PREFACE

This is a precise and concise solution guide to the past ten years of the G.C.E. Advanced level Pure mathematics with statistics Papers 1, 2 and 3 questions and answers provided for both teachers and students. This piece of work has been made available to help and guide students on how to tackle examination questions. Teachers are advised to use the workbook and revise with the students when the program has been covered.

The first edition of a workbook cannot be without errors and typos. You will certainly encounter some mishaps. Do not hesitate to report by contacting us at our email address given. That will be your own contribution in the struggle to demystify mathematics.

DEDICATIONS

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My profound gratitude to the **almighty God** for his endless mercy and graces. Special thanks to all my colleagues and friends especially Tchantchou Gerbault Ludovic, Sonkoue Stephanie, Donfack Zanguim Romeo,...

Contents

Chapter 1

PURE MATHEMATICS WITH STATISTICS PAPER 1 MCQ

1.1 GCE JUNE 2019 QUESTIONS

1. If $|x+1| < 2$, then
   A) $-2 < x < 2$ B) $x < -3$ or $x > 1$ C) $-3 < x < 1$ D) $x < -2$ or $x > 2$

2. Given that $x > 0$, the value of x for which $\log_4 x + \log_4(6x+10) = 1$ is
   A) $\frac{2}{3}$ B) $\frac{1}{3}$ C) $-\frac{2}{3}$ D) $\frac{3}{5}$

3. If $f(x) = x^3 - 2x - 1$ then $f(-2) =$
   A) -11 B) -13 C) -5 D) -3

4. The tangent of the acute angle between the lines $y - 4x - 3$ and $y = x - 5$ is
   A) $\frac{2}{5}$ B) $\frac{5}{2}$ C) $\frac{5}{3}$ D) $\frac{3}{5}$

5. The period of the function $f(x) = 3\cos 5x$ is
   A) $\frac{2\pi}{5}$ B) $\frac{5\pi}{2}$ C)$\frac{2\pi}{3}$ D) $\frac{3\pi}{2}$

6. $\frac{2x}{(x-3)(x-5)} \equiv$
   A) $\frac{3}{x-1} + \frac{5}{x-5}$ B) $-\frac{3}{x-3} + \frac{5}{x-5}$ C) $\frac{3}{x-3} - \frac{5}{x-5}$ D) $-\frac{3}{x-3} - \frac{5}{x-5}$

7. The function $f(x) = x^3 + 1$ is continuous in the interval $[-2, 4]$ and differentiable on the interval $(-2, 4)$. The value of the constant c, $-2 < c < 4$ for which $f'(c) = \frac{f(4)-f(-2)}{4-(-2)}$
   A) $\sqrt{7}$ B) $\frac{2}{\sqrt{3}}$ C) $\sqrt{3}$ D) 2

8. An arithmetic progression has first term $\ln 2$ and common difference $\ln 4$. The sum of the first four terms of this progression is
   A) $16\ln 4$ B) $16\ln 2$ C) $10\ln 4$ D) $8\ln 2$

9. If $x^2 + y^2 = 16x$, then $\frac{dy}{dx} =$ A) $\frac{8+x}{y}$ B) $\frac{8-x}{y}$ C)$-\frac{8+x}{y}$ D) $-\frac{8-x}{y}$

10. The radius of the circle with equation $x^2 + y^2 - 10x + 12y + 41 = 0$ is
    A) $5\sqrt{2}$ B) $3\sqrt{5}$ C)$2\sqrt{5}$ D) $4\sqrt{5}$

11.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
C	B	C		A	B	D	B	B	C	D	C	B	B	A	C	C	C	B	A	B	D	D	D
26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49
A	D	B	C	C	C	A	A	D	C	B	D	B	C	C	A	A	B	A	C	A	D	D	C

1.2 GCE JUNE 2018 QUESTIONS

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
C	B	C		A	B	D	B	B	C	D	C	B	B	A	C	C	C	B	A	B	D	D	D
26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49
A	D	B	C	C	C	A	A	D	C	B	D	B	C	C	A	A	B	A	C	A	D	D	C

Chapter 2

PURE MATHEMATICS WITH STATISTICS PAPER 2

2.1 GCE JUNE 2010 QUESTIONS

1. (a) The polynomial $f(x)$ where $f(x) = x^3 - ax^2 + bx + 8$, leaves a remainder of 6 when divided by $x - 1$ and a remainder of -70 when divided by $x + 3$. Find the values of the constants a and b. Hene factorise $f(x)$ completely.

   (b) Given that the equation $(k - 3)x^2 + 4x + k = 0$ has real roots, find

      i. the range of values of k,

      ii. the set of values of k for which the roots have the same sign.

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2. (a) Given that $2x\frac{dy}{dx} + 1 = y^2$ and that $y = 2$ when $x = 1/3$, show that $y = \frac{1+x}{1-x}$.

   (b) Sketch the graph of $y = \frac{1+x}{1-x}$, showing clearly the asymptotes and the points where the curve meets the coordinates axes.

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3. (a) A function f with domain $\{x : x \in \mathbb{R}, x \neq 0\}$ and codomain $\{x : x \in \mathbb{R}, x \neq b\}$, where $\mathbb{R}$ is the set of real numbers, is defined by $f : x \mapsto \frac{a+bx}{x}$ for some constants a and b. Given that $f(4) = 7$ and $f^{-1}(4) = -2$

      i. find the values of the constants a and b,

      ii. show that f is surjective

   (b) Find the set of values of x for which $|\frac{x^2-9}{x+1}| = \frac{9-x^2}{x+1}$

---

4. (a) A handball team of 7 players is to be selected from a group of 15 players in which there are 2 brothers. Find the number of ways in which the team can be selected if

      i. the 2 brothers must be included in the team,

      ii. at most one of the 2 brothers is to be included in the team.

   (b) The parametric equations of a curve are given by $x = at^2,\ quady = 2at$, where a is a constant and t a parameter. Show that the equation of the tangent to the curve at the point with parameter t is $ty = x + at^2$.

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5. (a) Evaluate $\int_{-1}^{0} \frac{2x^2+8}{(x-3)(x^2+1)} dx$ leaving your answer in terms of natural logarithms.

   (b) The area bounded by the curve $y = 3 + 2x - x^2$ and the line $y = 3$ is rotated completely about the line $y = 3$. Find the volume of the solid of revolution obtained.

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6. (a) i. Differentiate y with respect to x where $y = \cos\left(\frac{\pi}{4} - 3x^2\right)$

   ii. Find $\frac{dy}{dx}$ at the point $(-2, 2)$ given that $2x^2 - 5y^2 = 3xy$

   (b) Find the sine of the angle which the plane π, given by the equation $x + y - z = 24$ makes with the line passing through the points with position vectors $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $4\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$.

7. (a) Find the general solution of the equation $\cos\theta + 2\sin\theta = 0$

   (b) Express $f(x)$, where $f(x) = \cos x + \sin x$, in the form $R\cos(x - \lambda)$, where $R > 0$ and λ is an acute angle. Hence find the value of x for which the minimum value of $\frac{1}{f(x)}$ occurs.

8. (a) Prove by mathematical induction that $9^n - 1$ is divisible by 8 for all positive integers n.

   (b) deterrmine the value(s) of x for which $\sum_{r=0}^{\infty}\left(\frac{x-3}{x+1}\right)^r = \frac{3}{4}$.

9. (a) Express in the form $a + ib$, the complex number z, where a and b are real constants given that $\left(\frac{4-3i}{2-i}\right)z - (1 + 3i) = 1 - 2i$.

   (b) Verify that the comlex numbers $z_1 = 1 - i\sqrt{3}$ and $z_2 = 1 + i\sqrt{3}$ are roots of $p(z) = 0$ where $P(z) = z^4 - 3z^3 + 8z - 24$. Hence find the other roots of $P(z) = 0$.

10.

x	2	3	4	5	6	7
y	50	250	775	1875	390	7200

The table above shows corresponding values of x and y which approximately satisfy a relation of the form $y = ax^n$, where a and n are constants. By drawing a suitable linear graph, determine the values of a and n correct to one decimal place.

2.2 GCE JUNE 2011 QUESTIONS

1. A real-valued function f is defined by $f : x \mapsto \frac{2x-3}{x+1}, x \in \mathbb{R}, x \neq -1$.

   (a) Obtain the composite function ff and state its domain.

