Problem 1

(1)

Problem 1

Because it's a no state/control constraint problem

Form Hamiltonian: $H := g(x(t), u(t), t) + P^T(t)[a(x(t), u(t), t)]$

$$= \lambda + V^2(t) + w^2(t) + P_x \cdot \dot{x} + P_y \cdot \dot{y} + P_\theta \dot{\theta}$$

$$= \lambda + V^2(t) + w^2(t) + P_x \cdot V\cos\theta(t) + P_y V\sin\theta + P_\theta \cdot w(t)$$

$$\begin{cases} \dot{x}(t) = V\cos\theta(t) \\ \dot{y}(t) = V\sin\theta(t) \\ \dot{\theta}(t) = w(t). \end{cases}$$

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \quad u = \begin{bmatrix} V \\ w \end{bmatrix} \quad P = \begin{bmatrix} P_x \\ P_y \\ P_\theta \end{bmatrix}$$

Hamiltonian equations

$$\begin{cases} \dot{x}^*(t) = \frac{\partial H}{\partial P}(x^*(t), u^*(t), p^*(t), t) \\ \dot{p}^*(t) = -\frac{\partial H}{\partial x}(x^*(t), u^*(t), p^*(t), t) \\ 0 = \frac{\partial H}{\partial u}(x^*(t), u^*(t), p^*(t), t). \end{cases}$$

becomes

$$\begin{cases} \dot{x}^*(t) = V^* \cdot \cos\theta^*(t) \text{ ①} \overset{⑦}{\Rightarrow} \dot{x}^*(t) = -\frac{P_x^*\cos\theta^*(t) + P_y^*\sin\theta^*(t)}{2} \cdot \cos\theta^*(t) \\ \dot{y}^*(t) = V^* \cdot \sin\theta^*(t) \text{ ②} \overset{⑦}{\Rightarrow} \dot{y}^*(t) = -\frac{P_x^*\cos\theta^*(t) + P_y^*\sin\theta^*(t)}{2} \cdot \sin\theta^*(t) \\ \dot{\theta}^*(t) = w^*(t) \text{ ③} \overset{⑧}{\Rightarrow} \dot{\theta}^*(t) = -P_\theta^*/2 \\ \dot{P}_x^*(t) = 0 \text{ ④} \\ \dot{P}_y^*(t) = 0 \text{ ⑤} \\ \dot{P}_\theta^*(t) = -[P_x \cdot V \cdot (-\sin\theta(t)) + P_y \cdot V \cdot \cos\theta(t)] \text{ ⑥} \\ 0 = 2V(t) + P_x \cos\theta(t) + P_y \sin\theta(t) \text{ ⑦} \\ 0 = 2w(t) + P_\theta \text{ ⑧} \end{cases}$$

Because t_f is free, also have $H(x^*(t_f), u^*(t_f), p^*(t_f), t_f) + \overset{0}{\cancel{\frac{\partial h}{\partial t}}} = 0$

$$\lambda + V^{*2}(t) + w^{*2}(t) + P_x^* V^* \cos\theta(t) + P_y^* V \sin\theta(t) + P_\theta^* w(t) = 0 \text{ ⑨}$$

From ⑦ & ⑨ & ⑧ $\lambda + V^{*2}(t) + w^{*2}(t) + V^*(P_x\cos\theta(t) + P_y\sin\theta(t)) + P_\theta w^*(t) = 0$

$$\lambda + V^{*2}(t) + w^{*2}(t) - 2V^{*2}(t) - 2w^{*2}(t) = 0$$

$$\lambda - V^{*2}(t_f) - w^{*2}(t_f) = 0 \Rightarrow \lambda - \frac{1}{4}(\overset{0}{\cancel{P_x\cos\theta(t)}} + P_y\sin\theta(t))^2 - \frac{1}{4}P_\theta^2 \Big|_{t_f} = 0$$

$$\left(\lambda - \frac{1}{4}P_y^2 - \frac{1}{4}P_\theta^2\right)\Big|_{t_f} = 0.$$

From ⑦ $V(t) = -\frac{1}{2}(P_x\cos\theta(t) + P_y\sin\theta(t))$

Rescale time $\tau = t/t_f$ then $\tau \in [0,1]$

$$\dot{x}^*(\tau) = t_f \cdot -\frac{P_x^* \cos\theta^*(\tau) + P_y^* \sin\theta^*(\tau)}{2} \cdot \cos\theta^*(\tau)$$

$$\dot{y}^*(\tau) = t_f \cdot -\frac{P_x^* \cos\theta^*(\tau) + P_y^* \sin\theta^*(\tau)}{2} \cdot \sin\theta^*(\tau)$$

$$\dot{\theta}^*(\tau) = t_f \cdot -\frac{P_\theta^*}{2}$$

$$\dot{P}_x^*(\tau) = 0$$

$$\dot{P}_y^*(\tau) = 0$$

$$\dot{P}_\theta^*(\tau) = -t_f\Big[P_x^* \tfrac{1}{2}\sin\theta^*(t)\cdot(P_x\cos\theta(t) + P_y\sin\theta(t)) + P_y^* - \tfrac{1}{2}\cos\theta(t)\,(P_x\cos\theta(t) + P_y\sin\theta(t))\Big]$$

