Mathematics for Computer Science
Discrete Maths and Linear Algebra

Lecture 2: Basic Counting Principles

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Contents for today's lecture

- Basic counting;
- The product and sum rules;
- Factorials;
- Permutations and combinations;
- Exercises.

The product rule

Definition

Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 \times n_2$ ways to do the procedure.

Example: How many different passwords can be constructed using two (or k) symbols from a set of N distinct symbols?

For each of the N choices of the first symbol there are again N choices for the second symbol, so the answer is $N \times N$. For passwords consisting of k symbols the answer is $N \times N \times \ldots \times N = N^k$.

If we do not allow repetitions of symbols, the answer is $N \times (N-1)$, respectively $N \times (N-1) \times \ldots \times (N-k+1)$.

The sum rule

Definition

If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example: Suppose we have a set of N characters and a set of M integers. In how many ways can we choose one symbol which is either a character or an integer?

It is clear that we can choose the symbol in $N + M$ different ways.

In how many ways can we construct a sequence of three symbols where the first one is a character, the second one an integer, and the third one either of them?

A combination of the product and sum rules gives the answer: $N \times M \times (N + M)$.

Example: Counting IP addresses

- The Internet is made up of interconnected physical networks of computers.
- Each computer (actually, each network connection of a computer) is assigned an Internet address
- Version 4 of the Internet Protocol (IPv4) is still in use.
  - An address in IPv4 is a string of 32 bits (looks like 172.16.254.1 in decimal)
  - It consists of netid (network number) and hostid (host number)
  - There are three classes of addresses: class A, class B and class C
  - Class A is for large networks, B for medium-sized, and C for small
  - Actually, there are also classes E and D, for a separate purpose
- Class A address: 0 7-bit netid 24-bit hostid
  - Technical restriction: Class A netid cannot be 1111111
- Class B address: 10 14-bit netid 16-bit hostid
- Class C address: 110 21-bit netid 8-bit hostid
  - Technical restriction: hostid in any class cannot be all 0 or all 1

Q.: How many different IPv4 addresses are available for a computer on the Internet?

Counting IP addresses

Let x be the number we want to compute.

- Let x_A be the number of class A addresses,
- let n_A be the number of class A netids, and
- let h_A be the number of class A hostids;
- define x_B, n_B, h_B and x_C, n_C, h_C similarly.

- By the sum rule, $x = x_A + x_B + x_C$.
- By the product rule, $x_A = n_A \cdot h_A$.
  - By the product rule, $n_A = 2^7 - 1 = 127$ (since 1111111 is not available).
  - By the product rule, $h_A = 2^{24} - 2 = 16,777,214$.
  - Hence $x_A = 127 \cdot 16,777,214 = 2,130,706,178$.
- Similarly, $x_B = n_B \cdot h_B = 2^{14} \cdot (2^{16} - 2) = 1,073,709,056$.
- Also, $x_C = n_C \cdot h_C = 2^{21} \cdot (2^8 - 2) = 532,676,608$.
- All in all, $x = x_A + x_B + x_C = 3,737,091,842$ - this is a small number!

IPv4 addresses are exhausted now and 128-bit IPv6 addresses are in use.

Factorial function

Definition (Factorial)

The **factorial** of an integer $n \geq 0$, denoted $n!$, is defined by

$$\begin{aligned} 0! &= 1 \\ n! &= 1 \cdot 2 \cdot \ldots \cdot (n-1) \cdot n \quad n \geq 1. \end{aligned}$$

Example: How many different passwords of length 8 we can construct with the letters A, b, c, D, E, f, g, h if each symbol occurs exactly once?

The number is 8! (for the first symbol we have 8 possibilities; for each of these choices there are 7 possibilities for the second symbol, etc.)

Factorial function

Example:

$$
\begin{array}{lcll}
0! & = & 1 & \\
1! & = & 1 & \\
2! & = & 1 \cdot 2 & = & 2 \\
3! & = & 1 \cdot 2 \cdot 3 & = & 6 \\
4! & = & 1 \cdot 2 \cdot 3 \cdot 4 & = & 24 \\
5! & = & 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 & = & 120 \\
6! & = & 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 & = & 720 \\
10! & = & 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 & = & 3,628,800
\end{array}
$$

The factorial function $n!$ grows extremely fast with increasing n.