   (b) Show that f is injective.

   (c) Find the range of f and deduce that f is not surjective.

2. (a) Find $\frac{dy}{dx}$, , where

   i. $y = \ln\left(\frac{x^2-2}{x^2+3}\right)$,

   ii. $x + y + \cos xy = 0$

   (b) Given that $x = e^t\cos t$ and $y = e^t\sin t$, find $\frac{dy}{dx}$ when $t = \pi$.

3. (a) Show that $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$

   (b) Find, in radians, the general solution of the equation $\cos 2x + 1 = \sin 2x$.

4. Vector parametric equations of the lines L_1 and L_2 are given by $L_1 : \mathbf{r}_1 = 2\mathbf{i} + \mathbf{j} + \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ and $L_2 : \mathbf{r}_2 = 2\mathbf{i} + 2\mathbf{j} + t\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ where t is a constant. Find

   (a) the value of t for which L_1 and L_2 intersect,

   (b) the position vector of the point of intersection of L_1 and L_2,

   (c) the cosine of the acute angle between L_1 and L_2,

   (d) a vector parametric equation of the plane containing L_1 and L_2.

5. (a) Find the set of values of x for which the geometric series $\sum_{r=0}^{\infty}\left(\frac{x-1}{x+2}\right)^r$, $x \neq -2$ is convergent.
   (b) A woman has 10 closed friends. Find the number of ways in which she can invite 4 of them to the dinner if
      i. each of them is equally likely to attend,
      ii. two of the friends are married and will not attend separately,
      iii. two of them are not on speaking terms and will not attend together.

6. (a) Find the values of x for which $3^{2x+1} - 12(3^x) + 9 = 0$.
   (b) Given that p and q are the roots of the equation $9x^2 - 4ax = -(1+6x)$, where a is a real constant, show that the equation whose roots are $1/p$ and $1/q$ is $x^2 + (6-4a)x + 9 = 0$.

7. (a) Given the complex numbers $z_1 = 10 + 5i$ and $z_2 = \frac{2(1+i)}{(3-i)}$, express z_2 and $z_1 z_2^*$ in the form $a + ib$, where $a, b \in \mathbb{R}$ and z_2^* is the comlex conjugate of z_2.
   (b) Find the locus of points represented by the complex numbers z, such that $2|z-3| = |z-6i|$

8. (a) The table below shows the values of a continuous variable y corresponding to given values of x.

x	2	3	4	5	6
y	13.6	27.2	54.4	108.8	217.8

   Use the trapezium rule to find an estimate for $\int_2^6 y dx$.
   (b) Show that the equation $x^3 + x - 6 = 0$ has a root between 1 and 2.
   Using the Newton-Raphson's method and taking 1.6 as the first approximation, determine by means of two itterations, two other approximations for the root, giving your answer correct to 3 decimal places.

9. Define the concept of an equivalence relation.
   A relation R is defined on the set $\mathbb{Z}$, of integers, by $aRb \iff a^2 - a = b^2 - b$. Show that R is an equivalence relation . Write down 3 elements of the relation R, where aRb but $a \neq b$.

10. (a) Evaluate $\int_0^{\frac{\pi}{6}} x \cos x dx$
    (b) Express $\frac{2}{(1+x)(1+3x)}$ in partial fractions. Hence or otherwise, solve the differential equation $\frac{dy}{dx} = \frac{2(y+2)}{(1+x)(1+3x)}$ given that $y = -1$ when $x = 0$.

2.3 GCE JUNE 2012 QUESTIONS

1. (a) Given that the roots of the equation $x^2 - bx + c = 0$ are α and β, form quadratic equations whose roots are $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$.
   (b) Find the value(s) of the constant k for which the roots of the equation $x^2 - 2(1+3k)x + 7(2k+3) = 0$ are equal.

2. (a) The functions f and g are defined by $f : x \mapsto \frac{x+1}{2x+3}, x \in \mathbb{R}, x \neq -\frac{3}{2}$ $\quad g : x \mapsto 3x+2, x \in \mathbb{R}$.
      i. Express in a similar manner, the function $(gf)^{-1}$
      ii. Obtain an element in the domain of g which is invariant under g.

3. (a) Find the range of values of x for which $\frac{3x-10}{x-4} < 2$.
   (b) Solve the differential equation $(1+x^2)\frac{dy}{dx} = xy$, given that $y = 1$ when $x = 0$.
   (c) Show that if x is so small that x^3 and higher powers can be neglected, then $\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} = 1 + x + \frac{1}{2}x^2$.

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4. The expression $y = ax^2 + bx$, is an approximation to a relation connecting two variables x an y, where a and b are constants. By using the values given in the table below.Draw a suitable straight line graph and use it to estimate the values of a and b.

x	1	2	3	4	5	6
y	74	126	162	172	175	144

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5. (a) $f(x) = \frac{x+1}{(x-2)(x+3)}$.
      i. Express $f(x)$ in partial fractions.
      ii. Find $\int_3^5 f(x)dx$ leaving your answer in terms of natural logarithms.
   (b) Find $\int x \sin 2x dx$.

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6. (a) Given that $y = \cos^{-1}\left(\frac{a}{b}x\right)$, show that $\frac{dy}{dx} = \frac{-a}{\sqrt{b^2-a^2x^2}}$
   (b) Prove tha the curve $y = x^m(1-x)^n$, where $m, n \in \mathbb{Z}^+$, has a stationary point when $x = \frac{m}{m+n}$, show that whem $m = n = 2$, the stationary point is a maximum point.

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7. (a) Find the general solution of the equation $\sin 2\theta = \sin\theta$
   (b) $f(x) \equiv \cos x + 2\sin x$.
      i. Express $f(x)$ in the form $r\cos(x-\alpha)$, where $r > 0$ and $0 < \alpha < 90°$.
      ii. Find to the nearest tenth of a degree, the set of values of x which satisfy the equation $\cos x + 2\sin x = 1$.
      iii. Show that $-\sqrt{5} \leq \cos x + 2\sin x \leq \sqrt{5}$.

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8. (a) Find in the form $a + ib$, $a, b \in \mathbb{R}$, the complex number z such that $\frac{(2-i)z}{1+2i} - (3-4i) = 0$
   (b) Given that $z = x + iy$, $x, y \in \mathbb{R}$, find the locus of the point z, in an Argand diagram, for which the imaginary part of $z + \frac{1}{z}$ is zero.

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9. (a) A jury is to be chosen from 5 men and 7 women. Find
      i. the number of different ways in which the jury may be chosen.
      ii. the number of juries in which women are in the majority.
   (b) Prove by mathematical induction that $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$.

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10. Given that lines $L_1 : \frac{x-10}{3} = \frac{y-1}{1} = \frac{z-9}{4}$, $L_2 : \mathbf{r} = (-9\mathbf{i} + 13\mathbf{k}) + \mu(\mathbf{i} + 2\mathbf{i} - 3\mathbf{k})$ where μ is a parameter and $L_3 : \frac{x+3}{4} = \frac{y+5}{3} = \frac{z+4}{1}$.
    (a) Show that the point $P(4, -1, 1)$ is common to the lines L_1 and L_2.
    (b) Find the point of intersection of L_2 and L_3.
    (c) Find a vector parametric eqaution of the plane containing the lines L_2 and L_3.

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x	1.59	2.39	4.16	6.31	8.70	12.01
y	3.68	6.91	16.59	31.70	52.48	87.10

2.4 GCE JUNE 2013 QUESTIONS

1. (In this question you are adviced to wrok throughout with 2 decimal places). The table below shows corresponding values of x and y obtained in a certain experiment.
   The relation connecting x and y is given by $y = \lambda x^n$ where λ and n are constants. By drawing a suitable linear graph relating $\log_{10} x$ and $\log_{10} y$, calculating the values of λ and n.

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2. (a) When the polynomial $P(x)$ is divided by $x-3$ and $x+2$, the remainders are 5 and -5 respectively. Given that $P(x) = (x^2 - x - 6)Q(x) + ax + b$, Find the values of the constants a and b.
   (b) Find the range of values of x for which $x + 5 < |2x + 1|$.
   (c) A mixed delegation of 5 persons is to be selected from 5 women and 4 men. Find the number of ways in which the delegation can be selected if there must be more women than men.