$x^*(0) = 0$ $\quad$ $x^*(1) = 5$

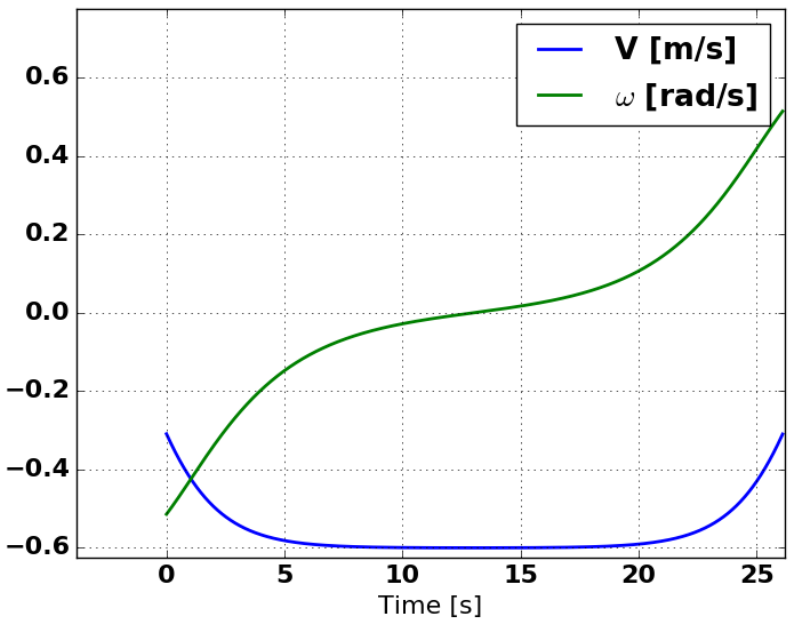
$y^*(0) = 0$ $\quad$ $y^*(1) = 5$

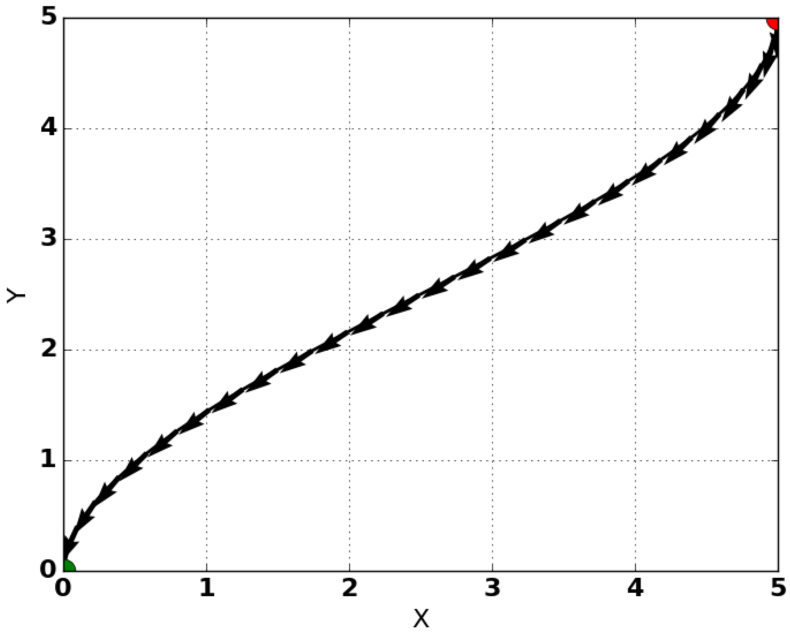
$\theta^*(0) = -\pi/2$ $\quad$ $\theta^*(1) = -\pi/2$.

$$\lambda = \frac{1}{4}P_y^{*2}(1) + \frac{1}{4}P_\theta^{*2}(1)$$

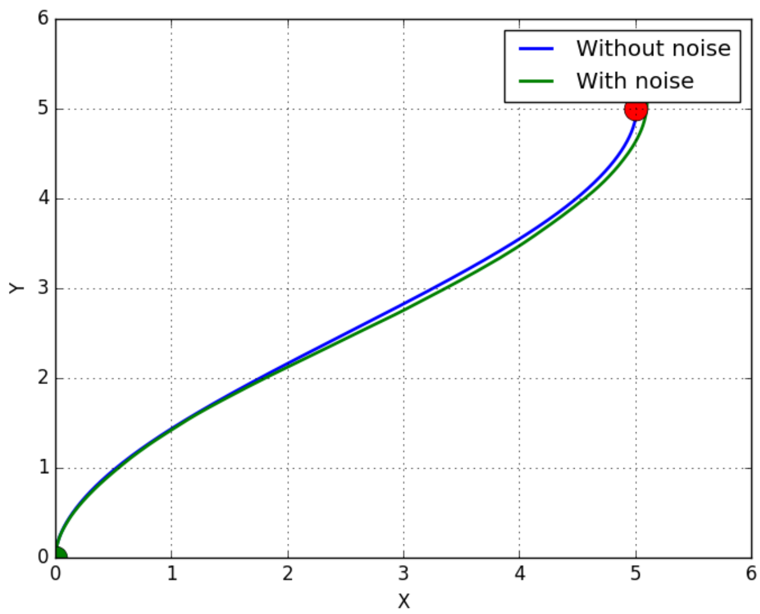
(3) The largest feasible lambda is essential because it's the weighting factor of time, the larger it is, the smaller the resulting time is for the optimization.

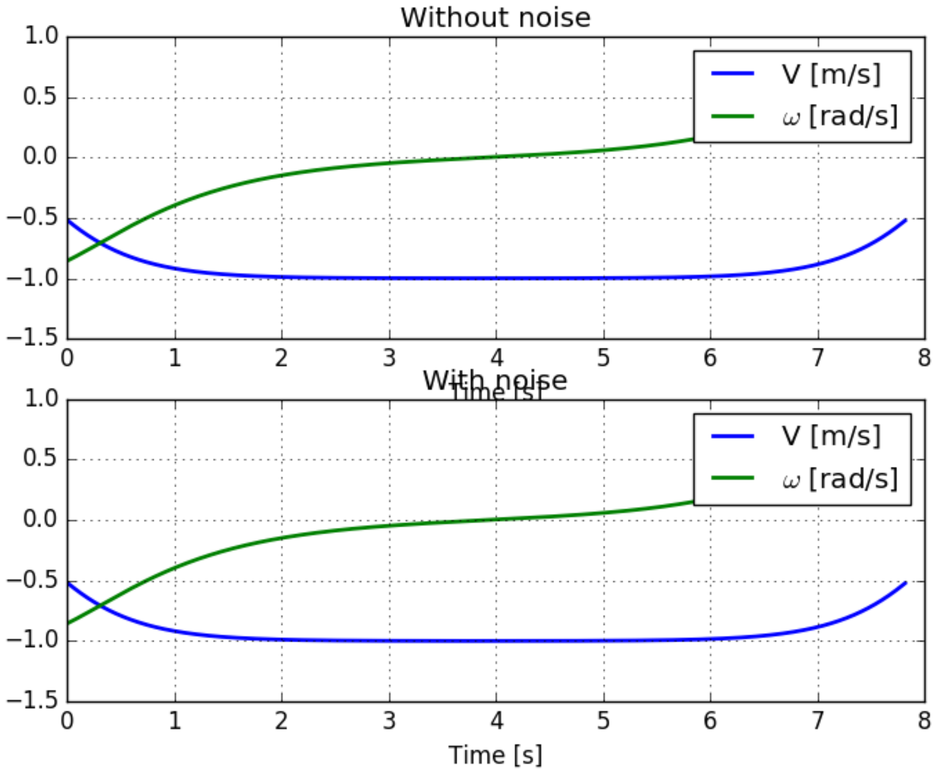
(4) Plots





Simulation Plots





Problem 2

P2 (a) **we can't set V(tf) = 0 because otherwise matrix J won't be invertible and it will lose the differential flatness characteristics when given any trajectory, we can find control function.**

