Images in Fourier Representation - Fourier Space I

1 Summary

The Fourier transform of an image is a global transform

Produces a representation of the image in Fourier space, (or frequency domain where there is no room for ambiguity as to the type of transform that was used)

Performed in images by applying the Discrete Fourier Transform (DFT) operator

Discrete Fourier Transform:

- Decomposes image into frequency components (sinusoidal functions)
- Typically. the frequency components are arranged in array the same size as the image (sometimes it is convenient to think of the output of the DFT as an image, even though the 'pixel values' are complex numbers)
- Major applications in image processing, analysis and compression

2 Terminology

Definition: Spatial domain (or real domain, or real space)

Images (signals) are represented by a spatial layour of the samples that are real numbers

Definition: Frequency domain (or Fourier domain, or frequency/Fourier space)

Images (signals) are represented by coefficients of sinusoidal basis frequencies

3 1D Fourier series

Our building block

$$A \cdot \sin(\omega \cdot x + \varphi)$$

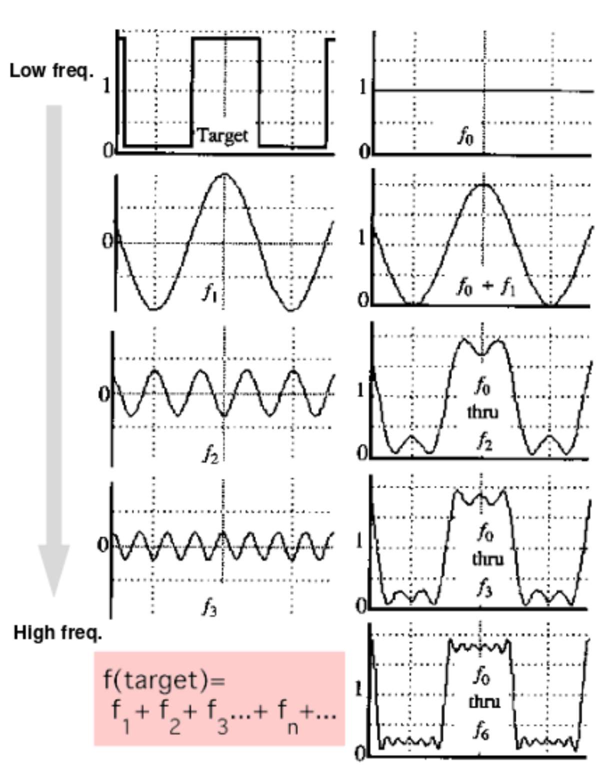
Add enough of them to get any signal

$$f(x) = f_1(x) + f_2(x) + f_3(x) + \dots$$

ω is the frequency. For each frequency we have one sinusoidal function in the sum.

A is the amplitude. It is the weight of that sinusoidal function in the sum

φ is the phase. Changes in the phase shift a sinusoidal function left or right



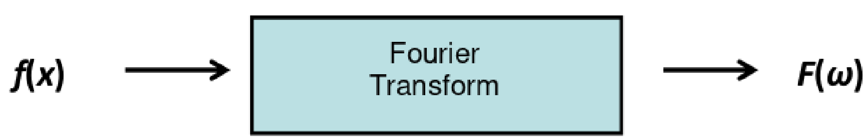
Low frequency components influence the coarse outline of the signal

Higher frequency components influence the fine detail of the signal

As we increase the number of frequency components the later, higher frequency components contribute less to the coarse outline of the signal and more to the fine detail of its shape

4 Fourier transform

We want to understand the signal as a sum of weighted and phase shifted frequencies. The Fourier transform reparametrises the signal by ω instead of x



$F(\omega)$ itself is called the Fourier transform of $f(x)$. The inverse Fourier transform applied on $F(\omega)$ parametrizes the signal again by x

