DYNAMIC HEDGING UNDER BLACK-SCHOLES MODEL

JIAYI YANG (1004244212), LUJIA YANG (1002955563), AND YUDA GAO (1002431727)

Abstract. Dynamic hedging strategy is widely used by an investment company for money management. This paper proposes the Black-Scholes model as a tool for delta and delta-gamma dynamic hedging in discrete time steps. We explore two different approaches in each of these hedging strategies: time-based and move-based hedging, under a specific case where the investment company underwrote a put option. Based on sensitivity analysis of the portfolio value concerning the change of external and internal factors, including transaction cost, realized volatility, time steps, risk-neutral volatility, and move-based rebalancing band, the results suggest a trade-off between selecting a more stable hedging methodology and targeting on a higher profit. Investment companies need to pay attention to the impact of transaction costs on portfolio returns. Moreover, it is necessary to understand the model risk caused by the difference in real-world volatility and risk-neutral volatility and adjust the rebalancing frequency and model assumptions in time according to market changes.

1. Introduction

Modern pricing theory treats the valuation of a derivative contract as a replication problem. Based on Black and Scholes (1973) and Merton (1973), the price of the financial contract is the cost to trade a self-financing portfolio with more simpler and liquid securities to match the option payoff at each time step.

The allocating strategies are usually dynamic and are selected to locally eliminate any model risk in the portfolio of derivative securities and replication strategies. A European option is a popular and well-known synthetic financial securities: a put (call) option gives the holder the right to sell (buy) the underlying asset at a predetermined strike price at a given expiry in the future.

From the perspective of the investment company, selling an option means simultaneously determining two problems: **(a.)** the price offered to customers for obtaining the right to exercise the option, and **(b.)** the dynamic hedging strategy for cash flow management, creating a portfolio (i.e., consist of equities and call option) and changing asset holding position periodically during the life of the option. The hedging strategy should make the portfolio value equal to the payoff amount to the customer at the time of expiration, regardless of the price fluctuating of the underlying assets of the option sold and the realized prices that construct the portfolio. In other words, the strategy should conduct a proper feedback action to eliminate the effects of the random variables that affect the market and hedge options against the uncertainties associated with price evolution.

This report introduced and compared several approaches for option pricing and dynamic hedging, including delta hedging and delta-gamma hedging. Within each hedging strategy, we discussed two approaches: time-based and move-based. The common goal of any approach is to minimize the risk of the hedging error where the value of the portfolio at the expiration date does not match the payoff. The report further addressed the effect of the inconsistency between realized volatility and risk-neutral volatility. Finally, we investigate the role of the rebalancing band on the delta for the move-based hedging strategies. To facilitate readers to understand the content of the report, we must first introduce some basic concepts in the next section before further explaining our research and algorithms.

2. **Methodology**

2.1. **Basic Concepts of Dynamic Hedging.**

Definition 2.1 (Financial Market) Within the probability space $(\Omega, \mathbb{P}, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$, consisting of event space Ω, probability measure $\mathbb{P}$, σ-algebra $\mathcal{F}$ and filtration $(\mathcal{F}_t)_{t\geq 0}$ we define:

- $S = (\mathcal{S}_t)_{t\geq 0}$ is a price process of the *risky asset*.
- $\alpha = (\alpha_t)_{t\geq 0}$ is the number of units of risky asset S held over the hedging period.
- $B = (\mathcal{B}_t)_{t\geq 0}$ is a *bank account* process.
- $\beta = (\beta)_{t\geq 0}$ is the number of bank account asset B held over the hedging period.
- $F = (F_t)_{t\geq 0}$ is the price process of the *traded claim* written on the source of uncertainty.
- $g = (g_t)_{0\leq t\leq T}$ is the price process of the *second contingent claim*, used in the gamma hedging.
- $\eta = (\eta)_{t\geq 0}$ is the number of the second contingent claim held over the hedging period.

Definition 2.2 (Self-Financing) A trading strategy (α, β), called *delta hedging* in this report, is self-financing if the change of the asset position does not influence the portfolio value: $(\alpha_t - \alpha_{t-1})S_t + (\beta_t - \beta_{t-1})B_t = 0$. If the value process of trading strategy $V = (V_t)_{t\geq 0}$,
where $V_t = \alpha_t S_t + \beta_t B_t$, a self-financing strategy satisfied:

$$V_t - V_{t-1} = \alpha_{t-1}(S_t - S_{t-1}) + \beta_{t-1}(B_t - B_{t-1})$$

A similar concept holds for the trading strategy (α, β, η), which is also called *delta-gamma hedging* and will be further discussed later.

Definition 2.3 (Option fundamentals Under the Black-Scholes Model) In this report, we assume the process of bank account $B = (\mathcal{B}_t)_{t\geq 0}$ and risky asset $S = (\mathcal{S}_t)_{t\geq 0}$ follow the assumptions under the Black-Scholes Model:

$$\frac{dS_t}{S_t} = {\mu_t}^S dt + {\sigma_t}^S dW_t$$

$$\frac{dB_t}{B_t} = r_t dt$$

In this process, the real-world drift of the risky asset is μ and the volatility is σ. The risk-free return of the bank account is r_t.

(a.) Option Price. Given a predetermined strike price K, the call option payoff at maturity is $(S_T - K)_+$, whereas the put option payoff at maturity is $(K - S_T)_+$. The investment company can price the options using the Black-Scholes Pricing formulas at each time t over the life of the option:

$$V_t^{call} = S_t\Phi(d_+) - Ke^{-r(T-t)\Phi(d_-)}$$

$$V_t^{put} = Ke^{-r(T-t)\Phi(-d_-)} - S_t\Phi(-d_+)$$

where

$$d_\pm = \frac{ln(S_t/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}$$

Under the Black-Scholes Pricing Model, the behavior of option price are showing below:

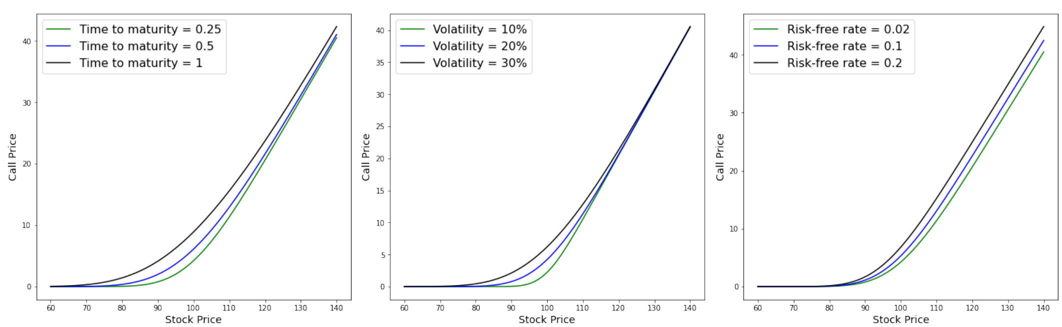

FIGURE 1. Call Price

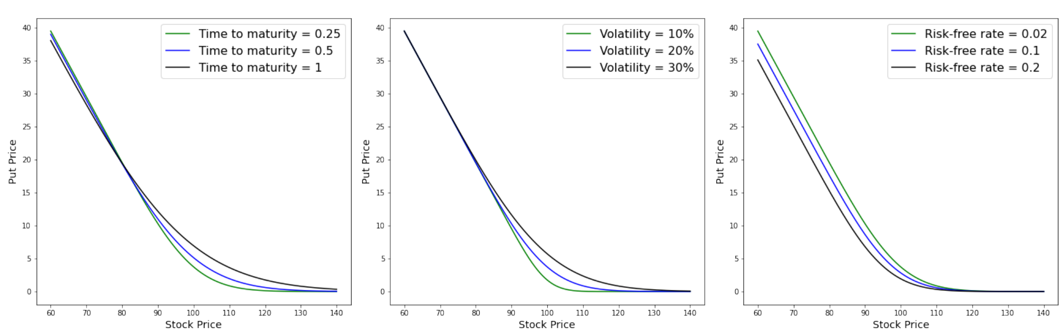

FIGURE 2. Put Price

We observed that as the time approaches maturity the value of both the call and put options decreases, since there is less uncertainty for a short period, meaning that the option becomes less attractive to investors. The effect of volatility for the call and put option is similar. The option price increase as the volatility increase. When the option is close or equal to the strike price (called at-the-money or ATM), the effect is much higher. When the option is unlikely to be exercised (called out-of-the-money or OTM) or achieves large profit (called deep-in-the-money or ITM), the effect is small. Additionally, the figure and the pricing formulas imply that as the risk-free rate increase the call option price increase, while the put option price decrease.

(b.) Delta. Delta is also known as a hedge ratio that compares the change in the price of an underlying asset with the change in the price of a derivative or option. For options traders, delta indicates the number of options contracts needed to hedge a long or short position in the underlying asset. Let $g(t, S_t)$ denote the price function of a call option and $f(t, S_t)$ denote the price function of a put option, assuming the same risk-free rate, strike price, and maturity. The Black-Scholes Pricing framework calculated delta as follow:

$$\Delta^{call} = \partial_s g(t, S_t) = \Phi(d_+);$$

$$\Delta^{put} = \partial_s f(t, S_t) = \Phi(d_+) - 1$$

Based on the behavior of option delta shown in the figure below, the option delta decrease as the risk-free rate decrease. The delta of a call option goes from 0 when it deep OTM and to 1 when it is ITM. Put options delta goes from -1 when it is deep ITM and 0 when it is deep OTM. When volatility and the time to maturity decrease, the curvature of the delta curve increase, which may increase the hedging error. When the asset price is close to the strike price (ATM), the slope of the delta curve is extremely large, because in this case, a small change in the asset price can influence the investor's decision of exercising the option or not, which lead to a significant change of the option value. To ensure a stable portfolio value, traders can use delta to construct a self-financing portfolio that replicates the underwriting option.

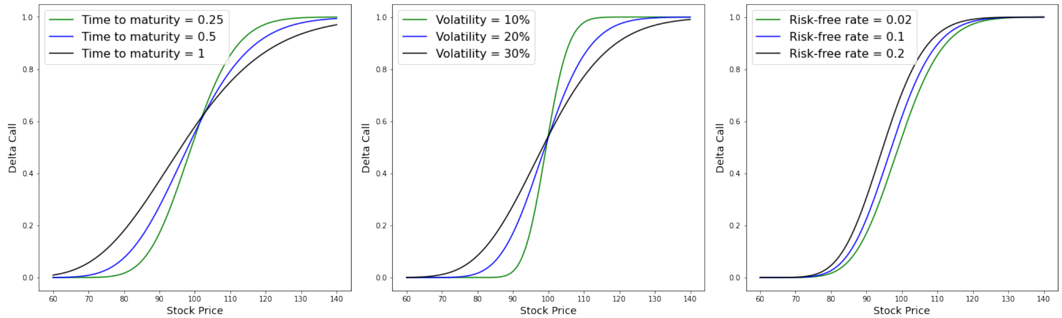

FIGURE 3. Call for Delta

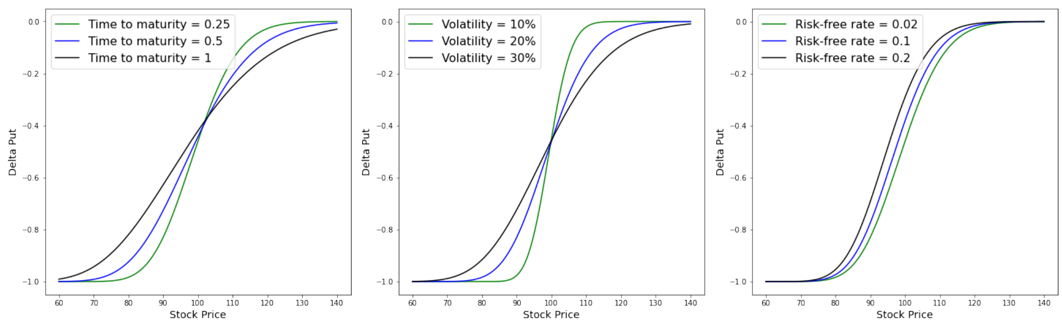

FIGURE 4. Put for Delta

(c.) Gamma. Gamma reflects the change of delta, which is the first derivative of delta and the second derivative of an option's price with respect to the price of an underlying asset. It is positive for a long position and negative for a short position. Gamma corrects for convexity issues in the delta hedging strategies. Some portfolio managers or traders who involved with extreme large value portfolios need to use gamma to achieve a more precision hedging strategy that makes the profit and loss more stable.