Problem 2: Differential Flatness

i)
$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} V\cos(\theta(t)) \\ V\sin(\theta(t)) \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2t \\ 3t^2 \end{bmatrix}$$

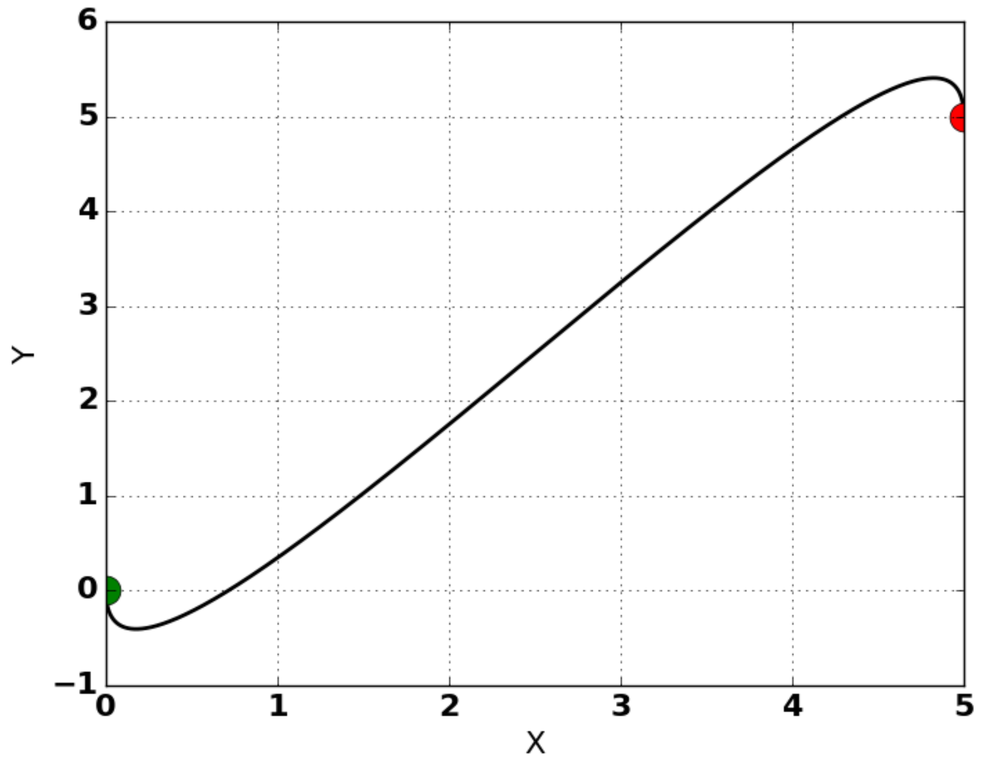
$$\begin{cases} x_1 + x_2 \cdot 0 + x_3 \cdot 0^2 + x_4 \cdot 0^3 = 0 \\ y_1 + y_2 \cdot 0 + y_2 \cdot 0^2 + y_4 \cdot 0^3 = 0 \\ x_1 \cdot 0 + x_2 + x_3 \cdot 2 \cdot 0 + x_4 \cdot 3 \cdot 0^2 = 0 \\ y_1 \cdot 0 + y_2 + y_3 \cdot 2 \cdot 0 + y_4 \cdot 3 \cdot 0^2 = -0.5 \\ x_1 + x_2 \cdot tf + x_3 \cdot tf^2 + x_4 \cdot tf^3 = 5 \\ y_1 + y_2 \cdot tf + y_3 \cdot tf^2 + y_4 \cdot tf^3 = 5 \\ x_1 \cdot 0 + x_2 + x_3 \cdot 2 \cdot tf + x_4 \cdot 3 \cdot tf^2 = 0 \\ y_1 \cdot 0 + y_2 + y_3 \cdot 2 \cdot tf + y_4 \cdot 3 \cdot tf^2 = -0.5 \end{cases}$$

if $V(tf) = 0$, Matrix J won't be invertible.

