More on Natural Deduction for Propositional Logic

1 More rules

- Rules for introducing disjunction

$$\frac{\varphi}{\varphi \vee \psi}\ \vee\mathrm{i1} \qquad\qquad \frac{\varphi}{\psi \vee \varphi}\ \vee\mathrm{i2}$$

$\vee$-introduction

- Rules for eliminating disjunction

$$\frac{\varphi \vee \psi \quad \boxed{\begin{array}{c}\varphi\\ \cdots\\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \cdots\\ \chi\end{array}}}{\chi}\ \vee\mathrm{e}$$

$\vee$-elimination

- In order to apply the rule $\vee$e, we use boxes as previously
  - But now there is a box starting with each disjunct φ and ψ
  - Each box needs to end with the same intended formula, χ

2 A proof using $\vee$ elimination

Here is a proof of the sequent $q \Rightarrow r \vdash (p \vee q) \Rightarrow (p \vee r)$

1.	$q \Rightarrow r$	premise
2.	$p \vee q$	assumption
3.	p	assumption
4.	$p \vee r$	$\vee$i 3
5.	q	assumption
6.	r	$\Rightarrow$e 1 5
7.	$p \vee r$	$\vee$i 6
8.	$p \vee r$	$\vee$e 2-i
9.	$(p \vee q) \Rightarrow (p \vee r)$	$\Rightarrow$e 2-8

- Assume that p is true, so the RHS is true
- $p \vee r$ is true
- Open box assuming q is true
- Eliminate the implies symbol from the LHS
- Shown that no matter if p or q is true, the statement $p \vee r$ will be true
- It has been shown that if $p \vee q$ is true, that $p \vee r$ is true, so $(p \vee q) \Rightarrow (p \vee r)$ will always be true

Here is a proof of the sequent $p \vee (q \vee r) \vdash (p \vee q) \vee r$

1.	$p \vee (q \vee r)$	premise
2.	p	assumption
3.	$p \vee q$	vi 2
4.	$(p \vee q) \vee r$	vi 3
5.	$q \vee r$	assumption
6.	q	assumption
7.	$p \vee q$	vi 6
8.	$(p \vee q) \vee r$	vi 7
9.	r	assumption
10.	$(p \vee q) \vee r$	vi 9
11.	$(p \vee q) \vee r$	ve 5-10
12.	$(p \vee q) \vee r$	ve 2-11

- Basically trying to build the RHS from the LHS

3 More rules

- Rules for negation

$$\frac{\perp}{\varphi}\ \perp e \qquad\qquad \frac{\varphi \quad \neg\varphi}{\perp}\ \neg e$$

$\perp$-elimination $\qquad$ $\neg$-elimination

- The symbol $\perp$, known as bottom, represents a contradiction, in natural deduction if one has a contradiction then one can infer **any** formula
- Rules for introducing negation

$$\frac{\boxed{\begin{array}{c}\varphi \\ \cdots \\ \perp\end{array}}}{\neg\varphi}\ \neg i$$

$\neg$-introduction

4 A proof using rules for negation

Here is a proof of the sequent $x \vee \neg y \vdash y \Rightarrow x$.

1.	$x \vee \neg y$	premise
2.	x	assumption
3.	y	assumption
4.	x	copy 2
5.	$y \Rightarrow x$	$\Rightarrow$i 3-4
6.	$\neg y$	assumption
7.	y	assumption
8.	$\perp$	$\neg$e 6 7
9.	x	$\perp$e 8
10.	$y \Rightarrow x$	$\Rightarrow$i 7-9
11.	$y \Rightarrow x$	$\vee$e 1 2-5 6-10

- $p \wedge \neg p \Rightarrow \varphi$ holds as $p \wedge \neg p$ will always be false, and if the LHS of an implication is false, then the whole statement will be true
- Lines $2 \rightarrow 5$ say $x \Rightarrow (y \Rightarrow x)$

Here is a proof of the sequent $x \Rightarrow (y \Rightarrow z), x, \neg z \vdash \neg y$.

