STA2503H Applied Probability for Mathematical Finance

The Heston Model

Final Project

Instructor: Prof. Jaimungal

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Authors: Lujia Yang, Xiangyu Li, Xinyue Sun
Master of Financial Insurance & Master of Mathematical Finance
University of Toronto

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Abstract

The Heston model was proposed to model stochastic stock price volatility, which related to the concept of stochastic volterra equation. It captures some important features observed in the financial market, including highly endogenous, metaorders, arbitrages prevention, and liquidity asymmetry. Different from stochastic differential equation, the stochastic volterra equation is extremely computationally expensive to simulate. Therefore, it is difficult to compute option prices and implied volatility under the Heston model by conventional Monte Carlo simulation. In this paper, we calculate the implied volatility and the confidence interval, applying Euler's discretization method, mixed Monte Carlo method, single and multilevel control variate methods, including option price, contingent claims (paying $\int_0^T v_s ds$ or $\int_0^T v_s^2 ds$), and stock price. The result indicates a superior performance of stock as control variate, multilevel control variate, and the mixing method, which have a high possibility of potential use in practice to enhance the computational efficiency in computing the implied volatility under Heston model.

1 Introduction

After the 1987 market crash, the Black-Scholes model came under heavy scrutiny for its incapability to price options. The underlying assumptions of constant volatility and normal distribution in the Black-Scholes model have proven to be inaccurate in empirical studies in equity markets. It is now a known fact that equity returns exhibit lognormal distribution with fat tails which is normality difficult to capture. Various empirical studies indicate that volatility is time-varying and has an inverse relationship with the stock price. Thus, proving again the incorrect assumption of the Black-Scholes model.

One of the famous methods to overcome the limitation of the Black-Scholes model mentioned above was to consider time-varying volatility, typically the Heston Model. It is considered to be a benchmark model in the field of stochastic volatility modeling. Heston model evaluates stochastic volatility which uses a statistical method of mathematical finance where volatility and dependence between variables can fluctuate over time. Meanwhile, it captures successive values of an independent random variable. It is widely used to price equity options since it can generate closed-form call prices and corresponding implied volatility (IV). As implied volatility has to be computed numerically using Monte Carol Method, analysts have to balance the calculation accuracy (which can be measured using the confidence width) and computational power needed (such as the number of simulation and time steps). Instead of increasing the simulation times blindly, **given fixed simulation and time steps**, a considerable amount of work has been done to improve the precision of implied volatility (IV) estimates and investigate their behavior.

This report researched the performance of various simulation methods to generating implied volatility, including **Euler's discretization method, mixed Monte Carlo method, single and multilevel control variate methods, such as option price, contingent claims (paying $\int_0^T v_s ds$ or $\int_0^T v_s^2 ds$, and stock price**. Section 2 demonstrated the basic concepts of the Heston Model, base case assumptions, various numerical methods used, and their related analytical formulas. Section 3 indicate the IV estimation result and the comparison of the Euler Milstein method and the mixing method. Section 4 focuses on variance reduction using the single control variate method, testing various control variates. Section 5 conduct further improvements by introducing multilevel control variate method and comparing with other methods, including the mixing method. The paper aims to investigate the **individual performance** of each method and **compared the quality of these methods** through three aspects: **1.)** Size of confidence width; **2.)** Potential over/underestimation of implied volatility in the case of the at-the-money (ATM) and away-from-the-money options; **3.)** Robustness of the IV estimation under various maturity.

2 Methodology

2.1 Heston Model and Base Assumptions

In this project, we assume that the asset price $S = (S_t)_{t\geq 0}$ and its instantaneous variance $v = (v_t)_{t\geq 0}$ follows the Heston model. Specifically, they satisfy the following stochastic differential equations (SDE):

$$dS_t = S_t\sqrt{v_t}dW_t^S \tag{1}$$

$$dv_t = \kappa(\theta - v_t)dt + \eta\sqrt{v_t}dW_t^v \tag{2}$$

where (W_t^S, W_t^v) are correlated risk-neutral Brownian motions with correlation ρ. Moreover, we use the following base model parameters throughout this report:

Model Parameters	**Base Value**
Interest rate	$r = 0$
Current Stock Price	$S_0 = 1$
Current Volatility	$\sqrt{v_0} = 20\%$
Long-term Volatility	$\sqrt{\theta} = 40\%$
Mean reversion rate	$\kappa = 3$
Correlation Parameter	$\rho = -0.5$
Volatility of variance	$\eta = 1.5$

2.2 Milstein Discretization of v_t

In order to simulate the instantaneous variance $v = (v_t)_{t\geq 0}$, we apply the Milstein discretization of SDE. Suppose a stochastic process $y = (y_t)_{t\geq 0}$ satisfies the following SDE:

$$dy_t = a(y_t)dt + b(y_t)dW_t \tag{3}$$

where a and b are some functions of y_t and W_t is a $\mathbb{P}$-measure Brownian motion. The Milstein scheme claims that we can simulate the a sample path of y_t based on the following equation:

$$\begin{aligned} y_{t+\Delta t} &= y_t + \int_t^{t+\Delta t} a(x_u)du + \int_t^{t+\Delta t} b(y_u)dW_u \\ &= y_t + a(y_t)\Delta t + b(y_t)\Delta W_t + b(y_t)\ b'(y_t)\cdot\frac{1}{2}[\Delta W_t^2 - \Delta t] \end{aligned}$$

We apply the Milstein discretization to $v = (v_t)_{t\geq 0}$ in the Heston model and obtain the following equation:

$$v_{t+\Delta t} = v_t + \kappa(\theta - (v_t)_+)\Delta t + \eta\sqrt{(v_t)_+\Delta t}\cdot\varepsilon_t^v + \frac{\eta^2}{4}((\varepsilon_t^v)^2 - 1)\Delta t \tag{4}$$

where $\varepsilon_t^v \sim \mathcal{N}(0,1)$

Figure 1 shows 5000 sample paths of the instantaneous asset price variance v_t from time 0 to time 1 with a fix step size $\Delta t = \frac{1}{1000}$ by Milstein discretization.

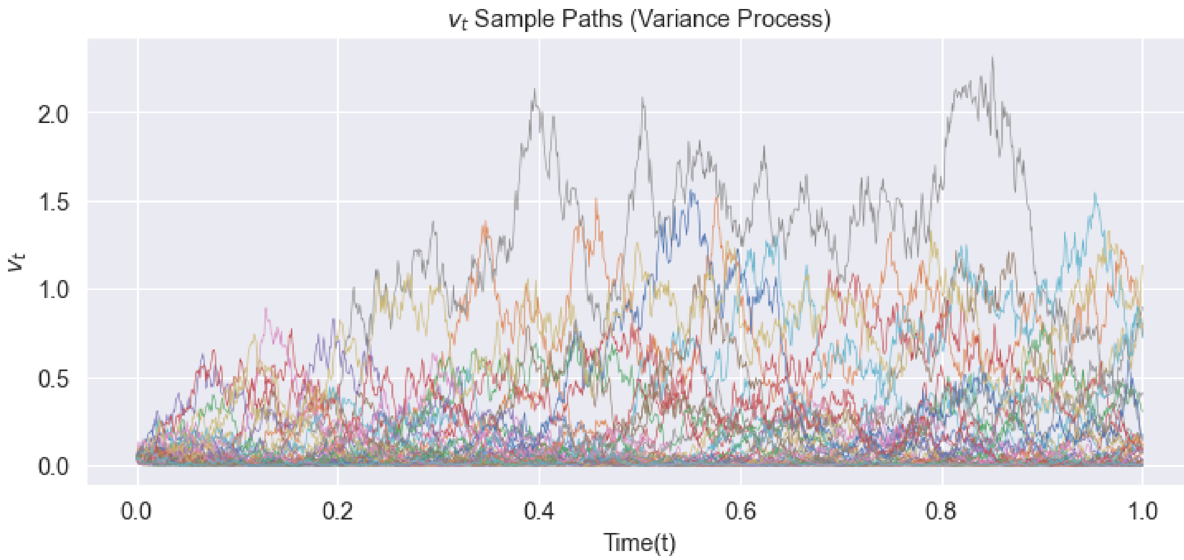


Figure 1: Simulated Instantaneous variance via Milstein Discretization

2.3 Euler discretization of $x_t = \log S_t$

After we simulate the sample paths of asset price variance, we apply Euler discretization to simulate the asset price. Suppose a stochastic process $y = (y_t)_{t\geq 0}$ satisfies the same SDE as (3), the Euler scheme claims that y_t follows the following distribution:

$$y_{t+\Delta t} = y_t + a(y_t) \cdot \Delta t + b(y_t)\sqrt{\Delta t}\ \varepsilon_t$$

where $\varepsilon_t \sim \mathcal{N}(0, 1)$.

Assuming $x_t = \log S_t$ and recalling that S_t satisfies SDE (1), by Ito's lemma:

$$\begin{aligned} dx_t &= \frac{1}{S_t} dS_t + \frac{1}{2}(-\frac{1}{S_t^2})d[S, S]t \\ &= \frac{1}{S_t} S_t \sqrt{v_t}\ dW_t^S - \frac{1}{2} v_t dt \\ &= -\frac{1}{2} v_t dt + \sqrt{v_t}\ dW_t^S \end{aligned} \tag{5}$$

Then the Euler discretization gives us the equation below:

$$x_{t+\Delta t} = x_t - \frac{1}{2} v_t \Delta t + \sqrt{v_t \cdot \Delta t}\ \varepsilon_t^S \tag{6}$$

where

$$\begin{pmatrix} \varepsilon_t^S \\ \varepsilon_t^v \end{pmatrix} \overset{\mathbb{Q}}{\sim} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right)$$

After simulating the sample paths of v_t by Milstein discretization, we can simulate sample paths of x_t by Euler discretization as (6) and obtain the stock price process S_t by taking exponential of x_t at each time point. Figure 2 shows 5000 sample paths of the asset price S_t from time 0 to time 1 with a fix step size $\Delta t = \frac{1}{1000}$ by Euler discretization.

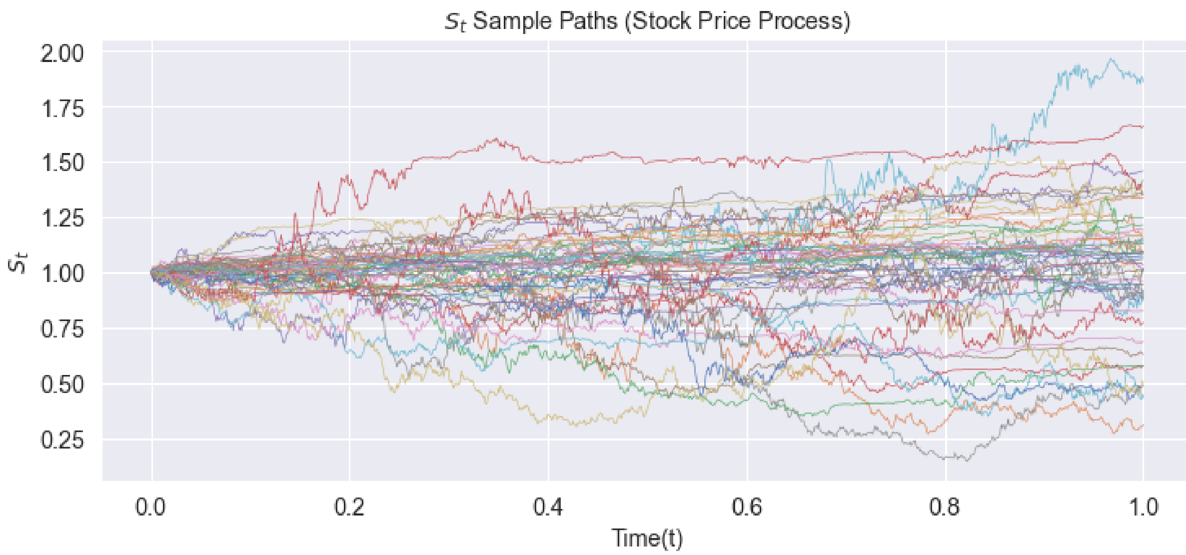


Figure 2: Simulated Asset Price via Euler Discretization

After generating 5000 possible scenarios of the asset price process, we compute the payoffs of the option at maturity for each scenario and take the average of them as the Monte Carlo estimate market option price. Since we assume zero interest rate, the average payoff is also the average price.

2.4 Mixing Method

In addition to the Euler discretization, we also use the mixing method to simulate the asset price S_t.

Recalling that (W_t^S, W_t^v) are correlated risk-neutral Brownian motions with correlation ρ, we first define $Z = \frac{W^S - \rho W^v}{\sqrt{1-\rho^2}}$ as a Brownian motion and $Z \perp W^v$. Thus, we could write W^S in terms of W^v and Z:

$$W^S = \rho W^v + \sqrt{1-\rho^2} Z \tag{7}$$

In the last section, we derived the SDE of $x_t = \log S_t$. Here, we replace W_t^S in SDE (5) with independent W_t^v and Z_t according to expression (7):

$$\begin{aligned} dx_t &= -\frac{1}{2}v_t dt + \sqrt{v_t}\ dW_t^S \\ &= -\frac{1}{2}v_t dt + \sqrt{v_t}\ (\rho dW_t^v + \sqrt{1-\rho^2} dZ_t) \\ &= (-\frac{1}{2}v_t dt + \sqrt{v_t}\sqrt{1-\rho^2}\ dZ_t) + \sqrt{v_t}\cdot \rho\ dW_t^v \end{aligned}$$

Integrating both sides from 0 to T, we obtain:

$$x_T - x_0 = -\frac{1}{2}\int_0^T v_u\ du + \sqrt{1-\rho^2}\int_0^T \sqrt{v_u}\ dZ_u + \rho \int_0^T \sqrt{v_u}\ dW_u^v$$

Notice that conditional on the sigma algebra $\mathcal{G}$ which contains all past information of the asset price variance v_t, $\int_0^T v_u\ du$ and $\int_0^T \sqrt{v_t}\ dW_u^v$ are deterministic and only $\int_0^T \sqrt{v_u}\ dZ_u$ is stochastic. Thus, conditional on $\mathcal{G}$, $x_T - x_0$ has the following distribution:

$$\begin{aligned} x_T - x_0|\mathcal{G} &\sim \mathcal{N}(-\frac{1}{2}\int_0^T v_u\ du + \rho\int_0^T \sqrt{v_u}\ dW_u^v\ ,\ (1-\rho^2)\int_0^T v_u\ du) \\ &\sim N(a, b^2) \end{aligned}$$

where $a = -\frac{1}{2}\int_0^T v_u\ du + \rho\int_0^T \sqrt{v_u}\ dW_u^v$ and $b^2 = (1-\rho^2)\int_0^T v_u\ du$.