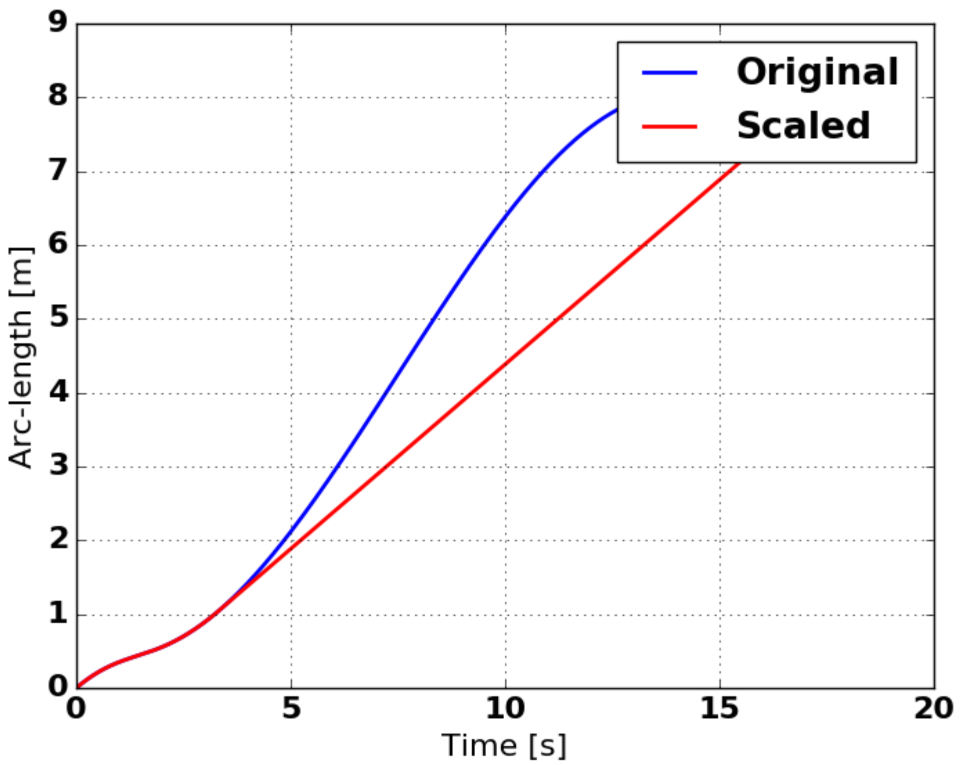
$$x(t) = x_1 + x_2 t + x_3 t^2 + x_4 t^3$$
$$y(t) = y_1 + y_2 t + y_3 t^2 + y_4 t^3$$
$$\dot{x}(t) = x_2 + 2x_3 t + 3x_4 t^2$$
$$\dot{y}(t) = y_2 + 2y_3 t + 3y_4 t^2$$
$$\ddot{x}(t) = 2x_3 + 3x_4 \cdot 2t$$
$$\ddot{y}(t) = 2y_3 + 3 \cdot y_4 \cdot 2t$$
$$\theta(t) = \tan^{-1}(\dot{x}(t)/\dot{y}(t))$$
$$\omega(t) \Rightarrow \ddot{x}(t) = a \cdot \cos\theta - \omega V \sin(\theta) \qquad \omega = \frac{a\cos\theta - \ddot{x}t}{V\sin\theta}$$
$$V(t) = \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}$$

