LDS Revision Notes

1 Introduction to Logic

Syntax - The definition of the well-formed formulae of the logic
Semantics - The association of meaning and truth to the formulae of the logic
Proof system - The manipulation of formulae according to a system of rules
Completeness - All the "true" semantics formulae should be "provable"
Soundness - Formula that is "provable" should be "true"

2 Fundamentals of propositional logic

If φ evaluates to T for every f then φ is a tautology
If φ evaluates to F for every f then φ is a contradiction

2.1 De Morgan's Laws

$$\neg(X \wedge Y) \equiv \neg X \vee \neg Y$$
$$\neg(X \vee Y) \equiv \neg X \wedge \neg Y$$

These can be generalised to

$$\neg\left(X_1 \vee X_2 \vee \ldots \vee X_n\right) \equiv \neg X_1 \wedge \neg X_2 \wedge \ldots \wedge \neg X_n$$
$$\neg\left(X_1 \wedge X_2 \wedge \ldots \wedge X_n\right) \equiv \neg X_1 \vee \neg X_2 \vee \ldots \vee \neg X_n$$

3 More on Propositional Logic

The Distributive Law of Disjunction over Conjunction

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

The Distributive law of Conjunction over Disjunction

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

These can be generalised to

$$X_{\wedge}\left(Y_1 \vee Y_2 \vee \ldots \vee Y_n\right) \equiv \left(X \wedge Y_1\right) \vee \left(X \wedge Y_2\right) \vee \ldots \vee \left(X \wedge Y_n\right)$$
$$X \vee \left(Y_1 \wedge Y_2 \wedge \ldots \wedge Y_n\right) \equiv \left(X \vee Y_1\right) \wedge \left(X \vee Y_2\right) \wedge \ldots \wedge \left(X \vee Y_n\right)$$

3.1 Functional Completeness

We say that a set C of logical connectives is functionally complete if any propositional formula is equivalent to one constructed using only the connectives from C

3.2 Conjunctive and Disjunctive Normal Form

Disjunctive normal form is of the form

$$\chi_1 \vee \chi_2 \vee \ldots \vee \chi_m$$

Where each χ_i is a conjunction of literals

Conjunctive normal form is of the form

$$\chi_1 \wedge \chi_2 \wedge \ldots \wedge \chi_m$$

Where each χ_i is a disjunction of literals

4 Natural Deduction for Propositional Logic

Modus Ponens

$$\frac{p \qquad p \Rightarrow q}{q}$$

Modus Tollens

$$\frac{\neg q \qquad p \Rightarrow q}{\neg p}$$

Hypothetical Syllogism

$$\frac{p \Rightarrow q \qquad q \Rightarrow r}{p \Rightarrow r}$$

Resolution

$$\frac{p \vee q \qquad \neg p \vee r}{q \vee r}$$

4.1 Natural Deduction

$\wedge$ introduction

$$\frac{a \qquad b}{a \wedge b}$$

$\wedge$ elimination 1

$$\frac{a \wedge b}{a}$$

$\wedge$ elimination 2

$$\frac{a \wedge b}{b}$$

$\vee$ introduction 1

$$\frac{a}{a \vee b}$$

$\vee$ introduction 2

$$\frac{b}{a \vee b}$$

$\vee$ elimination

$$\frac{a \vee b \qquad \boxed{\begin{array}{c} \text{A} \\ \vdots \\ \text{X} \end{array}} \quad \boxed{\begin{array}{c} \text{B} \\ \vdots \\ \text{X} \end{array}}}{X}$$

$\Rightarrow$ introduction

$$\frac{\boxed{\begin{array}{c} \text{a} \\ \vdots \\ \text{b} \end{array}}}{a \Rightarrow b}$$

$\Rightarrow$ elimination

$$\frac{a \Rightarrow b \qquad a}{b}$$

$\neg$ introduction

$$\frac{\boxed{\begin{array}{c} \text{a} \\ \vdots \\ \perp \end{array}}}{\neg a}$$

$\neg$ elimination

$$\frac{a \qquad \neg a}{\perp}$$

$\neg\neg$ elimination

$$\frac{\neg\neg a}{a}$$

$\perp$ elimination

$$\frac{\perp}{\varphi}$$

5 Sets

$x \in X$ - x is an element of the set X
$x \notin X$ - x is not a member of the set X
$\mathbb{N}$ - Natural Numbers (+ve inc 0)
$\mathbb{Z}$ - Integers (+ve and -ve)
$\mathbb{Q}$ - Rational Numbers (expressible as a fraction)
$\mathbb{R}$ - Real Numbers (not imaginary)