Since $S_t = \exp(x_t)$, the distribution of S_T is the following:

$$\begin{aligned} S_T &\overset{d}{=} S_0 \cdot e^{a+bZ^*} \qquad \text{where } Z^* \overset{\mathbb{Q}_Z}{\sim} \mathcal{N}(0,1) \\ &\overset{d}{=} S_0 \cdot e^{a+\frac{1}{2}b^2} \cdot e^{-\frac{1}{2}b^2+bZ^*} \end{aligned}$$

Thus, we conclude that the asset price S_T follows a log-normal distribution:

$$S_T \overset{d}{=} \widetilde{S_0} \cdot e^{-\frac{1}{2}b^2+bZ^*} \tag{8}$$

where $\widetilde{S_0} = S_0 \cdot e^{a+\frac{1}{2}b^2}$ and $Z^* \overset{\mathbb{Q}_Z}{\sim} \mathcal{N}(0,1)$.

Since S_T follows a log-normal distribution, the European call/put option price with underlying asset S and strike price K is:

$$C = E^{\mathbb{Q}}[(S_T - K)_+] = \widetilde{S_0}\Phi(d_+) - K\Phi(d_-) \tag{9}$$

$$P = E^{\mathbb{Q}}[(K - S_T)_+] = -\widetilde{S_0}\Phi(-d_+) + K\Phi(-d_-) \tag{10}$$

where $d_\pm = \frac{\log(\widetilde{S_0}/K)\pm b^2}{b}$.

In addition, after we simulate the sample paths of the variance process v_t via Milstein discretization, we could approximate the a and b via Riemann integral and Ito's integral, such that:

$$\int_0^T v_u du \approx \sum_{k=1}^{1000T} v_{t_{k-1}} \cdot (t_k - t_{k-1}) = \sum_{k=1}^{1000T} \frac{v_{t_{k-1}}}{1000}$$
$$\int_0^T \sqrt{v_u} dW_u^v \approx \sum_{k=1}^{1000T} \sqrt{v_{t_{k-1}}} \cdot (W_k^v - W_{k-1}^v) = \sum_{k=1}^{1000T} \sqrt{v_{t_{k-1}}} \cdot \varepsilon_t^v$$

where $t_{k-1} = \frac{k-1}{1000}$ for $k \in \{1, 2, \dots, 1000T\}$ and ε_t^v should be the same set of random standard normal realisations that we used to simulate sample paths of v_t.

Therefore, under the mixing method, we can estimate the call/put option price as long as we know S_0 and the sample paths of v_t via Milstein discretization.

2.5 Implied Volatility

The assumption of constant volatility in the Black-Scholes model is one of the drawbacks resulting in the phenomenon called the volatility smile. The implied volatility is introduced to demonstrate the volatility smile, where we calculate the volatility of an option given the option price, stock price, strike price, etc. The implied volatility is defined by finding the inverse of the Black-Scholes formula. The Black–Scholes model assumes that the asset price $S = (S_t)_{t\geq 0}$ process satisfies the following SDE:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where μ is the drift, σ is the volatility and W_t is a $\mathbb{P}$-measure Brownian motion. Thus, S_t follows the following distribution:

$$S_t = S_j \cdot e^{(\mu - \frac{1}{2}\sigma^2)(t-j) + \sigma\sqrt{t-j}\ Z},\ Z \sim \mathcal{N}(0,1)$$

and the price of the European put/call option on the asset S with strike price K and maturity T satisfies the following equations:

$$V_{0,\text{BS}}^{\text{call}}(S_0, K, T, r, \sigma) = S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-)$$
$$V_{0,\text{BS}}^{\text{put}}(S_0, K, T, r, \sigma) = Ke^{-rT}\Phi(-d_-) - S_0\Phi(-d_+)$$

where $d_\pm = \frac{log(S_0/K) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ and Φ is the CDF of standard normal distribution.

Since we assume $r = 0$ in this project, the above equations are simplified as:

$$V_{0,\text{BS}}^{\text{call}}(S_0, K, T, \sigma) = S_0\Phi(d_+) - K\Phi(d_-)$$
$$V_{0,\text{BS}}^{\text{put}}(S_0, K, T, \sigma) = K\Phi(d_-) - S_0\Phi(d_+)$$

where $d_\pm = \frac{log(S_0/K) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$.

The implied volatility σ_{imp} is the volatility that makes the Black-Scholes price equal to the market price, such that

$$V_{0,\text{market}}^{\text{call}} = V_{0,\text{BS}}^{\text{call}}(S_0, K, T, \sigma_{\text{imp}})$$
$$V_{0,\text{market}}^{\text{put}} = V_{0,\text{BS}}^{\text{put}}(S_0, K, T, \sigma_{\text{imp}})$$

Solving the above equations will give us the implied volatility of the call/put option with respect to different strike prices and maturities.

2.5.1 Newton's Method

One excellent way to solve for the implied volatility without deriving the exact analytical solution is Newton's method. Newton's method is an iterative method for finding the root of a real-valued function. Suppose we want to find the x such that $f(x) = 0$ and we assume an initial guess x_0 for the root of f, Newton's method claims that x_1 is a better approximation than x_0 such that,

$$x_1 = x_0 + \frac{f(x_0)}{f'(x_0)}$$

After running the initial case, the process will repeat based on the formula below:

$$x_{i+1} = x_i + \frac{f(x_i)}{f'(x_i)}$$

The process stop until $f(x_{i+1})$ falls into an acceptable tolerance level and x_{i+1} is a precise approximation of the root of $f(x) = 0$.

In this report, we will apply Newton's method to find the implied volatility of options with respect to different strike prices K and maturity T.

2.5.2 Confidence Interval of Implied Volatility

To find the confidence interval of implied volatility for an option with underlying asset S, we first find the confidence interval of the options market price by the Monte Carlo estimation (with the Heston model and base parameters). The steps are described below.

1. Generate 5000 sample paths of the asset price variance v_t from time 0 to maturity time T via Milstein scheme with base parameters and a fix step size $\Delta t = \frac{1}{1000}$.
2. Simulate 5000 sample paths of the asset price S_t via Euler's scheme with base parameters and sample paths of v_t as section 2.3. (Skip this step for mixing method)
3. For each of the 5000 possible S_T, compute the payoffs F of the option at maturity T and take the average of them as the market price. The average payoff $\bar{F}$ equals the average/market option price

since we assume zero interest rate in this project. Alternatively, for the mixing method, compute the option price according to formula (9) and (10) given each sample path of v_t and take the average of them as the market price.

4. Produce the 95% confidence interval $(V_{\text{lower}}, V_{\text{higher}})$ of the average market price as $\bar{F} \pm 1.96 \times \frac{\text{sd}(F)}{\sqrt{5000}}$.

Then, the confidence interval of the implied volatility $(\sigma_{\text{lower}}, \sigma_{\text{higher}})$ is constructed by finding the roots of the following equations with Newton's method:

$$V_{\text{lower}} = V_{0,\text{BS}}(S_0, K, T, \sigma_{\text{lower}})$$
$$V_{\text{lower}} = V_{0,\text{BS}}(S_0, K, T, \sigma_{\text{higher}})$$

2.6 Control Variates

A control variate is a technique to reduce the variance of Monte Carlo simulation which allows analysts to estimate the expectation of one random variable when they know the analytical solution of one or several closely related random variables (i.e. so called control variates). Suppose X and $Y_1, \ldots, Y_m$ are random variables and $(X^{(n)}, Y_1^{(n)}, \ldots, Y_m^{(n)})_{n=1,\ldots,N}$ are the Monte Carlo simulations of X and $Y_1, \ldots, Y_m$. The control variates method claims that we are able to estimate $g = \mathbb{E}(X)$ if we know the analytical results for $h_i = \mathbb{E}(Y_i)$, $i = 1, \ldots, m$, such that

$$\hat{g} = \frac{1}{N} \sum_{n=1}^{N} X^{(n)} + \sum_{i=1}^{m} \gamma_i \left(h_i - \frac{1}{N} \sum_{n=1}^{N} Y_i^{(n)} \right) \tag{11}$$

where $(\gamma_i)_{i=1,\ldots,N}$ are arbitrary constants.

This report introduces four different candidate assets as control variates and we will derive analytical calculation results for each of them later in this section.

- The option price price using the deterministic variance path which replaces v_t with $\bar{v}_t = \mathbb{E}(v_t)$ in the Heston model
- A claim paying $\int_0^T v_s ds$
- A claim paying $\int_0^T v_s^2 ds$
- The stock

2.6.1 Optimal Choice of γ_i

The usual optimal choice of γ_i is 0. However, the optimal choice is no longer zero when considering correlations between X and Y_i. The optimal choice of γ_i could be determined by minimizing the variance of the random variable $H = X + \sum_{i=1}^{m} \gamma_i (h_i - Y_i)$.

In this report, we estimate the implied volatility using each of the four control variates separately, and combined them when implement the all control variates method. In this section, we will derive the optimal choice of γ_i for both cases. Based on the underlying principle of control variate, this process is similar to **solving a certain least-squares system**, which is also called regression sampling. The method using a single control variate can be analogous to fit a one-factor regression model, while the method using multiple (four) control variates can be analogous to fit a multi-factor regression model. **The greater the absolute value of the correlation between the payoff of target option (X - generating based on Heston Model) and the payoff of control variate (Y), the greater the variance reduction achieved**.

2.6.1.1 Use One Control Variate

If we only use one control variate Y, the variance of H can be simplified as the following:

$$\begin{aligned} Var[H] &= Var[X + \gamma(h - Y)] \\ &= Var[X - \gamma Y] \\ &= Var[X] + \gamma^2 Var[Y] - 2\gamma \cdot Cov(X, Y) \end{aligned}$$

To find the γ that minimizes the $Var[H]$, we derive the derivative of $Var[H]$ with respect to γ and set it to zero:

$$\begin{aligned} \frac{\partial Var[H]}{\partial \gamma} &= 2\gamma Var[Y] - 2Cov[X, Y] = 0 \\ \gamma &= \frac{Cov[X, Y]}{Var[Y]} \end{aligned}$$

Thus, the optimal choice of γ is $\frac{Cov[X,Y]}{Var[Y]}$ if we only use one control variate.

2.6.1.2 Use Four Control Variates

If we combine four control variates Y_1, Y_2, Y_3, Y_4, the variance of H can be simplified as the following:

$$\begin{aligned} Var[H] &= Var[X + \sum_{i=1}^{4} \gamma_i (h_i - Y_i)] \\ &= Var[X - \sum_{i=1}^{4} \gamma_i Y_i] \\ &= Var[X] + \sum_{i=1}^{4} \gamma_i^2 Var[Y_i] - 2\sum_{i=1}^{4} \gamma_i \cdot Cov(X, Y_i) + \sum_{i=1}^{4}\sum_{j=1}^{4} \gamma_i \gamma_j Cov(Y_i, Y_j) \end{aligned}$$

To find the $(\gamma_i)_{i\in\{1,\ldots,4\}}$ that minimize the $Var[H]$, we derive the derivative of $Var[H]$ with respect to each $gamma_i$ and set them to zero:

$$\begin{cases} \frac{\partial Var[H]}{\partial \gamma_1} &= 2\gamma_1 Var[Y_1] - 2Cov[X,Y_1] + 2\gamma_2 Cov(Y_2,Y_1) + 2\gamma_3 Cov(Y_3,Y_1) + 2\gamma_4 Cov(Y_4,Y_1) = 0 \\ \frac{\partial Var[H]}{\partial \gamma_2} &= 2\gamma_2 Var[Y_2] - 2Cov[X,Y_2] + 2\gamma_1 Cov(Y_1,Y_2) + 2\gamma_3 Cov(Y_3,Y_2) + 2\gamma_4 Cov(Y_4,Y_2) = 0 \\ \frac{\partial Var[H]}{\partial \gamma_3} &= 2\gamma_3 Var[Y_3] - 2Cov[X,Y_3] + 2\gamma_1 Cov(Y_1,Y_3) + 2\gamma_2 Cov(Y_2,Y_3) + 2\gamma_4 Cov(Y_4,Y_3) = 0 \\ \frac{\partial Var[H]}{\partial \gamma_4} &= 2\gamma_4 Var[Y_4] - 2Cov[X,Y_4] + 2\gamma_1 Cov(Y_1,Y_4) + 2\gamma_2 Cov(Y_2,Y_4) + 2\gamma_3 Cov(Y_3,Y_4) = 0 \end{cases}$$

Thus, the optimal choice of $(\gamma_i)_{i\in\{1,\ldots,4\}}$ is obtained by solving the above system of equations. In this report, we utilize the function `numpy.linalg.solve` in Python to solve for $(\gamma_i)_{i\in\{1,\ldots,4\}}$.

2.6.2 Deterministic Variance Path

Recalling that the instantaneous variance of the asset price is itself a stochastic process, we can transform it into a deterministic volatility model by replacing v_t with $\bar{v}_t = \mathbb{E}[v_t]$ in the Heston model. One control variate idea is using the option price based on stock price generated by deterministic $\bar{v}_t$ in the Heston model and test if the payoff of this option is similar to the payoff of the target option (X).

2.6.2.1 Analytical Formula for Deterministic Variance Path

First, we will derive $\bar{v}_t$. Recall that the SDE of v_t in the Heston model is:

$$dv_t = \kappa(\theta - v_t)dt + \eta\sqrt{v_t}dW_t^v$$

Integrate both sides from 0 to t:

$$v_t - v_0 = \int_0^t \kappa(\theta - v_u)du + \eta\int_0^t \sqrt{v_u}dW_u^v$$

Take expectations on both sides:

$$\bar{v}_t - v_0 = \int_0^t \kappa(\theta - \bar{v}_u)du$$

The Brownian motion terms disappear since the expectation of an Ito integral is 0.

The above equation in differential form is an ordinary differential equation (ODE) such that:

$$\begin{cases} d\bar{v}_t &= \kappa(\theta - \bar{v}_t)dt \\ \bar{v}_0 &= v_0 \end{cases}$$

The solution to the above ODE is:

$$\bar{v}_t = v_0 \cdot e^{-\kappa t} + \theta \cdot (1 - e^{-\kappa t}) \tag{12}$$

Now the SDE of S_t is:

$$dS_t = S_t\sqrt{\bar{v}_t}dW_t^S$$

Let $x_t = \log S_t$. The Ito's lemma gives us the SDE of x_t:

$$dx_t = -\frac{1}{2}\bar{v}_t dt + \sqrt{\bar{v}_t}dw_t^S$$

Integrate on both sides from 0 to T:

$$x_T - x_0 = -\frac{1}{2}\int_0^T \bar{v}_u du + \int_0^T \sqrt{\bar{v}_u}dw_u^S$$

Thus $x_T - x_0$ is normally distributed as:

$$x_T - x_0 \overset{d}{=} \mathcal{N}(-\frac{1}{2}\int_0^T \bar{v}_u du, \int_0^T \bar{v}_u du) = \mathcal{N}(-\frac{1}{2}c, c)$$

where $c = \int_0^T \bar{v}_u du$. Solve the above integral, we obtain that:

$$c = \int_0^T \bar{v}_u du = \frac{(v_0 - \theta)(1 - e^{-\kappa T})}{\kappa} + \theta T$$

Since $x_T = \log S_T$, S_T is log-normally distributed:

$$S_T \overset{d}{=} S_0 \cdot e^{-\frac{1}{2}c+\sqrt{c}Z} \quad \text{where } Z \overset{\mathbb{Q}_S}{\sim} \mathcal{N}(0,1)$$

Thus the Black–Scholes formula gives us the price of call/put option with underlying asset S, strike price K and maturity T like the following:

$$\begin{aligned} V_0^{\text{call}} &= S_0\Phi(d_+) - K\Phi(d_-) \\ V_0^{\text{put}} &= -S_0\Phi(-d_+) + K\Phi(-d_-) \end{aligned}$$

where $d_\pm = \frac{\log(S_0/K)\pm\frac{1}{2}c}{\sqrt{c}}$.