$$\dot{\theta}(t) = \frac{d\theta(t)}{dt}$$

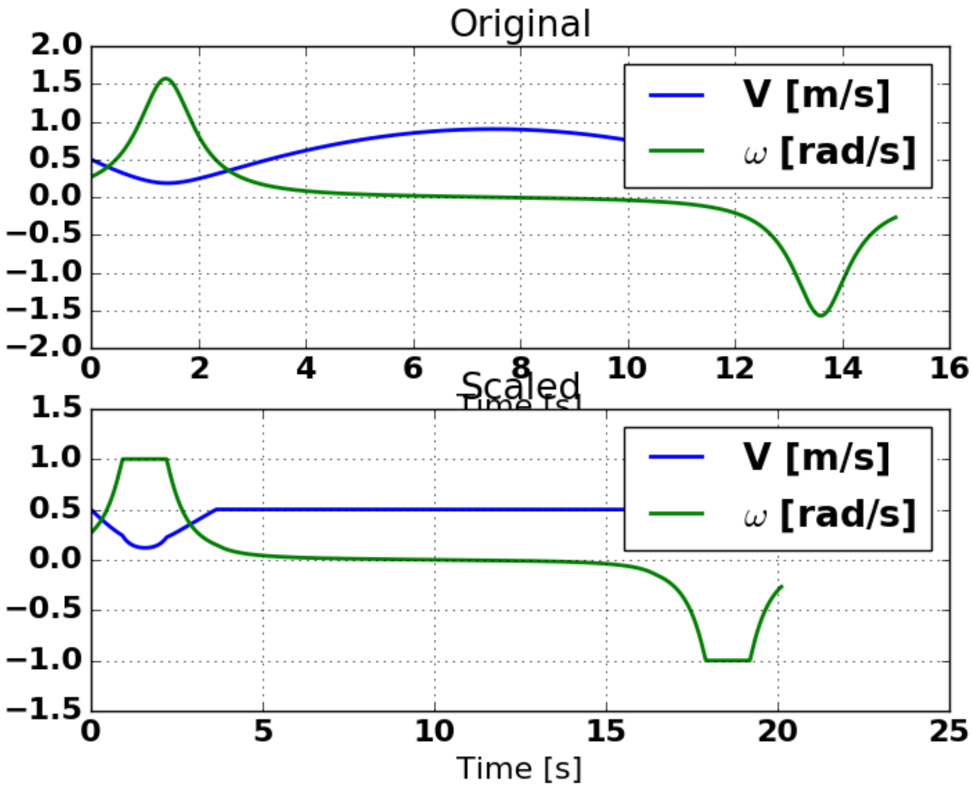
Trajectory as below



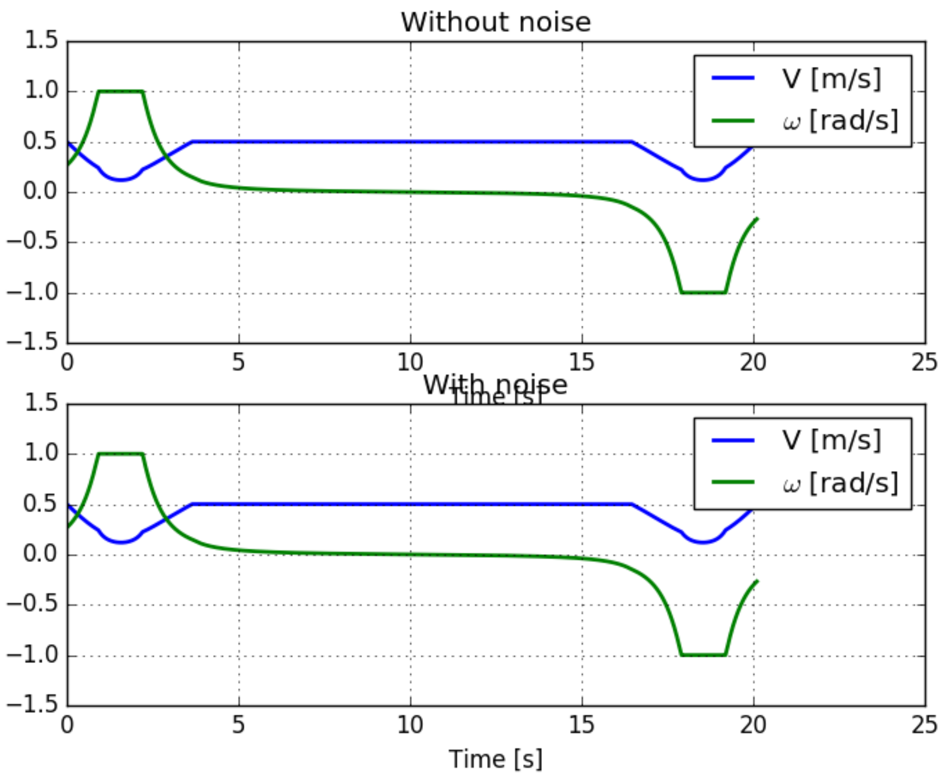
P2 (b) Arc Length

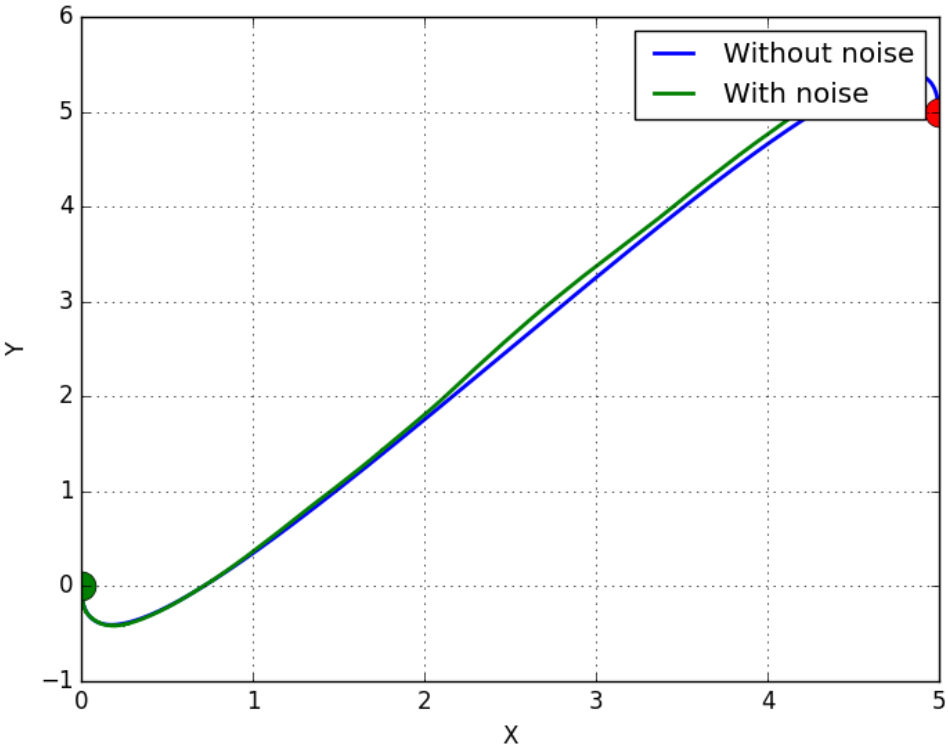


