DMLA Term 1

1 Mathematical Induction

Step 1 (**Basis Step**): Check that $S(n)$ is true for $n = j$; If this is not the case, then the statement cannot be true. If $S(j)$ is true, then proceed to Step 2
Step 2 (**Induction Step**): Prove the following **conditional** statement. If $S(n)$ holds for a fixed value $n = k \geqslant j$, then it also holds for $n = k + 1$

2 Basic Counting Principles

2.1 The product rule

If a procedure can be broken down into a sequence of 2 tasks, with n_1 ways of doing the 1st task and n_2 ways of doing the second, then there are $n_1 \times n_2$ ways to do the procedure

2.2 The sum rule

If the task can be done in either one of n_1 ways or in one of n_2 ways, where the two sets are distinct, there are $n_1 + n_2$ ways to do the task

2.3 Permutations

Permutation - A set of distinct objects in an ordered arrangement of these objects
r-permutation - An ordered arrangement of r elements of a set of at least r distinct objects

$$P(n,r) = \frac{n!}{(n-r)!}$$

For any set of n distinct objects, there are $n!$ permutations of the set

2.4 Combinations

r-combination - An unordered selection of r elements from a set of at least r elements

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

3 Basic Counting, Binomial Coefficients

3.1 Binomial Coefficients

The number of r-combinations of a set with n distinct elements (with $r \leqslant n$) is denoted by $C(n,r)$. It is also denoted by

$$\binom{n}{r}$$

Pascal's identity

$$\binom{n+1}{k} = \binom{n}{k+1} + \binom{n}{k}$$

3.2 Binomial Theorem

Let x and y be variables, and let n be a nonnegative integer.

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \\ + \binom{n}{n-2}x^2y^{n-2} + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

3.3 Permutations with indistinguishable objects

The number of different permutations of n objects, of with there are n_k indistinguishable objects of type k is

$$\frac{n!}{n_1!n_2!...n_k!}$$

3.4 Combinations with Indistinguishable Objects

In order to use this, use the stars and bars method, where each group is separated by bars, and the result is $C(Stars + Bars, Bars)$

4 Intro to Discrete Probability

4.1 Sample space, events and probability

Experiment - A procedure that yields one of a given set of possible outcomes
Sample Space - The set of possible outcomes
Event = A subset of the sample space

If S is a finite sample space of equally likely outcomes, and E is the event in it, then

$$p(E) = \frac{|E|}{|S|}$$

4.2 Probability of combinations of events

For events E_1, E_2 in a sample space S

- $E_1 \cup E_2$ is the event if at least one of E_1 and E_2
- Let $E_1 \cap E_2$ denote the event that occurs if both E_1 and E_2 occur

$$p(E_1 \cup E_2) = \frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S|} = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

5 Conditional Probability, Bernoulli Trials

5.1 Conditional Probability

Let E and F be events and let $p(F) > 0$. The conditional probability of E given F is:

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

5.2 Independence

The events E and F are independent if either

$$p(E \cap F) = p(E)p(F)$$

$$p(E) = p(E|F)$$

5.3 Bernoulli Trials

- This is an experiment with two possible outcomes, success and failure
- The corresponding probabilities are denoted p and q
- It is useful to determine the probability of k successes in n mutually independent Bernoulli trials

The probability of exactly k successes in n independent Bernoulli trials, with probability of success p and probability of failure $q = 1 - p$ is

$$b(k; n; p) = \mathrm{C}(n, k) \cdot p^k \cdot q^{n-k}$$

- When n independent trials are carried out, the outcome is a tuple $(t_1, ..., t_n)$ where each t_i is either success or failure
- There are $C(n, k)$ ways to have exactly k successes
- The probability of each of the ways is $p^k \cdot q^{n-k}$ (because of independence)
- Since no two ways can occur together, the overall probability is as in the theorem

The binomial distribution

- For a fixed n and p, assign the number $b(k; n; p)$ to the event "k successes"
- We have $0 \leqslant b(k; n; p) \leqslant 1$ for each k, and

$$b(0; n, p) + b(1; n, p) + \ldots + b(n; n, p) = \sum_{k=0}^{n} C(n, k) p^k q^{n-k} = (p+q)^n = 1$$

6 Bayes' Theorem, random variables

6.1 Bayes' Theorem

Let E and F be events in sample space S such that $p(E) \neq 0$ and $p(F) \neq 0$. Then

$$p(F|E) = \frac{p(E|F)p(F)}{p(E|F)p(F) + p(E|\overline{F})p(\overline{F})}$$

6.2 Random Variables

Random Variable - A function from the sample space of an experiment to the real numbers

Distribution of a random variable - With the random variable X on a sample space S is the set of pairs $(r, p(X = r))$ for all values $r \in X(S)$, where $p(x = r)$ is the probability that X takes value r

6.3 Expected Value

The expected value of a random variable X on a sample space S with possible outcomes $s_1, ..., s_n$ is equal to

$$E(X) = \sum_{i=1}^{n} p(s_i) X(s_i)$$

6.4 Linearity of expectation

If $X_i, i = 1, ..., n$ are random variables on a sample space S, and a and b are real numbers then

- $E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n)$
- $E(aX + b) = aE(X) + b$