2.6.2.2 Paths of Heston Model with $\bar{v}_t$

To investigate the performance of option price as a control variate, we simulate paths of the Heston model and the deterministic volatility model simultaneously, where the same set of random numbers and Milstein discretization are applied. **Stock price path generated using Heston model and deterministic variance only roughly matched when t is small**. Based on Figure 3 below, as the time increase, the simulated stock path for T = 1 is higher for the deterministic volatility model and there is a decreasing trend of the stock price. It is because the current volatility is lower than the long-term volatility in the Heston model, meaning that the volatility will increase to approach the long-term mean reversion level (40%) over time. Therefore, $\bar{v}_t = \mathbb{E}[v_t]$ is greater than v_0. Moreover, the change of stock price is a function of the current volatility and stock price and the correlation of two risk-neutral Brownian motions is negative ($\rho = -0.5$). An increase in volatility leads to a decrease in stock price. During the early stage (when v_t never exceed $\bar{v}_t$, or $t < 0.1$), although the change of the stock price is faster in the deterministic volatility model than the Heston model, as the change is small there is no significant gap between these two models. However, for the later stage ($t > 0.1$), the stock price and volatility become larger, so the decrease of the stock price in the Heston model is faster than the deterministic volatility model.

Additionally, the deterministic variance model, in this case, is comparable to the Black-Shocks model, with a similar assumption of constant volatility. We conclude that as the time increase, the constant variance model (e.g. Black-Shocks model) may overestimate the stock price and underestimate the market volatility since the stock price generated by a deterministic variance model is higher than that generated by the Heston model in the figure below. Using a constant volatility model is not conducive to effective risk management and long-term business resilience of financial institutions.

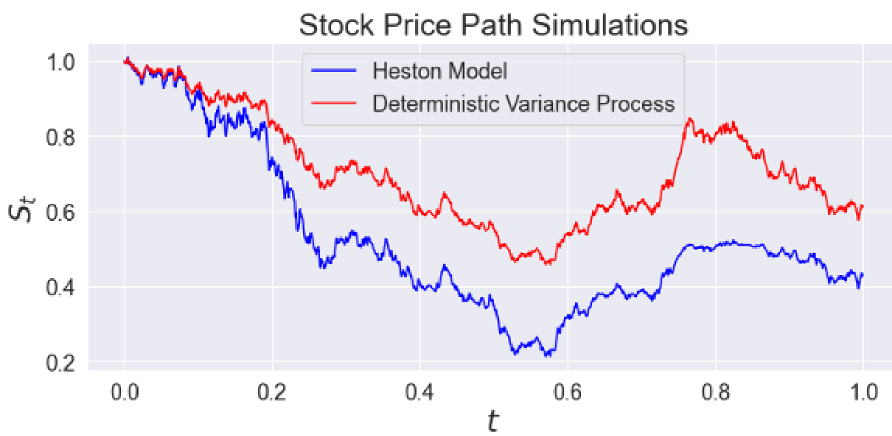


Figure 3: Simulated Asset Price under Heston and Deterministic Variance

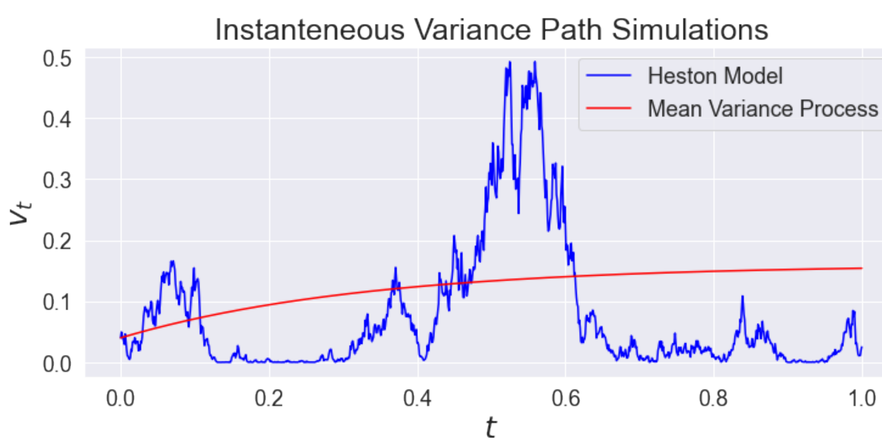


Figure 4: Simulated Volatility under Heston and Deterministic Variance

2.6.3 Value of a Claim paying $\int_0^T v_s ds$ and $\int_0^T v_s^2 ds$

The rationale behind using price of claim paying $\int_0^T v_s ds$ and $\int_0^T v_s^2 ds$ as control variate is the analytical formula introduced by the mixing method, where $a = -\frac{1}{2}\int_0^T v_u\ du + \rho \int_0^T \sqrt{v_u}\ dW_u^v$ and $b^2 = (1-\rho^2)\int_0^T v_u\ du$. We want to test how much the integration part of volatility in the mixing method can affect the target option payoff.

2.6.3.1 Analytical Formula for Claim paying $\int_0^T v_s ds$

Since we assume zero interest rate, the price of a claim equals to the expected value of its payoff, such that:

$$\begin{aligned} V &= \mathbb{E}[\int_0^T v_s ds] = \int_0^T \mathbb{E}[v_s]ds = \int_0^T \bar{v}_s ds \\ &= \frac{(v_0-\theta)(1-e^{-\kappa T})}{\kappa} + \theta T \end{aligned} \tag{13}$$

2.6.3.2 Analytical Formula for Claim paying $\int_0^T v_s^2 ds$

Similar to last section, the price of the claim equals to the expected value of its payoff:

$$V = \mathbb{E}[\int_0^T v_s^2 ds] = \int_0^T \mathbb{E}[v_s^2]ds \tag{14}$$

Let $g(v_t) = v_t^2$. Then by Ito's lemma:

$$\begin{aligned} dv_t^2 = dg &= \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial v}dv_t + \frac{1}{2}\frac{\partial^2 g}{\partial v^2}d[v,v]t \\ &= [(2\kappa\theta + \eta^2)v_t - 2\kappa v_t^2]dt + 2\eta v_t\sqrt{v_t}dW_t^v \end{aligned}$$

Integrate on both sides from 0 to t:

$$v_t^2 - v_0^2 = \int_0^t [(2\kappa\theta+\eta^2)v_s - 2\kappa v_s^2]ds + \int_0^t 2\eta v_s\sqrt{v_s}dW_s^v$$

Take expectation on both sides (the Brownian motion term disappears since the expectation of an Ito's integral is 0):

$$\mathbb{E}[v_t^2] = v_0^2 + \int_0^t [(2\kappa\theta+\eta^2)\bar{v}_s - 2\kappa\mathbb{E}[v_s^2]]ds$$

Let $\mathbb{E}[v_t^2] = m_t$ and write the above equation in differential form:

$$\begin{cases} m'(t) = (2\kappa\theta + \eta^2)\bar{v}_s - 2\kappa \cdot m(t) \\ m(0) = v_0^2 = 1.6 \times 10^{-3} \end{cases} \tag{15}$$

Solve this differential equation and plug in $\bar{v}_t = v_0 \cdot e^{-\kappa t} + \theta \cdot (1 - e^{-\kappa t})$ and the base model parameters:

$$m'(t) = 0.5136 - 0.3852 \cdot e^{-3t} - 6m(t)$$
$$m(t) = c_1 e^{-6t} - 0.1284 e^{-3t} + 0.0865$$

Substitute in the initial condition $m(0) = 1.6 \times 10^{-3}$ and we get $c_1 = 0.0444$. Thus,

$$\mathbb{E}[v_t^2] = 0.0444e^{-6t} - 0.1284e^{-3t} + 0.0856$$

Substitute $\mathbb{E}[v_t^2]$ into expression (14) and the value of the claim paying $\int_0^T v_s^2 ds$ is:

$$\begin{aligned} V &= \int_0^T \mathbb{E}[v_s^2] ds \\ &= \int_0^T 0.0444e^{-6t} - 0.1284e^{-3t} + 0.0856 ds \\ &= -0.0074(e^{-6T} - 1) + 0.0428(e^{-3T} - 1) + 0.0856T \end{aligned} \tag{16}$$

2.6.3.3 Comparison of Milstein simulation and Analytic Calculation

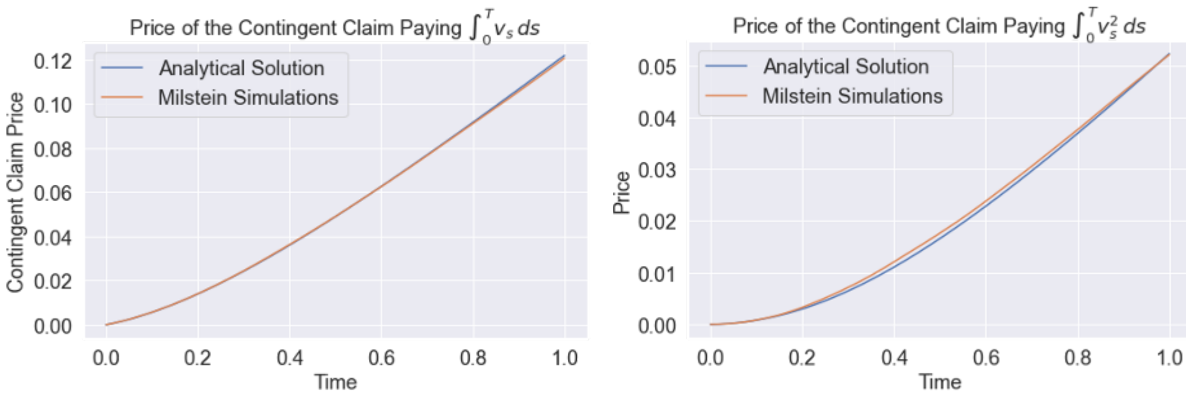
In this section, we compute the price of the contingent claims, paying $\int_0^T v_s ds$ and $\int_0^T v_s^2 ds$ via two methods: the analytical formula derived in the previous sections and Monte Carlo estimate via Milstein discretization. Since we assume zero interest rate in this project, the value of the claims and the value of the payoffs at maturity are equal.

For the Milstein discretization methods, we simulate 5000 sample paths of the instantaneous asset price variance v_t from time 0 to time 1 with a fix step size of $\Delta t = \frac{1}{1000}$ and compute the value/payoff of different maturity time $T \in 0.001, 0.002, \ldots, 1$ given each sample path of v_t via Riemann integral such that:

$$\int_0^T v_u du \approx \sum_{k=1}^{1000T} v_{t_{k-1}} \cdot (t_k - t_{k-1}) = \sum_{k=1}^{1000T} \frac{v_{t_{k-1}}}{1000}$$
$$\int_0^T v_u^2 du \approx \sum_{k=1}^{1000T} v_{t_{k-1}}^2 \cdot (t_k - t_{k-1}) = \sum_{k=1}^{1000T} \frac{v_{t_{k-1}}^2}{1000}$$

where $t_{k-1} = \frac{k-1}{1000}$ for $k \in \{1, 2, \ldots, 1000T\}$.

Finally, we take the average of each simulated claim price given different maturity T as the value of the contingent claim and plot the results in Figure 5. We notice that the analytical curve almost coincides with the Monte Carlo simulation curve for both contingent claims with some minor and negligible discrepancies as T increases. Therefore, we conclude that our analytical solutions of the claim values are correct. The rising discrepancies with big T might come from increasing errors of using Riemann integral to estimate claim payoffs as T increases. As the simulation error is in an acceptable tolerance, we can apply the price of these contingent claims as control variate in section 4.



(a) Claims paying $\int_0^T v_s ds$

(b) Claims paying $\int_0^T v_s^2 ds$

Figure 5: Monte Carlo Estimate Claim Price v.s. Analytical Claim Price

2.6.4 Expected Stock price

Since stock is a tradable asset so its relative price with respect to any numeraire asset is a martingale and the expected value of S_T given all past information is the current stock price S_0 times the current numeraire price $P_0(T)$ for any $T > 0$. Since we assume zero interest rate in this report, the expected value of S_T equals the current stock price S_0. We can use stock as one of the candidate control variate and test if the stock price is highly correlated with the target option payoff (X).

2.6.5 Estimate Implied Volatility via Control Variate Method

After deriving the analytical solution of the expected value of these four control variates, we will illustrate the estimating process of implied volatility with control variates in this section. Suppose we use control variate $Y_1, \ldots, Y_m$ (representing the price of the control variates), we estimate the implied volatility of call/put option in the following steps:

1. Compute the analytical expectation $h_1, \ldots, h_m$ of the price of control variates $Y_1, \ldots, Y_m$ based on the analytical formulae in section 2.6.2 - 2.6.4 given the base model parameters and choices of T and K.

2. Generate 5000 sample paths of the variance of asset price v_t via Milstein scheme and the asset price S_t via Euler's scheme from time 0 to maturity time T with base parameters and a fix step size

$\Delta t = \frac{1}{1000}$.

3. For each sample path of S_t, compute the option payoffs F at maturity given selected strike price K. Take the average of them as the average simulated option price given T and K. Specifically, $F = (K - S_T)_+$ for $K \leq S_0$ and $F = (S_T - K)_+$ for $K \geq S_0$. The average simulated payoff $\bar{F}$ equals to the average simulated option price (i.e. $\frac{1}{N}\sum_{n=1}^{N} X^{(n)}$) since we assume zero interest rate in this project.

4. Given each sample path of v_t and S_t, compute the optimal $(\gamma_i)_{i\in\{1,\ldots,m\}}$ according to section 2.6.1 and the simulated price of each control variate as $Y_m^{(1)}, \ldots, Y_m^{(n)}$ and take the average of them as the average simulated price of each control variate (i.e. $\frac{1}{N}\sum_{n=1}^{N} Y_i^{(n)}$).

5. Plug all h_i, γ_i, average simulated option price and average simulated price of control variates in expression (11) and return the estimate market price ($\hat{g}$) and its 95% confidence interval of the call/put option.

6. Produce the implied volatility and its 95% confidence interval via Newton's method.

3 Implied Volatility via Regular Monte Carlo Simulation

In this section, we simulate 5000 sample paths for variance v_t and stock price S_t via two approaches: the Euler and Milstein Discretization and the mixing method and plot the Monte Carlo estimate implied volatility of options of various maturities against strike price.