---

3. (a) Given that $f(x) = \sqrt{3}\cos x + \sin x$, express $f(x)$ in the form $R\cos(x - \lambda)$, where $R > 0$ and $0 < \lambda < \pi/2$. Hence find the general solution of the equation $f(x) = \sqrt{3}$.
   (b) Prove that $\frac{\sin 2x + \sin 4x}{1 + \cos 2x + \cos 4x} \equiv \tan 2x$.
   $$\text{Hence find the general solution of the equation } 1 - \frac{1 + \cos 2x + \cos 4x}{\sin 2x + \sin 4x} = 0.$$

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4. (a) Find the range of values of k for which the equation $x^2 + x + k^2 - 12 = 0$ has real and distinct roots.
   (b) Given that $f(x) = \frac{2-3x}{(1-x)(2-x)}$, express $f(x)$ as a series in ascending powers of x up to and including the term in x^3, giving the range of values of x for which the expansion is valid.

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5. (a) Given that $f(x) = x - 5 + x^2 \ln x$, show that the equation $f(x) = 0$ has a root in the interval $2 < x < 3$. Taking $x = 2$ as a first approximation to the root of the equation, use one itteration of the Newton-Raphson's procedure to find a second approxiamtion to this root, giving your answer correct to 2 decimal places.
   (b) The finction h is defined for all real values of x by $h(x) = ax^2 + bx - 5$. Given that $h(x)$ has a turning point at $\big(3/4, -49/8\big)$, find the value of the cosntants a and b..

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6. (a) The fucntions f and g are defined on $\mathbb{R}$, the set of real numbers by $f : x \mapsto \frac{x}{x-1}, x \neq 1$ $g : x \mapsto 2x - 3$. Find the composite function $f \circ g$ stating its domain.
   (b) A binary relation R is defined on the set of integers by $xRy \iff x - y = 3c$, where c is an integer. Prove that R is an equivalent relation.

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7. (a) The sum of the first, third and seventh term of an arithmetic progression is 25. If the thirtheenth term is three times the fourth term, find the first term and common difference of the progression.
   (b) Prove by mathematical induction that $\sum_{r=1}^{n}(2 + 3r) = \frac{n}{2}(3n + 7)$ for all positive integers n.

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8. Given that A is the point $(5, -1, 2)$, π is the plane with vector equation $r \cdot (2i + 6j + 9k) = 33$ and **O** is the origin, find

(a) the perpendicular distance of π from **O**.

(b) the vector equation of the line l which passes through A and is perpendicular to π.

(c) the coordinates of the point B where l meets π.

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9. (a) The complex number z is given by $z = \frac{(4+3i)(3+4i)}{(3+i)}$ Express z in the form $a + ib$ where $a, b \in mathbbR$.

   (b) Another complex number z_1 is such that $z_1 = \frac{i+1}{(\sqrt{3}+i)}$. Find $|z_1|$ and $arg(z_1^4.)$

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10. (a) Given that $y = 2(x-5)\sqrt{x+4}$, show that $\frac{dy}{dx} = \frac{3(x+1)}{\sqrt{x+4}}$. Hence evaluate $\int_5^{12} \frac{(x+1)}{\sqrt{x+4}} dx$.

    (b) Obtain in the form $y = f(x)$, a particular solution of the differential equation $\frac{dy}{dx} - x = 2xy$ given that $y = 0$ when $x = 0$.

2.5 GCE JUNE 2014 QUESTIONS

1. Express $\frac{6x+1}{(2x-3)(3x-2)}$ in partial fractions and hence prove that $\int_2^4 \frac{6x+1}{(2x-3)(3x-2)} dx = \ln 10$.

---

2. Express $\cos x + \sqrt{3}\sin x$ in the form $R\cos(x - \lambda)$, where $R > 0$ and $0 < \lambda < \pi/2$. Hence find

   (a) the general solution of the equation $\cos x + \sqrt{3}\sin x = \sqrt{3}$,

   (b) the maximum and minimum values of $\cos x + \sqrt{3}\sin x + 2$

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3. (a) The roots of the quadratic equation $x^2 - x + 2 = 0$ are α and β. prove that

   i. $\frac{1}{1+\alpha^2} + \frac{1}{1+\beta^2} = -\frac{1}{2}$

   ii. $\frac{1}{1+\alpha^2} \times \frac{1}{1+\beta^2} = \frac{1}{2}$

   $$\text{Hence form a quadratic equation whose roots are } \frac{1}{1+\alpha^2} \quad and \quad \frac{1}{1+\beta^2}.$$

   (b) Find the range of values of x for which $\frac{x^2-12}{x} > 1$.

---

4. (a) Given that $\mathbf{a} = i + 4j + 2k$, $\mathbf{b} = -2i + 2j + 6k$ and $\mathbf{c} = 3i - 3j - 2k$ find

   i. the vector $\mathbf{v}$, such that $\mathbf{v}$=$\mathbf{a}$+$\mathbf{b}$+$\mathbf{c}$,

   ii. the unit vector $\hat{v}$

   iii. the angle, to the nearest degree, which $\mathbf{v}$ makes with the y-axis

   (b) Find a vector equation of the line passing through the point $(4, -5, 1)$ and is parallel to the straight line with cartesian equation $\frac{x-2}{3} = \frac{y+3}{4} = \frac{1-z}{2}$.

   (c) Find a vector parametric equation of the plane containing the point $(5, 4, 3)$ and the vectors $-3i + 3k$ and $4i - 4j - 6k$

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5. (a) Given that

   i. $y = \ln\left(\frac{3+x}{3-x}\right)$, show that $\frac{dy}{dx} = \frac{6}{9-x^2}$,

   ii. $xy^2 + \cos 2y = 7x$, find $\frac{dy}{dx}$

   (b) Determine the stationary points on the curve $y = xe^{-x}$

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6. (a) Given the complex number z such that $\frac{(1+2i)(2+i)}{1+3i}$, express z in the form $a + bi$, $a, b \in \mathbb{R}$. Hence find

i. $z^2 - \left(1 + \frac{1}{2}i\right)$

ii. $\arg\left[z^2 - \left(1 + \frac{1}{2}i\right)\right]$

(b) Find the locus of w where $w = \left[x + (y-6)i\right]\left[(x+8) - yi\right]$ if

i. w is purely real

ii. w is purely imaginary

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7. (a) A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined as $f : x \mapsto \frac{3x}{2x+1}, x \neq -\frac{1}{2}$. Determine whether or not f is surjective.

(b) Prove by mathematical induction that $3^{2n} + 7$ is divisible by 8 for all integers $n > 0$.

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8. (a) Use one itteration of the Newton-Raphson method to find to 3 decimal places, a second approximation to the real roots of the equation $x^3 + 3x^2 - 1 = 0$, taking 0.2 as a first approximation to the root of the equation.

(b) Four men and three women are to be seated round a circular table. Find the number of arrangements if the 3 women must be together.

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9. (a) Given that $\left(1 + \frac{1}{3}x\right)^8 = 1 + \frac{2}{3}ax - \frac{4}{9}bx^2 + ...$, find the values of the constants a and b.

(b) The sum of the first n terms of a sequence S_n, is given by $S_n = \frac{n}{2}(3n+1)$ find the sum of the terms from the 10^{th} to the 30^{th} term.

(c) A geometric progression has its first term and common ratio as $\sin 2\theta$ and $\cos 2\theta$ respectively, where $|\cos 2\theta| < 1$. Prove that its sum to infinity is $\cot\theta$.

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10. Solve the differential equation $x(1-y)\frac{dy}{dx} = -2y$, given that $y = 2$ when $x = e$. Hence, show that $y = \ln\left(\frac{x^2y}{2}\right)$.

2.6 GCE JUNE 2015 QUESTIONS

1. (a) Given that $(x-1)$ is a factor of the polynomial $f(x)$, where $f(x) = ax^4 + x^3 - 12x^2 - x + 2$. Find the value of the constant a and verify that $f(-1)$

(b) Find the value of the constant k for which the equation $x^2 + (k+1)x + k = 0$ has one root double the other.