Black-Scholes Pricing formulas calculated delta as follow:

$$\Gamma^{call} = \partial_{ss} g(t, S_t) = \Gamma^{put} = \partial_{ss} f(t, S_t) = \frac{\phi(d_+)}{S_t \sigma \sqrt{(}T-t)}$$

Based on the figure, gamma is small when the option is deep in or out-of-the-money. As gamma is the slope of the delta curve, the largest gamma occurs when the option is near or at the money. Moreover, the gamma curve becomes sharper when the time to maturity and the volatility decrease. We also observed an increasing risk-free rate slightly right-skewed the gamma curve.

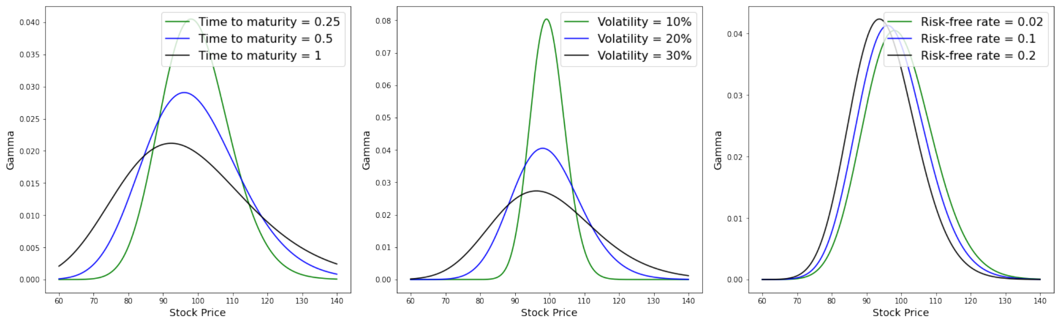

FIGURE 5. Gamma curve

Definition 2.4 (Four Dynamic Hedging Strategies) This report investigate four hedging strategies: **time-based delta** hedging, **move-based delta** hedging, **time-based delta-gamma** hedging, and **move-based delta-gamma** hedging. In other words, we investigate the performance of delta hedging and delta-gamma hedging under two different discrete-time version strategies. They are time-based hedging and move-based hedging. When using the time-based strategy, for both delta hedging and delta-gamma hedging, we are re-balancing the portfolio at equally sized, discrete-time steps Δt. On the other hand, under the move-based strategy, for both delta hedging and delta-gamma hedging, we set the corresponding re-balancing band around the delta. We only adjust the asset position (i.e.: α, η) when the movement of the portfolio delta exceeds a certain range.

2.2. Detailed Algorithm of Delta Hedging using Two Strategies.

Usually, the dynamic hedging argument was used in the continuous-time version. Now, we are trying to implement it in practice in the discrete-time version within the Black-Scholes model. In this section, we will introduce **Delta-hedging** and show how it is implemented under both **the move-based** and the **time-based hedging** strategy onto the given scenario.

To start the analysis, we begin with the overview of the general idea of **Delta-hedging**. Delta-hedging is a kind of trading strategy which aims to neutralize the risk associated with price movements in the underlying asset by establishing offsetting long and short positions. Like what has been shown in Figure 6, it is aiming to construct a zero cost, self-financing portfolio. In other words, we want to achieve zero profit and loss over the whole portfolio by applying delta hedging. Figure 6 shows the discrete delta hedging consists of a portfolio with components:

(i) Short put: cash inflow ≥ 0 [The red line for the bank account]

(ii) Long risky asset: cash outflow ≤ 0 [The green line for the stock holding]

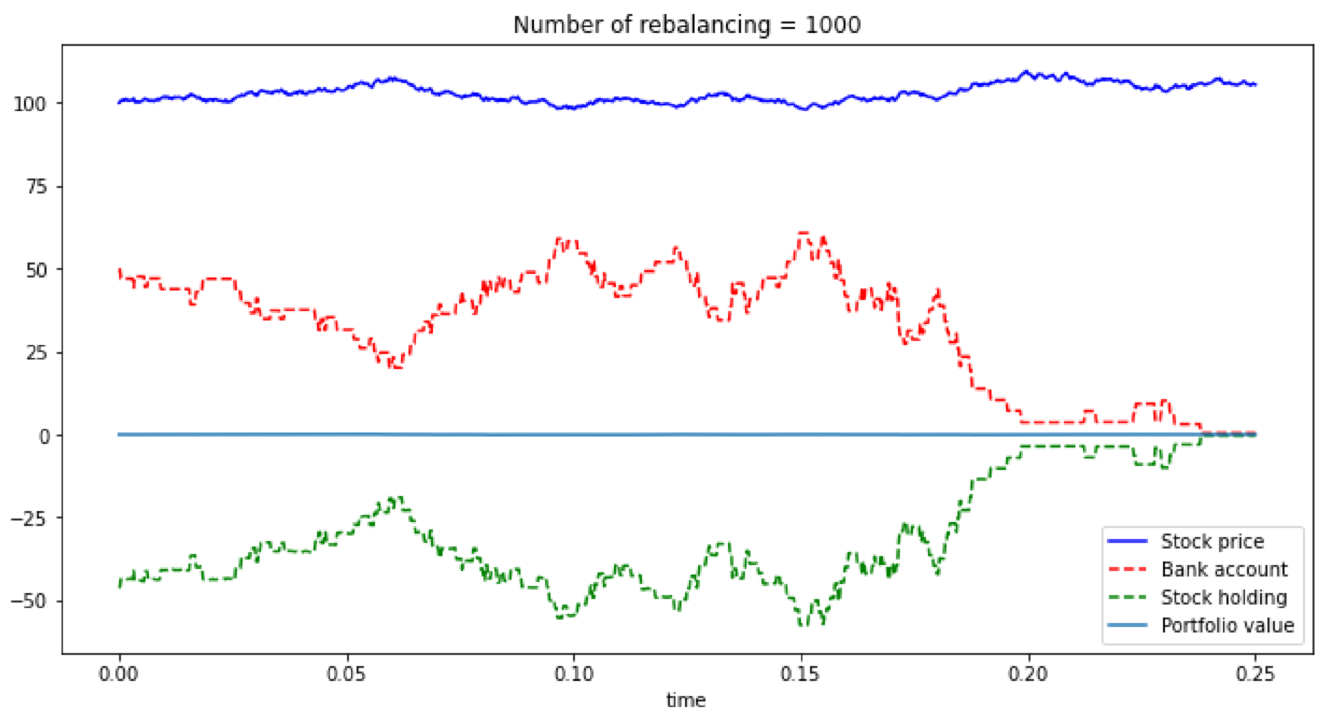

FIGURE 6. Delta Hedging Process

Then, we go on to explore the algorithm behind the **Delta-hedging** in detail. Before conducting the argument about it, we need to assume the following model for the economy:

(i) **A risky asset** pricing process given by the *Itô* process $S = (S_t)_{t\geq 0}$ satisfying the SDE:

$$\frac{dS_t}{S_t} = {\mu_t}^S dt + {\sigma_t}^S dW_t$$

* Here, $S = (S_t)_{t\geq 0}$ shows the value of 1 unit of risky asset at the corresponding time step. ${\mu_t}^S$ represents the drift process, ${\mu_t}^S = \mu^S(t, S_t)$, and ${\sigma_t}^S$ represents the volatility process, ${\sigma_t}^S = \sigma^S(t, S_t)$. $W = (W_t)_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion.

(ii) **A bank account** pricing process, $B = (B_t)_{t\geq 0}$, satisfying:

$$\frac{dB_t}{B_t} = r_t dt$$

* Here, r_t represents the risk free rate at the corresponding time step.

(iii) **A European contingent claim** written on S has price process $F = (F_t)_{0\leq t\leq T}$, where F is Markovian in S and $F_t = F(t, S_t)$. At the maturity time T, the claim will pay $F(S_T)$. Such F_t satisfies:

$$\frac{dF_t}{F_t} = {\mu_t}^F dt + {\sigma_t}^F dW_t$$

Based on these settings, we could start to consider the hedging process. The main idea of dynamic hedging is to construct a zero-cost, self-financing portfolio. For such an instantaneously risk-free portfolio, our strategy is to invest $(\alpha, \beta) = (\alpha_t, \beta_t)_{t\geq 0}$ units in the risky asset S and bank account B and short a unit of European contingent claim F.

Then, recall the value process of this trading strategy (α, β), we have $V_t = \alpha_t S_t + \beta_t B_t - F_t$. And on the condition of zero cost and self-financing, by using *Itô* lemma we have:

$$\begin{aligned}
dF_t &= dF(t, S_t) \\
&= \partial_t F(t, S_t)dt + \partial_s F(t, S_t)dS_t + \frac{1}{2}\partial_{SS} F(t, S_t)d[S, S]_t \\
dS_t &= \mu S_t dt + \sigma S_t dW_t \\
d[S, S]_t &= \sigma^2 {S_t}^2 d[W, W]_t = \sigma^2 {S_t}^2 dt
\end{aligned}$$

$$\Rightarrow dF_t = \underbrace{[\partial_t F(t,S_t) + \mu S_t \partial_s F(t,S_t) + \frac{1}{2}\sigma^2 {S_t}^2 \partial_{SS} F(t,S_t)]}_{\mu_t^F F_t} dt$$

$$+ \underbrace{\sigma S_t \partial_S F(t,S_t)}_{\sigma_t^F F_t} dW_t$$

And we can derive the portfolio value SDE like below:

$$\begin{aligned} dV_t &= \alpha_t dS_t + \beta_t dB_t - dF_t \\ &= (\alpha_t \mu S_t + \beta r B_t - \mu_t^F F_t)dt + \underbrace{(\alpha_t \sigma S_t - \sigma_t^F F_t)}_{\bigstar} dW_t \end{aligned}$$

To locally remove the risk, we need to choose a α_t such that the part $(\bigstar) = 0$:

$$\alpha_t \sigma S_t - \sigma_t^F F_t = 0$$

$$\Rightarrow \alpha_t^* = \frac{\sigma_t^F F_t}{\sigma S_t}$$

So, we find that when we have α^* equal to the ratio of the product of the volatility of the contingent claim F times its price to the product of the volatility of the underlying asset σ and its price, we can achieve an instantaneously risk-free portfolio.

Moreover, implied by the *Itô* lemma, we have:

$$\alpha_t = \partial_s F(t,s)|_{s=s_t}$$

$$\Rightarrow \Delta(t,x) = \partial_x F(t,x)$$

The quantity $\partial_s F(t,s)|_{s=s_t}$ above is known as the **"Delta"** of the option F, and it can be denoted as $\Delta(t,x)$. For delta hedging, all we need to do is to adjust the delta here.