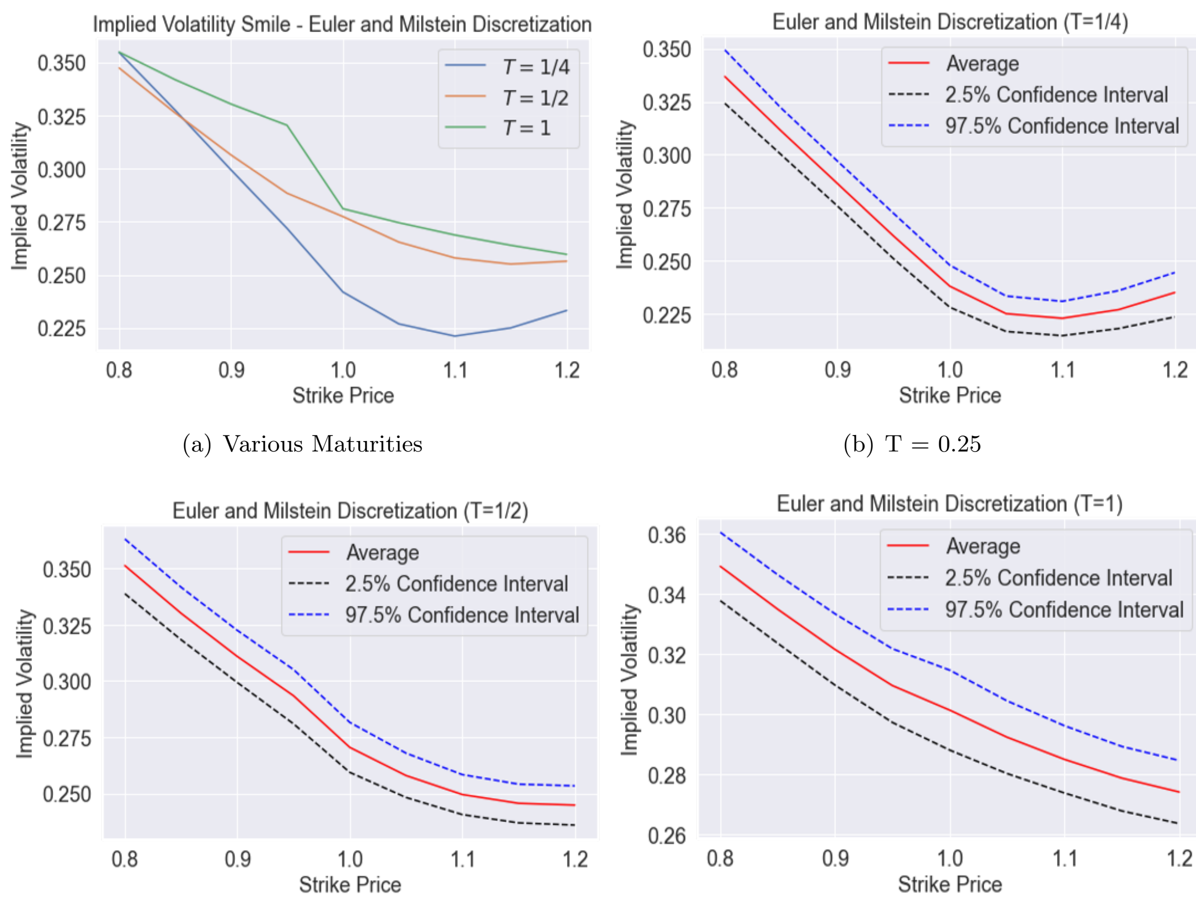
3.1 Monte Carlo Estimates via Euler and Milstein Discretization

In this section, we estimate the implied volatility of call/put options with maturity $T = 0.25, 0.5, 1$ and strike price $K = 0.8, 0.85, \ldots, 1.2$ through Monte Carlo simulation via Milstein discretization of v_t and Euler discretization of S_t. The steps are described as the following.

1. For each $T \in \{0.25, 0.5, 1\}$, generate 5000 sample paths of the variance of asset price v_t from time 0 to maturity time T via Milstein scheme with base parameters and a fix step size $\Delta t = \frac{1}{1000}$.
2. For each T, generate 5000 sample paths of the asset price S_t via Euler's scheme with base parameters and sample paths of v_t as section 2.3.
3. For each T, compute the option payoffs F at maturity given various strike price $K = 0.8, 0.85, \ldots, 1.2$. Take the average of them as the option market price given each T and K. Specifically, $F = (K - S_T)_+$ for $K \leq S_0$ and $F = (S_T - K)_+$ for $K \geq S_0$. The average payoff $\bar{F}$ equals to the estimate option price since we assume zero interest rate in this project.
4. Produce the implied volatility and its 95% confidence interval given each T and K via Newton's method.

Based on Figure 6 and Table 1, we notice that the implied volatility of long-term options is higher than those of short-term options at any strike price because the uncertainty of stock price decreases as the time approaches the time of expiration. Therefore, the volatility of long-term options is higher since they are exposed to the risk of price changes of the underlying asset for a longer duration. Moreover, the confidence bands are wider for options with longer maturity. One possible reason is that the accuracy of Monte Carlo estimates is better for the short-term options compared to the long-term options as a longer period causes higher uncertainty in terms of underlying asset price changes and thus worse estimates with the same method.

In addition, we notice that the implied volatility (IV) curves with respect to strike show a similar trend which looks like a smiling mouth, also known as a volatility smile. For example, the IV at $T = \frac{1}{4}$ decreases from about 0.35 to 0.225 as the strike price increases from 0.8 to 1.1 and then increases to about 0.25 as the strike price increases to 1.2. This phenomenon suggests a higher demand for the option as it is in-the-money (ITM) and out-of-the-money (OTM) compared to at-the-money (ATM). It also suggests that the demand further increases as the option becomes more and more OTM and ITM. Furthermore, we notice that the IV curves of $T = 0.5$ and $T = 1$ are less similar to a smiling mouth. One possible reason is that an option with a longer period may require a higher strike price to be ATM and smaller implied volatility to push it to the ITM position. Additionally, when maturity increases to T =1, there is a significant jump within the nearly ATM interval where strike price between 0.95 and 1. It seems the Euler Milstein Method is not stable and accurate enough for longer maturity.



(a) Various Maturities

(b) T = 0.25

(c) T = 0.5

(d) T=1

Figure 6: Implied Volatility Estimates and 95% Confidence Intervals via Euler and Milstein Discretization

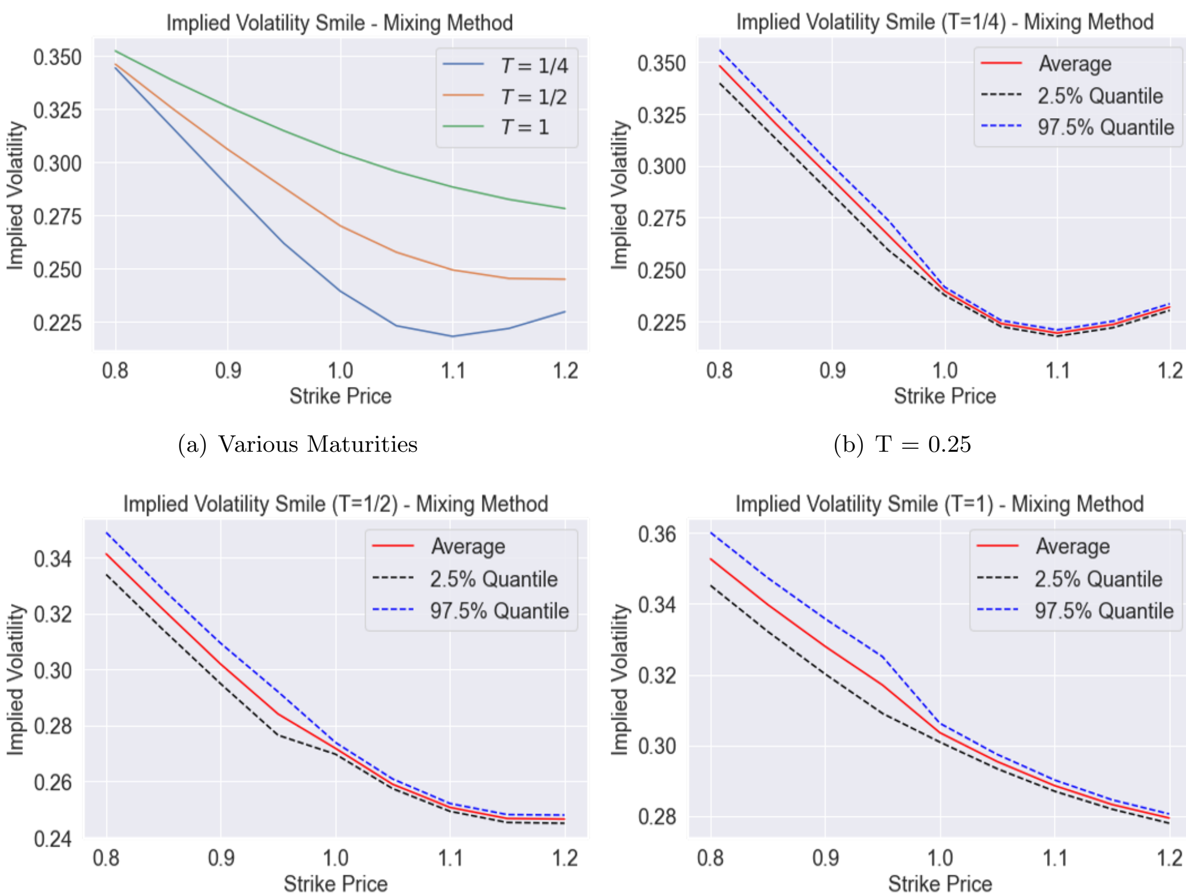
Strikes	**0.8**	**0.85**	**0.9**	**0.95**	**1**	**1.05**	**1.1**	**1.15**	**1.2**
$\mathbf{T = 1/4}$	0.0260	0.0232	0.0219	0.0219	0.0193	0.0160	0.0151	0.0157	0.0180
$\mathbf{T = 1/2}$	0.0250	0.0237	0.0234	0.0245	0.0227	0.0201	0.0183	0.0178	0.0181
$\mathbf{T = 1}$	0.0233	0.0235	0.0243	0.0251	0.0260	0.0234	0.0215	0.0204	0.0197

Table 1: Confidence Width: Euler and Milstein Discretization

3.2 Monte Carlo Estimates via Mixing Method

We also utilize the mixing method to draw the IV curves. The steps are described below.

1. For each $T \in \{0.25, 0.5, 1\}$, generate 5000 sample paths of the variance of asset price v_t from time 0 to maturity time T via Milstein scheme with base parameters and a fix step size $\Delta t = \frac{1}{1000}$.
2. compute the option price according to formula (9) and (10) given each sample paths of v_t and take the average of them as the market price given each T and K.
3. Produce the implied volatility and its 95% confidence interval given each T and K via Newton's method.



(a) Various Maturities

(b) T = 0.25

(c) T = 0.5

(d) T=1

Figure 7: Implied Volatility Estimates and 95% Confidence Intervals via Mixing Method

Following the discussed process, we plot the results in Figure 7 and Table 2. One can notice that the results of the implied volatility estimate (in the plot (a)) are similar to the Monte Carlo estimates via Euler and Milstein Discretization over various maturities. Compared to the Euler Milstein method, the mixing method performs better when maturity is large (T =1). The width of confidence bands of the mixing method is constantly narrower than that of the Euler and Milstein discretization. Especially for strikes greater than 1, the minimum width can be 0.0026 for T = 1 and strike = 1.2, which is about 1/10 of the width under the Euler Milstein method. The width of confidence bands depends on two factors: the sample size and the sampling standard error. Since we use the same sample size for both approaches, a narrower confidence interval implies smaller standard errors and more precise estimates.

Strikes	**0.8**	**0.85**	**0.9**	**0.95**	**1**	**1.05**	**1.1**	**1.15**	**1.2**
T = 1/4	0.0159	0.0141	0.0140	0.0142	0.0038	0.0029	0.0030	0.0031	0.0031
T = 1/2	0.0158	0.0152	0.0151	0.0161	0.0043	0.0032	0.0028	0.0028	0.0029
T = 1	0.0153	0.0154	0.0159	0.0163	0.0053	0.0041	0.0032	0.0026	0.0026

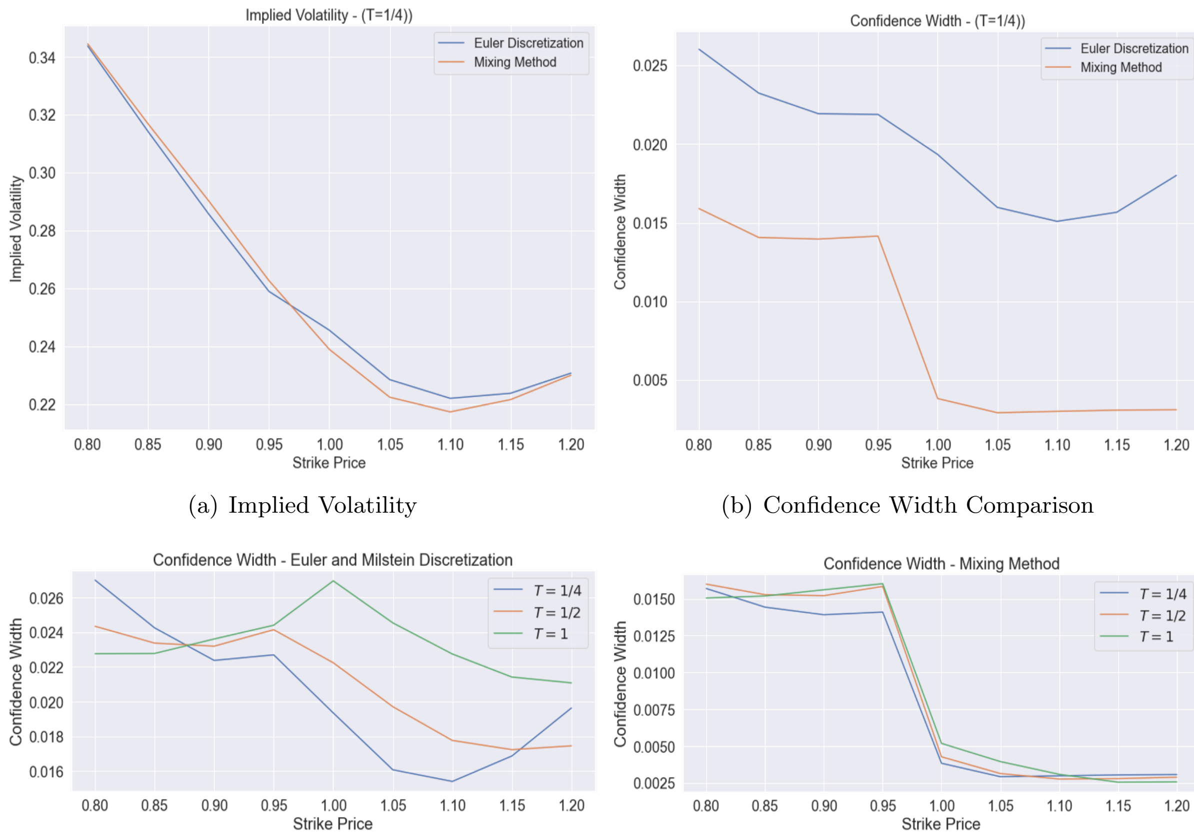
Table 2: Confidence Width: Mixing Method

The smile curve demonstrates the implied volatility of the OTM (out-of-the-money) put for the strike price less than 1, and that of the OTM call for the strike price greater than 1. We notice that the call options have narrower confidence intervals compared to the put options. One possible reason is that the deep OTM put option (LHS) has a higher volatility than the deep OTM call option (RHS) based on the Black-Scholes option pricing formula. We observe that the IV is in a decreasing trend for T =1. When the strike price is at a very low level, most likely the stock price is relatively high. As a result, We may expect a higher probability of sudden stock price changes in the future. On the other hand, when the stock price is low, the strike price is relatively high. The call option might be out-of-the-money due to the high strike price. As the behavior of stock depends on the current stock price and volatility, so we expect a lower probability of unexpected large stock price changes.