P2 (c) V & w control history



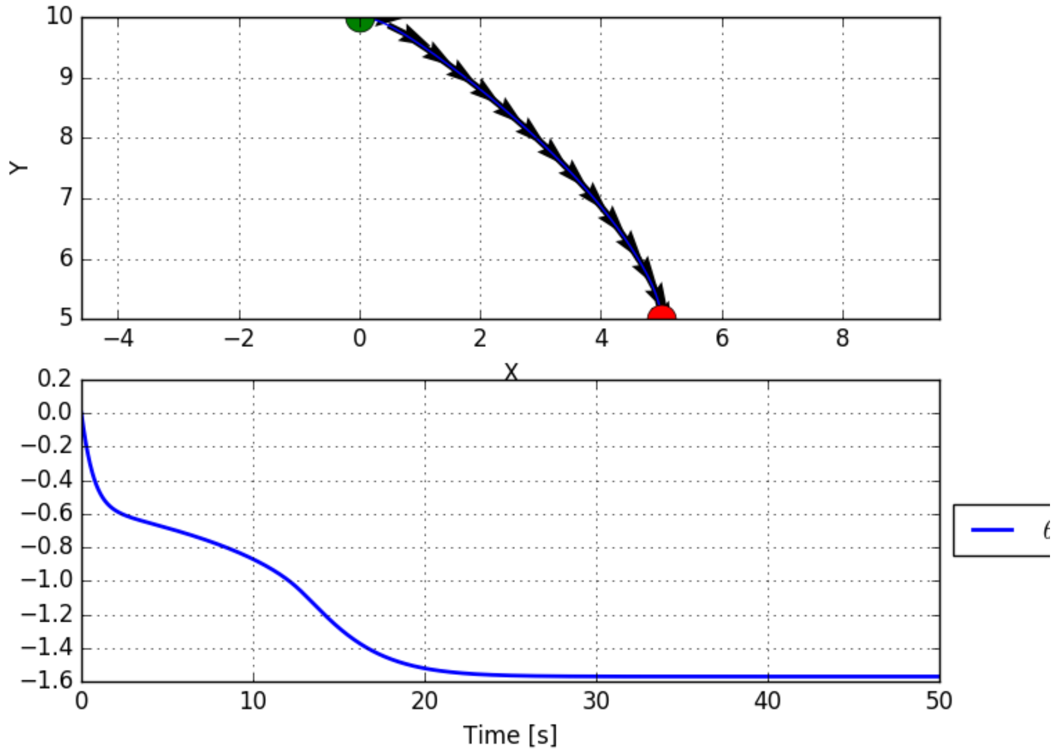
P2 (V) Trajectory with disturbance

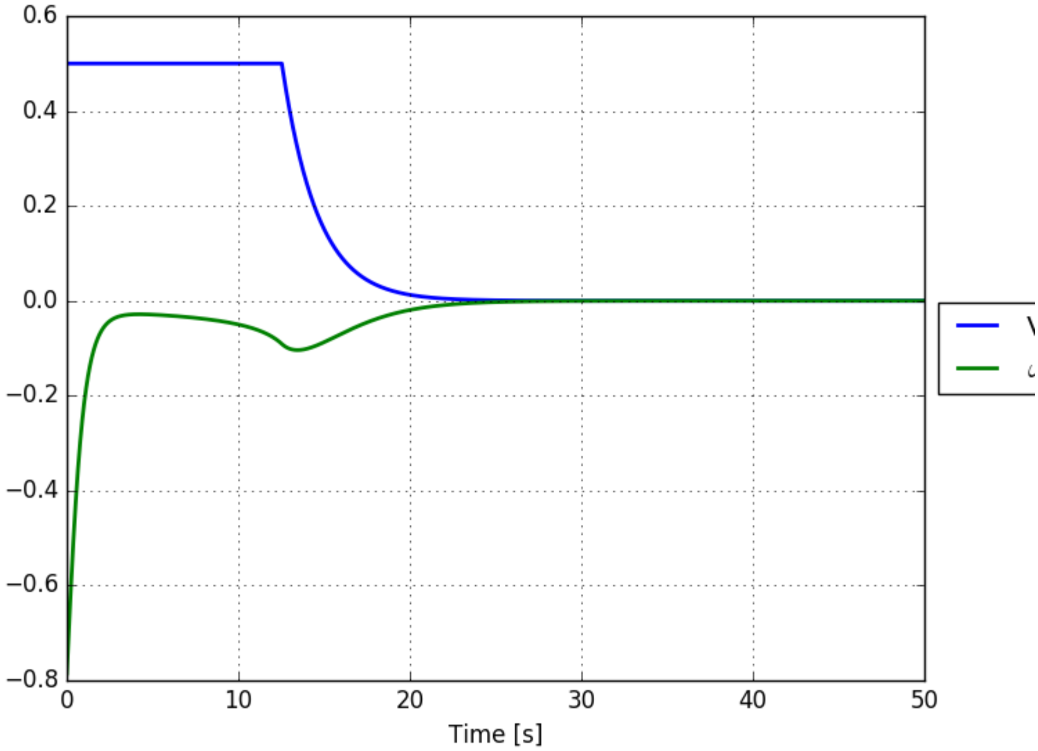




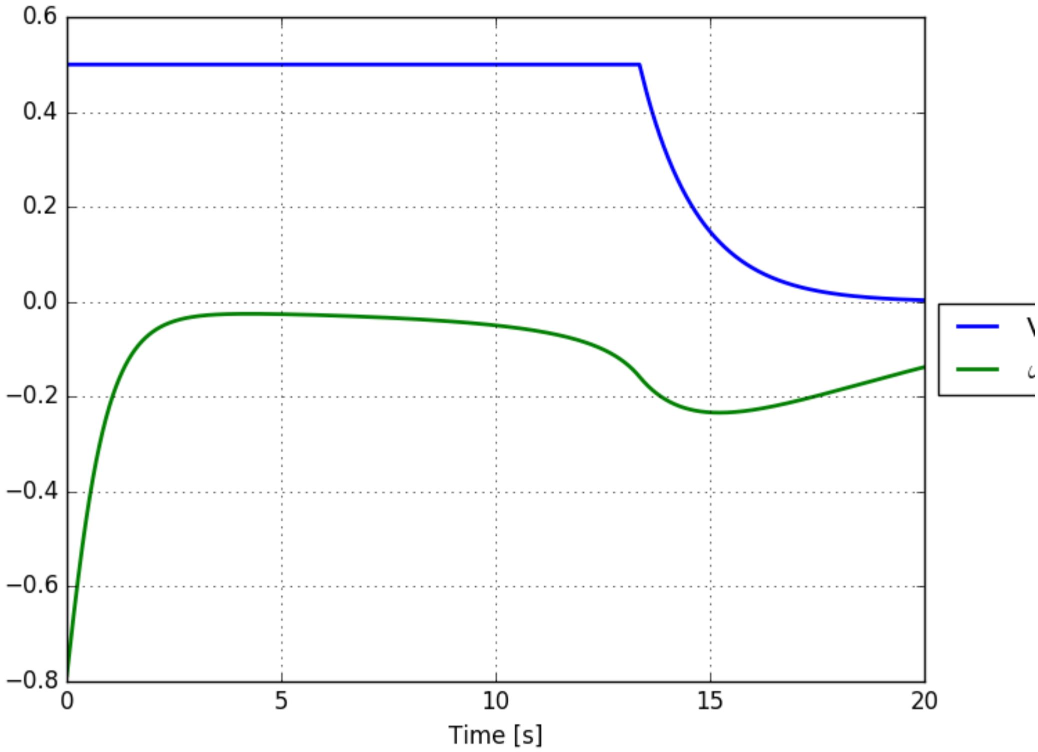
Problem 3

