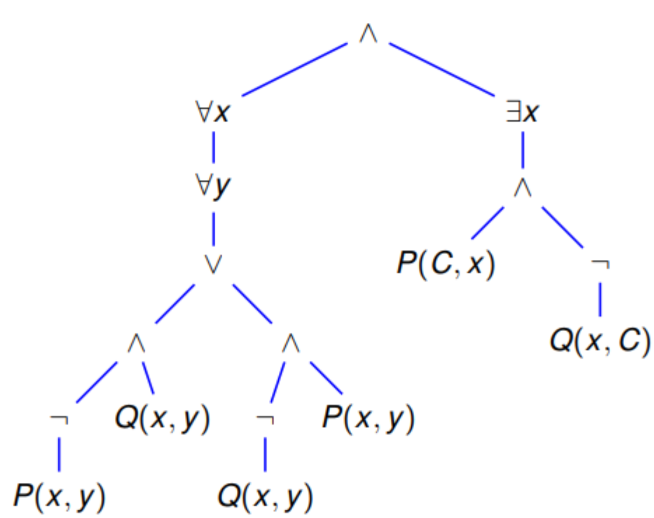
  are formulae, both with free variables $free(\phi) \backslash \{x\}$. The occurrence of x in both formulae is a bound occurrence

If a formula has no free variables then it is called a **sentence**

2 Parse trees

We can check that a formula is well formed using a parse tree (if the tree cannot be made then the formula is not well formed). We can illustrate with

$$\forall x(\forall y(P(x, y) \Leftrightarrow \neg Q(x, y))) \wedge \exists x(P(C, x) \wedge \neg Q(x, C))$$



Note that here $p \Leftrightarrow q$ has been replaced with $(p \wedge q) \vee (\neg p \wedge \neg q)$

3 Semantics for first-order logic

An interpretation or structure for a first order formula ϕ is:

- The domain of discourse D
- A value from D for every free variables of ϕ
- A relation over D for every relation symbol involved in ϕ
- A value from D for every constant symbol involved in ϕ

The semantics of a first order formula in some interpretation is as follows:

- We interpret atoms as propositional variables
- We interpret $\wedge, \vee$ and $\neg$ as propositional logic
- We interpret $\forall x\phi$ as true if ϕ is true for all values for x
- We interpret $\exists x\phi$ as true if there is at least one value for x making ϕ true

4 An illustration

Consider a **signature** consisting of two binary relation symbols P and Q and one constant symbol C. Let ϕ be defined as:

$$\forall x(\forall y(P(x,y) \Leftrightarrow \neg Q(x,y))) \wedge \exists x(P(C,x) \wedge \neg Q(x,C))$$

In order to decide whether ϕ evaluates to true or not we need an **interpretation**
Consider the interpretation:

$$\phi = \forall x(\forall y(P(x,y) \Leftrightarrow \neg Q(x,y))) \wedge \exists x(P(C,x) \wedge \neg Q(x,C))$$

where:

- The domain of discourse is the set of natural numbers $\mathbb{N}$
- The relation $P = \{(u,v) : u, v \in \mathbb{N}, u \leq v\}$
- The relation $Q = \{(u,v) : u, v \in \mathbb{N}, u > v\}$
- The constant $C = 0 \in \mathbb{N}$

So

- $(\mathbb{N}, p, Q, 0) \models \phi$ if and only if
  $(\mathbb{N}, P, Q, 0) \models \forall x \forall y(P(x,y) \Leftrightarrow \neg Q(x,y))$ and $(\mathbb{N}, P, Q, 0) \models \exists x(P(C,x) \wedge \neg Q(x,C))$
- if and only if for every $x, y \in \mathbb{N}, x \leqslant y \Leftrightarrow x \ngtr y$ and there exists $x \in \mathbb{N}$ such that $0 \leqslant x$ and $x \ngtr 0$

Both conjuncts are true. Thus $(\mathbb{N}, P, Q, 0)$ is a model of ϕ, i.e., $(\mathbb{N}, P, Q, 0) \models \phi$

4.1 Secondary interpretation

Now consider a **signature** consisting of two binary relation symbols P and Q and one constant symbol C. Let ϕ be defined as:

$$\forall x(\forall y(P(x,y) \Leftrightarrow \neg Q(x,y))) \wedge \exists x(P(C,x) \wedge \neg Q(x,C))$$

In order to decide whether ϕ evaluates to true or not we need an **interpretation**
Consider the interpretation:

$$\phi = \forall x(\forall y(P(x,y) \Leftrightarrow \neg Q(x,y))) \wedge \exists x(P(C,x) \wedge \neg Q(x,C))$$

where:

- The domain of discourse is the set of natural numbers $\mathbb{N}$
- The relation $P = \{(u,v) : u, v \in \mathbb{N}, u < v\}$

- The relation $Q = \{(u, v) : u, v \in \mathbb{N}, u > v\}$
- The constant $C = 0 \in \mathbb{N}$

So

- $(\mathbb{N}, p, Q, 0) \models \phi$ if and only if
  $(\mathbb{N}, P, Q, 0) \models \forall x \forall y(P(x, y) \Leftrightarrow \neg Q(x, y))$ and $(\mathbb{N}, P, Q, 0) \models \exists x(P(C, x) \wedge \neg Q(x, C))$
- if and only if for every $x, y \in \mathbb{N}, x < y \Leftrightarrow x \not> \text{y}$ and there exists $x \in \mathbb{N}$ such that $0 < x$ and $x \not> 0$

Both conjuncts are false. Thus $(\mathbb{N}, P, Q, 0)$ is not a model of ϕ, i.e., $(\mathbb{N}, P, Q, 0) \models \neg\phi$

5 A subtlety

Consider a signature consisting of two binary relation symbols P and Q and one constant symbol C. Let ϕ be defined as

$$\forall x(\forall y(P(x, y) \Leftrightarrow \neg Q(x, y))) \wedge \exists z(P(z, x) \wedge \neg Q(x, C))$$

This is a perfectly legal formula of first order logic, even though the variable x appears "differently" in the formula

- x appears bound in the first conjunct
- x appears free in the second conjunct

Consequently, it is more precise to speak of "free occurrences" or "bound occurrences" of variables rather than free or bound variables

6 Another subtlety

Consider the formula χ defined as

$$\forall x(\forall y(P(x, y) \Leftrightarrow \neg Q(x, y))) \wedge \exists y(P(y, x) \wedge \neg Q(x, y))$$

and the interpretation I for χ where:

- The domain D={1, 2, 3}
- $P = \{(1, 3), (2, 3), (3, 1)\}$ and $Q = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 2), (3, 3)\}$
- x=3

Not only does x appear both free and bound but y appears bound but within the scopes of two different quantifications. We clearly have $I \models \chi$ as

- For every $(x, y) \in D \times D, (x, y) \in P$ if and only if $(x, y) \notin Q$
- There exists a $y \in D$ such that $(y, 3) \in P$ and $(3, y) \notin Q$, namely y=1 (only need to show it for one value in this case as $\exists$)

If we amend the interpretation so that x is interpreted as x=2 then we have that $I \models \neg\chi$

7 More illustrations

Consider the well formed formula ϕ defined as $\forall x \exists y P(x, y)$
And consider the interpretation of ϕ where:

- The domain of discourse is the set $\mathbb{Z}$ of integers
- $P = \{(u, v) : u, v \in \mathbb{Z}, u > v\}$

So,

- $(\mathbb{Z}, P) = \forall x \exists y P(x, y)$
  if and only if for every $x \in \mathbb{Z}, (\mathbb{Z}, P) \models \exists y P(x, y)$
  if and only if for every $x \in \mathbb{Z}$, there exists $y \in \mathbb{Z}$ with $x > y$

For any $x \in \mathbb{Z}$, if we take $y = x - 1$ then this value of y **witnesses** that $x > y$; hence, $(\mathbb{Z}, P) \models \phi$
If we restrict the domain to the natural numbers $\mathbb{N}$ and where $P = \{(u, v) : u, v \in \mathbb{N}, u > v\}$, i.e. we have the restriction of $(\mathbb{Z}, p)$ to $\mathbb{N}$ then $(\mathbb{N}, p) \models \neg\phi$. This fails for example when a value is 0 as there is not a natural number smaller than it

Consider the well formed formula ϕ defined as $\exists y \forall x P(x, y)$
And consider the interpretation of ϕ where

- The domain of discourse if the set $\mathbb{Z}$ of integers
- $P = \{(u, v) : u, v \in \mathbb{Z}, u > v\}$

So,

- $(\mathbb{Z}, P) = \exists y \forall x P(x, y)$
  if and only if there exists $y \in \mathbb{Z}$ such that $(\mathbb{Z}, P) \models \forall x P(x, y)$
  if and only if there exists $y \in \mathbb{Z}$ such that for all $x \in \mathbb{Z}, x > y$

No matter which $y \in \mathbb{Z}$ we choose, putting $x = y - 1$ results in $x \leqslant y$
Hence $(\mathbb{Z}, P) \models \neg\exists y \forall x P(x, y)$

Take care with the **order** of quantifiers