In the optimal theory, we could trade continuously. By applying dynamic hedging, we would achieve zero profit-and-loss (PnL) as we can have $V_t = 0$ for all t which means that the value of the whole portfolio remains zero all the time. However, in real-world cases, trading to re-balance the hedging portfolio could only be done in a finite and discrete approach. There mainly have two methodologies to implement hedging in the discrete version, one is the **time-based strategy** and the other one is the **move-based strategy**.

For the **time-based delta-hedging**, traders adjust the "Delta" position in the option at every time step. To show how **Delta-hedging** works under the **time-based strategy** in detail, here we present the entire cash-flow for a general scenario: Suppose that we sold an option which pays $F(S_T)$ at the maturity time T, and we do hedging on the option at equally-sized discrete time steps $(0, \Delta t, 2\Delta t, \cdots, T)$. Additionally, the transaction fee for each unit of asset S is ϵ. Below is the entire cash flow for this scenario:

- At $t = 0$: We receive $F(0, S_0)$ in funds in the bank account by selling put. To locally remove the risk, we need to purchase $\Delta_0 = \Delta(0, S_0)$ units of risky asset S to hedge the risk. To do that, we need to pay $\epsilon|\Delta_0|$ transaction costs. The price of the risky asset S is S_0 at this time step. In this case, there remains $B_0 = F(0, S_0) - \Delta_0 S_0 - \epsilon|\Delta_0|$ in the bank account.

- At $t = \Delta t$: We have $B_0 e^{r\Delta t}$ in bank account as it grows at risk free rate r. To locally remove the risk, we need to re-balance the "Delta", $\Delta_{\Delta t} = \Delta(t, S_{\Delta t})$. Now we need to purchase additional $(\Delta_{\Delta t} - \Delta_0)$ units of risky asset S to hedge the risk. To do that, we need to pay $\epsilon|\Delta_{\Delta t} - \Delta_0|$ transaction costs. The price of the risky asset S is $S_{\Delta t}$ at this time step. In this case, there remains $B_{\Delta t} = B_0 e^{r\Delta t} - (\Delta_{\Delta t} - \Delta_0)S_{\Delta t} - \epsilon|\Delta_{\Delta t} - \Delta_0|$ in the bank account.

$\vdots$

- At $t = t_{k+1}$ where $t_{k+1} = (k+1)\Delta t$: We have $B_{t_k} e^{r\Delta t}$ in bank account as it grows at risk free rate r. To locally remove the risk, we need to re-balance the "Delta", $\Delta_{t_{k+1}} = \Delta(t_{k+1}, S_{t_{k+1}})$, and now we need to purchase additional $(\Delta_{t_{k+1}} - \Delta_{t_k})$ units of risky asset S to hedge the risk. To do that, we need to pay $\epsilon|\Delta_{t_{k+1}} - \Delta_{t_k}|$ transaction costs. The price of the risky asset S is $S_{t_{k+1}}$ at this time step. In this case, there remains $B_{t_{k+1}} = B_{t_k} e^{r\Delta t} - (\Delta_{t_{k+1}} - \Delta_{t_k}))S_{t_{k+1}} - \epsilon|\Delta_{t_{k+1}} - \Delta_{t_k}|$ in the bank account.

$\vdots$

- Finally, at $t = T$ where $T = n\Delta t$: We have $B_{T-\Delta t} e^{r\Delta t}$ in bank account as it grows at risk free rate r. To locally remove the risk, we need to re-balance the "Delta", $\Delta_T = \Delta(T, S_T)$, and now we need to purchase additional $(\Delta_T - \Delta_{T-\Delta t})$ units of risky asset S to hedge the risk. To do that, we need to pay $\epsilon|\Delta_T - \Delta_{T-\Delta t}|$ transaction costs. The price of the risky asset S is S_T at this time step. In this case, there remains $B_T = B_{T-\Delta t}e^{r\Delta t} - (\Delta_T - \Delta_{T-\Delta t})S_T - \epsilon|\Delta_T - \Delta_{T-\Delta t}|$ in the bank account.

- The profit and loss at this maturity T is : $PnL_T = B_T - F(S_T)$.

Similar to the **time-based strategy**, we achieve the hedging by adjusting the α position. However, compare to do re-balancing at every single time step, under the move-based strategy, traders only adjust "Delta" when Δ_t^F exceeds the preset re-balancing band. To show how **Delta-hedging** works under the **move-based strategy** in details, we use the same scenario as above, the only different is that we additionally preset a balancing bandwidth b here. And below is the algorithm behind the **move-based delta-hedging**:

- At $t = 0$: We receive $F(0, S_0)$ in funds by selling put option. To locally remove the risk, we need to purchase $\Delta_0 = \Delta(0, S_0)$ units of risky asset S to hedge the risk. To do that, we need to pay $\epsilon|\Delta_0|$ transaction costs.The price of the risky asset S is S_0 at this time step. In this case, there remains $B_0 = F(0, S_0) - \Delta_0 S_0 - \epsilon|\Delta_0|$ in the bank account. And the re-balancing band range is $[\Delta_0 - b, \Delta_0 + b]$ at present.

$\vdots$

- If Δ_{t_k} (where $t_k = k\Delta t$ and $t_k \leq T$) still in the range of the re-balancing band, the position of "Delta" will remain the same as before that $\Delta_{t_k} = \Delta_{t_k-\Delta t}$. We do not need to buy or sell extra units of risky asset S to hedge the risk. And the re-balancing band range also keeps the same as previous: $[\Delta_{t_k} - b, \Delta_{t_k} + b] = [\Delta_{t_k-\Delta t} - b, \Delta_{t_k-\Delta t} + b]$. We have $B_{t_k-\Delta t}e^{r\Delta t}$ in bank account as it grows at risk free rate r. In this case, there remains $B_{t_k} = B_{t_k-\Delta t}e^{r\Delta t}$ in the bank account.
- If $\Delta_{t_k} \notin [\Delta_{t_k-\Delta t} - b, \Delta_{t_k-\Delta t} + b]$ (where $t_k = k\Delta t$ and $t_k \leq T$), meaning that Δ_{t_k} exceeds the range of the re-balancing band, the position of "Delta" need to be adjusted to $\Delta_{t_k} = \Delta(t_k, S_{t_k})$. And now, we need to purchase additional $(\Delta_{t_k} - \Delta_{t_k-\Delta t})$ units of risky asset S to hedge the risk. To do that, we need to pay $\epsilon|\Delta_{t_k} - \Delta_{t_k-\Delta t}|$ transaction costs. The price of the risky asset S is S_{t_k} at this time step. We also have $B_{t_k-\Delta t}e^{r\Delta t}$ in bank account as it grows at risk free rate r. In this case, there remains $B_{t_k} = B_{t_k-\Delta t}e^{r\Delta t} - (\Delta_{t_k} -$

$\Delta_{t_k-\Delta t})S_{t_k} - \epsilon|\Delta_{t_k} - \Delta_{t_k-\Delta t}|$ in the bank account. Also, the re-balancing band range need to be adjusted to $[\Delta_{t_k} - b, \Delta_{t_k} + b]$.

- The profit and loss at the maturity time step T is : $PnL_T = B_T - F(S_T)$.

Roughly, we will expect that the move-based delta hedging is cheaper as it has fewer adjustments than the time-based delta hedging. And a more detailed comparative analysis would be carried out in the later section.

2.3. **Detailed Algorithm of Delta-Gamma Hedging using Two Strategies.**
Similar to the previous section, in this section, we will introduce **Delta-Gamma hedging** and show the implementation under both **the move-based** and the **time-based hedging** strategy.

Compare to the delta hedging, the **Delta-Gamma hedging** immunizes a portfolio not only from linear changes in the underlying asset's value but also considers the non-linear changes. To neutralize the risk and the portfolio value, we need to maintain both net-zero delta and gamma. To have a zero portfolio gamma, we need to introduce a new hedging option. Like what has been shown in Figure 7, the discrete delta-gamma hedging consists of a portfolio with components:

(i) Short put: cash inflow ≥ 0 [The red line for the bank account]

(ii) Long risky asset: cash outflow ≤ 0 [The green line for the stock holding]

(iii) Short call: cash inflow ≥ 0 [The black line for the option hedging]

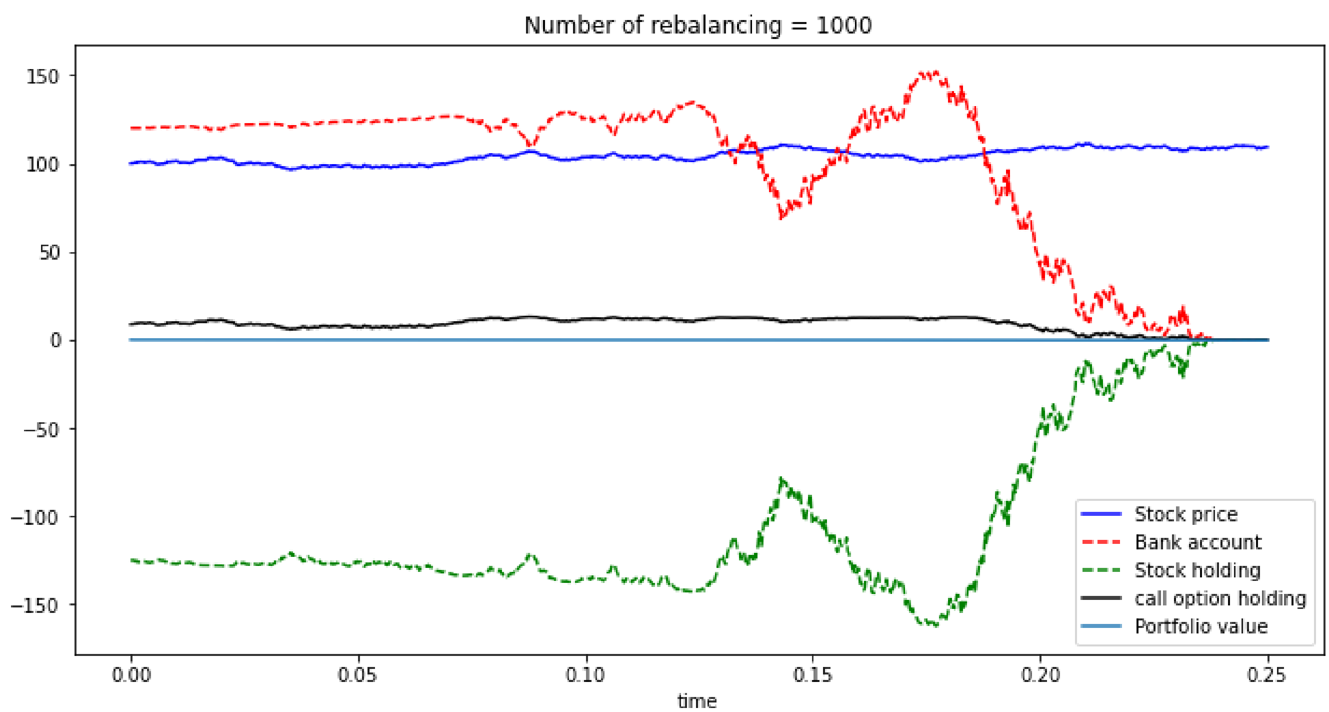

FIGURE 7. Delta-Gamma Hedging Process

Now, we go on to explore the algorithm of the **Delta-Gamma hedging** in detail. To find out the change in the value of the contingent claims F with respect to the change in the price of the underlying asset S, we could consider the Taylor expansion:

$$\begin{aligned} F(t, X_{t+\Delta t}) &= F(t, X_t + \Delta X_t) \\ &= F(t, X_t) + \Delta X_t \underbrace{\partial_x F(t, X_t)}_{\Delta_t^F} + \frac{1}{2}(\Delta X_t)^2 \underbrace{\partial_{xx} F(t, X_t)}_{\Gamma_t^F} + \cdots \end{aligned}$$

