Linear Maps

1 Linear Maps

1.1 Definition

Let V and W be vector spaces. A function $f : V \to W$ is called a linear map, or a linear transformation from V to W if, for all $u, v \in V, k \in \mathbb{R}$

$$f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v}) \text{ and } f(k\mathbf{u}) = k \cdot f(\mathbf{u})$$

If V=W then f is called a linear operator

1.2 Examples

- The map $f : V \to W$ such that $f(u) = 0$ for all u is linear
- If A is an $m \times n$ matrix then the map $f_A : \mathbb{R}^n \to \mathbb{R}^m$ given by $f_A(x) = Ax$ is linear. (Here x and Ax are column vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively)
  Indeed,
  $$f_A(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = f_A(\mathbf{u}) + f_A(\mathbf{v})$$
  and
  $$f_A(k\mathbf{u}) = A(k\mathbf{u}) = k(A\mathbf{u}) = kf_A(\mathbf{u})$$

2 Non-examples in $\mathbb{R}^2$

- The map $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $g(x, y) = (x, y + 1)$ is not linear . Indeed, any linear map f satisfies
  $$f(\mathbf{0}) = f(0\mathbf{x}) = 0f(\mathbf{x}) = \mathbf{0}$$
  and the above map g fails this property
- The map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (0, xy)$ is not linear. Indeed,
  $$f(e_1 + e_2) = f(1, 1) = (0, 1)$$
  while
  $$f(e_1) + f(e_2) = f(0, 1) + f(1, 0) = (0, 0)$$

3 Example in $\mathbb{R}^2$

3.1 Reflection

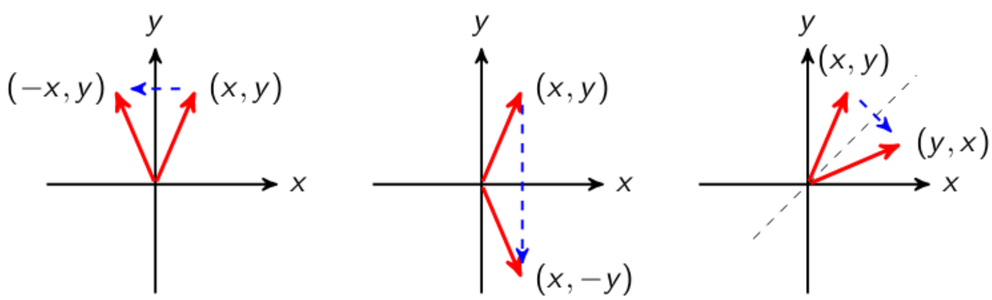
Consider linear operators f_A on $\mathbb{R}^2$ where A is one of the following matrices

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The corresponding linear maps f_A satisfy

$$f_A(x, y) = (-x, y), f_A(x, y) = (x, -y), f_A(x, y) = (y, x), \text{respectively}$$

They correspond to **reflections** of $\mathbb{R}^2$ about the y-axis. x-axis, and line $x = y$ respectively



3.2 Orthogonal Projection

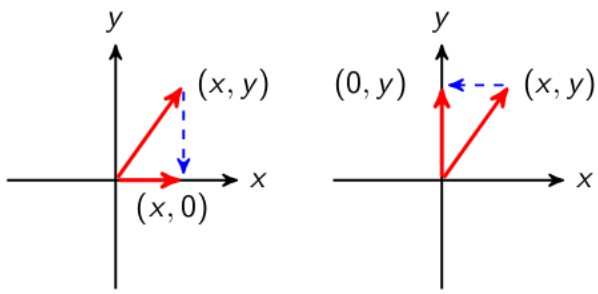
Consider linear operators f_A on $\mathbb{R}^2$ where A is one of the following matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The corresponding linear maps f_A satisfy

$$f_A(x, y) = (x, 0) \text{ and } f_A(x, y) = (0, y), \text{ respectively.}$$

They correspond to the orthogonal projections of $\mathbb{R}^2$ onto x-axis and y-axis respectively



3.3 Rotation

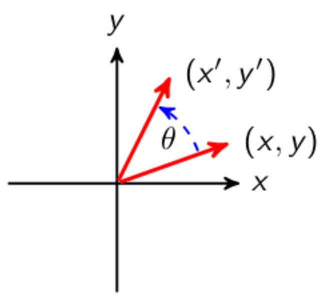
Consider the linear operator f_A on $\mathbb{R}^2$ where A is the following matrix:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