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2. (a) Show that $\frac{\sin\theta + \sin 2\theta}{1 + \cos\theta + \cos 2\theta} = \tan\theta$

(b) Find the general solution of the equation $\sin 4x + \cos x = 0$

(c) Solve for x, where $0 \leqslant x \leqslant 180$, the equation $\sin 3x + \cos x = 0$

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3. (a) The function $f : \mathbb{R} \mapsto \mathbb{R}$ is defined by $f(x) = x - \frac{x^3}{3}$. Find the monotonicity of f, showing clearly its variation table.

(b) Solve the differential equation $x\frac{dy}{dx} = y(2x^2 + 1)$

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4. (a) Given that $z = e^{i\theta}$, show that $z^n + z^{-n} = 2\cos n\theta$. Use this result to express $\cos^5\theta$ in terms of cosines of multiples of θ

(b) Given that $z_1 = 1 + i\sqrt{3}$ and $z_2 = -1 + i$, evaluate

i. $|z_1 z_2|^2$

ii. $\arg(z_1^4)$

---

5. The coordinates of the points A, B and C are $(0,1,3), (-1,0,1)$ and $(1,-1,2)$ respectively. Find
   (a) $\overrightarrow{AB} \times \overrightarrow{BC}$
   (b) the sine of the angle between $\overrightarrow{AB}$ and $\overrightarrow{BC}$
   (c) the value of the constant μ for which the line $\mathbf{r} = i + 2j - k + \mu(3\mu - j + 5k)$ is parallel to the plane containing A, B and C.

---

6. Given the matrix $\mathbf{A}$ where $\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ find
   (a) $\det(\mathbf{A})$, the determinant of $\mathbf{A}$
   (b) $\mathbf{A}^{-1}$, the inverse of $\mathbf{A}$

   Hence, or otherwise solve the system of equations

$$\begin{aligned} 2x + y + 2z &= 3 \\ 3x + y + 2z &= 3 \\ 2x + 2y + z &= 2 \end{aligned}$$

---

7. Express $\frac{1}{(3t+1)(t+1)}$ in partial fractions.
   By using the substitution $t = \tan x$, or otherwise, show that

$$\int_0^{\frac{\pi}{4}} \frac{dx}{3 + 5\sin 2x} = \frac{1}{8}\ln 3$$

---

8. (a) Find the set of values of x for which $\left|\frac{3x+4}{2x-3}\right| < 1$
   (b) Sketch the curve of $y = \frac{x+2}{x+1}, x \in \mathbb{R}, x \neq -1$, showing clearly the intercepts with the coordinate axes and the behavior of the curve as it approaches its asymptotes.

---

9. (a) The functions g and gof are defined by $g : x \mapsto x + 5, x \in \mathbb{R}$, and $gof : x \mapsto \frac{6(x-3)}{x-4}, x \in \mathbb{R}, x \neq 4$. find f and show that f is injective
   (b) If p is the statement: Eric plays golf and q the statement: Oscar plays tennis. Write down the statement represented by each of the following:
      i. $p \implies q$
      ii. $\sim q \implies p$
      iii. $\sim (p \vee q)$

---

10. (a) The first three terms in the series expansion of $\sqrt{\frac{1-x}{1+kx}}$ are $1, -2x$ and $4x^2$ respectively. Determine the value of k and state the range of values of x for which the expansion is valid.
    (b) Five cards are to be dealt out to a player from a standard pack of 52 playing cards. How many different possibilities are there if
       i. there is no ace
       ii. there are at least two aces

2.7 GCE JUNE 2016 QUESTIONS

1. (a) Given that the roots of the equation $x^2 - x + 2 = 0$ are α and β, find the quadratic equation whose roots are $\frac{1}{1+\alpha^2}$ and $\frac{1}{1+\beta^2}$
   (b) Given that the polynomial $P(x) = (2x-1)(x-3)Q(x) + 12x - 8$ of degree 3, is exactly divisible by $x = -1$ and that $P(0) = 10$, find $Q(x)$.

---

2. Given the matrices **M** and **N**, where $\mathbf{M} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \\ 5 & 1 & 0 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} -1 & 0 & 1 \\ 5 & 0 & -2 \\ 6 & 3 & -3 \end{pmatrix}$ find the matrix product **MN** and **NM**. Hence, find $\mathbf{M}^{-1}$ the inverse of **M**
   The transformation represented by the matrix **M** maps the points A, B and C to the points $(3,0,6), (0,5,3)$ and $(1,0,1)$ respectively. Find the coordinates of A, B and C

---

3. (a) Given that the function $f(x) = x^3$ is differentiable in the interval $(-2,2)$, use the mean value theorem to find the value of x for which the tangent to the curve is parallel to the chord through the point $(-2,-8)$ and $(2,8)$.
   (b) Express in the form $y = f(x)$, the general solution of the differential equation $y\frac{dy}{dx} = x(1+y^2)$.

---

4. (a) Use De Moivre's theorem to express $\cos 4\theta$ in terms of $\cos\theta$
   (b) Given that $z_1 = 2 + i, z_2 = -2 + 4i$ and $\frac{1}{z_3} = \frac{1}{z_1} + \frac{1}{z_2}$

---

5. The position vector of the points A, B and C are **a**,**b** and **c** respectively, where **a**=3**i**+6**k**, **b**=5**j**+3**k** and **a**=**i**+**k**. Find
   (a) the vector product
   (b) the vector equation of the plane ABC

---

6. Express $f(\theta) = 8\cos\theta - 15\sin\theta$ in the form $r\cos(\theta+\alpha)$ where r is positive and α an acute angle. Hence, find
   (a) the general solution of the equation $80\cos\theta - 150\sin\theta = 13$
   (b) the maximum and minimum values of $\frac{5}{f(\theta)+3}$

---

7. (a) Express $f(x) = \frac{5x-3}{(x+1)(x+3)}$, in partial fractions. Hence, evaluate $\int_3^5 f(x)dx$
   (b) Given that $f(x) = 5x^2 - 4\sqrt{x} - 6, x > 0$, and taking 1.5 as a first approximation to the root of $f(x) = 0$, use the Newton-Raphson procedure to obtain, to three decimal places, a second approximation to the root of the equation.

---

8. (a) Find the set of values of x for which $\frac{x+3}{x-1} < 3$
   (b) Given the function f, where $f(x) = \frac{3x-4}{x+2}, x \neq -2$
      i. find the range of f
      ii. sketch the graph of f

---

9. (In this question, you are required to work throughout with two decimal places.)The table below shows the values of x and y obtained in a certain laboratory work. The variables x and y are connected by the equation $y - 2 = b(x-1)^a$. By drawing a suitable graph of $\log_{10}(y-2)$ against $\log_{10}(x-1)$ estimate the values of the constants a and b.

---

x	2.659	4.801	6.248	9.708	17.595	20.96
y	10.317	6.569	5.63	4.512	3.585	3.38

10. (a) Find the term independent of x in the expansion of $\left(x^2 + \frac{1}{2x}\right)^6$

    (b) A geometric progression with positive terms has the sum of its first two terms as $\frac{20}{3}$ and its sum to infinity is 12. Find

    i. the first term and the common ratio of the progression

    ii. the sum of the first three terms

2.8 GCE JUNE 2017 QUESTIONS

1. (a) Find the set of values of k for which the quadratic equation $x^2 - kx + 2x + k - 2$ are real and different.

   (b) Given that $2^x = 3^y$ and that $x + y = 1$, show that $x = \frac{\log 3}{\log 6}$

2. (a) The first term of an arithmetic progression is a and the common difference is -1. If the sum of the first n terms is equal to the sum of the first $3n$ terms of the progression, express a in terms of n. Obtain the values of a when $n = 10$ and hence, find the sum of the first 30 terms of the progression.

   (b) Find the position of the term in x^{-12} in the expansion of $\left(x^3 - \frac{1}{x}\right)^{24}$

3. (a) i. Find $\frac{dy}{dx}$ if $y = \left(\sqrt{1 + 2x^2}\right)^5$

   ii. Given that $y = \ln\left(\frac{1+x^2}{1-x^2}\right)$ show that $\frac{dy}{dx} = \frac{4x}{1-x^4}$

   (b) The parametric equation of a curve are given by $x = ct$ and $y = \frac{c}{t}$, where t is a parameter and c is a constant. Show that an equation of the tangent to the curve at the point P with parameter t is given by $x + t^2y = 2ct$

4. (a) The table below shows corresponding values of x and y which approximately satisfy a relation of the form $y = an^x$, where a and n are constants.

x	2	3	4	5	6
y	13.6	27.2	54.4	108.8	217.6

   By drawing a suitable linear graph, determine the values of a and n, correct to 1 d.p.