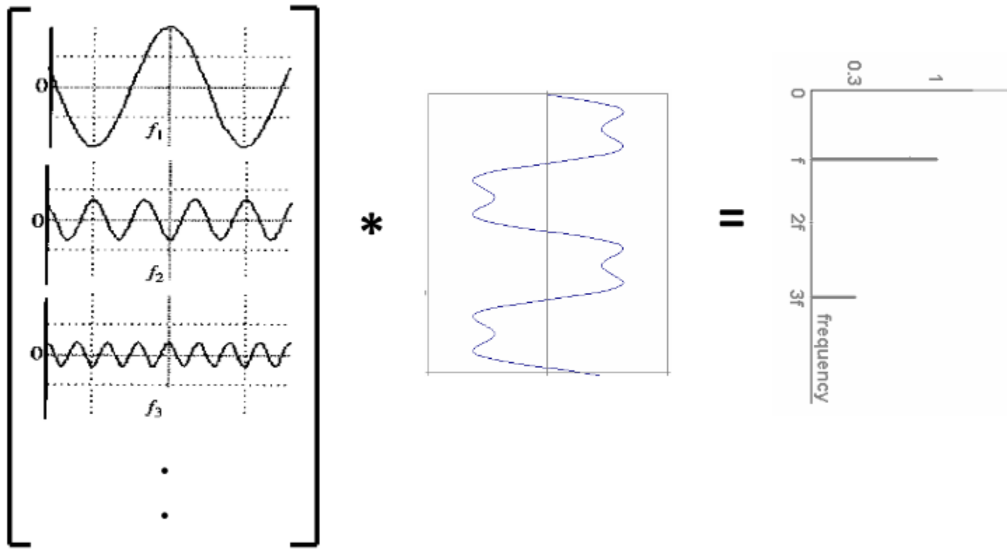
5 1D discrete Fourier transform

The Fourier transform $M * f(x) = f(\omega)$ is a change of mathematical basis

M: set of basis functions (N samples from each sinusoidal function form one row of the $N \times N$ matrix)

$f(x)$: spatial domain representation (N samples of a signal arranged as a vertical vector)

$F(\omega)$: Fourier domain representation (N frequency coefficients arranged as a vertical vector)



The inverse Fourier transform $M^{-1} * F(\omega) = f(x)$ is again a change of basis

6 Complex numbers in Fourier transform

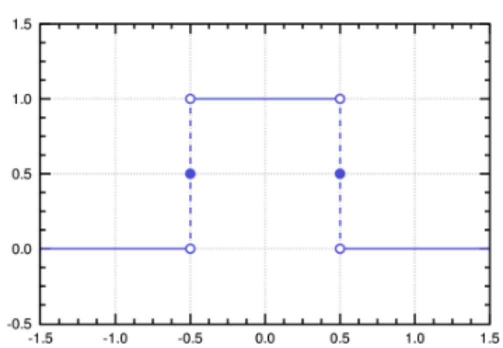
For every frequency ω, the Fourier transform at that point $F(\omega)$ is defined by the amplitude A and the phase φ of the corresponding sine

While in principle we can handle amplitude and phase separately, as two real functions, mathematically it is very convenient to use complex number

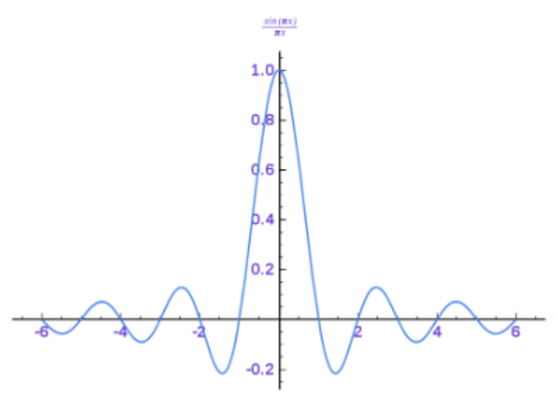
$$F(\omega) = R(\omega) + iI(\omega)$$
$$A = \pm\sqrt{R(\omega)^2 + I(\omega)^2}$$
$$\varphi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$$

7 1D continuous Fourier transform

By taking integrals instead of infinite sums, we can extend the Fourier transform to continuous functions



Box function



Fourier transform of the box function

8 2D sine functions

Any image can be reconstructed as a weighted sum of different frequencies aligned in different phases
The weights are the Fourier coefficients of the image

9 Discrete Fourier transform

The Fourier transform operator we apply on images is called the Discrete Fourier Transform (DFT)

Inverted by the Inverse Discrete Fourier Transform (DFT^{-1})

Why "discrete"? Because of the discrete nature of image sampling (pixels)

Computable efficiently via Fast Fourier Transform (FFT) algorithm

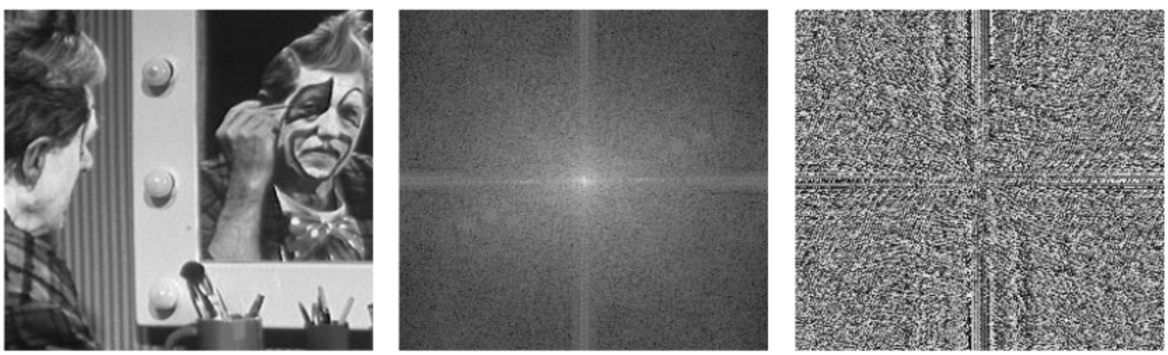
- 1D FFT is $\mathcal{O}(N \log_2 N)$
- 2D FFT is a series of 2N 1D FFTs

