Vector Spaces and Linear Independence

1 Vectors in $\mathbb{R}^n$

- You are familiar with vectors in two and three dimensions
- Such a vector can be identified with an (ordered) tuple of real numbers (a_1, a_2) or (a_1, a_2, a_3) respectively
- The numbers in the tuple are the **components** of the vector
- The sets of all 2D and 3D vectors are denoted by $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively
- Two vectors are equal iff all corresponding coordinates are equal
- Main operations on vectors: addition and multiplication by a scalar
  - If $a = (a_1, a_2)$, $b = (b_1, b_2)$ are vectors in $\mathbb{R}^2$ then $a + b = (a_1 + b_1, a_2 + b_2)$
  - If k is a scalar (i.e. real number) and $a = (a_1, a_2) \in \mathbb{R}^2$ then $ka = (ka_1, ka_2)$
- For example, if $a = (-1, 3)$ and $b = (2, 1)$ then $2a - 5b = (-12, 1)$
- All the above can be generalised to n-tuples of real numbers, for any fixed n
- Notation: $\mathbb{R}^n = \{(a_1, \dots, a_n) \mid \text{all } a_i \in \mathbb{R}\}$
- Note that one can view a vector in $\mathbb{R}^n$ as a $1 \times n$ (or $n \times 1$) matrix

2 Norm and dot product in $\mathbb{R}^n$

- The length (aka norm) of a vector $v = (v_1, v_2, ..., v_n) \in \mathbb{R}^n$ is defined by the formula

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

- It holds that
  - $||v|| \geqslant 0$, and $||v|| = 0$ iff $v = 0$
  - $||kv|| = |k| \cdot ||v||$
- A vector of length 1 is called a **unit vector**
- For any vector v, the vector $\frac{1}{||v||}v$ is a unit vector in the same direction as v. It is obtained by normalizing v
- The dot product (aka inner product) of vectors $u = (u_1, ..., u_n)$ and $v = (v_1, ..., v_n)$ in $\mathbb{R}^n$ is defined as

$$u \cdot v = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

  Note that $||v|| = \sqrt{v \cdot v}$
- For example, if $u = (-1, 3, 5, 7)$ and $v = (2, -1, 3, -5) \in \mathbb{R}^4$ then $\mathbf{u} \cdot \mathbf{v} = (-1) \cdot 2 + 3 \cdot (-1) + 5 \cdot 3 + 7 \cdot (-5) = -25$

3 Properties of dot product

If u,v and w are vectors in $\mathbb{R}^n$ then the following properties hold:

- $u \cdot v = v \cdot u$ (symmetry)
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$(Distributivity)
- $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot v$(Homogeneity)
- $\mathbf{v} \cdot \mathbf{v} \geq 0$ and $\mathbf{v} \cdot \mathbf{v} = 0$ iff $\mathbf{v} = \mathbf{0}$(Positivity)

3.1 Theorem (Cauchy-Schwarz inequality, without proof)

If u and v are vectors in $\mathbb{R}^n$ then $u \cdot v \leq ||u|| \cdot ||v||$

3.2 Corollary (Triangle Inequality)

If u and v are vectors in $\mathbb{R}^n$ then $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}||$

4 Orthogonality in $\mathbb{R}^n$

- Two vectors u and v in $\mathbb{R}^n$ are orthogonal (or perpendicular) if $u \cdot v = 0$
- Example: vectors $u = (-2, 3, 1, 4)$ and $v = (1, 2, 0. -1)$ in $\mathbb{R}^4$ are orthogonal because $u \cdot v = (-2) \cdot 1 + 3 \cdot 2 + 1 \cdot 0 + 4 \cdot (-1) = 0$

4.1 Theorem (projection theorem)

If u and $a \neq 0$ are vectors in $\mathbb{R}^n$ then u can be uniquely expressed as $u = w_1 + w_2$ where $w_1 = ka$ and a and w_2 are orthogonal

4.1.1 Proof

Let $k = (\mathbf{u} \cdot \mathbf{a})/||\mathbf{a}||^2$, $\mathbf{w}_1 = k\mathbf{a}$, and $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$
Check than $a \cdot w_2 = 0$
The vector w_1 is called the orthogonal projection of u on a