   (b) Given that $x = 0.2$ is a first approximation to the root of the quadratic equation $f(x) = 0$ where $f(x) = x^2 + 3x - 1$. Use one iteration of the Newton-Raphson procedure to obtain a second approximation to the root of the equation, giving your answer to two decimal places.

5. (a) Given that $(x + 1)$ and $(x - 2)$ are both factors of the expression $ax^3 - x^2 + bx - a$, find the value of the constants a and b. Hence, find also the remainder when the expression is divided by $(x - 4)$

(b) Express as a single fraction $\frac{5}{x+2} - \frac{7}{2x+3}$, simplifying the numerator.

6. (a) The functions f and g are defined by $f : x \mapsto \frac{3}{x-2}, x \in \mathbb{R}, x \neq 2$ and $g : x \mapsto \frac{x-1}{x+2}, x \in \mathbb{R}, x \neq -2$

   i. Find $fg(x)$ and $gf(x)$ stating their domains

   ii. Show that g is not surjective.

   (b) A relation R is defined on the set $A = \{1, 2, 3, 4, 5\}$ by $aRb \iff a + b = 2n, n \in \mathbb{N}$. List all the equivalent classes of A under R

7. (a) Find all the values of the $\theta, 0 \leqslant \theta \leqslant 2\pi$ for which $\sin 2\theta = \cos\theta$

   (b) Show that the matrix $\mathbf{M}$ is invertible, where $\mathbf{M} = \begin{pmatrix} 3 & -2 & 5 \\ 7 & 4 & -8 \\ 5 & -3 & -4 \end{pmatrix}$

   Hence, find $\mathbf{M}^{-1}$, the inverse of matrix $\mathbf{M}$

8. (a) Given that $\left(\frac{4-3i}{2-i}\right) z - (1 + 3i) = 1 - 2i$, express the complex number z in the form $a + ib$, where a and b are constants

   (b) The vector equation of two lines L_1 and L_2 are given by

$$L_1 : r = 13i + 4j + 11k + \lambda(3i - 8j - 6k), \qquad L_2 : r = 5i + 22j + 9k + \mu(7i - 17j - 5k) \quad (2.1)$$

   Find

   i. the position vector of the point of intersection of L_1 and L_2

   ii. the cosine of the acute angle between L_1 and L_2

9. (a) The equation of two circles S_1 and S_2 are given by

$$S_1 : x^2 + y^2 + 2x + 2y + 1 = 0 \qquad \textit{and} \qquad S_2 : x^2 + y^2 - 4x + 2y + 1 = 0 \qquad (2.2)$$

   Show that S_1 and S_2 touches each other externally and obtain the equation of the common tangent at the point of contact.

   (b) A father and a mother have 5 children. The family is to occupy a particular front-line bench in church on a special thanksgiving service. Given that this bench has a capacity of 7 persons, in how many ways can this family be seated on the bench

   i. if the parents must sit adjacent to each other,

   ii. if the parents must sit adjacent to each other at one end of the bench.

10. (a) Given that $f(x) = \frac{2}{x(x+1)(x+2)}$, express $f(x)$ in partial fractions.

   Hence, show that $\int_2^4 f(x)dx = \left(\frac{27}{25}\right)$

   (b) Sketch the graph of the curve whose equation is given by $y = \frac{2x-7}{x-4}$, showing clearly the point where the curve meets the coordinate axes and the behaviour of the curve near its asymptotes.

2.9 GCE JUNE 2018 QUESTIONS

1. $P(x) = ax^3 - 3x^2 - bx + 6$. When $P(x)$ is divided by $(x-1)$ the remainder is -6. Given that $(x+2)$ is a factor of $P(x)$, find the values of te constants a and b. Hence, solve the equation $P(x) = 0$

---

2. (a) Given that one root of the quadratic equation $x^2 - 8x + k = 0$ is a multiple of the other, find the value of the constant k and hence, solve the equation $x^2 - 8x + k = 0$

   (b) A relation R is defined on the set of integers $\mathbb{Z}$ by aRb if $a + 2b$ is a multiple of 3. Show that R is an equivalence relation.

---

3. (a) Differentiate with respect to x

      i. $\frac{\ln(1+x^2)}{x^5}$

      ii. $\sin^2 x$

   (b) $f(x) = x^3 - x^2 - x + 5$

   Find the set of values of x for which $f(x)$ is increasing.

---

4. (a) Find the first four terms in the expansion of $(1-2x)^{\frac{1}{2}}$ in ascending powers of x stating the range of values of x for which the expansion is valid.

   (b) Find the value of x for which

$$y = 2\log_2 x \qquad \textit{and} \qquad y + 4 = \log_2 2x \tag{2.3}$$

---

5. (a) The complex number z is given by

$$z = \frac{3-i}{2+1} \tag{2.4}$$

   Express z in the form $a + ib$ where $a, b \in \mathbb{R}$

   (b) Given that $z = \cos\theta + i\sin\theta$. Show that $z^3 + z^{-3} = 2\cos 3\theta$. Hence, find the general solution of the equation $z^3 + z^{-3} = \sqrt{3}$

---

6. Express $\frac{x}{(x+1)(x+2)}$ in partial fraction. Hence, solve the differential equation

$$(x+1)(x+2)\frac{dy}{dx} = x(y+1) \tag{2.5}$$

   for $x > -1$, given that $y = \frac{1}{2}$ when $x = 1$ expressing the solution in the form $y = f(x)$

---

7. (a) Find the coordinates of the centre and the length of the radius of the circle

$$x^2 + y^2 - 3x - 4 = 0 \tag{2.6}$$

   Show that the line $3x + 4y - 17 = 0$ is a tangent to the circle

   (b) M is a 3×3 matrix given by

$$M = ` \begin{pmatrix} 3 & 4 & 0 \\ 1 & x+2 & 1 \\ 2x-4 & 4 & x-4 \end{pmatrix} \tag{2.7}$$

   Given that M is a singular matrix, show that $3x^2 - 2x - 36 = 0$.

---

8. (a) Find the general solution of the equation

$$\sin\left(x + \frac{\pi}{6}\right) = 2\cos x \tag{2.8}$$

(b) The vector equations of two lines L_1 and L_2 are

$$\begin{aligned} \mathbf{L}_1 : \mathbf{r} &= i + 2j + \lambda(2i + j + 3k) \\ \mathbf{L}_1 : \mathbf{r} &= -j - 4k + \mu(3i + 2j + 5k) \end{aligned}$$

Find

i. the point of intersection of the lines L_1 and L_2

ii. the vector parametric equation of the plane containing L_1 and L_2

---

9. (a) Show that the equation $x \ln x + x - 3 = 0$ has a root between 1 and 2.
Given that $x = \frac{3}{2}$ is an approximate root, use one iteration of the Newton-Raphson procedure to obtain a second approximate root of the equation.

(b) A class is made up of 5 boys and 8 girls. Find the number of ways in which a mixed delegation of 4 students can be chosen from the class if it must include at least 2 boys.

(c) Two statements p and q are defined as follows:
p: the workers will go on strike.
q: there will be no salary
Writhe the following in correct English

i. $q \implies p$

ii. $\sim q \implies \sim p$

iii. $p \wedge q$

---

10. (a) Consider the table below

x	2	6	10	14	16
y	62	270	580	994	1248

The table above gives the values of a continuous variable y observed values of x. It is known that y and x are connected by the law of the form

$$y = ax^2 + bx, \qquad \textit{where} \qquad a, b \in \mathbb{R} \tag{2.9}$$

By drawing a suitable linear graph, estimate the values of a and b giving the answer correct to two decimal places.

(b) The first and last term of an arithmetic progression are 7 and 51 respectively.
Given that the sum of the terms of the progression is 348, find the number of terms and the common difference of this progression.

2.10 GCE JUNE 2019 QUESTIONS

1. The polynomial $F(x) = 2x^3 + px^2 + qx - 30$ leaves a remainder of -28 when divided by $(x - 1)$ and a remainder of 66 when divided by $(x - 3)$

(a) Find the values of the constants p and q

(b) Given that $x - 2$ is a factor of $F(x)$, factorize $F(x)$ completely.

