Supplemental note

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1 Binomial distribution → Normal distribution

Start from Eq.(C6) shown below.

$$\begin{aligned} P(n) &= \frac{M!}{n!(M-n)!}p^n(1-p)^{M-n} && (1)\\ \ln P(n) &= \ln M! - \ln n! - \ln(M-n)! + n\ln p + (M-n)\ln(1-p) && (2)\\ &= M(\ln M - 1) - n(\ln n - 1) - (M-n)(\ln(M-n) - 1) && (3)\\ &\quad + n\ln p + (M-n)\ln(1-p) && (4)\end{aligned}$$

Here we used Stirling's approximation valid for large n.

$$\ln n! \simeq n(\ln n - 1) \tag{5}$$

Take the 1st derivative of the above equation in terms of n.

$$\begin{aligned} \frac{d\ln P}{dn} &= -(\ln n + 1) + (\ln(M-n) + 1) + \ln p - ln(1-p) && (6)\\ &= -\ln n + \ln(M-n) + \ln p - \ln(1-p) && (7)\\ &= \ln\left[\frac{M-n}{n}\right] - \ln\left[\frac{1-p}{p}\right] && (8)\end{aligned}$$

When n and M are both large, the peak in $P(n)$ and also in $\ln P(n)$ is very sharp around the mean value $n = \langle N\rangle$, and thus

$$\begin{aligned} \left.\frac{d\ln P}{dn}\right|_{n=\langle N\rangle} &= \ln\left[\frac{M-\langle N\rangle}{\langle N\rangle}\right] - \ln\left[\frac{1-p}{p}\right] = 0 && (9)\\ \therefore \frac{M-\langle N\rangle}{\langle N\rangle} &= \frac{1-p}{p} && (10)\\ \langle N\rangle &= Mp && (11)\end{aligned}$$

Consider a Taylor expansion of $\ln P(n)$ around the mean $\langle N\rangle$ by defining $n = \langle N\rangle + \delta n$.

$$\ln P(n) = \ln(\langle N\rangle) + \left.\frac{d\ln P}{dn}\right|_{n=\langle N\rangle}\delta n + \frac{1}{2}\left.\frac{d^2\ln P}{dn^2}\right|_{n=\langle N\rangle}\delta n^2 + \cdots \tag{12}$$

From Eq.(7),

$$\begin{aligned} \frac{d^2\ln P}{dn^2} &= -\frac{1}{n} - \frac{1}{M-n} && (13)\\ \left.\frac{d^2\ln P}{dn^2}\right|_{n=\langle N\rangle} &= -\frac{1}{Mp} - \frac{1}{M-Mp} = -\frac{1}{Mp(1-p)} = -\sigma^{-2} && (14)\end{aligned}$$

Terminate Eq.(12) at the 2nd order in terms of δn, and use Eqs.(9) and (14).

$$P(n) = const. \times \exp\left(-\frac{\delta n^2}{2\sigma^2}\right) \tag{15}$$

Determine *const.* so that that $\int_{-\infty}^{\infty} P(n)dn = 1$, we finally obtain the following normal distribution function with $\langle N \rangle = Mp$ and $\sigma^2 = Mp(1-p)$.

$$P(n) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(n - \langle N \rangle)^2}{2\sigma^2}\right) \tag{16}$$

2 Binomial distribution → Poisson distribution

We now consider the limit of $M \to \infty$ while $\langle N \rangle = Mp = a$ remains constant. Notice that the following approximations hold.

$$M - n \simeq M \tag{17}$$

$$\frac{M!}{(M-n)!} = M(M-1)\cdots(M-n+1) \simeq M^n \tag{18}$$

Again start from Eq.(C6), we can derive Poisson distribution as shown below with $\langle N \rangle = Mp = a$ and $\sigma^2 = Mp(1-p) \simeq a$.

$$\begin{aligned} P(n) &= \frac{M!}{n!(M-n)!} p^n (1-p)^{M-n} &(19)\\ &\simeq \frac{1}{n!} M^n \left(\frac{a}{M}\right)^n \left(1 - \frac{a}{M}\right)^M &(20)\\ &\simeq \frac{a^n e^{-a}}{n!} &(21) \end{aligned}$$

Here we used

$$\lim_{M\to\infty} \left(1 - \frac{a}{M}\right)^M = e^{-a} \tag{22}$$