4.2 Theorem (Pythagoras' theorem in $\mathbb{R}^n$)

If u and v are orthogonal vectors in $\mathbb{R}^n$ then $||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$

4.2.1 Proof

Since u and v are orthogonal, we have $u \cdot v = 0$, hence

$$||\mathbf{u} + \mathbf{v}||^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = ||\mathbf{u}||^2 + 2(\mathbf{u} \cdot \mathbf{v}) + ||\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$$

5 General (real) vector spaces

5.1 Definition

Let V be a set equipped with operations of "addition" and "multiplication my scalars", that is, for every $u, v \in V$ and every $k \in \mathbb{R}$

- the "sum" $u + v \in V$ is defined, and
- the "product" $ku \in V$ is defined

V is called a (real) vector space, or linear space, if the following axioms hold:

1. $u + v = v + u$
2. $u + (v + w) = (u + v) + w$
3. there is an element $0 \in V$ such that $u + 0 = 0 + u = u$ for all u
4. For each $u \in V$, there is $-u \in V$ such that $u + (-u) = (-u) + u = 0$
5. $k(u + v) = ku + kv$
6. $(k + m)u = ku + mu$
7. $k(mu) = (km)u$
8. $1u = u$

The elements from V are called vectors

5.2 Examples of vector spaces

- $\mathbb{R}^n = \{(a_1, \ldots, a_n) \mid \text{all } a_i \in \mathbb{R}\}$
- The set $\mathbb{R}^\infty$ of all infinite sequences $u = (u_1, u_2, ..., u_n, ...)$ is a vector space with operations of point-wise addition and multiplication (just as in $\mathbb{R}^n$)

$$(u_1, u_2, \ldots, u_n, \ldots) + (v_1, v_2, \ldots, v_n, \ldots) = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n, \ldots)$$

$$k\,(u_1, u_2, \ldots, u_n, \ldots) = (ku_1, ku_2, \ldots, ku_n, \ldots)$$

- All matrices of size $m \times n$ form a vector space, denoted $\mathbb{M}_{mn}$, with the usual operations of matrix addition and multiplication by scalars

6 Subspaces

6.1 Definition

A subset W of a vector space V is called a **subspace** of V is W itself is a vector space, with operations inherited from V

- To verify that W is a subspace of V, we don't need to check all 8 axioms
- We only need to check that W is closed under the operations of V, that is, if $u, v \in W$ and $k \in \mathbb{R}$ then $u + 1 \in W$ and $ku \in W$

Examples:

- $\{0\}$ is always a subspace (the zero subspace) of any vector space
- For any fixed $a \in \mathbb{R}^n$, the set $\{ka|k \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^n$. Indeed, if $u = k_1a$ and $v = k_2a$ then $u + v = (k_1 + k_2)a$ and $ku = k(k_1a) = (kk_1)a$
- The solution set of a homogeneous linear system $Ax = 0$ with n variables is a subspace of $\mathbb{R}^n$. Indeed, if u and v are solutions, i.e. $Au = 0$ and $Av = 0$ then $A(u + v) = Au + Av = 0$ and, for any k, $A(ku) = k(Au) = 0$

Non example, Invertible $n \times n$ matrices do not form a subspace of $\mathbb{M}_{nm}$

6.2 Lemma

If $W_1, W_2, ..., W_r$ are subspaces of V then so is $W_1 \cap W_2 \cap ... \cap W_r$

6.3 Proof

If vectors u, v are in $W_1 \cap W_2 \cap ... \cap W_r$ then they belong to each W_i. Since each W_i is a subspace, $u + v$ belongs to W_i.
Hence $u + v \in W_1 \cap W_2 \cap ... \cap W_r$.
The proof for multiplication by scalars is similar

7 Linear Combinations

Say that a vector $w \in V$ is a linear combination of vectors $v_1, ..., v_r \in V$ if $w = k_1v_1 + k_2v_2 + ... + k_rv_r$ for some scalars $k_1, ..., k_r$

