Numerische Mathematik

Dynamischer Fall

$$\mu \ddot{w} + (EJw'')'' = q \quad , \quad \text{RB wie vorher}$$

$$\Leftrightarrow \int_0^L \mu \ddot{w} \psi \, dx + \int_0^L (EJw'')'' \psi \, dx = \int_0^L q \psi \, dx + (RB)$$

$$\ddot{w}(x,t) = \sum_{k=1}^{2n} \ddot{u}_k(t) \varphi_k(x) \quad , \psi = \varphi_j$$

$$\underbrace{\sum_k \left(\int_0^L \mu \varphi_j \varphi_k \, dx \right)}_{M_{jk}} \underbrace{\ddot{u}_k}_{(\ddot{\bar{u}}_k)} + \sum_k \underbrace{\left(\int_0^L EJ \varphi_j'' \varphi_k'' dx \right)}_{S_{ij}} \underbrace{u_k}_{\bar{u}_k} = \underbrace{\int_0^L q \varphi_j}_{(\bar{q})_j} dx + (RB)$$

$$M\ddot{\bar{u}} = S\bar{u} = \bar{q} - Q^0 \bar{e}^{(0)} - M^0 \bar{d}^{(0)} + Q^L \bar{e}^{(L)} + M^L \bar{d}^{(L)}$$

$$\overset{RB}{\Rightarrow} \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\bar{c}} \\ \ddot{\bar{\lambda}} \end{bmatrix} + \begin{bmatrix} S & C \\ C^T & 0 \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{\lambda} \end{bmatrix} = \begin{bmatrix} \bar{f} \\ \bar{a} \end{bmatrix} \qquad \bar{a} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Newmark-Methode

Die Funktion $u : [t_1, t_e] \Rightarrow \mathbb{R}^n$ sei eine Lösung der DGL

$$f(\ddot{u}(t), \dot{u}(t), u(t), t) = 0$$

$t_1 < t_2 < \ldots < t_{n-1} < t_n = t_e$ gesucht: Näherungswerte $u_j \approx u(t_j)$, $\dot{u}_j \approx \dot{u}(t_j)$, $\ddot{u}_j \approx \ddot{u}(t_j)$ mit $j = 1, \ldots, n$

Forderung $u_1 = u(t_1)$, $\dot{u}_1 = \dot{u}(t_1)$, $\ddot{u}_1 = \ddot{u}(t_1)$

und $f(\ddot{u}_{j+1}, \dot{u}_{j+1}, u_{j+1}, t_{j+1}) = 0 \quad j = 1, \ldots, n-1$

Herleitung der Berechnungsvorschrift: Taylorentwicklung

$$u(t_{j+1}) = u(t_j) + \dot{u}(t_j)\, h_j + \frac{\ddot{u}(\tau_1)}{2} h_j^2 \qquad h_j = t_{j+1} - t_j,\quad \tau_1 \in [t_j, t_{j+1}]$$

$$\dot{u}(t_{j+1}) = \dot{u}(t_j) + \ddot{u}(\tau_2)\, h_j \qquad \tau_2 \in [t_j, t_{j+1}]$$

$$u_{j+1} = u_j + \dot{u}_j \cdot h_j + \left[\left(\tfrac{1}{2}-\beta\right)\ddot{u}_j + \beta\, \ddot{u}_{j+1}\right] \cdot h_j^2$$

$$\dot{u}_{j+1} = \dot{u}_j + \left[(1-\gamma)\,\ddot{u}_j + \gamma \cdot \ddot{u}_{j+1}\right] \cdot h_j$$

$$\beta \in \left[0, \tfrac{1}{2}\right] \qquad \gamma \in [0,1]$$

$$u_j^* = u_j + \dot{u}_j h_j + \left(\tfrac{1}{2}-\beta\right)\ddot{u}_j h_j^2$$
$$\dot{u}_j^* = \dot{u}_j + (1-\gamma)\,\ddot{u}_j h_j$$