As we just do the approximation on the first two terms, those higher order terms could be ignored. So we have:

$$F(t, X_{t+\Delta t}) \approx F(t, X_t) + \Delta X_t \underbrace{\partial_x F(t, X_t)}_{\Delta_t^F} + \frac{1}{2}(\Delta X_t)^2 \underbrace{\partial_{xx} F(t, X_t)}_{\Gamma_t^F}$$

The second-order derivative here, $\partial_{xx} F(t, X_t)$, is known as **Gamma** of the option F, we could denoted as Γ_t^F. Like what we have mentioned before, the aim of **Delta-Gamma hedging** is to construct a portfolio that has both zero net deltas and zero net gamma. We introduce a second hedging option g to complete the hedging task. Therefore, under the Delta-Gamma hedging strategy, we need to invest $(\alpha, \beta, \eta) = (\alpha_t, \beta_t, \eta_t)_{t \geq 0}$ units in the risky asset S, bank asset B and hedging option g to hedge the unit short in F. Therefore, we can adjust our value process as follows:

$$V_t = \alpha_t S_t + \beta_t B_t + \eta_t g_t - F_t$$

The hedging logic behind is to construct a self-financing portfolio to match the value of the contingent claim F. In other words, we aim to figure out the coefficients α and η such that the **delta** and **gamma** of this hedging option g match the **delta** and **gamma** of the target claim F. Then, we have:

$$\alpha_t X = \alpha_t S_t + (X - S_t)\alpha_t$$

$$\eta_t g = \eta_t g(t, X) = \eta_t [g(t, S_t) + (X - S_t)\Delta^g(t, S_t) + \frac{1}{2}(X - S_t)^2 \Gamma^g(t, S_t) + \cdots]$$

Where X represents the arbitrary asset price.

To match the **delta** and the **gamma** with F, we first check the quadratic term:

$$\begin{aligned} Matching\ (X - S_t)^2 &\Rightarrow \eta_t \Gamma_t^g = \Gamma_f^F \\ &\Rightarrow \eta_t = \frac{\Gamma_t^F}{\Gamma_t^g} \end{aligned}$$

This implies that the number of the secondary option g we need to purchase is equal to the ratio of the gamma of the underwriting option to the gamma of a second option selected for hedging purposes.

Next, we check the linear contribution:

$$\begin{aligned} Matching\ (X - S_t) &\Rightarrow \alpha_t + \eta_t \Delta_t^g = \Delta_t^F \\ &\Rightarrow \alpha_t = \Delta_t^F - \frac{\Gamma_t^F}{\Gamma_t^g} \Delta_f^g \end{aligned}$$

After checking the linearity, the delta position is equal to the portfolio delta, which is the delta of the option that we hedged minus the delta of the other option times the ratio of their gamma.

Similar to the Delta hedging, there mainly have two methodologies to implement **Delta-Gamma hedging** in the discrete version. They are the **time-based** strategy and the **move-based** strategy.

To show how **Delta-Gamma hedging** works under the **time-based strategy**, we present the entire cash-flow for a general scenario: Suppose that we sold an option which pays $F(S_T)$ at the maturity time T, and we do hedging on the option at equally-sized discrete time steps $(0, \Delta t, 2\Delta t, \cdots, T)$. Additionally, the transaction fee of each unit of asset S is ϵ_1, and the transaction cost of each unit of hedging option g is ϵ_2. Below is the process of cash flow:

- At $t = 0$: We receive $F(0, S_0)$ in funds by selling put option. To locally remove the risk, we need to purchase α_0 units of risky asset S, and η_0 units of hedging option g to hedge the risk. To do that, we need to pay $(\epsilon_1|\alpha_0| + \epsilon_2|\eta_0|)$ transaction costs. At this time step, the price of the risky asset S is S_0, and the price of the hedging option g is g_0. In this case, there remains $B_0 = F(0, S_0) - \alpha_0 S_0 - \eta_0 g_0 - (\epsilon_1|\alpha_0| + \epsilon_2|\eta_0|)$ in the bank account.
- At $t = \Delta t$: We have $B_0 e^{r\Delta t}$ in bank account as it grows at risk free rate r. To locally remove the risk, we need to re-balance the asset position to $\alpha_{\Delta t}$ and also re-balance the

option position to $\eta_{\Delta t}$. Now we need to purchase additional $(\alpha_{\Delta t} - \alpha_0)$ units of risky asset S and $(\eta_{\Delta t} - \eta_0)$ units of hedging option g to hedge the risk. To do that, we need to pay $(\epsilon_1|\alpha_{\Delta t} - \alpha_0| + \epsilon_2|\eta_{\Delta t} - \eta_0|)$ transaction costs. At this time step, the price of the risky asset S is $S_{\Delta t}$, and the price of the hedging option g is $g_{\Delta t}$. In this case, there remains $B_{\Delta t} = B_0 e^{r\Delta t} - (\alpha_{\Delta t} - \alpha_0)S_{\Delta t} - (\eta_{\Delta t} - \eta_0)g_{\Delta t} - (\epsilon_1|\alpha_{\Delta t} - \alpha_0| + \epsilon_2|\eta_{\Delta t} - \eta_0|)$ in the bank account.

- At $t = t_{k+1}$ where $t_{k+1} = (k+1)\Delta t$: We have $B_{t_k} e^{r\Delta t}$ in bank account as it grows at risk free rate r. To locally remove the risk, we need to re-balance the asset position to $\alpha_{t_{k+1}}$, and also re-balance the option position to $\eta_{t_{k+1}}$. Now we need to purchase additional $(\alpha_{t_{k+1}} - \alpha_{t_k})$ units of risky asset S and $(\eta_{t_{k+1}} - \eta_{t_k})$ units of hedging option g to hedge the risk. To do that, we need to pay $(\epsilon_1|\alpha_{t_{k+1}} - \alpha_{t_k}| + \epsilon_2|\eta_{t_{k+1}} - \eta_{t_k}|)$ transaction costs. At this time step, the price of the risky asset S is $S_{t_{k+1}}$, and the price of the hedging option g is $g_{t_{k+1}}$. In this case, there remains $B_{t_{k+1}} = B_{t_k} e^{r\Delta t} - (\alpha_{t_{k+1}} - \alpha_{t_k}))S_{t_{k+1}} - (\eta_{t_{k+1}} - \eta_{t_k})g_{t_{k+1}} - (\epsilon_1|\alpha_{t_{k+1}} - \alpha_{t_k}| + \epsilon_2|\eta_{t_{k+1}} - \eta_{t_k}|)$ in the bank account.

$\vdots$

- Finally, at $t = T$ where $T = n\Delta t$: We have $B_{T-\Delta t} e^{r\Delta t}$ in bank account as it grows at risk free rate r. To locally remove the risk, we need to re-balance the asset position to α_T, and also re-balance the option position to η_T. Now we need to purchase additional $(\alpha_T - \alpha_{T-\Delta t})$ units of risky asset $S$and $(\eta_T - \eta_{T-\Delta t})$ units of hedging option g to hedge the risk. To do that, we need to pay $(\epsilon_1|\alpha_T - \alpha_{T-\Delta t}| + \epsilon_2|\eta_T - \eta_{T-\Delta t}|)$ transaction costs. At this time step, the price of the risky asset S is S_T, and the price of the hedging option g is g_T. In this case, there remains $B_T = B_{T-\Delta t} e^{r\Delta t} - (\alpha_T - \alpha_{T-\Delta t})S_T - (\eta_T - \eta_{T-\Delta t})g_T - (\epsilon_1|\alpha_T - \alpha_{T-\Delta t}| + \epsilon_2|\eta_T - \eta_{T-\Delta t}|)$ in the bank account.
- The profit and loss at this maturity T is : $PnL_T = B_T - F(S_T)$.

For the **move-based strategy**, we achieve the hedging by adjusting the α position and the η position when α_t exceeds the preset re-balancing band. To show how **Delta-Gamma hedging** works under the **move-based strategy**, we use the same scenario as above, the only difference is that instead of presetting the discrete-time steps, we preset a balancing bandwidth b here. Below is the algorithm behind the **move-based Delta-Gamma hedging**:

- At $t = 0$: We receive $F(0, S_0)$ in funds and put $F(0, S_0)$ into bank account. To locally remove the risk, we need to purchase α_0 units of risky asset S, and η_0 units of hedging option g to hedge the risk. To do that, we need to pay $(\epsilon_1|\alpha_0| + \epsilon_2|\eta_0|)$ transaction costs. At this time step, the price of the risky asset S is S_0, and the price of the hedging option g is g_0. In this case, there remains $B_0 = F(0, S_0) - \alpha_0 S_0 - \eta_0 g_0 - (\epsilon_1|\alpha_0| + \epsilon_2|\eta_0|)$ in the bank account. And the re-balancing band range is $[\alpha_0 - b, \alpha_0 + b]$ at present.

$\vdots$

- If α_{t_k} (where $t_k = k\Delta t$ and $t_k \leq T$) in the range of the re-balancing band, the position of α remains the same $\alpha_{t_k} = \alpha_{t_k - \Delta t}$, the position of η also keeps at $\eta_{t_k} = \eta_{t_k - \Delta t}$. We do not need to buy or sell extra units of risky asset S and hedging option g. And the re-balancing band range also remain unchanged: $[\alpha_{t_k} - b, \alpha_{t_k} + b] = [\alpha_{t_k - \Delta t} - b, \alpha_{t_k - \Delta t} + b]$. We have $B_{t_k - \Delta t} e^{r\Delta t}$ in bank account as it grows at risk free rate r. At this time step, the price of the risky asset S is S_{t_k} and the price of the hedging option g is g_{t_k}. In this case, there remains $B_{t_k} = B_{t_k - \Delta t} e^{r\Delta t}$ in the bank account.
- If $\alpha_{t_k} \notin [\alpha_{t_k - \Delta t} - b, \alpha_{t_k - \alpha t} + b]$ (where $t_k = k\Delta t$ and $t_k \leq T$), meaning that α_{t_k} exceeds the range of the re-balancing band, the position of α need to be adjusted to α_{t_k}, and the position of η need to be adjusted to η_{t_k}. And now, we need to purchase additional $(\alpha_{t_k} - \alpha_{t_k - \Delta t})$ units of risky asset S and $(\eta_{t_k} - \eta_{t_k - \Delta t})$ units of hedging portfolio g to hedge the risk. To do that, we need to pay $(\epsilon_1|\alpha_{t_k} - \alpha_{t_k - \Delta t}| + \epsilon_2|\eta_{t_k} - \eta_{t_k - \Delta t}|)$ transaction costs. At this time step, the price of the risky asset S is S_{t_k}, and the price of the hedging option g is g_{t_k}. In this case, there remains $B_{t_k} = B_{t_k - \Delta t} e^{r\Delta t} - (\alpha_{t_k} - \alpha_{t_k - \Delta t}) S_{t_k} - (\eta_{t_k} - \eta_{t_k - \Delta t}) g_{t_k} - (\epsilon_1|\alpha_{t_k} - \alpha_{t_k - \Delta t}| + \epsilon_2|\eta_{t_k} - \eta_{t_k - \Delta t}|)$ in the bank account. Also, the re-balancing band range need to be adjusted to $[\alpha_{t_k} - b, \alpha_{t_k} + b]$

$\vdots$

- The profit and loss at the maturity time step T is : $PnL_T = B_T - F(S_T)$.

3. **Analysis Results**

To implement the analysis, we set up a specific base case to work on, where we implemented sensitivity and scenario analysis to explore the effect of model factors, including volatility, time steps, transaction costs, and rebalancing band. The result section compared the shape of the profit and loss (PnL) distribution, 95% VaR and 95% CVaR among the four strategies mentioned in the methodology section.