3.3 Performance Comparison: Euler Milstein Discretization and Mixing Method

Based on figure 8, we observe that the Mixing Method outperforms the Euler Milstein Discretization Method. For strike price less than 1 (OTM put option), the estimated implied volatility under the mixing method is higher than Euler Milstein Discretization. However, the estimated implied volatility using the mixing method is lower than Euler Milstein Discretization. The confidence width of the Mixing Method is significantly smaller than the Euler Milstein Discretization. This conclusion last for all length of maturities and we select a specific case when T =0.25 as an example showing in Figure 8(a) and 8(b).

Based on Figure 8(c) and 8(d), we find the performance of the mixing method is more robust compared to Euler Milstein method when the maturity change for away-from-the-money options. Euler Milstein method performs well for smaller maturity, but become less reliable as maturity increase or the case for deep OTM put option. This method is less stable because of the simulation process of the stock price. In this case, we simulate the sample paths of the variance process v_t via Milstein discretization, then we simulate log return of stock by Euler discretization before taking exponential to obtain the stock price.



(a) Implied Volatility

(b) Confidence Width Comparison

(c) Various Maturities - Euler Milstein

(d) Various Maturities - Mixing

Figure 8: Performance Comparison: Euler Milstein Discretization and Mixing Method

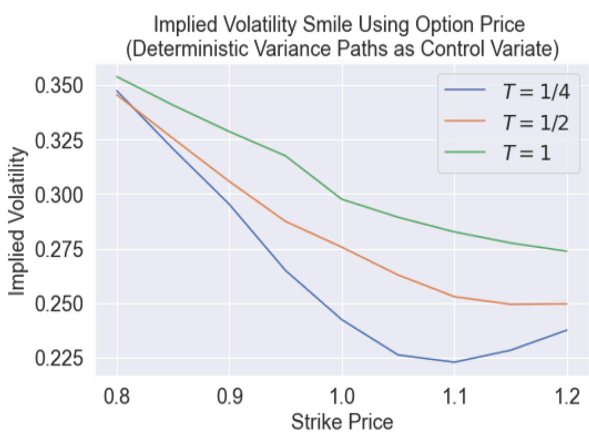
Simulation twice increases the volatility of the estimation and the Euler discretization method is hard to catch the non-linear price movement. Therefore, the overall performance of this method is poor.

However, the mixing method combined both **Monte Carol simulation** and **analytical calculation** of the stock price. **Reducing the dependence of the overall process on simulation by introducing analytical calculation can significantly improve the accuracy and stability of IV estimation**. The mixing method first simulates the variance process v_t via Milstein discretization, then uses Riemann integral and Ito's integral to solve the result of the analytical formula, so end up with a more precise result. The standard deviation of payoff generated using the mixing method is calculated by the Black-Scholes option pricing formula, changing the real world current stock price to $\widetilde{S_0} = S_0 \cdot e^{a+\frac{1}{2}b^2}$ and the risk-neutral variance to $b^2 = (1-\rho^2)\int_0^T v_u \, du$. Black-Scholes Implied volatility for a put option is obviously greater than the call option, which explained the jump point of the confidence width curve when the strike is close to 1. We observe that for strike price less than 1, the curve shows the OTM put option, with higher Black-Scholes Implied volatility, leading to higher corresponding standard deviation and higher confidence width. Similarly, for strike prices greater than 1, the curve shows the OTM call option, with lower Black-Scholes Implied volatility, leading to a smaller simulation error.

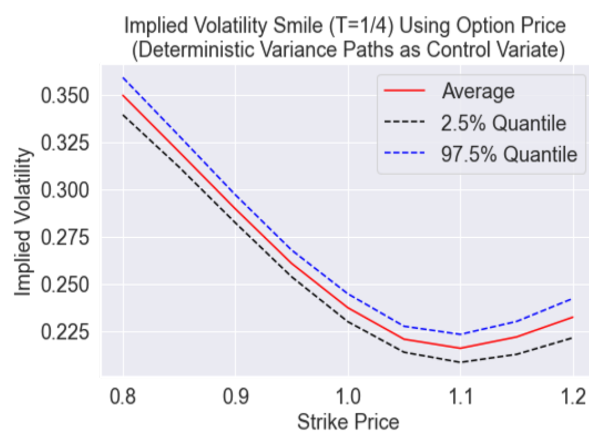
4 Implied Volatility via Monte Carlo Simulation with Control Variates

Monte Carlo integration typically has an error variance σ^2/n, which can be reduced by increasing simulation time n. To save computing time, we can instead find an optimum control variate with a known analytical payoff. So that, we construct a new Monte Carlo problem with the same answer with a narrower confidence width. As discussed before, the single control variate method can be analogous to fit a **one-factor regression** model. As this is an alternative way to **introduce analytical calculation** in the simulation process, it has **potential to reduce estimation error** but depends on the **correlation between selected control variate and the target option (X).** This section will analyze the individual performance and conduct comparison among four different control variates.

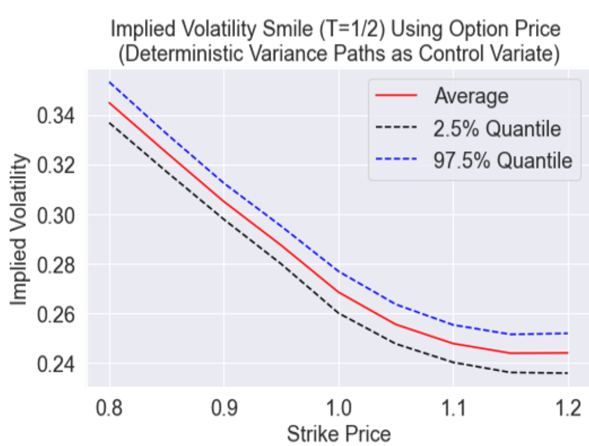
4.1 Control Variates - Option Price



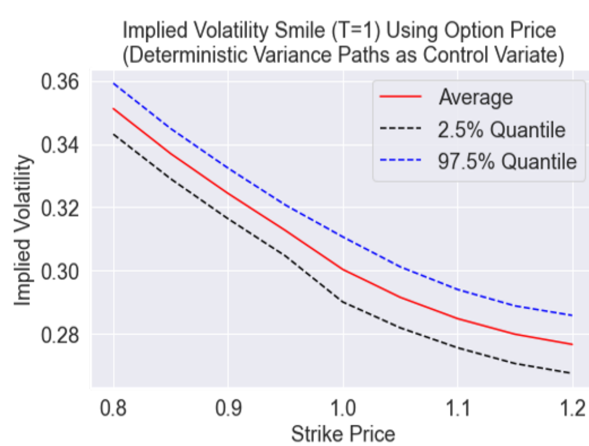
(a) Various Maturities



(b) T = 0.25



(c) T = 0.5



(d) T=1

Figure 9: Implied Volatility Estimates and 95% Confidence Intervals using Option as Control Variates

Based on Figure 9 and Table 3, the smaller the maturity the more obvious the smile shape. We find that the different IV curves in Figure 9(a) are similar to that of the mixing method and Euler Method, but it performs closer to the mixing method. The IV curves are relatively smooth and the confidence bands are narrow, which suggests using option price as a control variate can generate a precise result of the IV estimation, with the width in the range between 0.01 and 0.02. In other words, the estimated γ corresponding to this control variate is relatively close to the optimum value. Moreover, for various maturity (T = 0.25, 0.5, 1), the width of confidence bands is consistent and move smoothly over the x-axis (K), which is different from that of the mixing method. For a fixed maturity, the IV estimation seems more accurate for the ATM option than the away-from-the-money option. Overall, the width of the confidence interval increase as maturity increase.

Strikes	**0.8**	**0.85**	**0.9**	**0.95**	**1**	**1.05**	**1.1**	**1.15**	**1.2**
$\mathbf{T = 1/4}$	0.0204	0.0169	0.0144	0.0140	0.0137	0.0126	0.0134	0.0157	0.0182
$\mathbf{T = 1/2}$	0.0170	0.0156	0.0149	0.0154	0.0176	0.0164	0.0162	0.0171	0.0191
$\mathbf{T = 1}$	0.0158	0.0156	0.0157	0.0159	0.0197	0.0183	0.0174	0.0173	0.0171

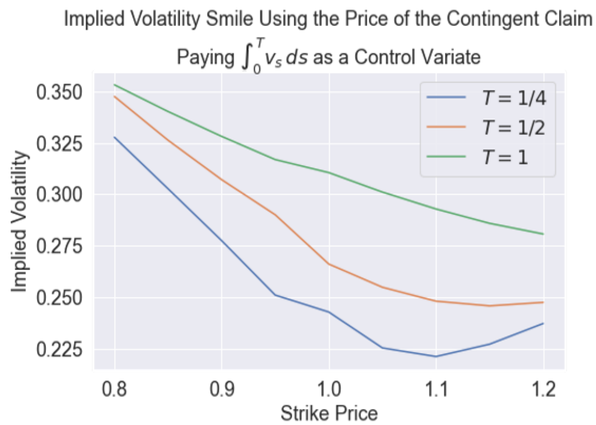
Table 3: Confidence Width: Option Price

4.2 Control Variate - Price of Claim paying $\int_0^T v_s ds$

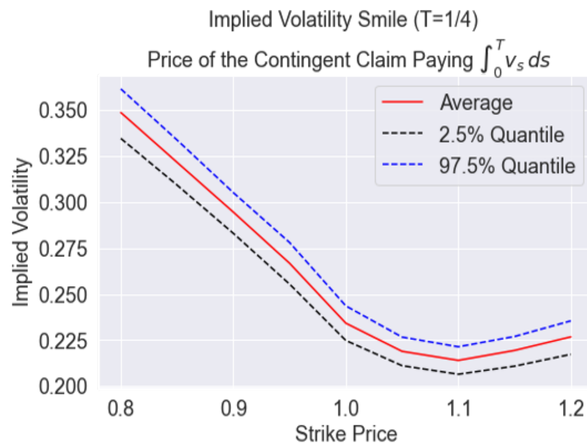
In this section, we estimate the implied volatilities of the put/call option with a control variate of the price of a contingent claim paying $\int_0^T v_s ds$. For each $T \in \{1, 0.5, 1\}$ and $K \in \{0.8, 0.85, \ldots, 1.2\}$, we estimate the implied volatility and its 95% confidence interval via the process described in section 2.6.5 and plot the results in Figure 10 and Table 4. We notice that the estimated IV curves in Figure 10(a) are similar to that of the mixing method in Figure 7(a) except that the IV curves of the mixing method are more smooth, and the confidence bands of the mixing method are narrower. It suggests mixing method is more precise. Moreover, the width of confidence width range from 0.0149 to 0.0263 based on the table, which is higher than the mixing method. According to the plot, the upper and lower band change smoothly over various K for the control variate method, but the mixing method shows smaller confidence bands for OTM call options. Similar to the mixing method, the confidence bands continue to grow wider as T increases which suggests that the method provides more precise estimates for short-period options than the long-period option.

Strikes	**0.8**	**0.85**	**0.9**	**0.95**	**1**	**1.05**	**1.1**	**1.15**	**1.2**
$\mathbf{T = 1/4}$	0.0260	0.0238	0.0221	0.0226	0.0186	0.0153	0.0149	0.0167	0.0195
$\mathbf{T = 1/2}$	0.0243	0.0231	0.0229	0.0242	0.0226	0.0203	0.0185	0.0182	0.0188
$\mathbf{T = 1}$	0.0228	0.0227	0.0237	0.0245	0.0263	0.0238	0.0219	0.0208	0.0202

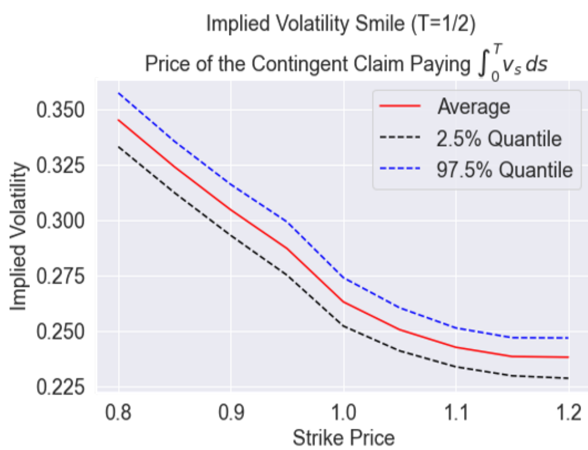
Table 4: Confidence Width: claim paying $\int_0^T v_s ds$