1.	$x \Rightarrow (y \Rightarrow z)$	premise
2.	x	premise
3.	$\neg z$	premise
4.	y	assumption
5.	$y \Rightarrow z$	$\Rightarrow$e 1 2
6.	z	$\Rightarrow$e 4 5
7.	$\perp$	$\neg$e 3 6
8.	$\neg y$	$\neg$i 4-7

5 A derived rule

- We can derive other rules in natural deduction
- Consider modus tollens $\varphi \Rightarrow \psi, \neg\psi \vdash \neg\varphi$

1.	$\varphi \Rightarrow \psi$	premise
2.	$\neg\psi$	premise
3.	φ	assumption
4.	ψ	$\Rightarrow$e 1 3
5.	$\perp$	$\neg$e 2 4
6.	$\neg\varphi$	$\neg$i 3-5

- Note that we can use derived rules just as if they were rules of natural deduction
  - e.g., in a proof with
    - a line reading $\varphi \Rightarrow \psi$

    * and another line reading $\neg\psi$
  - we could immediately infer $\neg\psi$ and write modus tollens as an explaining remark

6 More derived rules

- Proof by contradiction is the principle "if from $\neg\varphi$ I can prove $\perp$ then I can deduce φ
- Here is a proof that this principle can be applied in natural deduction

1.	$\neg\varphi \Rightarrow \perp$	premise
2.	$\neg\varphi$	assumption
3.	$\perp$	$\Rightarrow$e 1 2
4.	$\neg\neg\varphi$	$\neg$i 2-3
5.	φ	$\neg\neg$e 4

- We denote reductio ad absurdum by RAA

7 More derived rules

- The law of excluded middle states that either φ is true or $\neg\varphi$ is true
- Here is a proof of it

1.	$\neg(\varphi \vee \neg\varphi)$	assumption
2.	φ	assumption
3.	$\varphi \vee \neg\varphi$	$\vee$i 2
4.	$\perp$	$\neg$e 1 3
5.	$\neg\varphi$	$\neg$i 2-4
6.	$\varphi \vee \neg\varphi$	$\vee$i 5
7.	$\perp$	$\neg$e 1 6
8.	$\neg\neg(\varphi \vee \neg\varphi)$	$\neg$i 1-7
9.	$\varphi \vee \neg\varphi$	$\neg\neg$e 8

- We denote the law of excluded middle by LEM

8 Some facts about Natural Deduction

- Natural deduction is sound and complete
- Let $\varphi_1, \varphi_2, ..., \varphi_m$ and ψ be formulae
- Soundness
  - If the sequent $\varphi_1, \varphi_2, ...\varphi_m \vdash \psi$ is provable then the formula $\varphi_1 \wedge \varphi_2 \wedge ... \wedge \varphi_m \Rightarrow \psi$ is a tautology
- Completeness
  - If $\varphi_1 \wedge \varphi_2 \wedge ... \wedge \varphi_m \Rightarrow \psi$ is a tautology then the sequent $\varphi_1, \varphi_2, ...\varphi_m \vdash \psi$ is provable
- A **theorem** is a formula ψ for which the sequent $\vdash \psi$ is provable, thus, the soundness and completeness of natural deduction tells us that every theorem is a tautology and every tautology is a theorem

9 Proving Theorems

- Here is a proof that the sequent $(p \Rightarrow (\neg p \vee q)) \vee (p \Rightarrow \neg q)$ is a theorem

1.	$q \vee \neg q$	LEM
2.	q	assumption
3.	$\neg p \vee q$	$\vee$i 2
4.	p	assumption
5.	$\neg p \vee q$	copy 3
6.	$p \Rightarrow (\neg p \vee q)$	$\Rightarrow$i 4-5
7.	$(p \Rightarrow (\neg p \vee q)) \vee (p \Rightarrow \neg q)$	$\vee$i 6
8.	$\neg q$	assumption
9.	p	assumption
10.	$\neg q$	copy 8
11.	$p \Rightarrow \neg q$	$\Rightarrow$i 9-10
12.	$(p \Rightarrow (\neg p \vee q)) \vee (p \Rightarrow \neg q)$	$\vee$i 11
13.	$(p \Rightarrow (\neg p \vee q)) \vee (p \Rightarrow \neg q)$	$\vee$e 1 2-7 8-12