Q.: If $n! >$ the age of the universe in seconds, how large should n be?

A.: $n = 20$ is enough, $20! = 2.43290200817664 \cdot 10^{18} > \mathrm{age} \approx 4.32 \cdot 10^{17}$ sec.

Permutations

Definition

The **permutation** of a set of distinct objects is an **ordered** arrangement of these objects.

The different passwords in the previous example are all permutations of the 8 available symbols.

We are also interested in ordered arrangements of some of the elements of a set of objects (8 letter passwords from larger sets).

Definition

An **ordered** arrangement of r elements of a set of at least r distinct objects is called an ***r*-permutation**.

r-Permutations

Theorem

If n and r are integers with $1 \leq r \leq n$, then there are

$$P(n, r) = n \cdot (n-1) \cdot \ldots \cdot (n-r+1)$$

r-permutations of a set with n distinct elements.

Easy to prove using the product rule:

- n different choices for the first position;
- for each of these choices, $n-1$ choices for the second position,
- and so on, until the final position, for which there are $n-r+1$ different choices given any of the choices for the first $r-1$ positions.
- The product rule yields the given formula.

r-Permutations

Corollary

If n and r are integers with $1 \leq r \leq n$, then

$$P(n,r) = \frac{n!}{(n-r)!}.$$

This is also easy to prove by using the definition of the factorial function and writing out the expansions of $n!$ and the product of $P(n,r)$ and $(n-r)!$

Corollary

For any set of n distinct elements, there are $n!$ permutation of the set.

Examples

Suppose there are 8 runners in the final race, there can be no ties. How many ways are there to award the three medals if all outcomes are possible?

Answer: $P(8,3) = 8 \cdot 7 \cdot 6 = 336$

How many permutations of the letters ABCDEFGH contain the string ABC?

Answer: Since ABC must occur as a block, treat this string as one symbol. Then we need to count the number of permutations of symbols (ABC),D,E,F,G,H. It's $6! = 720$.

r-Combinations

Definition

An ***r*-combination** of elements of a set of at least r elements is an **unordered** selection of r elements from the set.

This means an r-combination is just a subset with r elements.

A classical example is the National Lottery where we could just as well pick all the six balls at once.

r-Combinations

Theorem

The number of r-combinations of a set with n elements, where n and r are integers with $0 \leq r \leq n$, equals

$$C(n, r) = \frac{n!}{r!(n-r)!}.$$

We can prove this as follows, using what we know about r-permutations:

- Each r-combination can be ordered in $r!$ different ways to obtain an r-permutation.
- So $P(n, r) = r! \cdot C(n, r)$. Writing this out and dividing by $r!$ gives the above formula.

Example

The department needs to form a committee by selecting 3 of 9 postdoctoral researchers and 4 of 11 PhD students. How many ways can this be done?

Answer: the order of selecting committee members does not matter, so

- $C(9,3) = \frac{9!}{3!6!} = 84$ ways to select postdocs and
- $C(11,4) = \frac{11!}{4!7!} = 330$ ways to select PhD students.
- By the product rule, there are $84 \cdot 330 = 27,720$ ways to select the committee.

Exercises

Exercise 1 A bookshelf contains 4 English books, 5 German books, and 3 Russian books. We assume all the books have different titles (no copies).

- In how many different ways can we choose one book?
- In how many different ways can we choose one book from each language? (Here we assume that the order in which the three books are chosen is not important.)
- What if the order is important?

Exercise 2: A multiple-choice test contains 10 questions. There are four possible answers for each question.

- How many ways can a student answer the questions on the test if the student answers every question?
- How many ways can a student answer the questions on the test if the student can leave answers blank?

Exercises

Exercise 3: A bit is a symbol with two possible values, namely 0 and 1. A bit string is a sequence of bits.

- How many different bit strings are there of length six?
- How many bit strings of length six start and end with 1s?

Exercise 4: Five people give presentations. How many different ways are there to timetable them?

Exercise 5: Twenty people meet in a room. Each person shakes hands once with each other person. How many hand shakes are there in all?