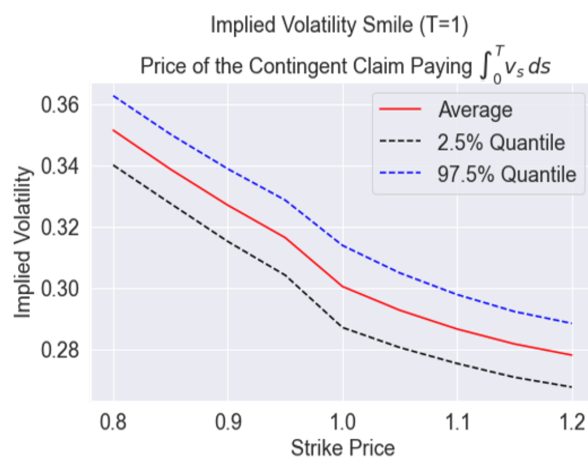
(a) Various Maturities



(b) T = 0.25



(c) T = 0.5

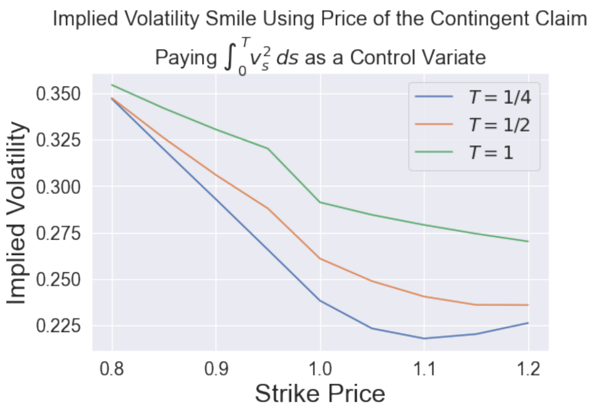


(d) T=1

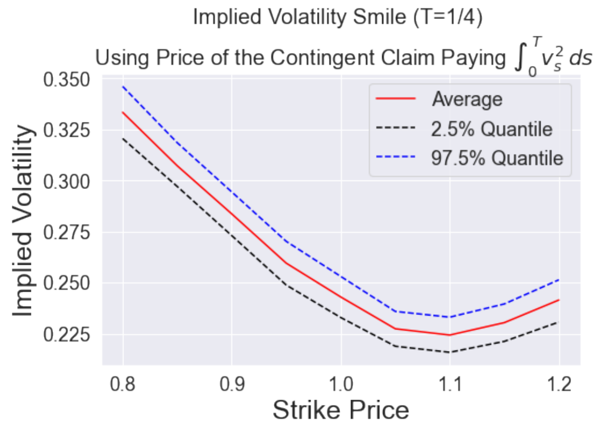
Figure 10: Implied Volatility Estimates and 95% Confidence Intervals using Claims paying $\int_0^T v_s ds$ as Control Variates

4.3 Control Variate - Price of Claims paying $\int_0^T v_s^2 ds$

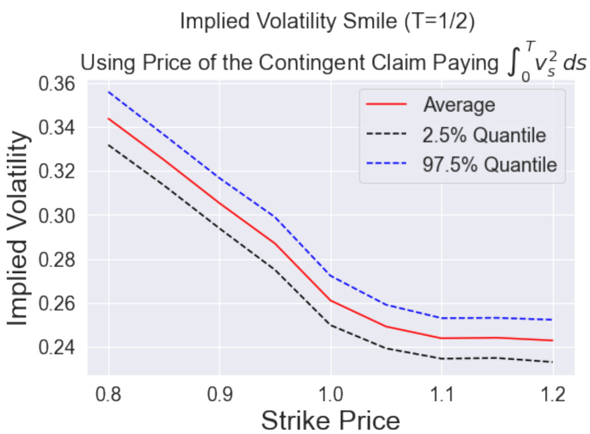
In this section, we estimate the implied volatilities of the put/call option with a control variate of the price of a contingent claim paying $\int_0^T v_s^2 ds$. We repeat the same process as the previous section except using a new control variate and plot the results in Figure 11. We notice that the IV curves in Figure 11(a) are similar to that of the mixing method in Figure 7(a) and more smooth than that of using the Claims paying $\int_0^T v_s ds$ in Figure 10(a). Moreover, similar to methods discussed previously, the confidence bands continue to grow wider as T increases which suggests that the method provides more precise estimates for short-period options than the long-period option. The width of confidence bands is similar to that of using the Claims paying $\int_0^T v_s ds$. Based on the Table 5, we found the overall performance of using control variate of claims paying $\int_0^T v_s^2 ds$ is slightly better than that of using claims paying $\int_0^T v_s ds$ in terms of the smoothness of the IV curves but still worse than the mixing method. Comparing Table 4 and Table 5, there is no obvious difference in the range of the confidence width between methods using $\int_0^T v_s^2 ds$ and $\int_0^T v_s ds$, as both are around 0.02.



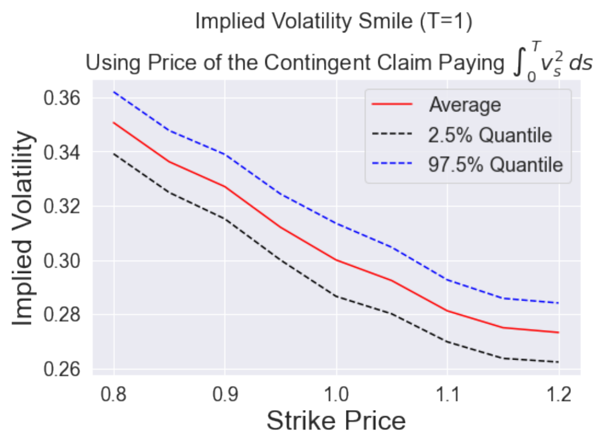
(a) Various Maturities



(b) T = 0.25



(c) T = 0.5



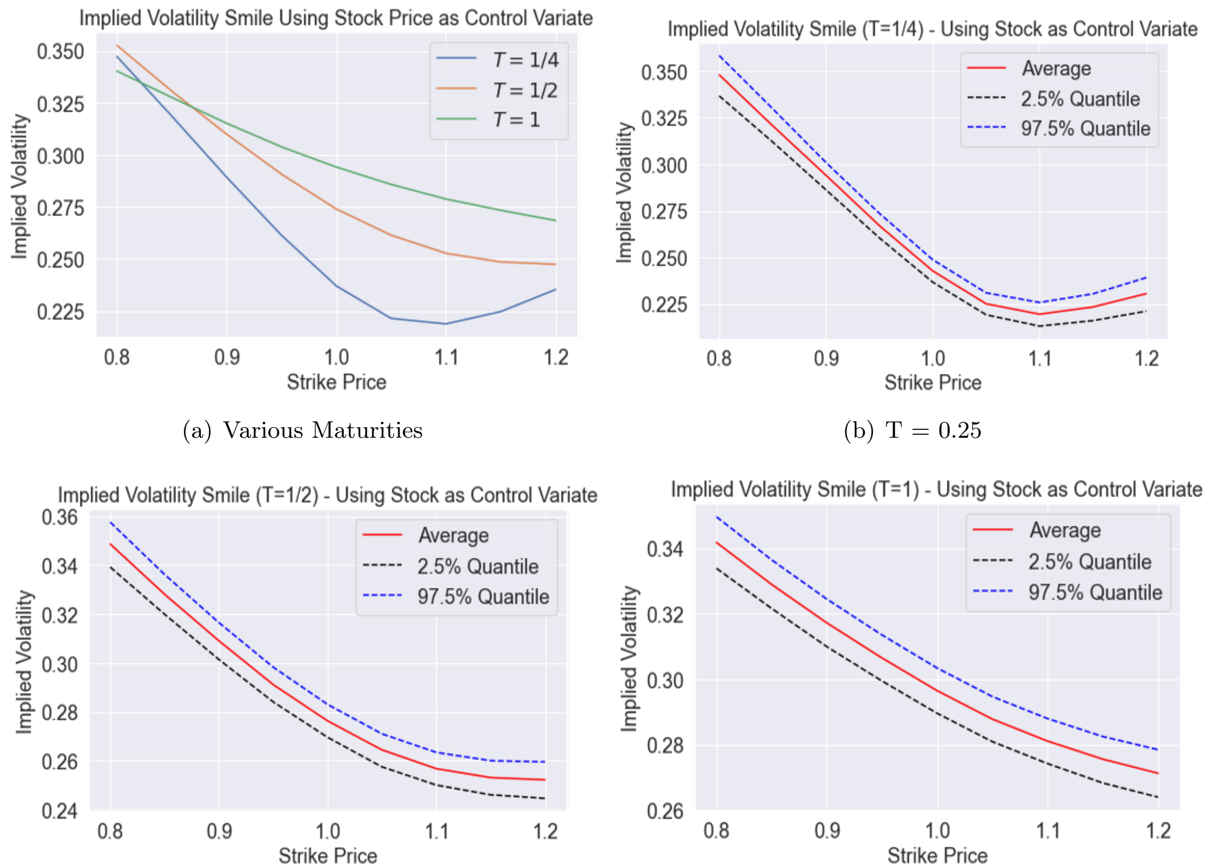
(d) T=1

Figure 11: Implied Volatility Estimates and 95% Confidence Intervals using Claims paying $\int_0^T v_s^2 ds$ as Control Variates

Strikes	**0.8**	**0.85**	**0.9**	**0.95**	**1**	**1.05**	**1.1**	**1.15**	**1.2**
T = 1/4	0.0233	0.0224	0.0220	0.0220	0.0191	0.01598	0.0156	0.0175	0.0200
T = 1/2	0.0239	0.0229	0.0229	0.0240	0.0223	0.0198	0.0179	0.0174	0.0186
T = 1	0.0230	0.0232	0.0239	0.0247	0.0262	0.0237	0.0218	0.0207	0.0198

Table 5: Confidence Width: claim paying $\int_0^T v_s^2 ds$

4.4 Control Variates - Stock price



(a) Various Maturities

(b) T = 0.25

(c) T = 0.5

(d) T=1

Figure 12: Implied Volatility Estimates and 95% Confidence Intervals Using Stock as a Control Variate

Strikes	**0.8**	**0.85**	**0.9**	**0.95**	**1**	**1.05**	**1.1**	**1.15**	**1.2**
T = 1/4	0.0202	0.0171	0.0141	0.0130	0.0121	0.0121	0.0132	0.0153	0.0191
T = 1/2	0.0181	0.0160	0.0147	0.0142	0.0134	0.0136	0.0135	0.0144	0.0156
T = 1	0.0163	0.0155	0.0150	0.0144	0.0140	0.0138	0.0137	0.0137	0.0143

Table 6: Confidence Width: Stock Price

Using stock price as a control variate, the above plot of implied volatilities and its 95% confidence interval shows the estimation results. The smaller the maturity the more obvious the smile shape. Similar to the previous case when we use options price as a control variate, IV curves and confidence width perform closer to the mixing method, compared to the performance of the Euler method. The IV curves are relatively smooth and the confidence bands are narrower than claims paying $\int_0^T v_s^2 ds$ and claims paying $\int_0^T v_s ds$. It means that the estimated γ corresponding to this control variate is closer to the optimum value. In other words, using stock price as a control variate can generate a more precise result. Different from the mixing method, for various maturity (T = 0.25, 0.5, 1), the width of confidence bands seems consistent and move smoothly over the x-axis (K). Additionally, the width of the confidence interval increase as maturity increase. For maturity T = 0.25 and 0.5, the IV estimation seems more accurate for the ATM option than the away-from-the-money option, but this feature is not obvious for large maturity when T = 1. Table 6 shows the confidence width is around 0.015 and achieves the smallest value when T = 0.25 and Strike = 1.05.

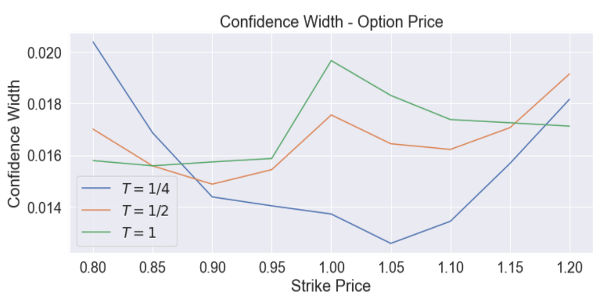
4.5 Performance Comparison: Four Different Control Variates

For the control variate selection, analysts need to find an appropriate asset with a known analytical payoff to replicate or explain the target option payoff (X). It is similar to finding the most significant factor when fitting the regression model. The higher correlation between the payoff of the asset as control variate (Y) and the target option payoff (X), the better the control variate selected. In this section, we compare control variates according to performance robustness, IV estimation, and confidence width.

(a.) Performance Robustness under Various Maturities

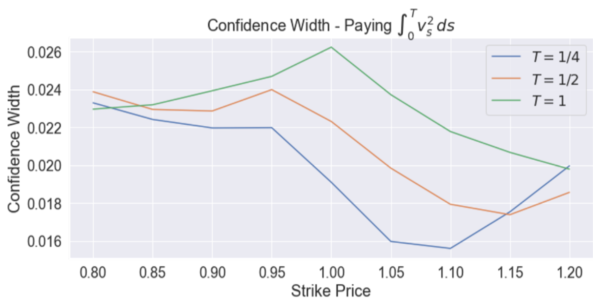
First, we analyze the performance **robustness over different maturity** based on Figure 13. It is obvious that the **most robust control variate is the stock price when the time of maturity increases**, especially for the **ATM option**. It is because the longer the maturity the greater the movement of stock price based on the Heston model and the target option payoff is highly dependent on the stock price. On the other hand, for the target option ATM, its payoff is extremely sensitive to the stock price movement. A slight change of stock price affects investors' decision of exercises the option or not, thus the correlation of stock price and target option payoff is high when the option is ATM.

Second, we found that **option price** based on stock price generated by deterministic $\bar{v}_t$ as control variate **perform well only at short maturity** (T =0.25). Based on the comparison of the stock price path generated using the Heston model and deterministic variance in Section 2.6.2.2, we know that only when time duration is small, the option payoff based on deterministic variance can closely represent the target option payoff. As time increase, the gap between the simulation result from the Heston model and deterministic variance model increase, so use option prices based on $\bar{v}_t$ as control variate does not well-behaved for larger maturity (T = 0.5 and T = 1). Moreover, this method is **not stable** for the ATM option and option that is far away from the money **if the maturity change**.