The corresponding linear map f_A satisfies

$$f_A(x, y) = (x', y') = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$$

This corresponds to the rotation of $\mathbb{R}^2$ by angle of θ counter clock-wise



3.4 Contraction/Dilation

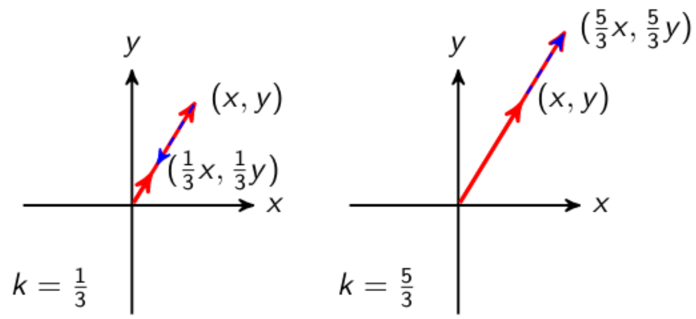
Consider linear operators f_A on $\mathbb{R}^2$ where A is the following matrix

$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

The corresponding linear map f_A satisfies

$$f_A(x, y) = (kx, ky)$$

This is **contraction** (if $0 < k < 1$)) or **dilation** (if $k > 1$) of $\mathbb{R}^2$



3.5 Compression/Expansion

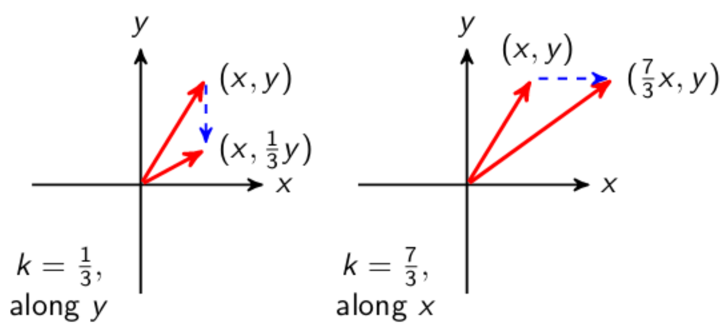
Consider linear operators f_A on $\mathbb{R}^2$ where A is one of the following matrices:

$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$$

The corresponding linear maps f_A satisfy

$$f_A(x,y) = (kx, y) \text{ and } f_A(x,y) = (x, ky), \text{ respectively.}$$

They correspond to **compressions** (if $0 < k < 1$) and **expansions** (if $k > 1$) of $\mathbb{R}^2$ along x-axis and y-axis respectively



4 Bases and Linear Maps

4.1 Theorem

Let $f : V \to W$ be a linear map where V is finite dimensional. If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V then the image of any vector $v \in V$ can be expressed as

$$f(\mathbf{v}) = c_1 f(\mathbf{v}_1) + c_2 f(\mathbf{v}_2) + \ldots + c_n f(\mathbf{v}_n)$$

where $c_1, ..., c_n$ are the coordinates of v relative to S

4.2 Proof

Express $v \in V$ as $v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ and use the linearity of f

4.3 Theorem

Conversely, if $f_0 : S \to W$ is any map then the map $f : V \to W$ defined by

$$f(\mathbf{v}) = c_1 f_0(\mathbf{v}_1) + c_2 f_0(\mathbf{v}_2) + \ldots + c_n f_0(\mathbf{v}_n)$$

where $c_1, ..., c_n$ are the coordinates of v relative to S, is a linear map

5 Exercise

Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in $\mathbb{R}^3$ where $v_1 = (1,1,1), v_2 = (1,1,0), v_3 = (1,0,0)$ let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map such that

$$f(\mathbf{v}_1) = (1,0), f(\mathbf{v}_2) = (2,-1), f(\mathbf{v}_3) = (4,3)$$

Find a formula for $f(x_1, x_2, x_3)$ and use it to decide whether $f(2,-3,5) = (9,23)$

5.1 Solution

First express $x = (x_1, x_2, x_3)$ as $x = c_1v_1 + c_2v_2 + c_3v_3$ From this we get

$$\begin{aligned} c_1 + c_2 + c_3 &= x_1 \\ c_1 + c_2 &= x_2 \\ c_1 &= x_3 \end{aligned}$$