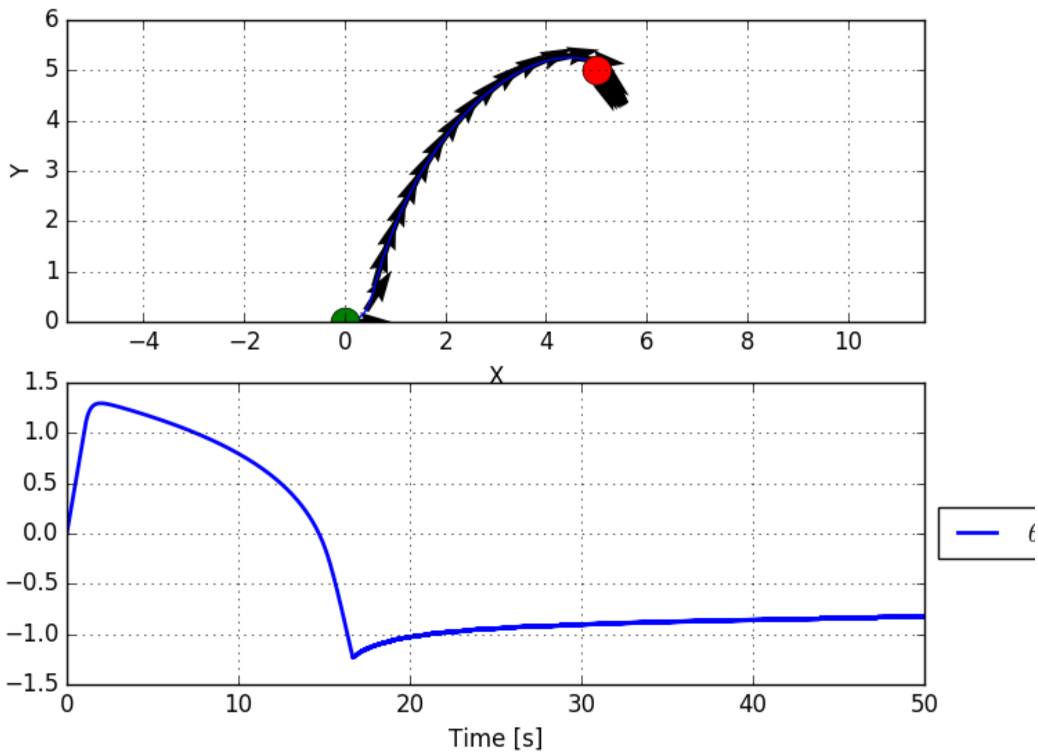
Forward Parking Plots



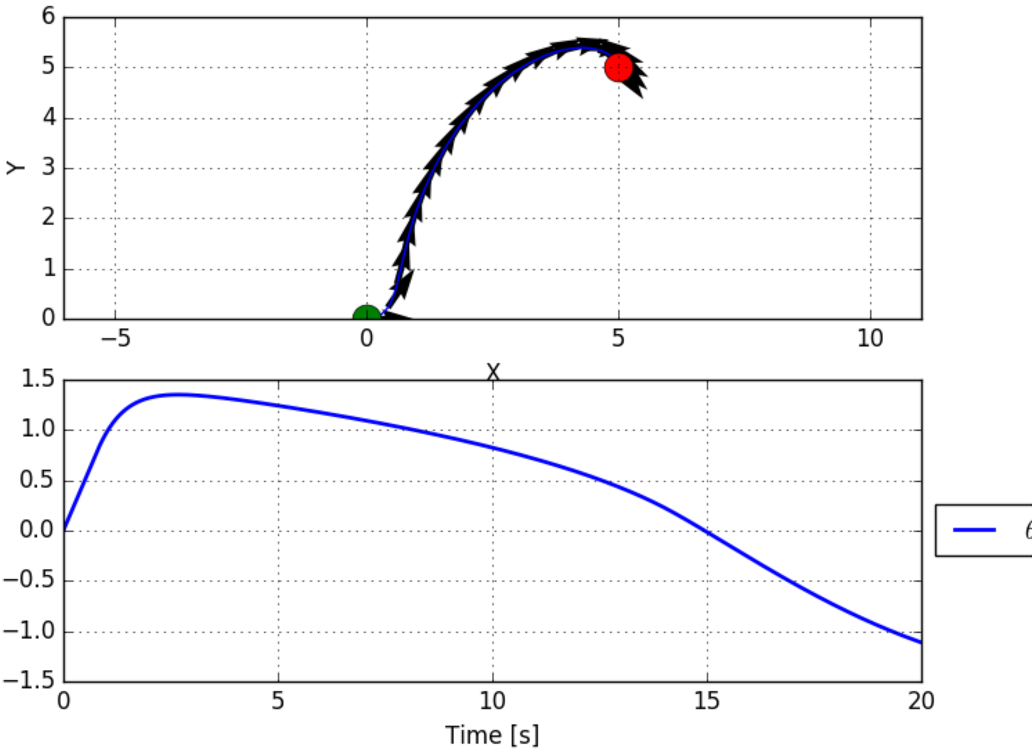


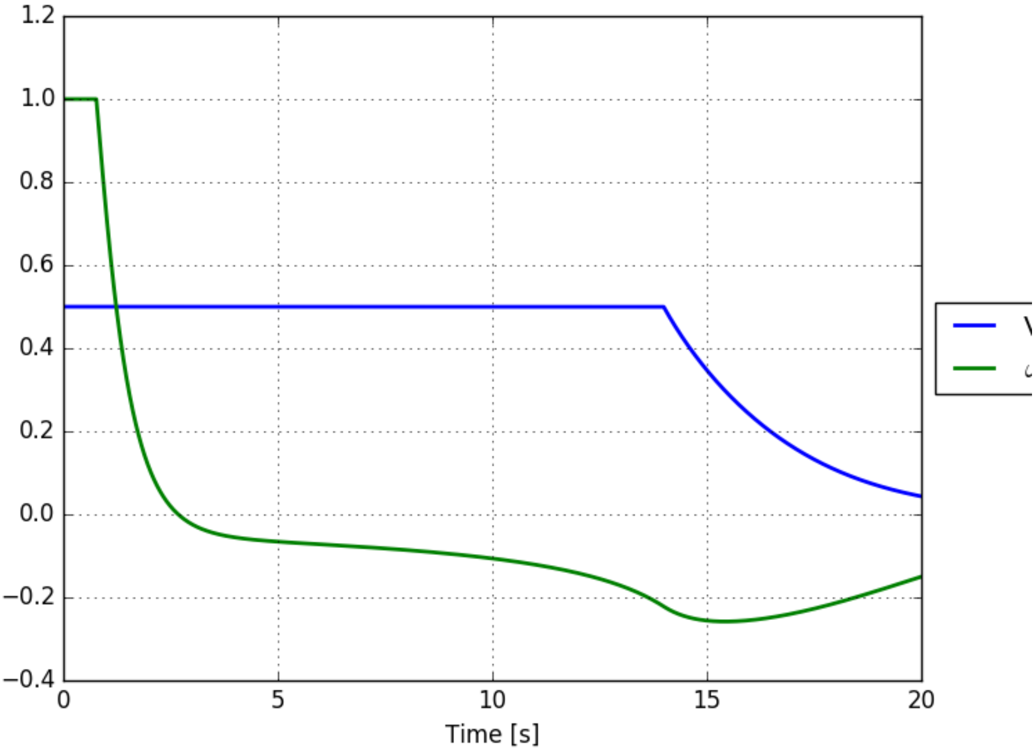
Reverse Parking Plots





Parallel Parking





Problem 4

Problem 4.