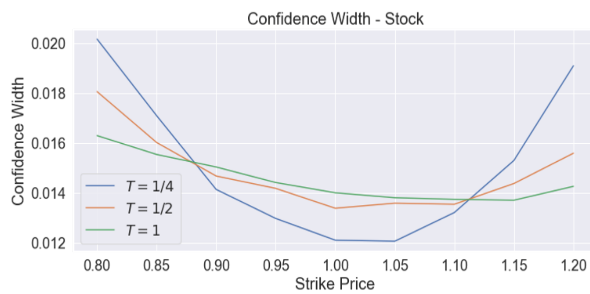


(a) Various Maturities - Option as Control Variate

(b) Various Maturities - Claims paying $\int_0^T v_s ds$



(c) Various Maturities - Claims paying $\int_0^T v_s^2 ds$



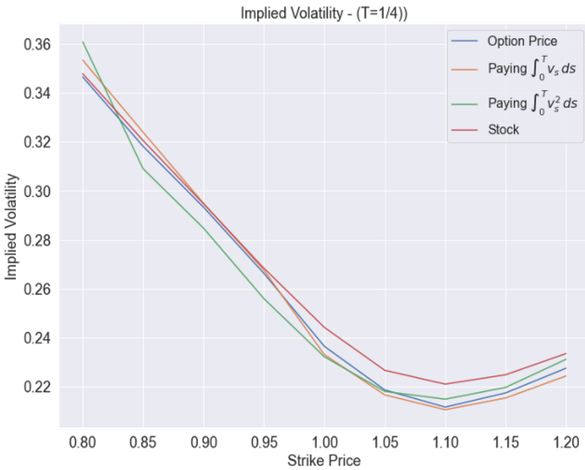
(d) Various Maturities - Stock as Control Variate

Figure 13: Performance Comparison: Robustness under various maturities

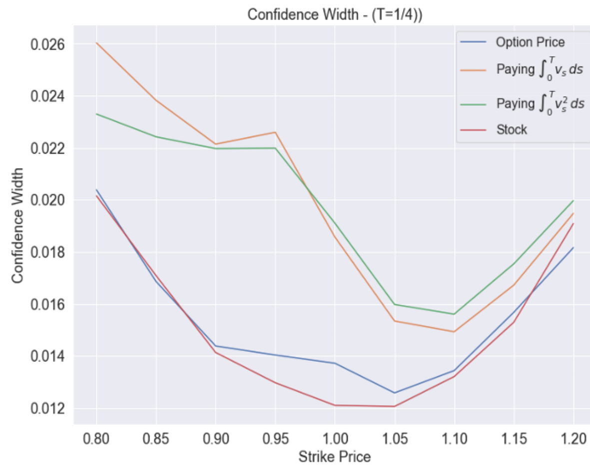
Moreover, the performance robustness of methods using claims paying $\int_0^T v_s^2 ds$ and claims paying $\int_0^T v_s ds$ is similar. It seems that taking a square of volatility (v_s) has no obvious improvements on the estimation result. The confidence width increases significantly when maturity increase for the option near the ATM point (strike between 0.95 and 1.15). An interesting observation is that the **behavior of the confidence width curve ((b) - $\int_0^T v_s ds$; (c) - $\int_0^T v_s^2 ds$)** for different maturity have **similar pattern as the mixing method**: the confidence width is larger for OTM put option (strike less than 1), followed by a sharp decrease when the strike is close to 1, and eventually remain at a lower level for OTM call option (strike greater than 1). It is reasonable because $\int_0^T v_s ds$ is part of the analytical formula when calculating the stock price based on the Heston model. However, the performance of this method is **not as stable as the mixing method for deep OTM call option** and the estimation error is relatively large over all other control variates.

(b.) Confidence Width and IV Estimation

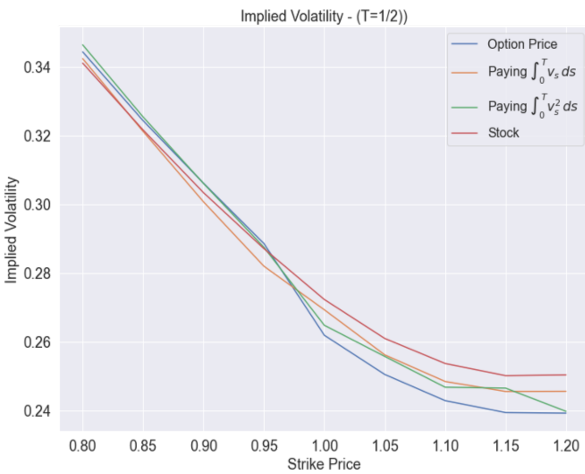
Figure 14 aims to compare the **width of confidence interval** (Figure 14(b), 14(d),14(f)) and the **IV estimation value** (Figure 14(a), 14(c),14(e)). The **performance** of each control variate **change** as the **maturity change** (T=0.25, 0.5, 1). First, based on the plot showing the **confidence width**, we conclude that the method **using stock as a control variate outperform the other methods** for different maturities (T=0.25, 0.5, 1). We observed that the confidence width for stock as a control variate maintain the lowest, and it is stable over various strike price and maturities.



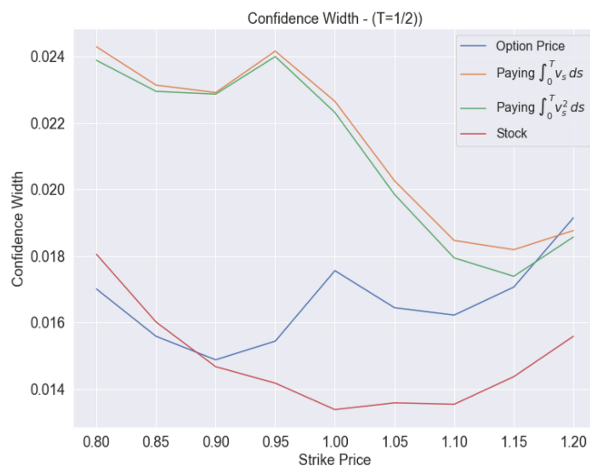
(a) Implied Volatility (T = 0.25)



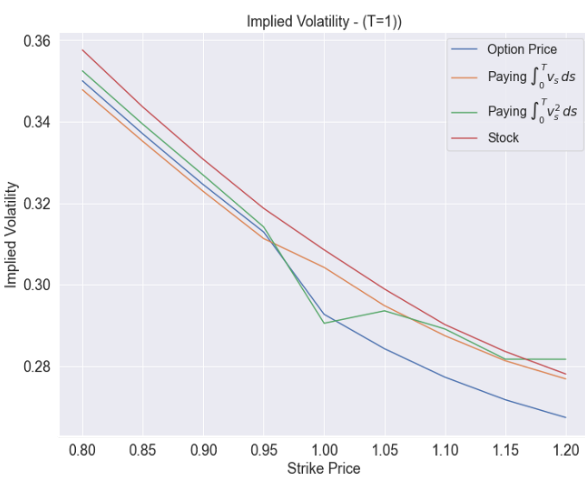
(b) Confidence Width Comparison (T = 0.25)



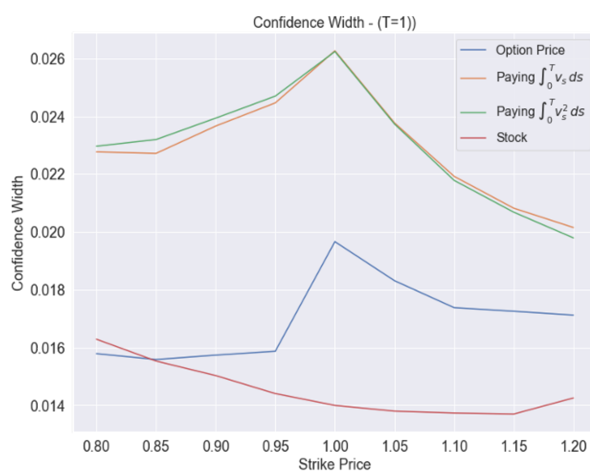
(c) Implied Volatility (T = 0.5)



(d) Confidence Width Comparison (T = 0.5)



(e) Implied Volatility (T = 1)



(f) Confidence Width Comparison (T = 1)

Figure 14: Performance Comparison: Estimated IV and Confidence width

Additionally, **the method of using option price can be as superior as using stock for T = 0.25** although it is less robust for ATM option and OTM call option. As the maturity (T) increase (plot (d) and (e)), the simulation gap between the Heston model and deterministic variance model increase, which reduce the estimation accuracy of methods using option price and result in a larger confidence width. We observed there is a large gap of confidence width between method using option and stock for OTM call with (T = 0.5, 1). OTM call implies the underlying stock price is low and longer maturity (T), therefore there is a large simulation gap between the Heston model and deterministic variance model, which explained the poor performance of the method of using option price.

For ATM option, only the method using stock as a control variate maintains a low confidence width since the option payoff is very sensitive to the stock price change for the ATM option. The stock price is the most significant factor that influences the option payoff compared to other types of assets. **For deep OTM put option**, the method using option and stock as control variate achieved a much smaller confidence width, meaning these methods are better than the methods using claim paying $\int_0^T v_s^2 ds$ and $\int_0^T v_s^2 ds$. **For deep OTM call option** (strike = 1.2), the difference between the confidence width is small when T = 0.25. Increasing the maturity improved the performance of the method using stock, but have a negative impact on the performance of the method using the other three assets as a control variate, especially for claims paying $\int_0^T v_s^2 ds$ and $\int_0^T v_s^2 ds$. When T = 1, the confidence width for claims paying $\int_0^T v_s^2 ds$ and $\int_0^T v_s^2 ds$ is much greater than the other two methods.

Another finding is based on the confidence width curve for claim paying $\int_0^T v_s^2 ds$ and $\int_0^T v_s^2 ds$. As maturity increase, the gap between these two methods decrease. Based on plot (f), these two methods have nearly the same behavior when T = 1.

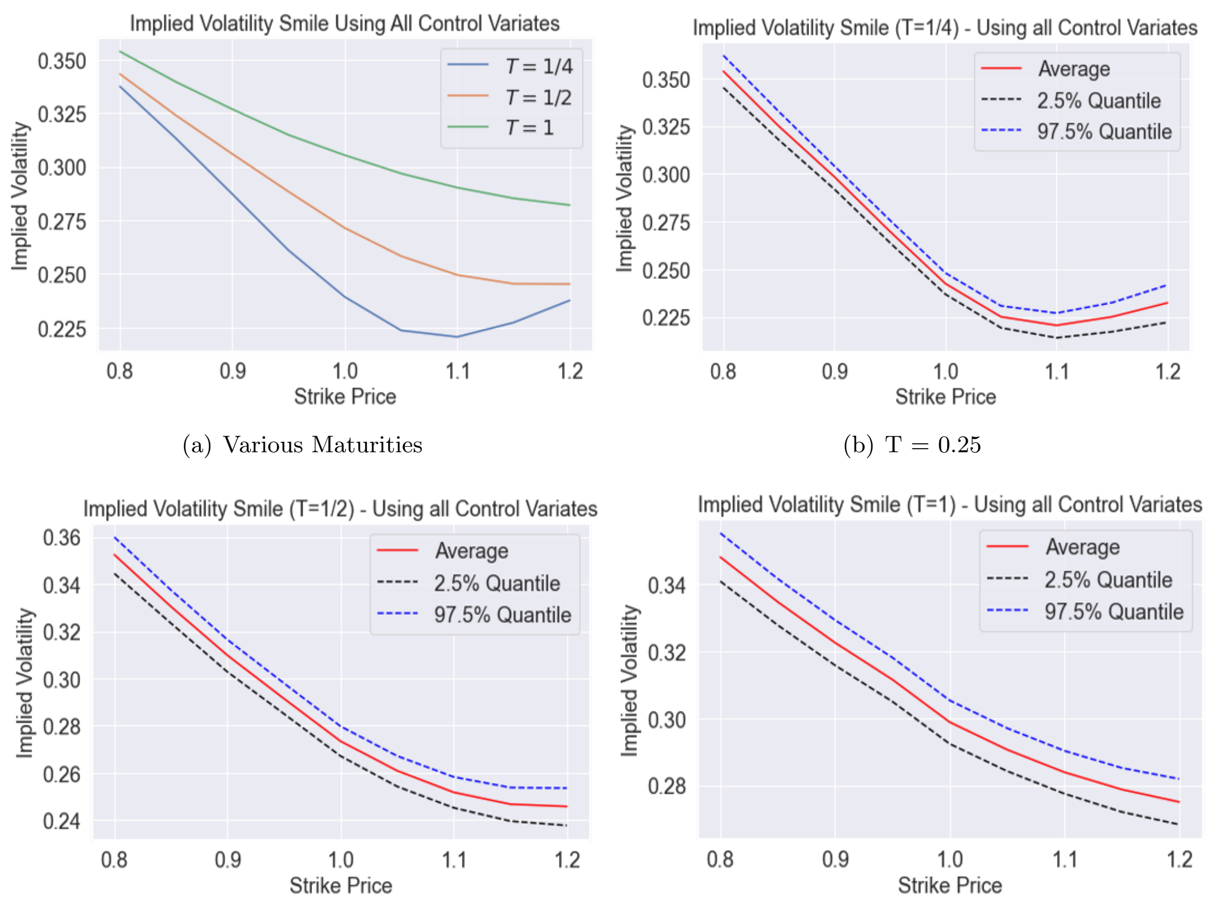
Next, based on the plot showing the **estimated IV** (Figure 14(a), 14(c),14(e)), the estimated implied volatility for each methods are relatively close for OTM put option when T = 0.25 and 0.5. However, there are obvious gaps in IV estimation for larger T = 1. For the ATM option, there is a significant jump in the IV curve, especially for larger T or the cases using option price and claim to pay $\int_0^T v_s^2 ds$ as a control variate.

On the other hand, the gaps of the IV estimation for the OTM call option based on these four methods keep large even for shorter T = 0.25 (Figure 14(a)). By comparing all of the IV estimation plot (14(a), 14(c),14(e)). The estimated IV curve for claim paying $\int_0^T v_s^2 ds$ and $\int_0^T v_s^2 ds$ keep fluctuating, which proved our previous conclusion about performance robustness. The estimated IV is the highest if using stock price as a control variate, and it is the lowest if using option as a control variate. **Considering the fact that the method using stock as a control variate provides the most accurate result, the other methods are likely to underestimate the implied volatility**. Finally, we conclude that stock price is the best control variate under the single control variate method.

5 Further Improvements via All Control Variates and Performance Comparison

Compared to the single control variate method, the **multilevel method further improved the IV estimation accuracy. It integrates different types of assets (Y1, Y2, Y3, Y4) to explain the behavior of option value (X) based on the Heston model**. The corresponding γ is in vector form. Intuitively, this process can be analogous to adding new explanatory variables on the basis of the univariate regression model, thereby improving the estimation performance. Therefore, we achieve a greater variance reduction.

5.1 All Control Variates



(a) Various Maturities

(b) T = 0.25

(c) T = 0.5

(d) T=1

Figure 15: Implied Volatility Estimates and 95% Confidence Intervals Using All Control Variates

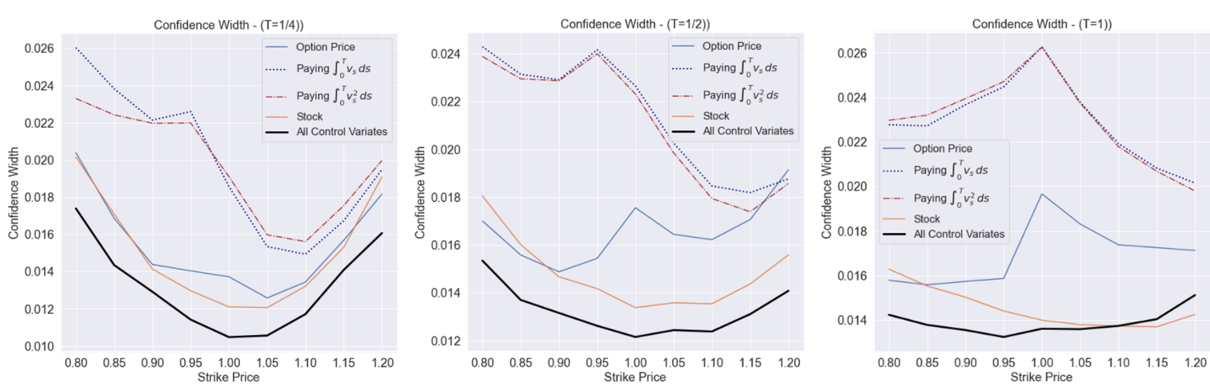
Strikes	0.8	0.85	0.9	0.95	1	1.05	1.1	1.15	1.2
T = 1/4	0.0174	0.0143	0.0129	0.0114	0.0105	0.0106	0.0117	0.0141	0.0161
T = 1/2	0.0153	0.0137	0.0132	0.0126	0.0122	0.0124	0.0124	0.0131	0.0141
T = 1	0.0142	0.0138	0.0136	0.0132	0.0136	0.0136	0.0137	0.0140	0.0151

Table 7: Confidence Width: All Control Variate

Using all control variates (including option, claims to pay $\int_0^T v_s^2 ds$ and claims paying $\int_0^T v_s ds$, and stock price), Table 7 and plot of implied volatilities with its 95% confidence interval shows the individual performance of all control variate method. The lowest confidence width can be 0.0105 for short-term ATM option (T = 0.25 and strikes = 1). We observe that the overall performance is as precise as the mixing method and the method using stock price as a control variate, especially for OTM put option (Strike < 1). The combination of various assets can provide a more accurate estimation result. The estimated vector γ corresponding to these control variate is relatively close to the optimum value. Similar to all other control variate methods in the previous section the width of confidence bands seems to move smoothly over strike price. The smaller the maturity the more obvious the smile shape. Increased maturity increases the width of the confidence interval. For maturity T = 0.25 and 0.5, the IV estimation seems more accurate for the ATM option than the away-from-the-money option.