10 Fast Fourier Transform

DFT admits description in terms of multiplication by the Fourier matrix

The special structure of the DFT matrix means that the multiplication of a 1D signal (an $N \times 1$ vector) by the DFT matrix can be done faster than general matrix multiplication via the FFT algorithm

11 Visualising Fourier images



Magnitude (logarithmic transformed) Phase

The output F_{nm} of the DFT of an input image I_{input} is a complex number valued output image containing the coefficients of the DFT of I_{input}

Known as the **Fourier spectrum** of I_{input}, its dimension is $n \times m$ (the same as I_{input})

It can be displayed as the real and imaginary parts of the complex image

$$F_{nm} = G_{nm} + iH_{nm}$$

where

$$G_{nm} = \text{real part of} F_{nm}$$

$$H_{nm} = \text{imaginary part of} F_n m$$

Components commonly used for visualising the Fourier spectrum

Amplitude (magnitude) spectrum

$$|F_{nm}| = \sqrt{G_{nm}^2 + H_{nm}^2}$$

Phase spectrum:

$$\varphi_{nm} = \tan^{-1}\left(\frac{H_{nm}}{G_{nm}}\right)$$

Power spectrum (where $|\cdot|$ is the norm of a complex number)

$$|F_{nm}|^2$$

The image mean intensity, which is given by the Fourier coefficient F(0,0) and is also known as the DC-component, is at the centre (by convention)

The highest frequency present is $F(N-1, N-1)$

The magnitude is presented on a logarithmic scale $\ln(1+|F_{nm}|)$ as the floating point range is vary large

12 Understanding Fourier spectrum

Magnitude and phase contain **all frequencies**

Lower frequencies (close to the centre of magnitude image) are larger than the **higher frequencies** (near the boundary of the magnitude image). Thus, **more information in lower frequencies**.

Two predominant directions in the magnitude image (horizontally and vertically correspond to patterns of the original image)

The **phase spectrum** contains the phase part of the frequencies

Vertical and horizontal features correspond to image patterns

In general, the phase spectrum does not contain much structural information about the original image. However, it is crucial in reconstructing the original via inverse Fourier transform. Signal reconstruction needs phase and amplitude

13 DFT: individual frequency components

Removing frequencies in the Fourier domain by setting their magnitude to zero results in frequency filtering in the spatial domain.

14 Discrete Convolution Theorem

Convolution in the spatial domain reduces to multiplication in the frequency domain

$$I_A * I_B = \mathrm{DFT}^{-1}\left(\mathrm{DFT}\left(I_A\right)\cdot \mathrm{DFT}\left(I_B\right)\right)$$

Where $*$ denotes convolution of two matrices and $\cdot$ denotes component-wise multiplication of the elements of two matrices

15 Applications of DFT

Fourier space filtering, suppressing certain frequencies whilst leaving others unchanged

Example:

- Compute and inspect the amplitude spectrum of the image
- Hand-craft a binary mask P
  - 1 to keep a frequency
  - 0 to remove that frequency
- Component wise multiplication of amplitude spectrum and mask P
- Inverse Fourier transform

JPEG compression algorithm based on a Fourier related transform called Discrete Cosine Transform (DCT) Efficient implementation of spatial filtering:

- Make image and mask the same dimension by zero padding
- Apply FFT to the image and to the mask
- Component wise multiplication of the corresponding Fourier spectra
- Return to the spatial domain by applying inverse FFT

If the mask is large, for example larger than 20×20, the above algorithm is more efficient than doing to convolution directly in the spatial domain.

Some image processing / computer vision libraries switch automatically to this algorithm when the mask is large.

Using the convolution theorem, predict the behaviour of a filter against variations in the frequency of the noise

The behaviour of the mean filter against variations in noise frequency is erratic

The Fourier transform of a Gaussian is again a Gaussian.

16 DFT - a few asides

The DFT is only an approximation to the Fourier transform of the continuous image from which the digital image was obtained

Edge effects: DFT assumes the digital image is one period of a periodic function (in both directions)

If the values at the opposite edges of the function are not the same, the result is additional edge discontinuities

Frequency aliasing: DFT inaccuracy from under-sampling causes frequency aliasing