Which yields $c_1 = x_3, c_2 = x_2 - x_3, c_3 = x_1 - x_2$, so

$$\mathbf{x} = (x_1, x_2, x_3) = x_3\mathbf{v}_1 + (x_2 - x_3)\mathbf{v}_2 + (x_1 - x_2)\mathbf{v}_3$$

Hence

$$f(x) = x_3 f(v_1) + (x_2 - x_3) f(v_2) + (x_1 - x_2) f(v_3) = (4x_1 - 2x_2 - x_3, 3x_1 - 4x_2 + x_3)$$

6 The matrix of a linear map

- Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map
- Let A be the $m \times n$ matrix $\left[f(\mathbf{e}_1) \middle| f(\mathbf{e}_2) \middle| \dots | f(\mathbf{e}_n)\right]$ whose columns are vectors $f(e_i) \in \mathbb{R}^m$. For example, if $f : \mathbb{R}^3 \to \mathbb{R}^2$ is linear and $f(1,0,0) = (2,3)$

$$f(0,1,0) = (0,0), f(0,0,1) = (-1,1) \text{ then } A = \begin{pmatrix} 2 & 0 & -1 \\ 3 & 0 & 1 \end{pmatrix}$$

- Note that $f(e_i) = Ae_i = f_A(e_i)$ for all i, For example

$$f(\mathbf{e}_2) = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} = A\mathbf{e}_2 = f_A(\mathbf{e}_2)$$

- Since f and f_A agree on all vectors in a basis, we have $f = f_A$
- Hence, every linear map $f : \mathbb{R}^n \to \mathbb{R}^m$ of the form f_A for some matrix A
- This matrix A is called the (standard) matrix of linear map f
- Thus, linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ are in 1-to-1 correspondence with $m \times n$ matrices (The same works for any pair of finite dimensional spaces)

7 Exercise

Find the standard matrix of the linear map $f : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$f(x_1, x_2, x_3, x_4) = (7x_1 + 2x_2 - x_3 + x_4, x_2 + x_3, -x_1)$$

7.1 Solution

We have

$$f(1,0,0,0) = (7,0,-1)$$
$$f(0,1,0,0) = (2,1,0)$$
$$f(0,0,1,0) = (-1,1,0)$$
$$f(0,0,0,1) = (1,0,0)$$

Hence the matrix is

$$\begin{pmatrix} 7 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

8 The kernel and range of a linear map

8.1 Definition

Let $f : V \to W$ be a linear map
The **kernel** of f, denoted by $\ker(f)$ is defined by $\ker(f) = \{x \in V | f(x) = 0\}$
The **range** of f is defined as $\text{range}(f) = \{\mathbf{u} \in W | \mathbf{u} = f(\mathbf{x}) \text{ for some } \mathbf{x} \in V\}$

- Let A be the standard matrix of a linear map $f : \mathbb{R}^n \to \mathbb{R}^m$(so $f(\mathbf{x}) = A\mathbf{x}$)
- The $\ker(f)$ is the null space of A and range(f) is the column space of A
- Use algorithms for null space and column space to find $\ker(f)$ and range(f)

9 Dimension theorems for matrices and linear maps

9.1 Definition

The **rank** of a linear map, denoted by rank(f), is the dimension of range(f)
The **nullity** of f, denoted by nullity(f), is the dimension of ker(f)

Recall that, for a matrix A, rank(A) and nullity(A) are the dimensions of the column space and the null space of A. If A is the standard matrix of f then rank(A)=rank(f) and nullity(f)=nullity(A)

9.2 Theorem (Dimension theorem for Matrices)

For any matrix A with n columns, rank(A)+nullity(A)=n

9.3 Theorem (Dimension Theorem for Linear Maps)

If f is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ then rank(f)+nullity(f)=n

10 Exercise

If $f : \mathbb{R}^5 \to \mathbb{R}^3$ is a linear map, what are the possible pairs (rank(f),nullity(f))?

10.1 Solution

$(0,5), (1,4), (2,3), (3,2)$. The pairs $(4,1)$ and $(5,0)$ are not possible because $\mathbb{R}^3$ does not have a subspace of dimension > 3