---

2. (a) The roots of the quadratic equation $2x^2 - x + 6 = 0$ are α and β. Find the quadratic equation with integral coefficients whose roots are $\alpha - 2\beta$ and $\beta - 2\alpha$
   (b) Find the value of the constant k for which the quadratic equation $x^2 + (2-k)x + 2(2-k) = 0$ has imaginary roots

---

3. (a) Express $\frac{x-1}{(x+2)(x+1)}$ in partial fraction
   (b) Of Peter's 13 friends, 7 are older than him. In how many ways can he invite 6 friends including at least 4 older friends.

---

4. Express $\cos 3\theta - \sqrt{3}\sin 3\theta$ in the form $R\cos(3\theta + \beta)$ where $R > 0$ and β is an acute angle. Hence, or otherwise, find the general solution of the equation $\cos 3\theta - \sqrt{3}\sin 3\theta = \sqrt{2}$.
   Determine the minimum and maximum values of $\frac{5}{2\cos 3\theta - 2\sqrt{3}\sin 3\theta + 7}$

---

5. (a) Given that $\frac{z-1}{z+1} = i$, express z in the form $a + ib$ where a and b are real constants. Hence, find $|2z|$
   (b) The ninth term of an arithmetic progression is three times the third term. If the sum of the first four terms is 30, find the first term and common difference of the progression

---

6. Find $\frac{dy}{dx}$ if
   (a) $y = \frac{x^2-1}{x^2+1}$
   (b) $y = \cos^4 x$
   (c) $x^2 + y^2 = 16x$

7. Evaluate
   (a)
$$\int_1^3 \frac{x^2}{x+1}dx$$
   (b)
$$\int_0^\pi e^{\cos x}\sin x dx$$

---

8. (a) Given the statements
   p: John is sick ,
   q: John will go to school,
   translate into ordinary English the statements
      i. $\sim p \wedge q$
      ii. $\sim (p \wedge \sim q)$
   (b) A relation R is defined on the set of ordered pair of real numbers by
   $(a, b)R(c, d)$ if and only if $a^2 + b^2 = c^2 + d^2$.
   Prove that R is an equivalence relation.

---

9. (a) The vector equation of two lines L_1 and L_2

$$\begin{aligned}\mathbf{L}_1 : \mathbf{r} &= 2i + j + \lambda(i + j + k)\\ \mathbf{L}_2 : \mathbf{r} &= 2i + 2j + \mathbf{a}k + \mu(i + 2j + k) \quad \textit{where } \mathbf{a} \textit{ is a constant}\end{aligned}$$

   Find the value of a if L_1 and L_2 intersect and the position vector of the point of intersection.

(b) The matrices $\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 5 & 2 \\ 2 & -1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 2 & 0 \\ 1 & 3 & 2 \\ 2 & 0 & 1 \end{pmatrix}$.

Calculate the matrix product $\mathbf{AB}$.

Hence, find $\mathbf{B}^{-1}\mathbf{A}^{-1}$

---

10. (a) A function

$$f(x) = \begin{cases} x^2 - 3 & for \quad 0 \leqslant x < 2 \\ 4x - 7, & for \quad 2 \leqslant x < 4 \end{cases}$$

is such that $f(x) = f(x \pm 4)$

Find, $f(27)$ and $f(-106)$

(b) Solve the differential equation

$$(x^2 - 1)\frac{dy}{dx} + xy = 3y$$

given that $y = 1$ when $x = 0$, expressing the result in the form $y = f(x)$.

2.11 GCE JUNE 2020 QUESTIONS

1. (a) Given that $(x + 1)$ is a factor of $f(x)$, where $f(x) = x^3 + 6x^2 + 11x + 6$, factorize $f(x)$ completely.

   (b) Let λ be a real constant. Show that the roots of the quadratic equation $3x^2+(-4-2\lambda)x+2\lambda = 0$ are always real.

---

2. (a) Given that $y = ln(4 + x^2)$, find

   i. $\frac{dy}{dx}$,

   ii. the equation of the tangent and normal to the curve $y = ln(4 + x^2)$ at the point where $x = 1$.

   (b) Solve the differential equation $\frac{dy}{dx} = xy - x$, given that $y = 2$ when $x = 0$, expressing y in terms of x.

---

3. (a) Draw the truth table for each of the proposition $p \Longrightarrow q$ and $\sim p \vee q$ and show that they are identical

   (b) Given that $\sin^{-1}(x) = \alpha$ and $\cos^{-1}(x) = \beta$, show that $\sin(\alpha + \beta) = 1$

---

4. (a) The function f is given by $f(x) = \frac{x+1}{(x-1)(x^2+1)}$. Express $f(x)$ in partial fraction and hence find $\int f(x)dx$.

   (b) By using the substitution $u = \sin(x)$, find $\int \left(\frac{\cos x}{1+\sin^2 x}\right)dx$.

---

5. Given the circles $C_1 : x^2 + y^2 - 6x - 4y + 9 = 0$ and $C_2 : x^2 + y^2 - 2x - 6y + 9 = 0$, Find

   (a) the equation of the circle C_3 which passes through the center of C_1 and through the point of intersection of C_1 and C_2.

   (b) the equation of the two tangents from the origin to C_1 and the length of each tangent.

---

6. (a) Determine whether the function $f : \mathbb{R} \longrightarrow \mathbb{R}$, $\quad f(x) = \frac{x+1}{x-3}, \quad x \neq 3$ is surjective.

   (b) A periodic function $f : \mathbb{R} \longrightarrow \mathbb{R}$, of period 4 is defined in the interval $-2 \leq x \leq 2$ by

$$f(x) = \begin{cases} -x^2 + k, & -2 \leq x \leq 0 \\ x^2 + 4, & 0 \leq x < 2. \end{cases}$$

i. Find the value of k for which f is continuous and the value of $f(-5)$

ii. Sketch the curve of $y = f(x)$ for $-2 \leq x \leq 10$.

---

7. (a) Find the image of the line $2y = x$ under the transformation with matrix operator $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

   (b) Find the inverse of the matrix A, where $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix}$.

   Hence solve the equations
   $$\begin{aligned} x - y + z &= 7 \\ 2x + y - 3z &= -6 \\ x + y + z &= 4 \end{aligned}$$

---

8. (a) Evaluate $\sum_{r=1}^{\infty} 3\left(\frac{1}{4}\right)^r$

   (b) The sum of the first n terms of a series is given by $S_n = 2n^2 + n$

   i. Find an expression for the n^{th} term of the series

   ii. Show that the series is an Arithmetic progression

---

9. (a) Prove by Mathematical induction that for $n \in \mathbb{N}$
   $$\sum_{r=1}^{n} \frac{1}{4r^2 - 1} = \frac{n}{2n+1}$$

   (b) A relation R, is defined on $\mathbb{Z}$, the set of integers by $aRb \iff a - b$ is even. Show that R is transitive.

---

10. (a) Find the range of values of x for which

    i. $\frac{2x-1}{x+2} < -1$

    ii. $|2x - 4| < x + 1$

    (b) the complex numbers Z_1 and Z_2 are such that $Z_1 = -2 + 2i$ and $Z_2 = -2 - 2i$. Find $argZ_1$ and $|Z_2|^6$.

End

2.12 JUNE 2015 SOLUTION

2.13 JUNE 2016 SOLUTION

2.14 JUNE 2017 SOLUTION

1. (a) **Method 1**: $x^2 - kx + 2x + k - 2 = 0 \implies x^2 - (k-2)x + k - 2 = 0$ and for real and different roots, $\Delta = b^2 - 4ac > 0$. That is $\left[-(k-2)\right]^2 - 4(1)(k-2)(k-2-4) > 0 \implies (k-2)(k-6) > 0$ with 2 and 6 as zeros of the quadratic equation in k.

   Hence, the set of values of k is $\{k : k < 2\} \cup \{k : k > 6\}$

   Method 2: $x^2 - kx + 2x + k - 2 = 0 \implies x^2 + (2-k)x + k - 2 = 0$ and for real and different roots, $\Delta = b^2 - 4ac > 0$. That is $(2-k)^2 - 4(1)(k-2)(k-2-4) > 0 \implies (k-2)(k-6) > 0$ with 2 and 6 as zeros of the quadratic equation in k.