5.2 Performance Improvements by Multilevel Control Variate

The previous sections has discussed the IV estimation and robustness of various control variate. This section will focusing on the comparison of the confidence width.



(a) T = 0.25) (b) T = 0.5 (c) T = 1

Figure 16: Confidence Width: Single Control Variates and All Control Variates

Based on the Figure 16, we observe that the confidence width curve for all control variate is the lowest (<u>the black curve</u>) for T = 0.25 and 0.5. As we previously conclude stock is the best control variate under the single control variate method, we compared the gap of confidence width between stock control variate and all control variate in the table above. The larger the gap implies greater improvements when replacing stock control variate to all control variate. Figure 16 and Table 8 shows that when T = 1, stock control variate (the yellow curve) can outperform all control variate, achieving smallest confidence width. For T =1 and strike = 1.2, the confidence width of the stock control variate is 0.0009 lower than all control variate. It is reasonable because, in the case of the OTM call option, the performance of the method using stock as control variate improves when maturity increase. Overall change stock control variate to all control variate can achieve a decrease of confidence width around 0.0014. The improvement is greater for OTM put (strike = 0.8). For example, the average decrease for strike = 0.8 over different maturities is 0.0025, which is the highest among all other strike values. Moreover, the smallest average decrease in width is 0.0019 at a smaller maturity (T = 0.25). Additionally, if consider the control variate method as a regression system, **the stock factor (control variate) seems to have the greatest contribution** of the overall accuracy under the all control variate method, compared to the other three assets (option, $\int_0^T v_s^2 ds$ and $\int_0^T v_s ds$).

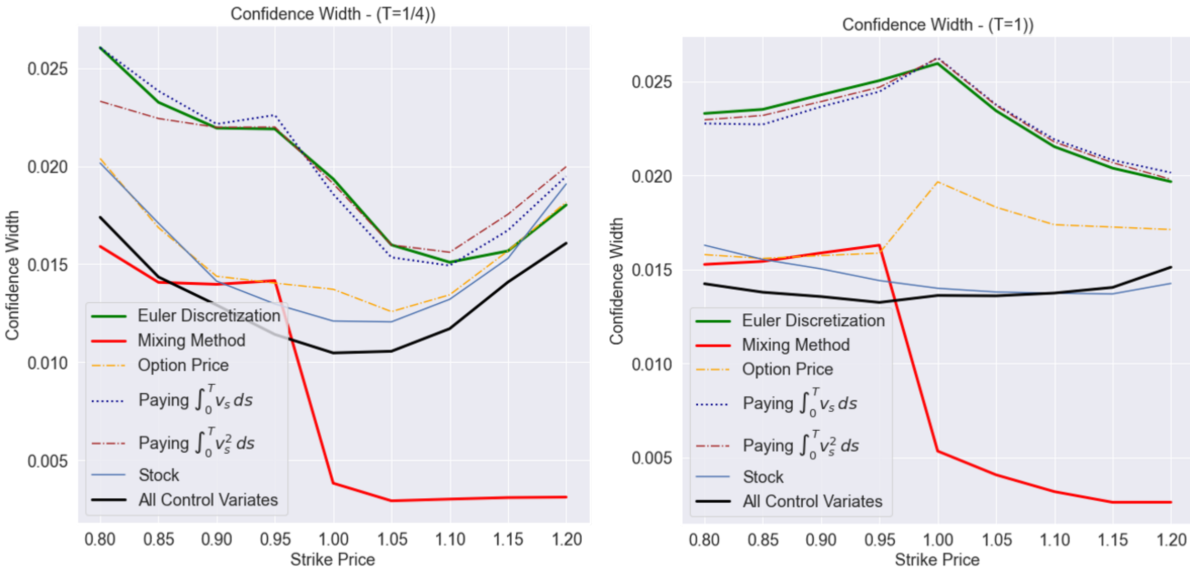
Strikes	**0.8**	**0.85**	**0.9**	**0.95**	**1**	**1.05**	**1.1**	**1.15**	**1.2**	**Average**
T = 1/4	0.0028	0.0027	0.0012	0.0016	0.0016	0.0015	0.0015	0.0012	0.0030	0.0019
T = 1/2	0.0027	0.0023	0.0015	0.0016	0.0012	0.0011	0.0012	0.0013	0.0015	0.0016
T = 1	0.0021	0.0017	0.0015	0.0012	0.0004	0.0002	0.0000	**-0.0003**	**-0.0009**	0.0006
Average	0.0025	0.0023	0.0014	0.0014	0.0011	0.0009	0.0009	0.0007	0.0012	0.0014

Table 8: Decrease of Confidence Width = Stock - All Control Variate

5.3 Performance Comparison: Euler Milstein, Mixing Method, Single and Multi-level Control Variates Method

In this section, we focus on evaluating performance based on the confidence width among all methods discussed in previous sections, including Euler Milstein, Mixing Method, and Various Control Variates. As there is no significant difference in the confidence width curve between T = 0.5 and T = 1, we discussed cases for shorter maturity (T = 0.25) and longer maturity (T =1) based on the Figure 17.

According to the confidence width curve, the shape of the confidence width curve for these three methods is similar to the shape of the curve using the mixing method when the maturity is small (see Figure (a) T = 0.25). However, **the performance of method using claims paying $\int_0^T v_s^2 ds$ and claims paying $\int_0^T v_s ds$ converge to the Euler Milstein method as maturity increase** from T = 0.25 to T = 1. In other words, the payoff of $\int_0^T v_s^2 ds$ and $\int_0^T v_s ds$ is not highly correlated with the target option payoff for large maturity, especially for the ATM and OTM call option (when strikes > 1). The reason is that the payoff of $\int_0^T v_s^2 ds$ and $\int_0^T v_s ds$ increase as maturity increase, while the underlying stock price of the target payoff decrease over time, since the correlation of two risk-neutral Brownian Motion for volatility and stock price is negative based on the Heston model. For a longer time period, the payoff of $\int_0^T v_s^2 ds$ and $\int_0^T v_s ds$ no longer represent the payoff of the target option, so there is no variance reduction benefit if using these two methods, resulting in higher remaining uncertainty and a poor performance like Euler Milstein method.



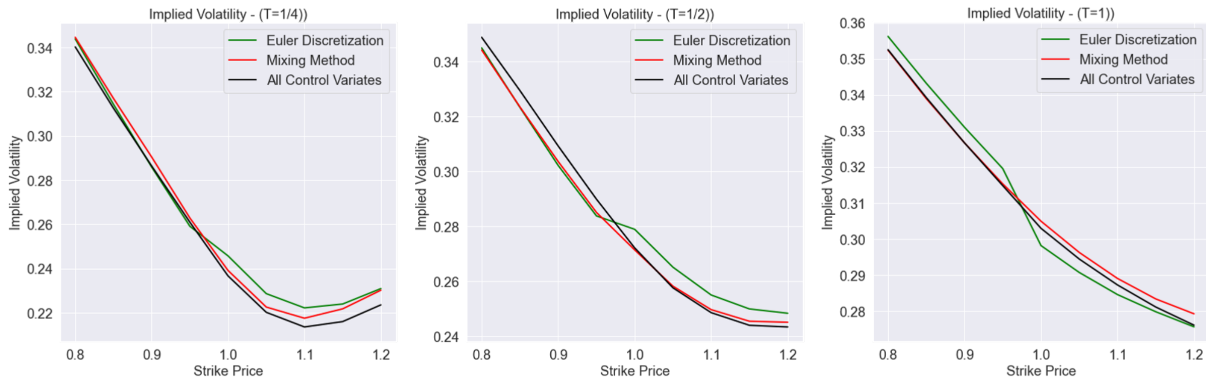
(a) Confidence Width Comparison (shorter maturity) (b) Confidence Width Comparison (longer maturity)

Figure 17: Performance Comparison among Various Control Variates and Mixing Method

The behavior of confidence width for **method using the option, stock, and all control variate are similar** based on the range and shape of the curve. The all control variate usually outperforms the other two, but they can be classified as a well-performed control variate. The options control variate is less stable when maturity increases, which have discussed in the previous sections. Overall, these three methods are superior to the Euler Milstein method and single control variate using $\int_0^T v_s^2 ds$ and $\int_0^T v_s ds$.

Moreover, for both long and short maturity, if the strike is less than 1, the confidence width curve for the mixing method is closer to a well-performed control variate (such as option, stock, and all control variate). Once the strike becomes greater than 1, the estimation accuracy of the mixing method increases sharply, leading to a shrink of the confidence width. Therefore, we can see a **jump point in the curve for mixing method** when option ATM.

Based on the previous analysis, it is worth conducting further investigation of the IV estimation (see Figure 18) focusing on three typical methods, including Euler Milstein, Mixing Method, and All Control Variate Method. For Euler Milstein (the green curve), We observed the shape of the estimated implied volatility curve keep changing as maturity changes, indicating a wider confidence interval for the estimation. On the other hand, the curve for the mixing method and all control variate methods maintain smooth and robust. However, the **estimated IV for OTM call option seems higher under mixing method** compared to the all control variate method.



(a) T = 0.25) (b) T = 0.5 (c) T = 1

Figure 18: IV Estimation: Three Typical Methods

Based on the Figure 17 and Figure 18, we conclude that the confidence width is relatively small and the IV estimation is stable as maturity changes for both mixing methods and all control variate methods. Therefore, we further analyzed whether the IV estimation accuracy will improve when changing all control variate methods to the mixing method. The narrowing amount of the confidence width is calculated in the table below:

Strikes	**0.8**	**0.85**	**0.9**	**0.95**	**1**	**1.05**	**1.1**	**1.15**	**1.2**	**Average**
T = 1/4	0.0015	0.0003	-0.0011	-0.0027	0.0066	0.0076	0.0087	0.0110	0.0129	0.0050
T = 1/2	-0.0005	-0.0015	-0.0020	-0.0035	0.0078	0.0093	0.0096	0.0103	0.0111	0.0045
T = 1	-0.0010	-0.0016	-0.0023	-0.0030	0.0083	0.0095	0.0106	0.0114	0.0125	0.0049
Average	0.0000	-0.0009	-0.0018	-0.0031	0.0076	0.0088	0.0096	0.0109	0.0122	0.0048

Table 9: Decrease of Confidence Width = All Control Variate - Mixing Method

For strike less than 1 (OTM put option), the all control variate method outperforms the mixing method. The confidence width of all control variate is around 0.003 less than the mixing method when strike = 0.95 (near ATM), while there is no difference between these methods for the deep OTM put option (strike = 0.8). Although the all control variate method is better for strikes less than 1, the improvements of using this method instead of the mixing method are relatively small. On the other hand, the analyst can use the mixing method to achieve superior accuracy when the strike price is greater than 1 (OTM call option). The mixing method reduces around 0.005 in the IV estimation confidence width over different maturities, and the variance reduction reaches the maximum for deep OTM call (strike = 1.2).

The mixing method applied analytical calculation in the simulation process, which reduce the simulation error significantly. While the single/ multilevel control variate method is another way of introducing analytical calculation in the simulation process, its performance depends on the selection of the control variate asset. Only when the payoff of the selected control variate assets is highly correlated with the target option payoff, can the methods of all control variate obviously improve the IV estimation accuracy.

As stock price is the strongest factor in the all control variate system (discussed in the previous section), **in the case of OTM call option**, the stock price is small and the option is unlikely to be exercised, **the stock price no longer highly correlated with the target option payoff**, the all control variate method become ineffective. Therefore, the performance of using control variables is worse than using analytical calculations. It explains the inferior performance of all control variate methods for the strike price greater than 1, compared to the mixing method.

In practice, if the analyst cares about the consistency of the estimation variance, using all control variate is a reasonable choice, because the confidence width curve of this method is smooth without jump point and become more stable as maturity increase. However, selecting the all control variate method sacrifices the opportunity to obtain a more precise result when the strike price greater than 1. One reasonable solution is to combine these two methods, using the all control variates method for strikes less than 1 and using the mixing method for strikes greater than 1. According to the average decrease of confidence width by using mixing method over various maturities, which is 0.0048, **mixing method is more accurate** when strike change from 0.8 to 1.2. Considering the improvements of switching to all control variate is small, we can simply apply the mixing method for the entire IV estimation.

6 Conclusion

Using a model with the assumption of constant variance (e.g. Black-Scholes mode) can overestimate return but underestimate market volatility, leading to an ineffective risk control for investment management companies. Therefore, the Heston model is widely applied to estimate option price and implied volatility (IV). The numerical calculation of IV relies on Monet Carol Method. Given limited computation resources in the real-world situation, four different types of methodologies are investigated to improve the IV estimation accuracy, including Euler's discretization method, mixed Monte Carlo method, single and multilevel control variate methods. Two times of simulation in Euler's discretization method introduce extra uncertainty in the estimation. Fortunately, introducing analytical calculation in the simulation process can improve the simulation accuracy. One way is using control variate (single or multiple) with known analytical payoff. It can be analogous to solving a certain least-squares system or fitting a regression model. The effectiveness of this method highly depends on the selection of the control variate. The higher the correlation between the payoff of the control variate and the target option, the greater the variance reduction achieved. Four candidate control variates are investigated, including, option based on deterministic variance model, claims paying $\int_0^T v_s^2 ds$, $\int_0^T v_s ds$, stock. The performance evaluation suggests stock price is the best control variate under the single control variate method, while all control variate can further improve the IV estimation accuracy. The other better way is using the mixing method, where the estimation accuracy boost for strike greater than 1 (OTM call option). The mixing method is the most precise methodology compared to all other methods discussed in this paper. Overall, the stock control variate, all control variate, and mixing method are well-performed In the IV estimation, with lower confidence width, and their performance is robust over various maturities. Analysts need to select appropriate simulation methodologies based on the business purpose, data availability, and technical resource in practice.