Damit ist

$$u_{j+1} = u_j^* + \beta \cdot \ddot{u}_{j+1} \cdot h_j^2$$
$$\dot{u}_{j+1} = \dot{u}_j^* + \gamma \cdot \ddot{u}_{j+1} \cdot h_1$$

$$f\left(\ddot{u}_{j+1},\ \dot{u}_j^* + \gamma\, \ddot{u}_{j+1} h_j,\ u_j^* + \beta\, \ddot{u}_{j+1} h_j^2,\ t_{j+1}\right) = 0$$

$$f(\ddot{u}, \dot{u}, u, t) = \tilde{M}\ddot{u} + \tilde{D}\dot{u} + \tilde{S}u - \tilde{p}(t)$$

$$\left(\tilde{M} + \gamma\, h_j \tilde{D} + \beta h_j^2 \tilde{S}\right)\ddot{u}_{j+1} = \tilde{p}(t_{j+1}) - \tilde{D}\dot{u}_j^* - \tilde{S}u_j^*$$

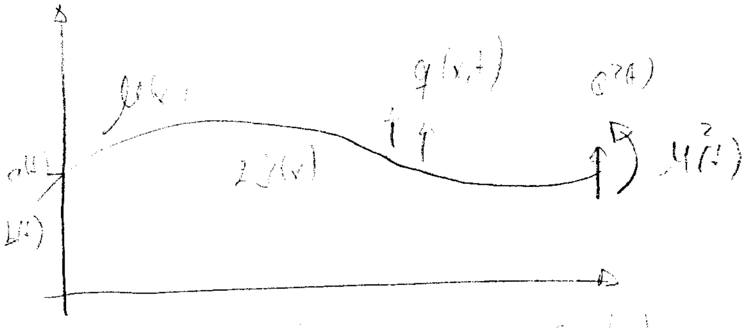
unser System:

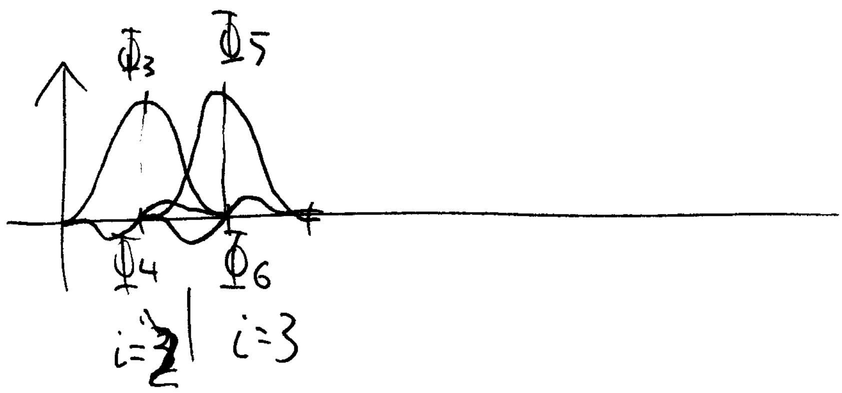
$$\left(\tilde{M} + \beta\, h_j^2 \cdot \tilde{S}\right)\ddot{u}_{j+1} = \tilde{p}(t_{j+1}) - \tilde{S}u_j^*$$

$\beta = \frac{1}{4}$, $\gamma = \frac{1}{2}$ stabil für alle h_j, aber Phasenverschiebung

$\beta = \frac{1}{12}$, $\gamma = \frac{1}{2}$ minimale Phasenverschiebung, aber nur für genügend kleine h_j stabil

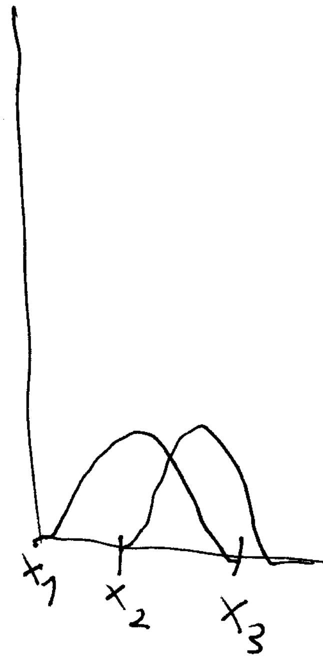
h_j - Zeitschritt


q(x,t)



$$\int_0^L E(x)\, I(x)\, \varphi_j''\, \varphi_k''$$

Stückweise konstant



$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$\begin{bmatrix} 0\ 1\ 2 & 2\ 3\ 4 \\ 4\ 5\ 6 & 7\ 8\ 9 \end{bmatrix}$$