Recap

1 Polynomials

1.1 Definition

Let $n \geqslant 0$ be an integer, and let $a_0, a_1, ..., a_n$ be real numbers, $a_n \neq 0$ the function:

$$f(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + ... + a_1 \cdot x + a_0$$

is called a **polynomial**

The numbers $a_0, ..., a_n$ are called **coefficients**

We say this is a polynomial of **degree** n

Note: If f(x)=0, then the degree of f(x) is $-\infty$

1.2 Types of polynomials

Degree	Name
0	constants
1	linear
2	quadratic
3	cubic

1.3 Proposition

Let

$$f(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \ldots + a_1 \cdot x + a_0$$
$$g(x) = b_m \cdot x^m + b_{m-1} \cdot x^{m-1} + \ldots + b_1 \cdot x + b_0$$

be polynomials of degrees n and m respectively

- the **sum** of the polynomials f(x)+g(x) is a polynomial of **degree** max{n,m}
- the **product** of the polynomials $f(x) \cdot g(x)$ is a polynomial of degree n+m. Product is multiplying two functions together
- the **composition** of the polynomials $f(g(x))$ is a polynomial of degree $n \cdot m$. Composition is replacing the x terms in f(x) with g(x). Remember $f(g(x)) \neq g(f(x))$
- The degree is the important part, as most other parts are insignificant as x becomes large

2 Positive integer powers

2.1 Definition

For a positive integer n and a real number a,

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n}$$

The number a is called the **base** and n is called the **exponent** or the **power**

2.2 Basic rules

For positive integers n,m and a real number a

$$a^n \cdot a^m = a^{n+m}$$
$$(a^n)^m = a^{n \cdot m}$$

3 Rational Powers

3.1 Definitions

3.1.1 Definition 1

For a real number $a \neq 0$ (because 0^0 is undefined)

$$a^0 = 1$$

3.1.2 Definition 2

For a positive integer n and a real number $a \neq 0$

$$a^{-n} = \frac{1}{a^n}$$

3.1.3 Defintion 3

For a positive integer n and a real number $a \geqslant 0$, we define $a^{\frac{1}{n}}$ as the **n-th root** of a

That is $a^{\frac{1}{n}}$ is a real number x with the property $x^n = a$ ($a^{\frac{1}{n}} = x \Leftrightarrow x^n = a$)

We also write $a^{\frac{1}{n}} = \sqrt[n]{a}$

3.2 More on Rational Powers

When $a > 0$ and n is even the equation

$$x^n = a$$

may have more than one real solution

For example, the equation

$$x^2 = 4$$

has two solutions, 2 and -2

By convention, we normally consider the **positive solution** as the value of the n-th root of a

Notice that we assumes that $a > 0$
If n is an **odd** integer, then we can extend the definition of the n-th root to **negative** bases a because the equation

$$a^n = a$$

still has real solutions
For example:

$$(-8)^{\frac{1}{3}} = -2$$

because

$$(-2)^3 = -8$$

3.3 Definition

Let m be an integer and let n be a positive integer. For a real number $a > 0$,

$$a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = (a^{\frac{1}{n}})^m$$

For example

$$8^{\frac{2}{3}} = 4$$

$$8^{-\frac{2}{3}} = \frac{1}{4}$$

4 Real Powers

Because the set of rational numbers is a dense subset (belong in or limit points) of the real numbers we can also define real powers. That is, we can define a^x for any positive real number a and any real number x.

The formal technique to do this is by taking the **limit**.

That means that for any real number x, we can find a rational $\frac{m}{n}$ arbitrarily close to x, so that $a^{\frac{m}{n}}$ is also arbitrarily close to a^x.

