Natural Deduction for Propositional Logic

1 Proof Systems for Propositional Logic

- What we would like from a proof system
  - **Completeness** - Using our proof system, we should be able to prove all of the tautologies
  - **Soundness** - All theorems proved by our proof system should be tautologies
- A **proof system** defines the proofs (valid mathematical arguments) of the system - it is a collection of **rules of inference**
- These rules of inference can be applied to infer new formulae from old
- Henceforth, we consider propositional logic to consist only of those formulae build using the connectives $\wedge \vee \neg \Rightarrow$
  - With other connectives, such as $\Leftrightarrow$, abbreviations
- An **argument form** in propositional logic is a sequence of formulae
  $\varphi_1, \varphi_2, ..., \varphi_n, \psi$
  and such an argument form is valid if:
  - Whenever a truth assignment f is s.t. $\varphi_1, \varphi_2, ..., \varphi_n$ evaluate to true under f then ψ necessarily evaluates to true under f
- An argument form can also be written in the form $\varphi_1, \varphi_2, ..., \varphi_n \vdash \psi$ when it is referred to as a **sequent**
- The rule of inference corresponding to the above argument form is:
  $\varphi_1, \varphi_2, ..., \varphi_n \Rightarrow \psi$
  and if the above argument form is valid then this rule of inference is a **tautology**
- The most well known rule of inference for propositional logic is the law of detachment

2 Applying rules of inference

- Of course, when applying a rule of inference we can substitute arbitrary formulae for p and q

	$(p \wedge q) \Rightarrow \neg r$	X
and		
	$((p \wedge q) \Rightarrow \neg r) \Rightarrow ((q \wedge r) \vee s)$	$X \Rightarrow Y$
yields		
	$((q \wedge r) \vee s$	Y

- Similarly, given any rule of inference $\varphi_1, \varphi_2, ..., \varphi_n \Rightarrow \psi$
  - We can apply this rule by substituting **any** formula for **any** propositional variable, so long as the same formula is substituted for the same variable
  - Thus, a valid argument form yields an infinite collection of tautologies

3 Other rules of inference

Modus tollens
$$\frac{\neg q \qquad p \Rightarrow q}{\neg p}$$

Hypothetical syllogism
$$\frac{p \Rightarrow q \qquad q \Rightarrow r}{p \Rightarrow r}$$

Resolution
$$\frac{p \vee q \qquad \neg p \vee r}{q \vee r}$$

For all these diagrams, if the two statements on the top are true, then the statement on the bottom must be true.

4 Rules of inference in action

5 An alternative approach

- We could write down all possible truth assignments on A,W,I,P,S and D, and:
  - Retain only those for which $A \wedge W \Rightarrow I, A \vee P, W \vee S, \neg I$ and $D \Rightarrow \neg(P \vee S)$ are true
  - Then check to see that for all of these retained truth assignments we have that $\neg D$ is true
- However, this would mean that $2^6 = 64$ different truth assignments need to be checked
- Consequently, the proof-theoretic approach can be significantly more efficient than the truth table approach, especially when there is a large number of propositional variables
- Of course knowing which rules of inference to apply which formulae so that we get a speedy proof is another difficulty that needs to be overcome

6 Natural Deduction

- The proof system **natural deduction** consists of a collection of valid rules of inference and is used to obtain proofs of sequence of the form:
  $\varphi_1, \varphi_2, ..., \varphi_n \vdash \psi$
- We assume that we are given $\varphi_1, \varphi_2, ..., \varphi_n$ as **premises**. We hope to apply our rules of inference from the proof system to obtain ψ
- Rules for conjunction:

$$\frac{\varphi_1 \qquad \varphi_2}{\varphi_1 \wedge \varphi_2} \wedge i \qquad\qquad \frac{\varphi_1 \wedge \varphi_2}{\varphi_1} \wedge e1 \qquad\qquad \frac{\varphi_1 \wedge \varphi_2}{\varphi_2} \wedge e2$$

$\wedge$-introduction $\qquad$ $\wedge$-elimination

- Rules for double negation:

$$\frac{\varphi}{\neg\neg\varphi} \neg\neg i \qquad\qquad \frac{\neg\neg\varphi}{\varphi} \neg\neg e$$

$\neg\neg$-introduction $\qquad$ $\neg\neg$-elimination

- Note
  - In general φ_1 and φ_2 are formulae and not necessarily propositional variables
  - All of our rules are valid

7 A simple proof

- Here is the proof of the sequent $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$ using the rules we have introduced so far

1	p	premise
2	$\neg\neg(q \wedge r)$	premise
3	$\neg\neg p$	$\neg\neg i1$
4	$q \wedge r$	$\neg\neg e2$
5	r	$\wedge e24$
6	$\neg\neg p \wedge r$	$\wedge i35$

- Note that the validity of the rules means that
  - If p and $\neg\neg(q \wedge r)$ are true under some truth assignment then $\neg\neg p \wedge r$ is necessarily true under this truth assignment
- We often say that a sequent is **valid** if it can be proved

8 More rules

- Rule for eliminating implication

$$\frac{\varphi_1 \qquad \varphi_1 \Rightarrow \varphi_2}{\varphi_2} \Rightarrow e$$

$\Rightarrow$-elimination

- Rule for introducing implication

$$\frac{\boxed{\begin{array}{c}\varphi_1 \\ \cdots \\ \varphi_2\end{array}}}{\varphi_1 \Rightarrow \varphi_2} \Rightarrow i$$

$\Rightarrow$-introduction

- The box doesn't imply φ_1 is true, just that the stuff below the line is true if the stuff above the line is true
- In order to apply the rules $\Rightarrow i$
  - To start with the intended premise φ_1, as the first line of the box
  - Continue until we prove φ_2
  - Close the box and write our implication $\varphi_1 \Rightarrow \varphi_2$
- Thereafter we are not allowed to use any formula in the box. Once a box has closed then the formula within it are no longer available to us

9 Proof using boxes

- Here is a proof of the sequent $p \Rightarrow q, q \Rightarrow r \vdash p \Rightarrow r$

1.	$p \Rightarrow q$	premise
2.	$q \Rightarrow r$	premise
3.	p	assumption
4.	q	$\Rightarrow$e 1 3
5.	r	$\Rightarrow$e 2 4
6.	$p \Rightarrow r$	$\Rightarrow$i 3-5

- Note that it is possible
  - For a proof to involve more than one box

  - For boxes to be nested within each other

- Note that boxes cannot overlap
  - We cannot **open** a box and then **open** another box, then **close** the first box before **closing** the second box

10 More than one box

- Here is a proof of the sequent $(p \land q) \Rightarrow r \vdash p \Rightarrow (q \Rightarrow r)$

1.	$(p \land q) \Rightarrow r$	premise
2.	p	assumption
3.	q	assumption
4.	$p \land q$	$\land$i 2 3
5.	r	$\Rightarrow$e 1 4
6.	$q \Rightarrow r$	$\Rightarrow$i 3-5
7.	$p \Rightarrow (q \Rightarrow r)$	$\Rightarrow$i 2-6

- Note that the structure of the formula we wish to prove helps to determine the structure/tactics of our proof