7.1 Theorem

If $S = \{v_1, ..., v_r\}$ is a non empty subset of a vector space V then

- The set $W = \{\sum_{i=1}^{r} k_i\mathbf{v}_i | k_i \in \mathbb{R}\}$ of all linear combinations of the vectors in S is a subspace of V
- This set W is the (inclusion wise) smallest subspace of V that contains S

Inclusion wise minimal - The set in the collection that is not a superset of any other set in the collection The set W is called a **span** of S, it is denoted by $\text{span}(S)$ or $\text{span}(v_1, ..., v_r)$

8 Spanning $\mathbb{R}^n$

- The standard unit vectors in $\mathbb{R}^n$ are

$$\mathbf{e}_1 = (1,0,0,\ldots,0), \mathbf{e}_2 = (0,1,0,\ldots,0), \ldots, \mathbf{e}_n = (0,0,0,\ldots 1)$$

  They span $\mathbb{R}^n$ because any vector $(a_1, a_2, ..., a_n) \in \mathbb{R}^n$ can be represented as

$$(a_1, a_2, \ldots, a_n) = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \ldots + a_n\mathbf{e}_n$$

- How do we test whether a given set of n vectors spans $\mathbb{R}^n$? Let's take n=3
  - Vectors v_1, v_2, v_3 span $\mathbb{R}^3$ iff vectors e_1, e_2, e_3 can be expressed as linear combinations of the $v_i's$
  - Let $A = [v_1|v_2|v_3]$ be the matrix whose columns are the vectors v_1, v_2, v_3
  - The identity matrix I_3 can be written as $I_3 = [e_1|e_2|e_3]$
  - The vectors e_1, e_2, e_3 can be expressed as a linear combination of v_i's iff there is a 3×3 matrix B such that $AB = I_3$
  - So, v_1, v_2, v_3 span $\mathbb{R}^3$ iff the matrix $A = [v_1|v_2|v_3]$ is invertible
  - Hence, we only need to check whether $det(A) \neq 0$

9 Linear in(dependence)

9.1 Definition

Vectors $v_1, ..., v_r$ are called linearly independent if

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r = \mathbf{0} \Rightarrow k_1 = k_2 = \ldots = k_r = 0$$

Otherwise, they are linearly dependent

9.2 Explanation

- Standard unit vectors in $\mathbb{R}^n$ are linearly independent. Indeed, if $k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + \ldots + k_n\mathbf{e}_n = (k_1, k_2, \ldots, k_n) = 0$ then $k_1 = k_2 = \ldots = k_r = 0$
- Determine whether vectors $v_1 = (1,-2,3), v_2 = (5,6,-1)$, and $v_3 = (3,2,1)$ in $\mathbb{R}^3$ are linearly independent
  Assume that $k_1v_1 + k_2v_2 + k_3v_3 = 0$ This can be written as the linear system

$$\begin{aligned} k_1 + 5k_2 \quad +3k_3 &= 0 \\ -2k_1 + 6k_2 \quad +2k_3 &= 0 \\ 3k_1 - k_2 \quad +k_3 &= 0 \end{aligned}$$

  Let A be the matrix of this system. By Theorem bout invertible matrices, the system has only the trivial solution $k_1 = k_2 = k_3 = 0$ iff $det(A) \neq 0$, hence, the vectors are linearly dependent

9.3 Theorem

A set S of two or more vectors is linearly dependent iff at least one of the vectors is expressible as a linear combination of the other vectors in S

9.4 Proof

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$. Let $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r = \mathbf{0}$ and $k_i \neq 0$ for some i. Let k_3 be the first non zero coefficient. Then $\mathbf{v}_s = -\frac{k_{s+1}}{k_s}\mathbf{v}_{s+1} - \ldots - \frac{k_r}{k_s}\mathbf{v}_r$. The other direction of very easy

9.5 Theorem

Let $S = \{v_1, ..., v_r\}$ be a subset of $\mathbb{R}^n$. If $r > n$ then S is linearly dependent

9.6 Proof

Assume that $k_1v_1 + k_2v_2 + ... + k_rv_r = 0$
As in the example in the previous section, this can be written as a linear system. This is a homogeneous linear system with more variables than equations. Hence it has a non-trivial solution, so the vectors in S are linearly dependent.