Because $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -V\sin\theta \\ \sin\theta & V\cos\theta \end{bmatrix} \begin{bmatrix} a \\ w \end{bmatrix}$ and then

$$\begin{cases} u_1 = \dot{V}\cos\theta - wV\sin\theta \\ u_2 = \dot{V}\sin\theta + wV\cos\theta \end{cases}$$

$$(u_1 + wV\sin\theta)^2 + (u_2 - wV\cos\theta)^2 = \dot{V}^2 (\sin^2\theta + \cos^2\theta)$$

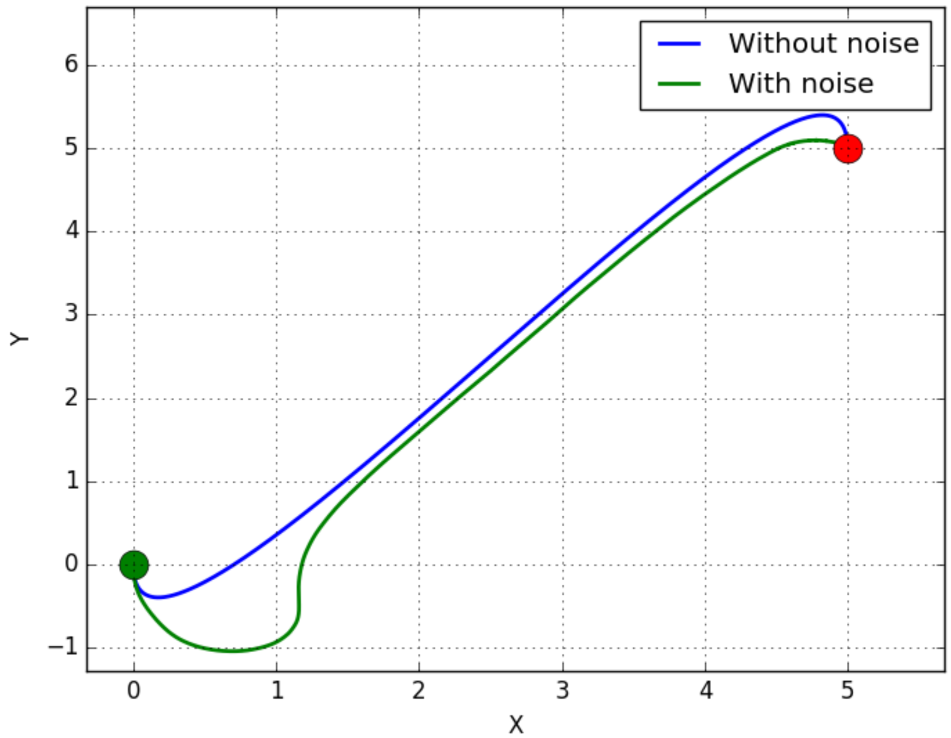
$$\Rightarrow \frac{dV}{dt} = \sqrt{(u_1 + wV\sin\theta)^2 + (u_2 - wV\cos\theta)^2}$$

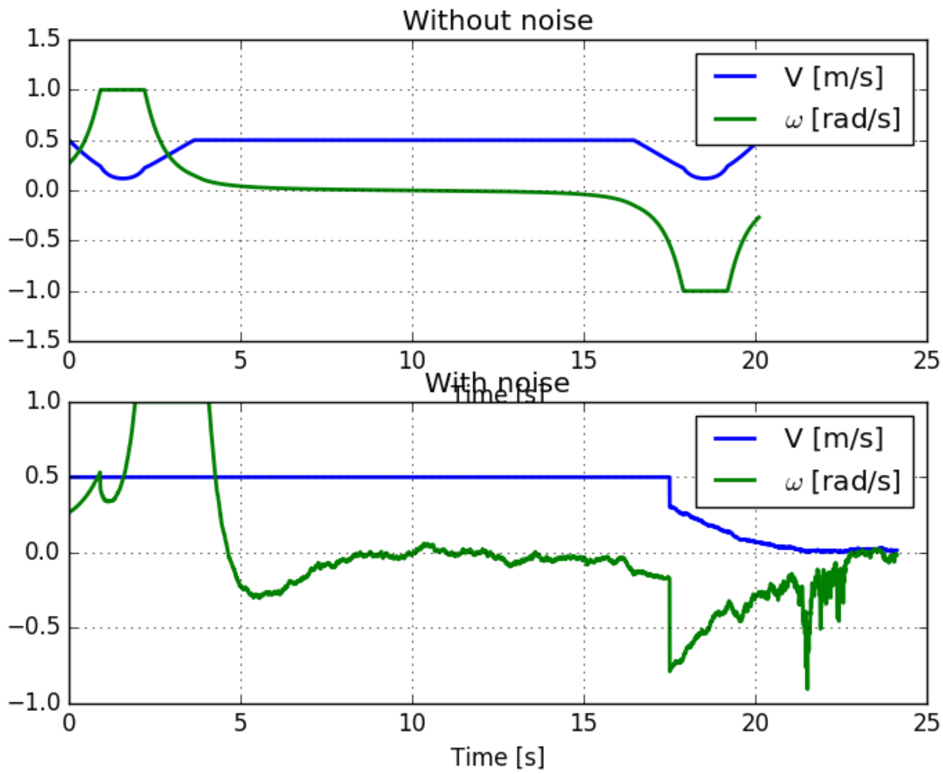
$$\frac{u_1 + wV\sin\theta}{u_2 - wV\cos\theta} = \frac{\cancel{\dot{V}}\cos\theta}{\cancel{\dot{V}}\sin\theta}$$

$$\Rightarrow w = \frac{1}{V}(u_2\cos\theta - u_1\sin\theta)$$

Closed form ODE system is $\begin{cases} \dfrac{dV}{dt} = \sqrt{(u_1 + wV\sin\theta)^2 + (u_2 - wV\cos\theta)^2} \\ w = \dfrac{1}{V}(u_2\cos\theta - u_1\sin\theta) \end{cases}$

Simulation Plots


Without noise
With noise
Y
X
6
5
4
3
2
1
0
−1
0
1
2
3
4
5


Without noise
V [m/s]
ω [rad/s]
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0
5
10
15
20
25
Time [s]
With noise
V [m/s]
ω [rad/s]
1.0
0.5
0.0
−0.5
−1.0
0
5
10
15
20
25
Time [s]

Problem 5

To verify the contents of rosbag:

#Play Back a rosbag file

Rosbag play <filename.bag>

#Using rostopic to listen to the playback log

rostopic echo random_strings