5.1 Cardinality

$|S|$ - The size of the set S
$\varnothing$ - Empty set

5.2 Set equality

$X = Y$ - They have the same elements
Note that sets are objects and they can have sets of elements

$$\{\varnothing\} \neq \varnothing$$

5.3 Subsets

$X \subseteq Y$ - X is a subset of Y
$X \subsetneq Y$ - X is not a subset of Y
$X \subset Y$ - A is a proper subset of Y

5.4 The power set

The power set of S is the set of all subsets of S

5.5 The Cartesian Product

The cartesian product $X \times Y$ is the set

$$\{(x, y) : x \in X \text{ and } y \in Y\}$$

Example:
The cartesian product of $\{0, 1, 2\}$ and $\{a, b\}$ is

$$\{(0, a), (1, a), (2, a), (0, b), (1, b), (2, b)\}$$

The cartesian product of $\{a, b\}$ and $\{0, 1, 2\}$

$$\{(a, 0), (a, 1), (a, 2), (b, 0), (b, 1), (b, 2)\}$$

5.6 Union and Intersection

$A \cup B$ - Union of A and B, the set that contains all elements in A, in B, and in both
$A \cap B$ - Intersection of A and B, the set of all elements in both A and B
Disjoint - Two sets that have a union of the empty set

5.7 Difference and Complement

$A - B$ or $A \backslash B$ - The difference of A and B. The set that contains all elements in A and not in B
$\overline{A}$ - All the elements that are not in A

6 Discrete Structures - Functions

In the case of a function $A \to B$

- The set A is known as the domain (or source)
- The set B is known as the codomain (or target)

If $f(a) = b$ then b is the image of a (under f)
The pre-image of $b \in B$ (under f) is the subset $\{a : f(a) = b\}$ of A
The image (or range) of f is the set of images of elements of A

6.1 Partial Functions

A partial function $f : A \to B$ is such that either $f(a) \in B$ or $f(a)$ is undefined

6.2 Special Types of function

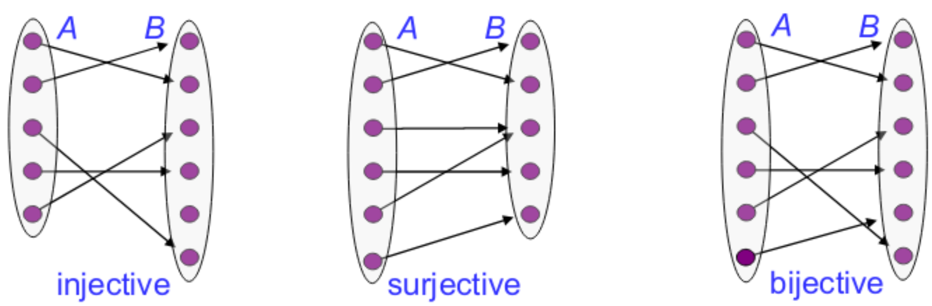
Injective - One to one function

$$\forall a \in A \text{ and } a' \in A\ f(a) = f(a') \Rightarrow a = a'$$

Surjective - Onto

$$\forall b \in B \exists a \in A\ s.t.\ f(a) = b$$

Bijective - Both injective and surjective



6.3 Compositions of functions

Suppose that $f : A \to B$ and $g : B \to C$ are functions. We can define the composition of g and f as the function $g \circ f : A \to C$ defined as $(g \circ f)(x) = g(f(x))$

6.4 Inverses

The inverse of the function $f : A \to B$ is the function $f^{-1} : B \to A$

6.5 Cardinality Revisited

Two sets A and B have some cardinality iff there is a bijection from A to B
A set is countable if it is finite or has the same cardinality as $\mathbb{N}$
A set is uncountable if it does not have cardinality $\aleph_0$

6.6 Uncountable sets

There exist uncountable sets, such as $\mathbb{R}$

7 Discrete Structures - Relations

A binary relation R from A to B is a subset of the cartesian product $A \times B$

- We write $(a, b) \in R$ or say that $R(a, b)$ holds if the ordered pair (a, b) is in the binary relation R
- We write $(a, b) \notin R$ or say that $R(a, b)$ does not hold

