Network flows

1 Flow networks, flow, maximum flow

- Material is transferred in a network from a "source" to a "sink"
- Source produces material at a steady rate, sink consumes at same rate

Edges have a given capacity

Vertices (other than source/sink) are junctions

- Material flows through them without collecting in them
- Entering rate = exiting rate

2 Definitions

Definition: Maximum-flow problem

We wish to compute the greatest possible rate of transportation from source to sink

Definition: Flow network

- G=(V,E)
- Two distinguished vertices: source s and sink t
- Each edge $(u, v) \in E$ as non-negative capacity $c(u, v) \geqslant 0$
- If$(u, v) \notin E$, we assume $c(u, v) = 0$
- For each $v \in V$, there is a path $s \rightarrow v \rightarrow t$

2.1 Flow constraints

Definition: Capacity constraint

For all $u, v \in V$, we require $f(u, v) \leqslant c(u, v)$
Flow from one vertex to another must not exceed given capacity

Definition: Skew symmetry

For all $u, v \in V$, we require $f(u, v) = -f(v, u)$
Flow from vertex u to vertex v is negative of flow in reverse direction

Definition: Flow conservation

For all $u \in V - \{s,t\}$ we require

$$\sum_{c \in V} f(u,v) = 0$$

Total flow out of a vertex is 0, likewise for total flow into a vertex (just saying what goes in, comes out), this doesn't apply to the source or drain

2.2 Total flows

Definition: Total positive flow

The total positive flow entering vertex v is

$$\sum_{u \in V : f(u,v) > 0} f(u,v)$$

The total positive flow leaving vertex u is

$$\sum_{v \in V : f(u,v) > 0} f(u,v)$$

Definition: Total net flow

The total net flow at a vertex v is
total positive flow leaving v - total positive flow entering v

Definition: Flow value

The value of flow f is defined as the total flow leaving the source (and thus entering the sink)

$$|f| = \sum_{v \in V} f(s,v)$$

Note that $|\cdot|$ does not mean absolute value

If there is an arrow only in one direction on the graph, then the capacity in the other direction is 0. There is no assumption of symmetric capacities.

3 Technical tools

Implicit summation
Let $X, T \subseteq V$. Then

$$f(X,Y) = \sum_{x \in X} \sum_{y \in Y} f(x,y)$$

Commonly occurring identities

1. For all $X \subseteq V$, we have $f(X,X) = 0$. Because each $f(u,v)$ and $f(v,u) = -f(u,v)$ cancel each other
2. For all $X, Y \subseteq V$, we have $f(X,Y) = -f(Y,X)$. Generalisation of $f(X,X) = 0$, with the same reasoning

3. For all $X, Y, Z \subseteq V$ with $X \cap Y = \varnothing$, we have
$$f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$$
$$f(Z, X \cup Y) = f(Z, X) + f(Z, Y)$$
Split summation into two: one over X, one over Y

Three important ideas:

1. Residual networks
2. Augmenting paths
3. Cuts

Method is iterative:

1. Start with $f(u, v) = 0$ for all $u, v \in V$
2. At each iteration, increase flow value by finding an augmenting path (a path from source to sink along which we can increase flow) and then augment flow along this path
3. Repeat until no augmenting paths can be found

4 Residual networks

Idea: Residual network consists of edges that can admit more flow

Formally: Consider vertices u and v. Amount of additional flow we can push from u to v before exceeding capacity $c(u, v)$ is residual capacity of (u, v)
$$c_f(u, v) = c(u, v) - f(u, v)$$
Note that when flow $f(u, v)$ is negative, then residual capacity $c_f(u, v)$ is greater than $c(u, v)$
Interpretation:

- Flow of $-x$ from u to v (i.e. $f(u, v) = -x < 0$)
- Implies flow of x from v to u (i.e. $f(v, u) = x > 0$)
- Can be cancelled by pushing x units from u back to v
- Can then push another $c(u, v)$ from u to v
- We can push in total $c_f(u, v) = c(u, v) + x > c(u, v)$ from u to v