Suppose we sold an at-the-money put option on an underlying asset S and constructed a portfolio by trading this asset and a call option. In Table 1 below, we list the relevant data for this scenario in detail.

Table 1. Base Case Assumptions

Put Option Maturity	T = 0.25
Call Option Maturity	T = 0.5
Current Price	$S_0 = 100$
Real-World Drift	$\mu = 0.1$
Risk-free Rate	$r = 0.02$
Strike Price	K = 100
Volatility	$\sigma = 0.2$
Equity Transaction Cost	$fee_1 = 0.005$
Option Transaction Cost	$fee_2 = 0.01$

3.1. **Delta Hedging: Comparison between two strategies.**

3.1.1 Volatility.

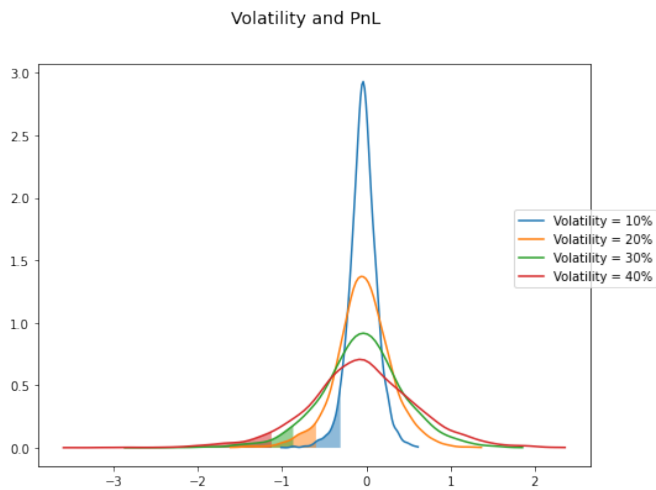

FIGURE 8. Time-based Detla

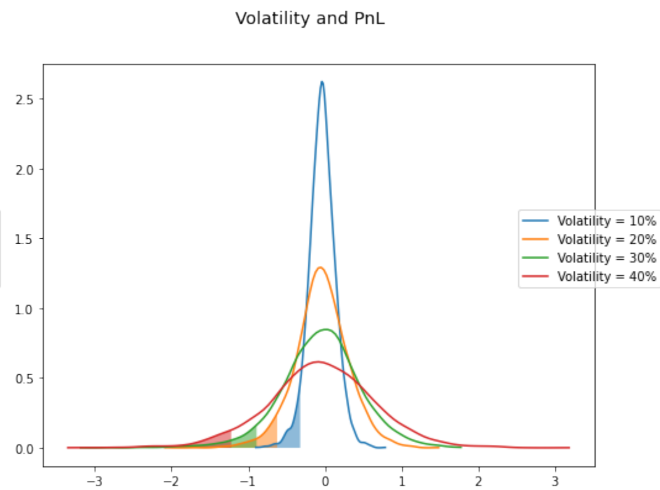

FIGURE 9. Move-based Delta

Parameter	Time-based Delta				Move-based Delta				Diff=Time-Move			
σ	10%	20%	30%	40%	10%	20%	30%	40%	10%	20%	30%	40%
Mean	-0.041	-0.038	-0.046	-0.061	-0.038	-0.042	-0.049	-0.048	-0.002	-0.005	0.003	-0.013
Std	0.166	0.338	0.505	0.659	0.174	0.350	0.522	0.700	-0.008	-0.012	-0.017	-0.041
VaR	-0.307	-0.597	-0.868	-1.126	-0.322	-0.623	-0.891	-1.222	0.016	0.025	0.023	0.096
CVaR	-0.430	-0.830	-1.195	-1.567	-0.436	-0.828	-1.261	-1.589	0.006	-0.002	0.066	0.023
Skewness	-0.292	-0.197	-0.211	-0.206	-0.161	-0.137	-0.335	-0.095	-0.131	-0.059	0.124	-0.111
Kurtosis	1.911	1.447	1.333	1.384	1.264	1.016	1.573	0.852	0.647	0.431	-0.240	0.532

TABLE 2. Different Variation: Time-based delta VS Move-based delta hedging

First of all, we studied the difference in the result of PnL from delta hedging under the influence of these two discrete-time strategies, time-based and move-based, from the perspective of volatility.

From the above table and figures, we can see that if we hold low volatility (i.e: 10%, 20%, 30%), there is no significant difference between move-based strategy and time-based strategy in the performance of $E(PnL)$. But as volatility continues to grow up, the expected profit and loss for the time-based delta hedging have a gradual downward trend, in contrast, the expected profit and loss for the move-based delta hedging remains stable. This is probably because with the increase of price fluctuation, using a time-based strategy will need large quantities of adjustments in the portfolio every day, resulting in huge transaction costs in total. In this case, a move-based strategy is preferred.

Similarly, as the volatility increase, the values of the standard deviation increase, and the absolute values of VaR and $CVaR$ also increase. This implies the raise of tail loss. Generally, for these four volatility levels, the time-based delta hedging strategy has a more desirable standard deviation and variance. But the gap between the two is not very big.

In short, when the volatility is low, the time-based delta hedging strategy will be better as it has lower variation, it is more stable. While as the volatility increases, the move-based delta hedging strategy will be more preferable. It probably can bring a higher PnL with variation not much different from the time-based one.

3.1.2 Time step.

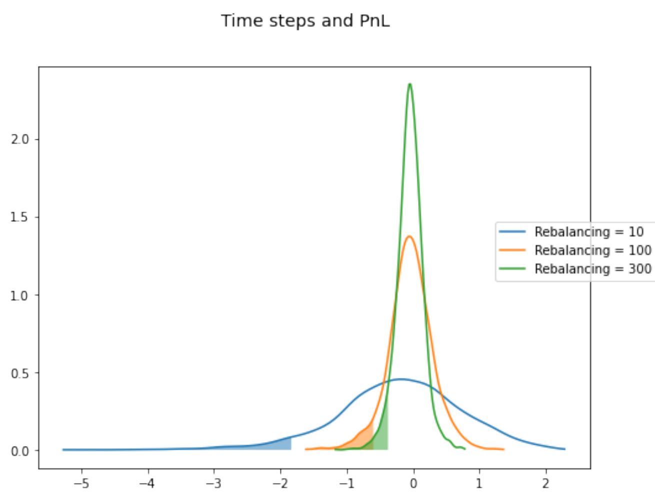

FIGURE 10. Time-based Delta

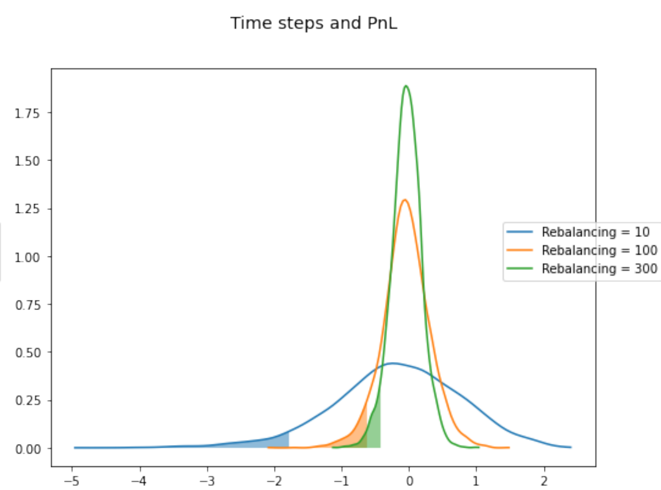

FIGURE 11. Move-based Delta

As the time step increase, we rebalance more frequently and reduce hedging error, thus the volatility of profit and loss decrease in both approaches. From the table, for 100 time-steps per quarter year, the time-based strategy outperforms the move-based strategy. Time-based $E(PnL)$ is slightly greater than move-based mean profit and loss, and time-based strategy has a better $CVaR$ value, indicating a smaller expected tail loss. However, when we increase the time steps

Parameter	Time-based Delta			Move-based Delta			Diff=Time-Move		
Time Steps N	10	100	300	10	100	300	10	100	300
Mean	-0.209	-0.038	-0.046	-0.182	-0.042	-0.038	-0.027	0.005	-0.008
Std	0.935	0.338	0.200	0.947	0.350	0.227	-0.011	-0.012	-0.027
VaR	-1.834	-0.597	-0.375	-1.780	-0.623	-0.420	-0.054	0.025	0.045
CVaR	-2.446	-0.830	-0.506	-2.407	-0.828	-0.549	-0.039	-0.002	0.044
Skewness	-0.486	-0.197	-0.157	-0.436	-0.137	-0.144	-0.050	-0.059	-0.014
Kurtosis	0.962	1.447	1.611	0.663	1.016	0.879	0.299	0.431	0.732

TABLE 3. Different Time Steps: Time-based delta VS Move-based delta hedging

to 300, the $E(PnL)$ level of the move-based strategy is greater than the time-based strategy, but the time-based strategy still has a better VaR and $CVaR$. It is because the time-based strategy rebalances more frequently and the accumulated transaction costs reduce portfolio profit. The time-based method maintains a lower hedging error and reduces loss risk by sacrificing potential profits. On the other hand, if time steps drop to 10, the rebalance frequency of both the time-step and move-based method is low, so the time-steps method no longer ensures a lower hedging error. It seems the move-based approach can obtain a higher profit and lower risk. Intuitively, there might be some time step N, where these two methods have the same mean profit and value at risk.

3.1.3 Transaction costs.

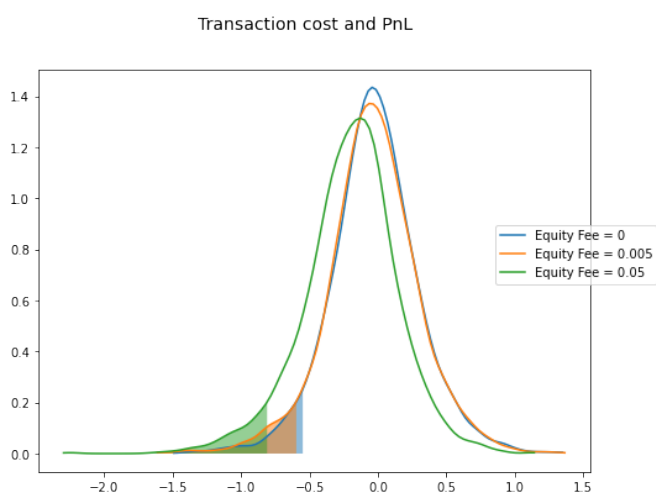

FIGURE 12. Time-based Delta

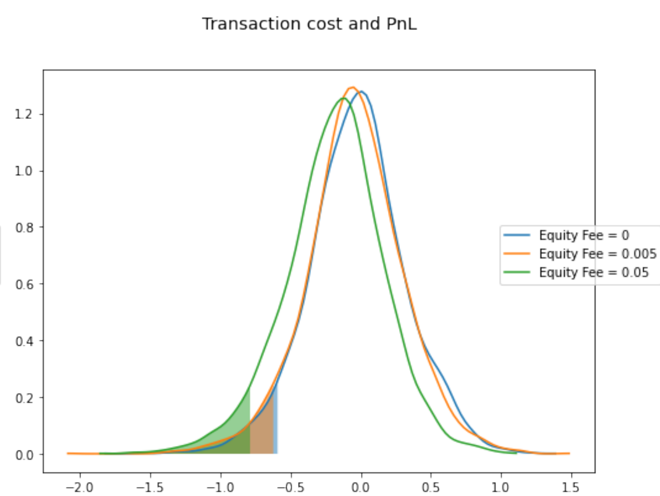

FIGURE 13. Move-based Delta

Parameter	Time-based Delta			Move-based Delta			Diff=Time-Move		
Transaction Cost	No cost	Cost=0.5%	Cost=5%	No cost	Cost=0.5%	Cost=5%	No cost	Cost=0.5%	Cost=5%
Mean	-0.025	-0.038	-0.203	-0.023	-0.042	-0.187	-0.003	0.005	-0.015
Std	0.326	0.338	0.348	0.346	0.350	0.353	-0.020	-0.012	-0.005
VaR	-0.550	-0.597	-0.810	-0.592	-0.623	-0.788	0.042	0.025	-0.023
CVaR	-0.762	-0.830	-1.039	-0.784	-0.828	-0.995	0.023	-0.002	-0.044
Skewness	-0.060	-0.197	-0.424	-0.095	-0.137	-0.261	0.034	-0.059	-0.163
Kurtosis	1.324	1.447	1.567	0.720	1.016	0.720	0.604	0.431	0.847

TABLE 4. Different Transaction Cost: Time-based delta v.s. Move-based delta hedging

Based on the graphs above, adding transaction fees to our hedging strategies results in a lower expected return together with a lower VaR and $CVaR$. However, the standard deviation, skewness, and kurtosis are roughly the same. On the other hand, when we add transaction fees especially when the transaction fee rate reaches 5%, the expected return of the Move-based strategy is higher than the time-based strategy. Consistently, the VaR and $CVaR$ are also better compared to the time-based strategy. It can be explained by the fact that time-based strategies require much more frequent hedging than Move-based strategies, resulting in a higher transaction cost. However, if the transaction fee rate is only 0.5% or less, there is no significant difference between the move-based and time-based strategies.

