Eigenvalues and Eigenvectors

1 Eigenvalues and Eigenvectors

1.1 Definition

Let A be an $n \times n$ matrix. A non-zero vector $x \in \mathbb{R}^n$ is called an eigenvector of A if, for some scalar λ

$$Ax = \lambda x$$

In this case, λ is called an eigenvalue of A and x is an eigenvector corresponding to λ

- The assumption $x \neq 0$ is necessary to avoid the case $A0 = \lambda 0$ which always holds
- The meaning of the notion is that when x is multiplied by A it does not change direction (up to reversal)

1.2 Example

Vector $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$ corresponding to the eigenvalue 3. Indeed,

$$A\mathbf{x} = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3\mathbf{x}$$

2 Characteristic equation of a matrix

2.1 Theorem

If A is an $n \times n$ matrix then λ is an eigenvalue of A iff it satisfies $\det(\lambda I - A) = 0$

The equation $\det(\lambda I - A) = 0$ is called the characteristic equation of A

2.2 Proof

By definition, λ is an eigenvalue of A iff $Ax = \lambda x$ for some $x \neq 0$
The equation $Ax = \lambda x$ can be re-written as $Ax = \lambda Ix$, and then as $(\lambda I - A)x = 0$
By theorem about invertible matrices, the last equation has solution $x \neq 0$ iff $\det(\lambda I - A) = 0$

2.3 Example

Find the eigenvalues of the matrix $A = \begin{pmatrix} 2 & -1 \\ 10 & -9 \end{pmatrix}$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 1 \\ -10 & \lambda + 9 \end{vmatrix} = (\lambda - 2) \cdot (\lambda + 9) - 1 \cdot (-10) = \lambda^2 + 7\lambda - 8$$

So, the characteristic equation of A is $\lambda^2 + 7\lambda - 8 = 0$
Its solutions are $\lambda_1 = 1$ and $\lambda_2 = -8$ are the eigenvalues of A

3 Characteristic polynomial of a matrix

- In general, the expression $\det(\lambda I - A)$ is a polynomial

$$p(\lambda) = \lambda^n + c_1\lambda^{n-1} + \ldots + c_{n-1}\lambda + c_n$$

- Solving the equation $p(\lambda) = 0$ is difficult in general (no closed formula). In practice, it is solved numerically.

3.1 Example

Find the eigenvalues of $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$. We have

$$\det(\lambda I - A) = \begin{pmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda - 8 \end{pmatrix} = \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

The solutions are $\lambda = 4, 2 + \sqrt{3}$ and $2 - \sqrt{3}$

4 Eigenspaces and their bases

- Let λ_0 be an eigenvalue of A and consider the equation $(\lambda_0 I - A)x = 0$
- The solution set of the equation is a subspace of $\mathbb{R}^n$, it is the null space of the matrix $\lambda_0 I - A$
- It is called the eigenspace of A corresponding to λ_0 because the non-zero vectors in this subspace are the eigenvectors of A corresponding to λ_0
- To find the basis in this subspace, use the algorithm for finding basis in null space of a matrix

4.1 Example

Find (a basis of) the eigenspace of $A = \begin{pmatrix} 2 & -1 \\ 10 & -9 \end{pmatrix}$ corresponding to $\lambda = 8$

Form the equation $(-8I - A)x = 0$, or

$$\begin{pmatrix} -10 & 1 \\ -10 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{array}{c} -10x_1 + x_2 = 0 \\ -10x_1 + x_2 = 0 \end{array}$$

The subspace consists of all vectors of the form $(x, 10x)$. One basis is $\{(1, 10)\}$

5 Similarity of matrices

5.1 Definition

Square matrices A and B are called **similar** if $A = P^{-1}BP$ for some invertible P

Not that if $A = P^{-1}BP$ then $B = Q^{-1}AQ$ where $Q = P^{-1}$
Similar matrices have many features in common, including determinant, rank, nullity, characteristic polynomial, eigenvalues etc.

5.2 Lemma

If A and B are similar then $\det(A) = \det(B)$

5.3 Proof

$\det(A) = \det\left(P^{-1}BP\right) = \det\left(P^{-1}\right)\det(B)\det(P) = \frac{1}{\det(P)}\det(B)\det(P) = \det(B)$

5.4 Definition

A square matrix is called **diagonalisable** if it is similar to a diagonal matrix

6 Diagonalisation

6.1 Theorem

An $n \times n$ matrix is diagonalisable iff it has n linearly independent eigenvectors

6.2 Proof

We prove only ($\Rightarrow$)-direction. Assume that there is an invertible matrix P and a diagonal matrix $D = \text{diag}(\lambda_1, ..., \lambda_n)$ such that $D = P^{-1}AP$, or $AP = PD$.
Denote the column vectors of P by $\mathbf{p}_1, ..., \mathbf{p}_n$, so that $P = [p_1|...|p_n]$. Then

$$AP = A\left[\mathbf{p}_1|\ldots|\mathbf{p}_n\right] = \left[A\mathbf{p}_1|\ldots|A\mathbf{p}_n\right]$$

On the other hand

$$PD = \left[\lambda_1\mathbf{p}_1|\ldots|\lambda_n\mathbf{p}_n\right]$$

Since $AP = PD$, we can conclude that $A\mathbf{p}_i = \lambda_i\mathbf{p}_i$ for all $1 \leqslant i \leqslant n$.
Since P is invertible the vectors $\mathbf{p}_1, ..., \mathbf{p}_n$ are linearly independent and in particular, none of $\mathbf{p}_1, ..., \mathbf{p}_n$ is $\mathbf{0}$, so each of them is a linearly independent eigenvector.