6.5 Variance and standard deviation

Let X be a random variable on a sample space S. The variance of X is given by

$$V(X) = \sum_{i=1}^{n} (X(s_i) - E(X))^2 \cdot p(s_i)$$

The standard deviation of X, denoted $\sigma(X)$, is defined as $\sqrt{V(X)}$
If X is a random variable then $V(X) = E\left(X^2\right) - E(X)^2$

6.6 Chebyshev's Inequality

Let X be a random variable on a sample space S with probability distribution p. If $r > 0$ is a real number then

$$p(|X(s) - E(X)| \geq r) \leq V(X)/r^2$$

In words, the probability that a random variable is far from its expectation by at least r is not greater than the variance divided by r^2

6.7 Markov's inequality

Let X be a random variable on a sample space S with $X(s) \geqslant 0$ for all s. Then, for any real number $a > 0$

$$p(X(s) \geq a) \leq E(X)/a$$

7 The basics of graph theory

7.1 Formal definitions

A graph G is a pair $(V(G), E(G))$ where $V(G)$ is a nonempty set of vertices (or nodes) and $E(G)$ is a set of unordered pairs $\{u, v\}$ with $u, v \in V(G)$ and $u \neq v$, called the edges of G

7.2 Types of graphs

- Directed graphs (digraphs) - edges can have directions
- Multigraphs - Multiple edges allowed between two vertices
- Pseudographs - edges of the form uu, called loops, are allowed
- Vertex or edge weighted graphs - vertices and/or edges can have weights

7.3 Terminology

Let G be a graph and uv an edge in it. Then

- u and v are called endpoints of the edge uv
- u and v are called neighbours or adjacent vertices
- uv is said to be incident to u(and v)
- if vw is an edge and $w \neq u$ then uv and vw are called adjacent

Let $G = (V, E)$ be a graph. The neighbourhood of a vertex $v \in V$, notation $N(v)$, is the set of neighbours of v, i.e. $N(v) = \{u \in V | uv \in E\}$
The degree of a vertex $v \in V$, notation $\deg(v)$, is the number of neighbours of v
With $\delta(G)$ or δ we denote the smallest degree in G, and with $\Delta(G)$ or Δ the largest degree
A vertex with degree 0 will be called an isolated vertex
A vertex with degree 1 an end vertex or a pendant vertex

A subgraph $G' = (V', E')$ of $G = (V, E)$ is a graph with $V' \subseteq V$ and $E' \subseteq E$; this subgraph is called proper if $G' \neq G$ and spanning if $V' = V$

7.4 Handshaking lemma

Let $G = (V, E)$ be a graph. Then $\sum_{v \in V} \deg(v) = 2|E|$

7.5 Graph classes

- P_n is a path on n vertices for some integer $n \geqslant 1$
- C_n is a cycle on n vertices for some integer $n \geqslant 3$
- $K_{p,q}$ is called a complete bipartite graph. Any subgraph of a $K_{p,q}$ is called a bipartite graph
- K_n is a complete bipartite graph and contains all the possible edges between pairs of vertices1

8 Paths, Cycles, Connectivity

8.1 Walks, paths, cycles and distances

Walk - A sequence of edges
Path - A walk where all vertices are distinct
Circuit/Closed walk - A walk where the start vertex is the same as the last vertex
Cycle - A closed walk where all vertices are distinct apart from the first and last
Directed Graph - A graph where ea ch edge is directed to the next
Length - The number of edges in a path or cycle
Distance - The length of the shortest path between two vertices if a path exists, and ∞ otherwise
Diameter - The largest distance between two vertices in a graph

8.2 Strong Connectivity

Weakly connected - The graph obtained from the digraph G forgetting directions is connected
Strongly connected - Any two distinct vertices are connected by directed paths in both directions
Strongly connected component - A maximal strongly connected subgraph of G

9 Paths, Cycles, Trees

9.1 Eulerian Circuits

A connected graph with at least two vertices has an Eulerian circuit iff each of its vertices has an even degree

9.2 Travelling Salesman Problem

Given a graph G with set V of vertices ($|V| = n$) and set E of edges

- For each vertex v, create a city c_v
- For each pair of distinct $u, v \in V$, set $d(c_u, c_v) = 1$ if $uv = E$ and $d(c_u, c_v) = 2$ otherwise

Then detecting a Hamiltonian cycle in G can be viewed as TSP:

- If G has a Hamiltonian cycle then the cycle is a route of cost exactly n
- If there is a route of cost n then it can't use pairs with cost 2 and so goes through edges of G and hence is a Hamiltonian cycle

9.3 Trees

Forest - An acyclic graph (graph without cycles) **Tree** - A connected forest

9.4 Spanning trees

A subgraph $G' = (V', E')$ of a graph $G = (V, E)$ is spanning if $V' = V$
Every connected graph contains a spanning tree, i.e. a spanning subgraph that is a tree

9.5 Leaves in trees

A leaf in a tree is a vertex of degree 1

Every tree on at least 2 vertices contains a leaf

9.6 Edges of trees

A connected graph on n vertices is a tree iff it has n-1 edges

9.7 Paths in trees

Let T be a tree and $u, v \in V(T)$ with $u \neq v$ then there is a unique path in T between u and v

9.8 Rooted trees

Rooted tree - A tree in which one vertex is fixed as the root and every edge is directed away from this root
Let v be a vertex in a rooted tree T

- The neighbours of v in the next level are called the children of v
- The unique neighbour of v in the previous level (if v is not the root) is called the parent of v
- If v has no children then it is called a left of T
- If v has children, then it is an internal vertex