5 Exponential Functions

5.1 Definition

For a fixed positive real number a, the function

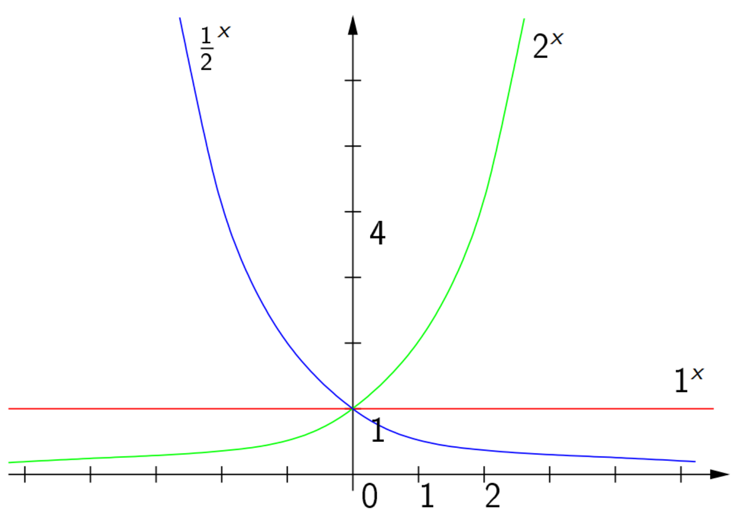
$$f(x) = a^x$$

is called **exponential function** with base a

5.2 More on exponential functions

Exponential functions are defined over the set of real number, so the above function is defined for every real number x

Their values are positive real numbers



- The exponential functions are positive everywhere
- At zero their value is 1
- For base $a > 1$ the function f(x)=a^x increases monotonically (never constant, never decreases, rate of increase is continually increasing). It grows fast compared to many other functions

5.3 Proposition 1

Let a, b, x, y be real numbers with $a, b > 0$. Then

- $a^x \cdot a^y = a^{x+y}$

- $a^{-x} = \frac{1}{a_x}$
- $(a^x)^y = a^{x \cdot y}$
- $(ab)^x = a^x \cdot b^x$

5.4 Proposition 2

Let a, x, y be real numbers with $a > 1$. Then for $x \leqslant y$, $a^x \leqslant a^y$

We say that the exponential function with $a > 1$ increases **monotonically**

6 Logarithms

6.1 Definition

For real positive number x, a with $a \neq 1$, the **logarithm** of x to the **base** a, written $\log_a x$ as the unique real number y that satisfies $a^y = x$

That is, if we raise a to the power of $\log_a x$ we get x:

$$a^{\log_a x} = x$$

6.2 Examples

$$log_a 1 = 0$$
$$\log_a a = 1$$
$$\log_a a^2 = 2$$
$$\log_a \frac{1}{a} = -1$$

6.3 Properties of logarithms

6.3.1 Proposition

Let a, x, y be positive real numbers $a \neq 1$ we have

- $\log_a xy = log_a x + log_a y$
- $\log_a \frac{x}{y} = \log_a x - \log_a y$
- $\log_a x^s = s \cdot \log_a x$ for any real s

Proof

- $a^{\log_a x + \log_a y} = a^{\log_a x} \cdot a^{\log_a y} = x \cdot y$
- $a^{\log_a x - \log_a y} = a^{\log_a x} \cdot a^{-\log_a y} = \frac{x}{y}$
- $a^{s \log_a x} = (a^{\log_a x})^s = x^s$

6.3.2 Proposition

Let a, b, x be positive real numbers, $a, b \neq 1$, Then

$$\log_a x = \frac{log_b x}{\log_b a}$$

So logarithms to different constant bases only differ by a constant

Proof
By the definition

$$x = a^{\log_a x}$$

It follows that

$$\log_b x = \log_b a^{\log_a x} = \log_a x \cdot \log_b a$$

6.3.3 Natural Logarithms

The natural logarithm is denoted $\ln x$
2,e and 10 are the ”special” bases in CS

6.4 Logarithmic Function

6.4.1 Definition

Let a be a positive real number $a \neq 1$. The function

$$f(x) = \log_a x$$

Defined for positive real numbers is called **logarithmic**

- Logarithmic functions are **inverses** of exponential functions
- They are only defined on positive real numbers
- For any base, the logarithm of 1 to that base is 0
- For $a > 1$ logarithms to the base a increase monotonically.