		2		6	
$k-2$	$-$		$+$		$+$
$k-6$	$-$		$-$		$+$
$(k-2)(k-6)$	$+$		$-$		$+$

		2		6	
$k-2$	$-$		$+$		$+$
$k-6$	$-$		$-$		$+$
$(k-2)(k-6)$	$+$		$-$		$+$

(b) Given that $2^x = 3^y$ and that $x+y=1$, we show that $x = \frac{\log 3}{\log 6}$
Method 1: $2^x = 3^y \implies \log 2^x = \log 2^y$ and $x+y=1 \implies x = 1-y$, thus $\log 2^x = \log 2^{1-x}$ so $x\log 2 = (1-x)\log 3 \implies x\log 2 = \log 3 - x\log 3 \implies x(\log 2 + \log 3) = \log 3 \implies x\log 6 = \log 3 \implies x = \frac{\log 6}{\log 3}$
Method 2: $2^x = 3^y$ and $x+y=1 \implies x = 1-y$, thus $2^x = 3^{1-x} \implies \log 2^x = \log 2^{1-x}$ so $x\log 2 = (1-x)\log 3 \implies x\log 2 = \log 3 - x\log 3 \implies x(\log 2 + \log 3) = \log 3 \implies x\log 6 = \log 3 \implies x = \frac{\log 3}{\log 6}$

2. (a) The first term of an arithmetic progression is a and the common difference is -1. If the sum of the first n terms is equal to the sum of the first $3n$ terms of the progression, then we express a in terms of n.
Using $S_n = \frac{n}{2}[2a+(n-1)d]$, $S_n = S_{3n} \implies \frac{n}{2}[2a+(3n-1)(-1)] = \frac{n}{2}[2a+(3n-1)(-1)]$. Thus, $2a+(n-1)(-1) = 3[2a+(3n-1)(-1)] \implies 2a-n+1 = 6a-9n+3 \implies -4a = -8n+2 \implies a = 2n-\frac{1}{2}$.
Obtain the values of a when $n=10$ and hence, find the sum of the first 30 terms of the progression. When $n=10, a = \frac{4(10)-1}{2} = \frac{39}{2}$

(b)

2.15 JUNE 2018 SOLUTION

1. (a) If $P(x) = ax^3 - 3x^2 - bx + 6$ and the remainder when $P(x)$ is divided by $x-1$ is -6, then by the remainder theorem $P(1) = -6$. If $(x+2)$ is a factor of $P(x)$ then $P(-2) = 0$. Thus, we have two equations with two unknowns

$$\begin{aligned} P(1) &= -6 \implies a-3-b+6 = -6 \\ P(-2) &= 0 \implies a(-2)^3 - 3(-2) - b(-2) + 6 = 0 \end{aligned}$$

Hence, $a=2$ and $b=11$ and consequently, $P(x) = 2x^3 - 3x^2 - 11x + 6$. Given that $(x+2)$ is a factor of $P(x)$, we proceed by long division to obtain the other factors as follows

$$
\begin{array}{r}
2x^2 \quad - 7x + 3 \\
x+2 \overline{) \; 2x^3 - 3x^2 - 11x + 6} \\
\underline{- 2x^3 - 4x^2 \qquad\qquad} \\
- 7x^2 - 11x \quad \\
\underline{7x^2 + 14x \quad} \\
3x + 6 \\
\underline{- 3x - 6} \\
0
\end{array}
$$

So that $P(x) = (x+2)(2x^2 - 7x + 3) = (x+2)(2x-1)(x-3)$. Solving $P(x) = 0$ we have that

$$x = -2, \quad \frac{1}{2}, \quad 3$$

2. (a) If one root of the equation $x^2 - 8x + k = 0$ is three times the other, then if the roots are α and β, we have that $\alpha = 3\beta$.

   The sum of roots of the equation is $\alpha + \beta = 8$ and the product of roots is $\alpha\beta = k$. That is

$$\begin{cases} SOR = 8 \\ POR = k \end{cases}$$

   Since one root is three times the other,

$$\begin{cases} \alpha + \beta = 8 \implies 3\beta + \beta = 8 \implies \beta = 2 \\ \alpha\beta = (3\beta)\beta = 3\beta^2 = k \implies k = 12 \end{cases}$$

   Hence, the equation becomes $x^2 - 8x + 12 = 0$ and we have

$$x^2 - 8x + 12 = 0 \implies (x-6)(x-2) = 0 \implies x = 2, 6 \tag{2.10}$$

   (b) R is defined by aRb if $a + 2b$ is a multiple of 3 on the set of integers $\mathbb{Z}$. To show that R is an equivalence relation, we show that R is reflexive, symmetric and transitive.

   - Reflexivity: Let $a \in \mathbb{Z}$ then, we have that $a + 2a = 3a$ which is a multiple of 3. Hence aRa and R is reflexive
   - Symmetry: Let $a, b \in \mathbb{Z}$ then suppose aRb then $a + 2b = 3k$ for some $k \in \mathbb{Z} \implies a = 3k - 2b$ and $b + 2a = b + 2(3k - 2b) = b + 6k - 3b = 6k - 3b$
   - Transitivity: Let $a, b, c \in \mathbb{Z}$ and suppose aRb and bRc then we have

$$aRb \implies a + 2b = 3k, \quad k \in \mathbb{Z} \tag{2.11}$$

$$bRc \implies b + 2c = 3m, \quad m \in \mathbb{Z} \tag{2.12}$$

     Adding the two equations above, we have $a + 2b + b + 2c = 3k + 3m \implies a + 2c + 3b = 3k + 3m \implies a + 2c = 3k + 3m - 3b \implies a + 2c = 3l$ where $l = k + m - b \in \mathbb{Z}$

   Consequently, R is an equivalence relation on $\mathbb{Z}$

3. (a) We differentiate with respect to x

   i. $y = \frac{\ln(1+x^2)}{x^5}$

   We let $U = \ln(1 + x^2)$ and $V = x^5$. Then, we have $\frac{dU}{dx} = \frac{2x}{1+x^2}$ and $\frac{dV}{dx} = 5x^4$. Hence, by the quotient rule, we have

$$\frac{d(\frac{U}{V})}{dx} = \frac{V\frac{dU}{dx} - U\frac{dV}{dx}}{V^2} = \frac{x^5(\frac{2x}{1+x^2}) - 5x^4\ln(1+x^2)}{(x^5)^2} = \frac{2x^{-4}}{1+x^2} - 5x^{-6}\ln(1+x^2) \tag{2.13}$$

ii. $y = \sin^2(x+1)$. To differentiate y, we let $U = x+1$ and use the chain rule. $\frac{dU}{dx} = 1$ and $\frac{dy}{dU} = 2\sin U \cos U$

$$\frac{dy}{dx} = \frac{dy}{dU} \times \frac{dU}{dx} = 2\sin(x+1)\cos(x+1) \times 1 = 2\sin(x+1)\cos(x+1) \qquad (2.14)$$

(b) The function $f(x) = x^3 - x^2 - x + 5$ is increasing in the interval $[x = a, x = b]$ if $f'(x) = 3x^2 - 2x - 1 > 0$. Thus we have the inequality $3x^2 - 2x - 1 > 0 \implies (3x+1)(x-1) > 0$ We draw a sign table to determine where the solution lie.