3.2. **Delta-Gamma Hedging: Comparison between two strategies.**

3.2.1 Volatility.

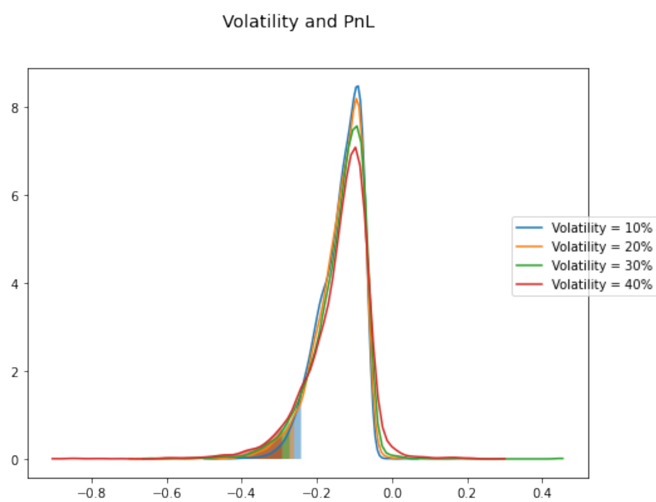

FIGURE 14. Time-based Delta-Gamma

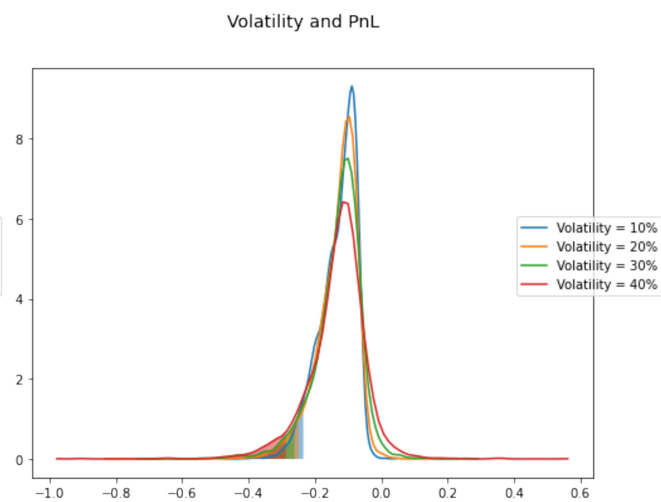

FIGURE 15. Move-based Delta-Gamma

Parameter	Time-based Delta				Move-based Delta				Diff=Time-Move			
σ	10%	20%	30%	40%	10%	20%	30%	40%	10%	20%	30%	40%
Mean	-0.138	0.140	0.140	-0.142	0.007	-0.132	-0.131	-0.133	-0.144	-0.009	-0.010	-0.009
Std	0.056	0.064	0.071	0.085	0.137	0.062	0.073	0.089	-0.081	0.002	-0.002	0.004
VaR	-0.242	-0.261	-0.273	-0.293	-0.167	-0.250	-0.262	-0.287	-0.075	-0.011	-0.011	-0.006
CVaR	-0.277	-0.314	-0.334	-0.377	-0.265	-0.298	-0.328	-0.369	-0.011	-0.016	-0.006	-0.009
Skewness	-0.955	-1.492	-1.233	-1.697	1.154	-1.433	-1.210	-1.166	-2.109	-0.059	-0.024	-0.531
Kurtosis	1.090	4.600	5.938	8.501	23.390	6.496	5.068	8.602	-22.300	-1.896	0.870	-0.100

TABLE 5. Different Variation: Time-based vs. Move-based delta-gamma hedging

Similar to the previous section, we again start the analysis over two discrete-time strategies from the perspective of volatility.

From the above table and graph, we can see that if we hold low volatility at 10%, time-based delta-gamma hedging has a much lower $E(PnL)$ than the move-based delta-gamma hedging. This could be explained by the transaction costs in some way that such low volatility does not necessarily need daily adjustment, daily adjustments will cause waste on transaction costs. Besides the 10% volatility, there is no significant difference between move-based strategy and time-based strategy in the performance of $E(PnL)$. But on the whole, the move-based delta-gamma hedging performs slightly better than the time-based hedging. And the similar trend could be found for both of them that when volatility increase, the values of the standard deviation increase, and the absolute values of VaR and Std also increase.

One point worth noting is that compare to the delta hedging, volatility has a much smaller impact on $E(PnL)$, Std, and VaR of delta-gamma hedging. This is because the delta-gamma hedging is designed to hedge the changes that happened in "delta". In this case, it could solve some convexity issues. Thus, delta-gamma hedging is a more precise method.

Generally, for these four volatility levels, the time-based delta hedging strategy has a more desirable standard deviation. But the gap between the two is not very big. In short, if we do not take the impact of transaction costs into consideration, delta-gamma hedging will be a better hedging strategy than delta-hedging. And under the delta-gamma hedging, the move-based strategy will perform a little bit better than the time-based strategy, especially at the lower volatility (i.e: 10%).

3.2.2 Time step.

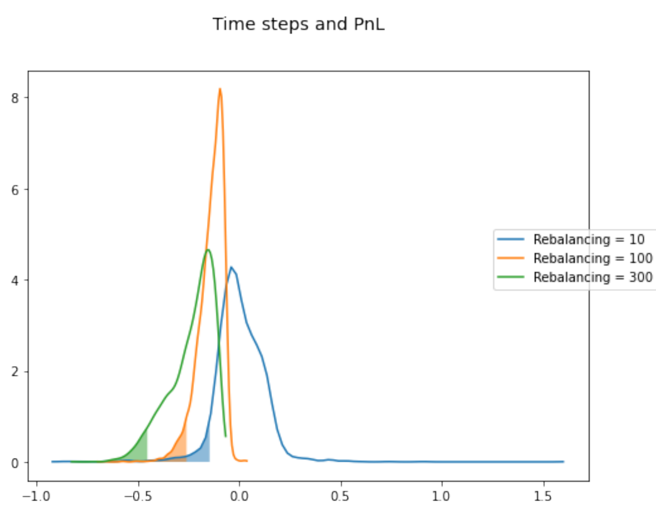

FIGURE 16. Time-based Delta-Gamma

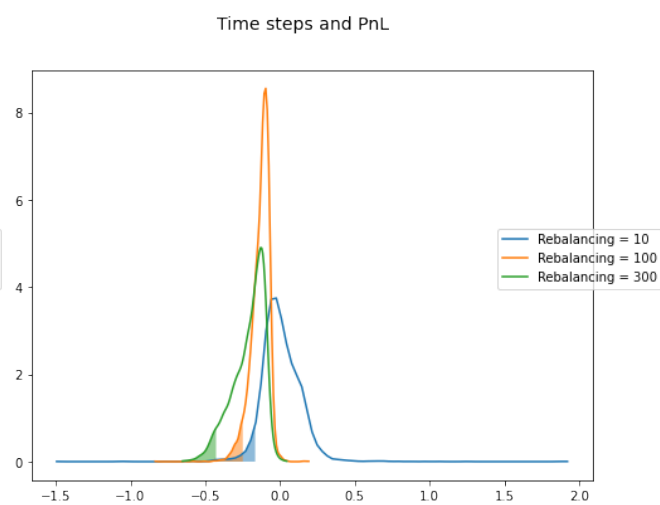

FIGURE 17. Move-based Delta-Gamma

Since the introduction of gamma solves the convexity issue of delta hedging, delta-gamma hedging has reduced the hedging error and volatility of profit and loss to a very low level. The additional time steps (or rebalancing frequency) will no longer significantly improve the overall

Parameter	Time-based Delta			Move-based Delta			Diff=Time-Move		
Time Steps N	10	100	300	10	100	300	10	100	300
Mean	0.003	-0.140	-0.235	0.007	-0.132	-0.211	-0.004	-0.009	-0.024
Std	0.122	0.064	0.110	0.137	0.062	0.108	-0.015	0.002	0.002
VaR	-0.147	-0.261	-0.454	-0.167	-0.250	-0.428	0.020	-0.011	-0.026
CVaR	-0.272	-0.314	-0.508	-0.265	-0.298	-0.477	-0.006	-0.016	-0.031
Skewness	0.087	-1.492	-0.957	1.154	-1.433	-0.900	-1.067	-0.059	-0.058
Kurtosis	13.393	4.600	0.426	23.390	6.496	0.308	-9.997	-1.896	0.118

TABLE 6. Different Time Steps: Time-based vs. Move-based delta-gamma hedging

performance, which is different from delta hedging. In terms of mean profit and loss, $CVaR$ and VaR, the move-based approach generally outperforms the time-based approach, with higher average profit and lower tail risk. Due to expensive option transactions, for delta-gamma hedging strategy, blindly increasing rebalancing time will only increase unnecessary transaction costs. Since this method has maintained the stability of the delta, investment companies can choose the move-based approach to save costs, without rebalancing for small movement of portfolio delta.

