STA 2503 / MMF 1928 Project - 1 American Options

Suppose that an asset price process $S = (S_{t_k})_{k\in\{0,1,\dots,N\}}$ (with $t_k = k\,\Delta t$ and $\Delta t = \frac{T}{N}$, for a fixed N) are given by the stochastic dynamics

$$S_{t_k} = S_{t_{k-1}}\, e^{r\,\Delta t + \sigma\sqrt{\Delta t}\,\epsilon_k}\,,$$

where ϵ_k are iid rv with $\epsilon_k \in \{+1, -1\}$ and

$$\mathbb{P}(\epsilon_k = \pm 1) = \tfrac{1}{2}\left(1 \pm \frac{(\mu - r) - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right).$$

Here, $r \geq 0$ and $\sigma > 0$ are constants.

Moreover, let $B = (B_{t_k})_{k\in\{0,1,\dots,N\}}$ denote the bank account with $B_t = e^{r\,t}$.

1. Let $X^{(N)}$ denote the random variable $X^{(N)} := \log(S_T/S_0)$. Prove that

$$X^{(N)} \xrightarrow[N\to\infty]{d} (\mu - \tfrac{1}{2}\sigma^2)\,T + \sigma\sqrt{T}\,Z, \qquad \text{and} \qquad Z \overset{\mathbb{P}}{\sim} \mathcal{N}(0,1)$$

2. Derive the probabilities $\mathbb{Q}(\epsilon_k = \pm 1)$ and $\mathbb{Q}^S(\epsilon_k = \pm 1)$, as well as the $\mathbb{Q}$ and $\mathbb{Q}^S$ distribution of S_T in the limit as $N \to \infty$.
[Recall that $\mathbb{Q}$ refers to the martingale measure induced by using the bank account B as a numeraire, and $\mathbb{Q}^S$ refers to the martingale measure induced by using the asset S as a numeraire.]

3. In this part, you will evaluate an American option. Assume that $T = 1$, $S_0 = 10$, $\mu = 5\%$, $\sigma = 20\%$, and the risk-free rate $r = 2\%$. Use $N = 5000$.

   (a) Implement a binomial tree to value the American put option with strike $K = 10$.

      i. Plot the exercise boundary as a function of t.

      ii. Show how the various plots vary as volatility and risk-free rate change.

   (b) Assume you have purchased the American option with the base set of parameters.

      i. Simulate 10,000 sample paths of the asset and obtain a kernel density estimate of (i) profit and loss you will receive, and (ii) the distribution of the time at which you exercise the option. Explore how the various model parameters effect these distributions.

      ii. Suppose that the realized volatility is $\sigma = 10\%, 15\%, 20\%, 25\%, 30\%$, but you were able to purchase the option with a volatility of $\sigma = 20\%$ and you use the $\sigma = 20\%$ exercise boundary in your trading strategy. Explore how the distributions of profit and loss and exercise time vary in this case.