7.1 Functions as binary relations

Functions can be viewed as binary relations
If $f : A \to B$ then the graph of the function f is the binary relation

$$\{(a, f(a)) : a \in A\} \subseteq A \times B$$

7.2 Properties of relations

Reflexive - $(a,a) \in R, \forall a \in A$
Irreflexive - $(a,a) \notin R \forall a \in A$
Symmetry - Whenever $(a,b) \in R$, then $(b,a) \in R \forall a, b \in A$
Antisymmetry - Whenever $(a,b)(b,a) \in R$, then $a = b \forall a, b \in A$
Transitivity - When $(a,b)(b,c) \in R$ then $(a,c) \in R, \forall a, b, c \in A$

7.3 Combining relations

Let $R \subseteq A \times B$ and $S \subseteq B \times C$ be relations. The composite relation $S \circ R \subseteq A \times C$ is defined as

$$\{(a,c) : a \in A, c \in C, \exists b \in B \text{ s.t. } (a,b) \in R \text{ and } (b,c) \in S\}$$

7.4 Projections

Projections are things, I don't understand them

7.5 Closures of relations

Let $R \subseteq A \times A$
Reflexive closure - The smallest reflexive relation that contains R. It is obtained by adding to R all the pairs (x,x) tat do not already lie in R
Symmetric closure - The smallest symmetric relation that contains R. It is obtained by adding to R all the pairs (x,y) for which (y,x), but not (x,y), lies in R
Transitive closure - The smallest transitive relation that contains R. It is the relation defined as

$$\{(a,b) : a, b \in A, (a,b) \in R^n, \text{ for some } n \geq 1\} = \bigcup_{n=1}^{\infty} R^n$$

7.6 Equivalence Relations

The relation $R \subseteq A \times A$ is called an equivalence relation if it is reflexive, symmetric and transitive.
This can be denoted by $a \equiv b$ or $a \sim b$ or $b \equiv a$

7.7 Partial Orders

A binary relation that is reflexive, anti-symmetric and transitive
A set S together with a partial order R on S is called a partially ordered set (or **poset**) and written (S,R)
We denote a partial order relation in a poset by $\leq$ even though we may not be referring to the usual ordering on numbers, and write $a \leq b$ rather than $\leq (a,b)$
If $(S, \leq)$ is some poset then two elements of S are comparable if either $a \leq b$ or $b \leq a$ and incomparable otherwise

7.8 Total and Well orders

If $(S, \leq)$ is a poset and every two elements in S are comparable then S is a **totally ordered set** or **linearly ordered set**
If $(S, \leq)$ is a poset and $\leq$ is a total ordering and every non-empty subset of S has a least element (under $\leq$) then $(S, \leq)$ is a well-ordered set

8 An overview of first-order logic

Predicate Symbol - A symbol with an associated arity

8.1 Quantifiers

A formula with free variables can be quantified using the ∃ quantifier
A formula with bound variables can be quantified using the ∀ quantifier

9 Formal Syntax and Semantics

9.1 Syntax of first order logic

Every well formed formula of first order logic is constructed from atoms. We define the syntax of first order logic by defining what we mean by atoms and constructions we are allowed to use

Signature - The finite set of predicate (relation) and constant symbols
Sentence - A formula with no free variables

9.2 Parse trees

If a formula can be written in a parse tree then it is well formed

9.3 Semantics of first-order logic

An interpretation or a structure for first-order formula ϕ is:

- A domain of discourse D
- A value from D for every free variable of ϕ
- A relation over D for every relation symbol involved in ϕ
- A value from D for every constant symbol involved in ϕ

1 The semantics of a first-order formula in some interpretation is as follows:

- We interpret atoms as propositional variables
- We interpret $\wedge, \vee$, and $\neg$ as in propositional logic
- We interpret $\forall x \phi$ as true if ϕ is true for all values for x
- We interpret $\exists x \phi$ as true if there is at least one value for x making ϕ true

10 First Order Logic - Logical Equivalence

Two formulas ϕ and ψ are logically equivalent if they are true for the same set of models, in which case we write $\phi \equiv \psi$

All logical equivalences from propositional logic give rise to equivalences in first-order logic

11 Prenex Normal Form

To write an expression in Prenex Normal Form, first draw it into a parse tree.
We say that a first-order formula is in prenex normal form if it is written in the form

$$Q_1 x_1 Q_2 x_2 \ldots Q_k x_k \phi$$

Where:

- Each Q_i is a quantifier
- Each x_i is a variable
- The formula ϕ is quantifier free

To build the prenex normal form from the tree we start at the leaves and work up the tree repeatedly constructing prenex normal form formulae that are equivalent to the formulae corresponding to sub-trees of the parse trees