3.2.3 Transaction costs.

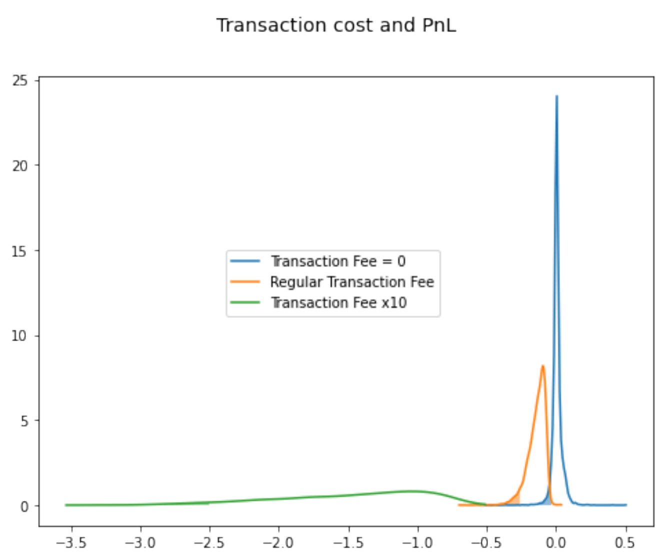

FIGURE 18. Time-based Delta-Gamma

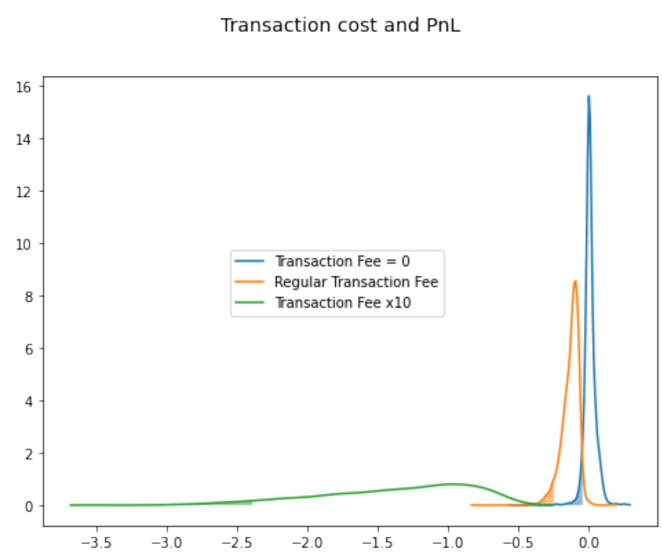

FIGURE 19. Move-based Delta-Gamma

Investment companies targeting a more accurate dynamic hedging may prefer the delta-gamma strategies. Unfortunately, by analyzing different levels of trading expenses, we found that high

Parameter	Time-based Delta-Gamma			Move-based Delta-Gamma			Difference=Time-Move		
Transaction Cost	No cost	Cost=0.5%	Cost=5%	No cost	Cost=0.5%	Cost=5%	No cost	Cost=0.5%	Cost=5%
Mean	0.007	-0.140	-1.466	0.007	-0.132	-1.364	0.000	-0.009	-0.102
Std	0.037	0.064	0.543	0.041	0.062	0.543	0.005	0.002	0.000
VaR	-0.035	-0.261	-2.503	-0.047	-0.250	-2.389	0.011	-0.011	-0.114
CVaR	-0.075	-0.314	-2.731	-0.082	-0.298	-2.625	0.007	-0.016	-0.106
Skewness	-1.154	-1.492	-0.692	-0.409	-1.433	-0.680	-0.745	-0.059	-0.011
Kurtosis	40.814	4.600	-0.162	16.373	6.496	-0.119	24.441	-1.896	-0.043

Table 7. Different Transaction Cost: Time-based vs. Move-based delta-gamma hedging

transaction costs strongly affect the performance of delta-gamma hedging. Although the differences between both re-balancing approaches under delta-gamma hedging are small, the move-based approach outperforms the time-based approach as long as transaction costs exist.

According to Table 7, the higher the transaction cost, the larger the difference in average return between the time-based and move-based approaches. It is obviously because of the higher hedging frequency in the move-based strategy, associating with higher transaction fees. The findings concerning the average return are consistent with the VaR and $CVaR$. It seems higher transaction cost introduce a more unexpected tail loss in a time-based approach. The move-based approach is better in terms of cost control.

Parameter	Time-based			Move-based		
Diff = Delta - Gamma	No cost	Cost=0.5%	Cost=5%	No cost	Cost=0.5%	Cost=5%
Mean	-0.032	0.103	1.264	-0.030	0.090	1.177
Std	0.290	0.274	-0.195	0.305	0.288	-0.191
VaR	-0.515	-0.337	1.692	-0.545	-0.373	1.601
CVaR	-0.687	-0.516	1.693	-0.702	-0.531	1.630
Skewness	1.094	1.295	0.267	0.315	1.295	0.419
Kurtosis	-39.489	-3.153	1.729	-15.653	-5.480	0.838

Table 8. Difference in Profit and Loss Between Delta and Delta-Gamma Hedging under Various Transaction Fee in Two Rebalancing Approach

Finally, it is worth mentioning an interesting observation from Table 8. When transaction fees equal 0, delta-gamma hedging outperforms delta hedging under both rebalancing approaches (time-based and move-based). However, when transaction fees reach 5%, delta hedging is preferred. In the high-cost scenario, the delta-gamma strategy has a lower expected return together with a higher absolute value of VaR and $CVaR$. Delta-Gamma strategy seems more expensive while perfectly hedging the risk exposure.

Similarly, given the rebalancing band = 0.05, the mean PnL in delta hedging outperform that in delta-gamma hedging as long as transaction costs exist. On the other hand, we can conclude that as the transaction cost grows the percentage change of standard deviation in the Delta-Gamma strategy grows more rapidly than the Delta strategy, which is consistent concerning the kurtosis. It means as the transaction cost grows, the distribution of the PnL of the Delta-Gamma strategy can be more volatile than the Delta strategy.

Moreover, under lower cost (0% or 0.5%), the improvement of VaR and $CVaR$ created by adding gamma hedging for the move-based approach is greater than the time-based approach. For example, if there is no cost, adding gamma hedging in a time-based strategy only improves CVaR by 0.687, but adding gamma hedging to a move-based strategy can improve CVaR by 0.702. However, under extremely high cost (5%), it is not worth the money to implement gamma hedging.

3.3. **Effect of Lower Real-world Volatility.**

In this section, we are going to explore the effect of different realized volatility have on our hedging strategies. Assuming the realized volatility remains the same as the expected volatility we used in the model cannot always be true in the real financial market, which will usually cause a great discrepancy between the expected PnL and the realized PnL. Therefore, it is beneficial to consider the different distributions of realized PnL under different realized distributions. We applied the Kernel Density Estimation(KDE) techniques to estimate the distributions.

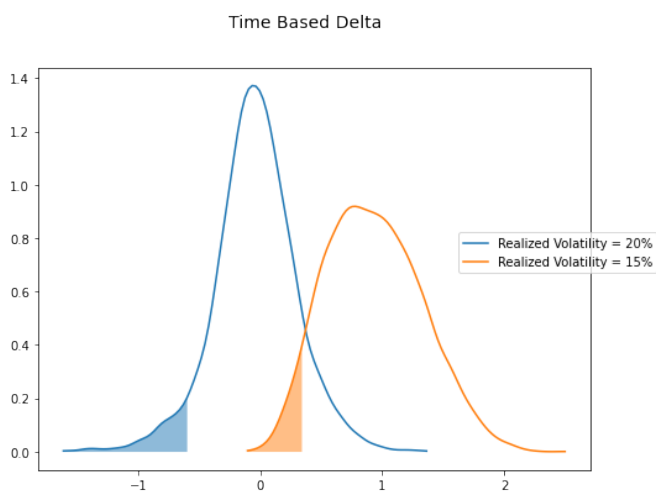

FIGURE 20. Time-based Delta

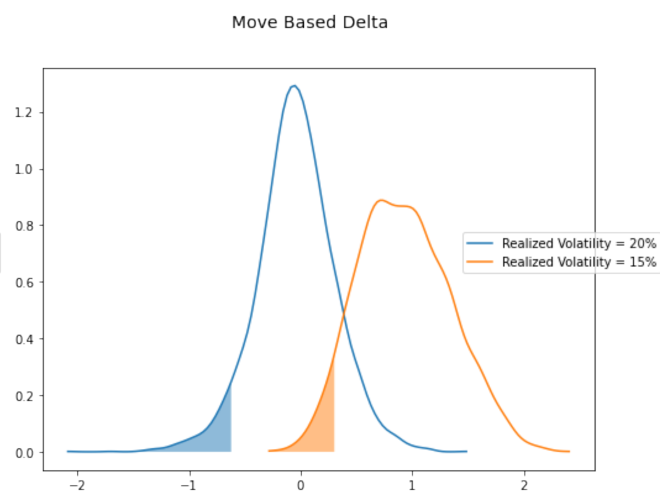

FIGURE 21. Move-based Delta

	Time-based Delta		Move-based Delta		Diff=Time-Move	
Realized-σ	15%	20%	15%	20%	15%	20%
Mean	0.9370	-0.0375	0.9324	-0.0422	0.0046	0.0046
Std	0.3933	0.3381	0.4134	0.3500	-0.0201	-0.0119
VaR	0.3430	-0.5973	0.3016	-0.6226	0.0415	0.0252
CVaR	0.2389	-0.8301	0.1719	-0.8282	0.0670	-0.0018
Skewness	0.2915	-0.1969	0.2382	-0.1374	0.0533	-0.0595
Kurtosis	-0.3494	1.4471	-0.3375	1.0157	-0.0119	0.4314

TABLE 9. Lower Real-world Volatility: Time-based v.s. Move-based Delta Hedging

Under the delta hedging, we can find various inspiring and interesting results under a lower realized volatility of 15%. According to the above figure and Table 9, we can compare the PnL distribution and key statistics among different strategies.

Initially, the expected PnL is much higher under the 15% volatility, since we captured option volatility premium by selling the options and construing corresponding hedging portfolio under the realized 15% volatility. Investors might benefit from this situation. Besides, there is no significant difference between the time-based strategy and the move-based strategy.

Secondly, the standard deviation is generally higher for the 15% realized volatility. Meantime, we have a lower kurtosis, which is consistent with the difference in the standard deviation. The discrepancies imply that the PnL under 15% volatility is less concentrated.

Lastly, under the smaller 15% volatility, the skewness of the distribution is much higher, which indicates the PnL has changed from normal distributed to right-skewed distributed. If we have higher volatility(e.g. 25% volatility), we may draw a contrary conclusion. Therefore, the portfolio manager should not neglect the potential model risks considering the potential mismatch of the real-world and risk-neutral volatility.

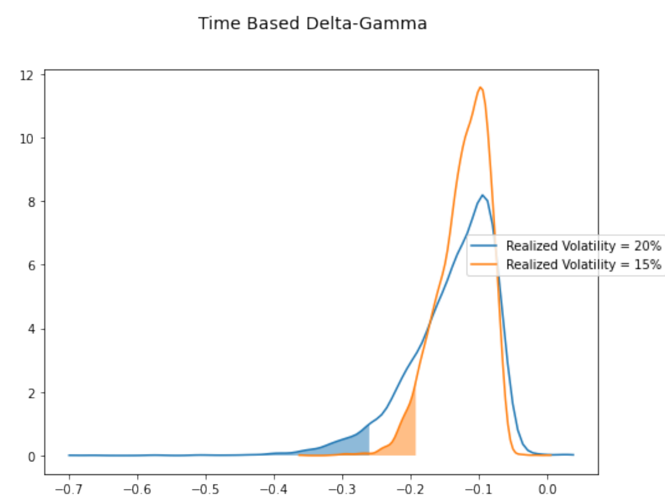

FIGURE 22. Time-based

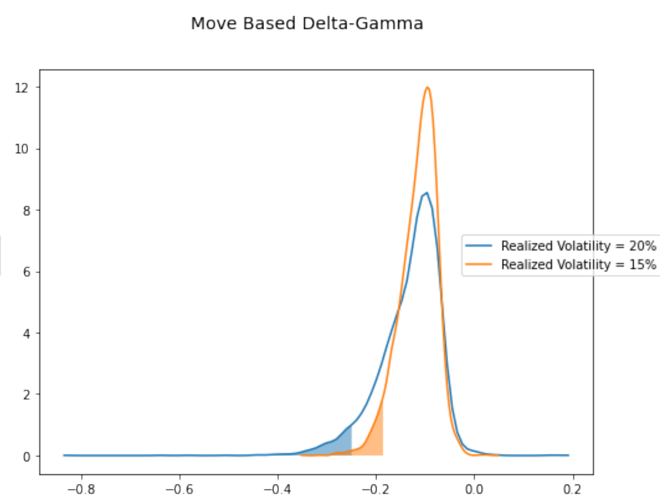

FIGURE 23. Move-based

Surprisingly, the findings under the delta-gamma strategy are obviously inconsistent with the findings under the delta strategy, which can be attributed to the fact that the delta-gamma strategy is more complicated with more adjustments and higher associated transaction fees. The related analysis is based on the Figure 22, Figure 23 and Table 10. Overall, the difference between the time-based and the move-based strategy under delta-gamma hedging is small.

