Linear Temporal Logic

1 Intuition

We have a Boolean state (or world), in which a number of atomic propositions (AP) are true or false

We'd like to reason about discrete linear time, which is an infinite sequence of states $A_0, A_1 ....$ with state A_0 at time 0 being the current state (thus any propositional formula over the AP talks about A_0)

We'd next line to add temporal modalities, such as:

- always A - $\Box A$
- eventually A - $\Diamond A$
- next A - $\circ A$
- etc

These let us express other natural temporal properties such as infinitely often a - $\Box \Diamond a$

2 Syntax of LTL

We are given a finite set AP of atomic propositions (boolean variables), Boolean connectives plus two temporal modalities:

- $\circ$ - next
- U - until

A formula in LTL is defined by the following grammar (in which brackets are omitted)

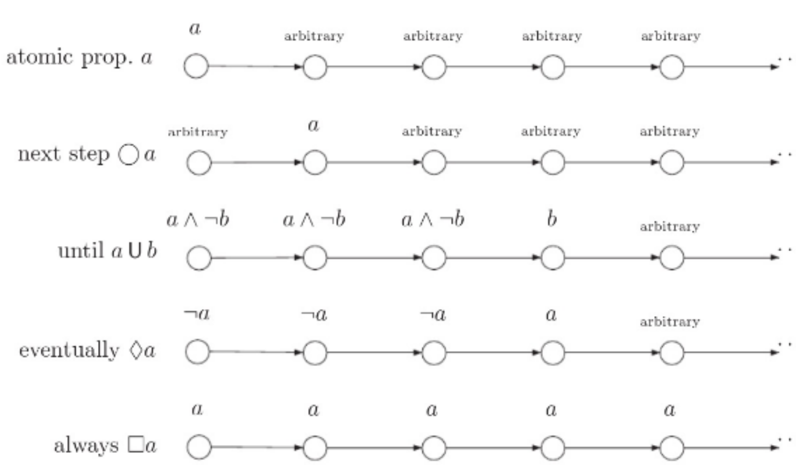
$$\varphi := \text{ true } |a|\varphi_1 \wedge \varphi_2|\neg\varphi| \bigcirc \varphi|\varphi_1 U \varphi_2$$

where $a \in AP$ and φ_1, φ_2 are LTL formulae

Other modalities can be expressed, e.g.

$$\Diamond a \overset{\text{def}}{=} \text{trueUa}$$
$$\Box a \overset{\text{def}}{=} \neg \Diamond \neg a$$

3 Intuitive statements



4 Formal Semantics

A world is labelled by the AP that are true in it, so it is just a letter from the alphabet 2^{AP} (the set of all subsets of AP)

A word σ is an infinite sequence of worlds, i.e. $\sigma \in (2^{AP})^{\omega}$

The satisfaction relation, $\sigma \models \varphi$ where $\sigma = A_0 A_1 ...$ is a word and φ is a formula, recursively defined by

$$
\begin{array}{ll}
\sigma \models \mathit{true} & \\
\sigma \models a & \mathit{iff}\ a \in A_0 \\
\sigma \models \varphi_1 \wedge \varphi_2 & \mathit{iff}\ \sigma \models \varphi_1 \text{ and } \sigma \models \varphi_2 \\
\sigma \models \neg\varphi & \mathit{iff}\ \sigma \not\models \varphi \\
\sigma \models \bigcirc\varphi & \mathit{iff}\ A_1 \ldots \models \varphi \\
\sigma \models \varphi_1 U \varphi_2 & \mathit{iff} \text{ there is } i \geq 0 \text{ s.t. } A_i \ldots \models \varphi_2 \text{ and } A_j \ldots \models \varphi_2 \text{ for all } 0 \leq j < i.
\end{array}
$$

The set of all words that satisfy a formula φ is called $Words(\varphi)$

5 Transition Systems

A transition system TS has:

1. A finite set of states S
2. A transition relation $\rightarrow \subseteq S \times S$, which is left-total (for every $s_1 \in S$, there is $s_2 \in S$ such that $s_1 \rightarrow s_2$)
3. A set of initial states $I \subseteq S$
4. A finite set of atomic propositions AP
5. A labelling function $L : S \rightarrow 2^{AP}$

The transitions may be labelled by a finite set of actions Act, in which case the transition relation becomes $\rightarrow \subseteq S \times Act \times S$

6 Execution of a TS

A run of TS is an infinite sequence of states

$$S_0 \rightarrow s_1 \rightarrow ...$$

where $s_0 \in I$, which produces an infinite trace $\sigma \in (2^{AP})^{\omega}$, $\sigma = L(s_0)L(s_2...)$

The set of all possible traces of the TS is called $Traces(TS)$

Finally, TS satisfies φ, $TS \models \varphi$, if $Traces(TS) \subseteq Words(\varphi)$ i.e. if each trace of the TS satisfies the formula φ

Thus is is possible that $TS \not\models \varphi$ and $TS \not\models \neg\varphi$