	Time-based Delta-Gamma		Move-based Delta-Gamma		Diff=Time-Move	
Realized-σ	15%	20%	15%	20%	15%	20%
Mean	-0.1239	-0.1403	-0.1147	-0.1317	-0.0092	-0.0086
Std	0.0375	0.0639	0.0384	0.0620	-0.0008	0.0020
VaR	-0.1927	-0.2606	-0.1857	-0.2496	0.0070	-0.0111
CVaR	-0.2148	-0.3142	-0.2102	-0.2977	-0.0047	-0.0164
Skewness	-0.8345	-1.4919	-0.8047	-1.4327	-0.0298	-0.0592
Kurtosis	0.8939	4.5999	1.3582	6.4958	-0.4643	-1.8958

Table 10. Lower Real-world Volatility: Time v.s. Move-based Delta-gamma Hedging

The expected PnL under the based case is similar to the PnL under the 15% volatility because a higher transaction fee has offset the potentially higher return generated by the option volatility premium. However, the lower realized volatility leads to a significantly higher absolute value of VaR, because of the increasing standard deviation of the PnL distribution. The investor should favor this situation but also need to consider the model risk associated with another scenario where the realized volatility is higher. Additionally, the kurtosis under the 15% realized volatility is much lower than the kurtosis under the expected 20% realized volatility.

Compared with the Delta hedging strategy, the reduction in real-world volatility has less impact on the PnL distribution shape of the Delta-Gamma strategy, since the Delta-Gamma hedging strategy is more accurate when replicating the underwritten put option.

3.4. **Effect of Rebalancing band in Move-based Strategies.**

In this section, we analyze the effect of the rebalancing band under two move-based hedging strategies. The risk tolerance of the portfolio manager is the maximum price movement accepted before rebalancing, which is also the value of the net portfolio delta. When the value of the portfolio's delta hits the upper or lower bound of the rebalancing band, we rebalance the position of all portfolio assets, including stocks in delta hedging and call option in gamma hedging, to reset the portfolio delta and gamma back to zero.

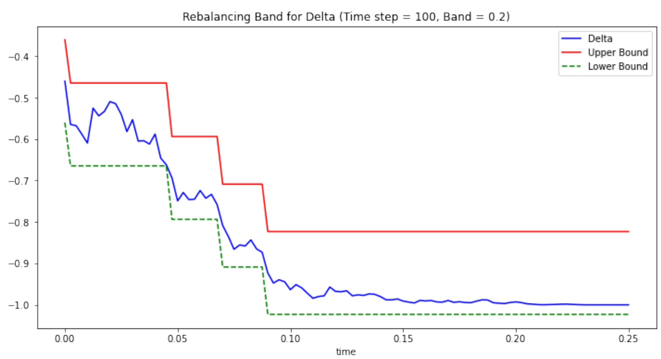

FIGURE 24. Band = 20%, N=100

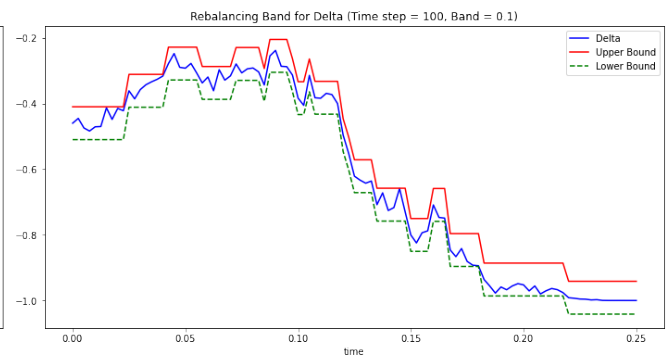

FIGURE 25. Band = 10%, N=100

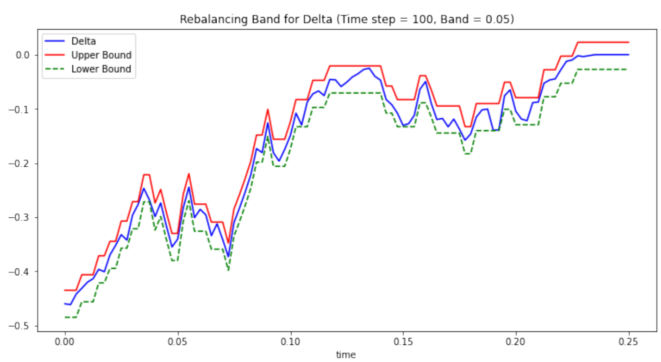

FIGURE 26. Band = 5%, N=100

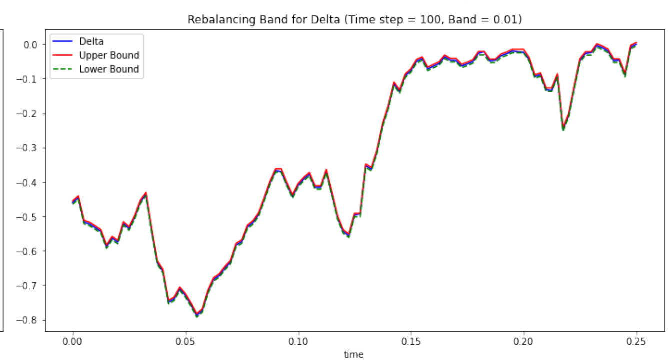

FIGURE 27. Band = 1%, N=100

First, we analyzed the movement of the rebalancing band and the portfolio delta. As we checked the portfolio delta at each equal-sized, discrete-time step N ($\Delta t = T/N$), and rebalancing only when delta moved large enough, decreasing the time step increased the number of checking frequency and then increase the rebalancing times, which tracked the option delta better. Next, by fixing time step N =100, we investigated the effect of bandwidth in the hedging process. Based on the figure above, the delta fluctuated within the predetermined band. As the bandwidth shrink, the frequency of rebalancing increases, which implies a smaller hedging error. We can also conclude that as the bandwidth approaches 0, the move-based strategy is nearly behaving the same as the time-based strategy.

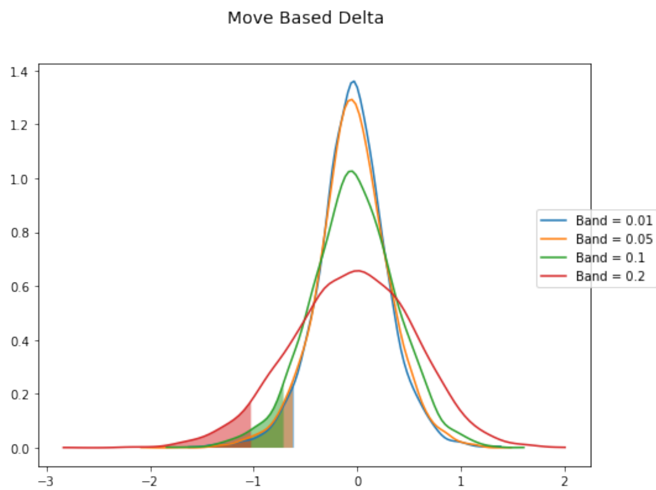

FIGURE 28. Delta

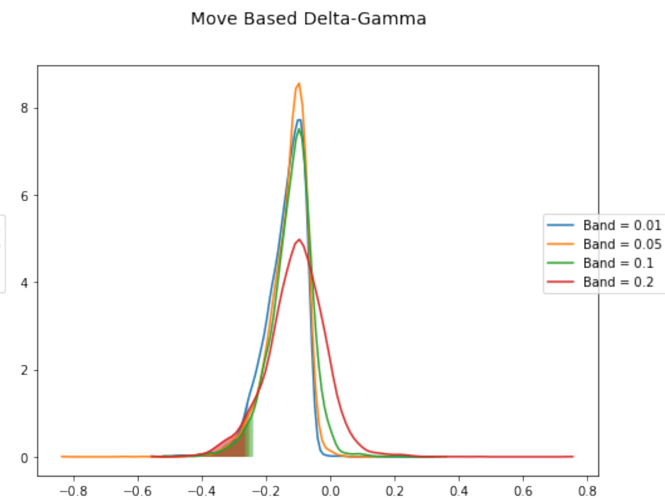

FIGURE 29. Delta-Gamma

	Move-based Delta-Gamma				Move-based Delta-Gamma				Diff = Delta-Gamma			
Band	1%	5%	10%	20%	1%	5%	10%	20%	1%	5%	10%	20%
Mean	-0.047	-0.042	-0.038	-0.047	-0.141	-0.132	-0.122	-0.105	0.094	0.090	0.084	0.059
Std	0.338	0.350	0.415	0.589	0.063	0.062	0.067	0.093	0.276	0.288	0.348	0.496
VaR	-0.610	-0.623	-0.711	-1.024	-0.255	-0.250	-0.239	-0.264	-0.354	-0.373	-0.472	-0.760
CVaR	-0.819	-0.828	-0.938	-1.303	-0.304	-0.298	-0.291	-0.316	-0.515	-0.53	-0.648	-0.987
Skewness	-0.156	-0.137	-0.100	-0.143	-1.106	-1.433	-0.643	0.065	0.950	1.295	0.543	-0.207
Kurtosis	1.292	1.016	0.396	0.071	3.255	6.496	3.129	2.935	-1.963	-5.480	-2.733	-2.864

TABLE 11. Move-based Rebalancing band: Delta v.s. Delta-gamma Hedging

Next, we analyze how the risk tolerance affects the profit and loss under the move-based hedging, assuming transaction fees same as the base case. The density of PnL shows that the standard deviation of the profit and loss increase as the band increase. This impact is more significant for delta hedging because the delta-gamma strategy initially provides more precise hedging. Based on Table 11, we find that although the average loss of delta strategy is less than the delta-gamma strategy, delta hedging seems to have higher tail losses. Specifically, under a wider band = 20%, the 95% VaR and $CVaR$ reached -1.024 and -1.303 respectively for the delta hedging, while that of delta-gamma hedging is only -0.264 and -0.316. In other words, the tail loss of using delta hedging can reach three to four times that of delta-gamma hedging.

4. Conclusions

Delta and delta-gamma dynamic hedging strategies in discrete time steps based on the Black-Scholes framework can effectively stabilize the portfolio profit and loss (PnL) around zero, where investment companies achieve profit neutral. Compared to the move-based approach, the time-based approach ensures a higher rebalancing frequency and lower volatility of PnL. When the rebalancing band of delta in move-based hedging becomes small enough, the rebalancing frequency increase and the move-based hedging can behave just like the time-based hedging, tracking the delta at almost each discrete time step. Moreover, by solving the convexity problem in a delta hedging strategy, delta-gamma hedging can achieve more precise hedging of the underwritten put option, especially when there is a difference between realized and risk-neutral volatility. In exchange, investors have to bear a higher transaction cost in delta-gamma hedging. Investment companies can control hedging costs by using the move-based delta-gamma strategy instead of a time-based strategy, while still maintaining a low hedging error, small PnL volatility, and reduced tail loss. Finally, even though a potentially lower realized volatility generates extra premiums, companies need to be aware of the model risks caused by the mismatching of real-world and risk-neutral volatility. Underestimate the real-world volatility might lead to higher unexpected loss. Therefore, to enhance the portfolio performance and control tail loss, it is worth adjusting model assumptions periodically and selecting appropriate hedging strategies based on the market and investment needs.