		$-\frac{1}{3}$		1	
$3x+1$	$-$		$+$		$+$
$x-1$	$-$		$-$		$+$
$(3x+1)(x-1)$	$+$		$-$		$+$

Hence, the set of values of x for which $f(x)$ is increasing is $S = \{x : x < \frac{1}{3}$ **and** $x > 1\}$

4. (a) To expand $(1-2x)^{\frac{1}{2}}$ we use the formula

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1(n-2))}{3!}x^3 + \cdots$$

$$\begin{aligned}\implies (1-2x)^{\frac{1}{2}} &= 1 + \frac{1}{2}(-2x) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(-2x)^2 + \frac{n(n-1(n-2))}{3!}x^3 + \cdots \\ &= 1 - x - \frac{4}{8}x^2 + \frac{8}{12}x^3 + \cdots = 1 - x - \frac{1}{2}x^2 + \frac{2}{3}x^3 + \cdots\end{aligned}$$

The range of values of x for which the expansion is valid is $|2x| < 1$ and thus $-1 < 2x < 1 \implies -\frac{1}{2} < x < \frac{1}{2}$

(b) The value of x for which $y = 2\log_2 x$ and $y + 4 = \log_2 2x$ is obtained by substituting the value of y in the second equation. $y = 2\log_2 x \implies 2^y = x^2$

$$y + 4 = \log_2 2x \implies 2^{y+4} = 2x \implies 2^y \times 2^4 = 2x \implies 2^4x^2 = 2x \implies 16x^2 = 2x$$

$\implies x = 0 \quad or \quad x = \frac{1}{8}$

5. (a) $z = \frac{3-i}{2+i} = \frac{(3-i)(2-i)}{(2+i)(2-i)} = \frac{6+i^2-3i-2i}{4-i^2} = \frac{5-5i}{1+4} = 1 - i$

(b) $z = \cos\theta + i\sin\theta$,
$z^3 + z^{-3} = (\cos\theta + i\sin\theta)^3 + (\cos\theta + i\sin\theta)^{-3} = \cos 3\theta + i\sin 3\theta + \cos 3\theta - i\sin 3\theta = 2\cos 3\theta$ by De Moivre's theorem. Hence we have $z^3 + z^{-3} = \sqrt{3} \implies 2\cos 3\theta = \sqrt{3} \implies \cos 3\theta = \frac{\sqrt{3}}{2}$. Thus the general solution is $3\theta = 360n \pm \cos^{-1}(\frac{\sqrt{3}}{2}) \implies 360n \pm \cos^{-1}(30) \implies \theta = 120n \pm 10$

6. We express the function into partial fraction of type 1

$$\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \implies x = A(x+1) + B(x+2) \qquad (2.15)$$

$x = -2 \implies -2 = -B \implies B = 2$ and $x = -1 \implies -1 = A \implies A = -1$. Thus we have

$$\frac{x}{(x+1)(x+2)} = \frac{-1}{x+1} + \frac{2}{x+2}$$

Hence, we solve the differential equation $(x+1)(x+2)\frac{dy}{dx} = x(y+1)$

$$\frac{dy}{dx} = \frac{x(y+1)}{(x+1)(x+2)} \implies \int \frac{dy}{y+1} = \int \frac{xdx}{(x+1)(x+2)} = \int \frac{2dx}{x+2} - \int \frac{dx}{x+1}$$

Thus $\ln|y+1| = 2\ln|x+2| - \ln|x+1| + C$. Given that $y(1) = \frac{1}{2}$, we have $\ln|\frac{1}{2}+1| = 2\ln|1+2| - \ln|1+1| + C \implies C = -\ln 3$. We have that

$$\ln|y+1| = 2\ln|x+2| - \ln|x+1| - \ln 3 = \ln\left[\frac{(x+2)^2}{3(x+1)}\right] \implies y(x) = \frac{1}{3}\left[\frac{(x+2)^2}{x+1}\right] - 1$$

7. (a) To find the centre and the radius of the circle $x^2+y^2-3x-4=0$, we complete the square of the equation as follows $x^2-3x+y^2-4 = (x-\frac{3}{2})^2 - \frac{9}{4} + y^2 - 4 = (x-\frac{3}{2})^2 + y^2 - \frac{25}{4}$. This implies $(x-\frac{3}{2})^2 + y^2 = \frac{25}{4}$ and so the centre is $(\frac{3}{2}, 0)$ and the radius is $r = \frac{5}{2}$.
   To show that the line $L : 3x+4y-17$ is a tangent to the circle, we show that the distance from the centre of the circle to the line L is equal to the radius of the circle

$$d(C,L) = \left|\frac{3(\frac{3}{2}) + 4(0) - 17}{\sqrt{3^2+4^2}}\right| = \left|\frac{-25/2}{5}\right| = \frac{5}{2} = radius$$

   (b) If the matrix M is defined by $\begin{pmatrix} 3 & 4 & 0 \\ 1 & x_2 & 1 \\ 2x-4 & 4 & x-4 \end{pmatrix}$. Then M is singular if it has determinant zero. Thus

$$\begin{vmatrix} 3 & 4 & 0 \\ 1 & x+2 & 1 \\ 2x-4 & 4 & x-4 \end{vmatrix} = 0 \implies 3\begin{vmatrix} x+2 & 1 \\ 4 & x-4 \end{vmatrix} - 4\begin{vmatrix} 1 & 1 \\ 2x-4 & x-4 \end{vmatrix} + 0 = 0$$

   We have the following $3[(x+2)(x-4)-4] - 4[(x-4)-(2x-4)] = 0 \implies 3[x^2+2x-4x-8] - 4[-x] = 0 \implies 3x^2 - 6x - 34 + 4x = 0 \implies 3x^2 - 2x - 36 = 0$

8. (a) To find the general solution of the equation $\sin(x+\frac{\pi}{6}) = 2\cos x$ we expand the composite angle to obtain

$$\sin(x+\frac{\pi}{6}) = \sin x \cos\frac{\pi}{6} + \sin\frac{\pi}{6}\cos x = \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x \tag{2.16}$$

   Hence $\sin(x+\frac{\pi}{6}) = 2\cos x \implies \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x = 2\cos x \implies \frac{\sqrt{3}}{2}\sin x = 2\cos x - \frac{1}{2}\cos x$. Thus we have $\frac{\sqrt{3}}{2}\sin x = \frac{3}{2}\cos x$ and consequently, $\tan x = \sqrt{3} \implies \theta = 180n + \tan^{-1}(\sqrt{3})$ $\theta = 180n + 60$

   (b) The point of intersection of the two lines L_1 and L_2 is obtained by equating the two lines and solving for the parameters λ and μ

$$i + 2j + \lambda(2i + j + 3k) = -j - 4k + \mu(3i + 2j + 5k) \implies \begin{cases} 1 + 2\lambda = 3\mu \\ 2 + \lambda = -1 + 2\mu \\ 3\lambda = -4 + 5\mu \end{cases}$$

      i. We have that $\mu = 5$ and $\lambda = 7$. Thus the point of intersection of the lines L_1 and L_2 is $15i + 9j + 21k = 0A$

ii. The vector parametric equation of the plane containing L_1 and L_2 is given by a point on the plane and the normal $\vec{n}$ to the plane.

$$\vec{n} = \begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{vmatrix} = -i - j + k \tag{2.17}$$

Thus, the equation of the plane π is given by $\vec{r}.\vec{n} = 0A.\vec{n}$, Implies that $\pi : -x - y + z = (15i + 9j + 21k).(-i - j + k) = -3 \implies \pi : -x - y + z = -3$

9. (a) Let $f(x) = x \ln x + x - 3$, then to show that a root lie between 1 and 2, we find $f(1) = 1 \ln 1 + 1 - 3 = 0 + 1 - 3 = -2$ and $f(2) = 2 \ln 2 + 2 - 3 = \ln 4 - 1 = 0.39$. Thus $f(1) \times f(2) < 0$ so there is a sign change moving from 1 to 2 and hence a solution exist between 1 and 2.
To obtain a second approximation, we differentiate the function. $f'(x) = \ln x + 2 \implies f'(\frac{3}{2}) = 2.4$ and $f(\frac{3}{2}) = -0.9$

$$x_2 = x_1 - \frac{f(x_1)}{f(x_2)} = \frac{3}{2} + \frac{0.9}{2.4} = 1.5 + 0.375 = 1.875 \tag{2.18}$$

(b) If a class is made up of 5 boys and 8 girls then we can choose. A mixed delegation of 4 can be chosen with at least two boys in the following ways

$$\begin{aligned} {}^5C_2 \times^8 C_2 +^5 C_3 \times^8 C_1 &= \frac{5!}{3!2!} \times \frac{8!}{6!2!} + \frac{5!}{3!2!} \times \frac{8!}{7!1!} = \frac{5 \times 4 \times \times 8 \times 7}{2 \times 2} + \frac{5 \times 4 \times 8 \times 1}{2 \times 1} \\ &= 5 \times 56 + 80 = 280 + 80 = 360 \end{aligned} \tag{2.19}$$

Thus, the committee can be chosen in 360 ways

(c) If p is the statement the workers will go on strike and q the statement there will be no salary then

i. $q \implies p \equiv$ If the workers go on strike, then there will be no salary.

ii. $\sim p \implies \sim p \equiv$ If there will be salaries, then the workers will not go on strike.

iii. $p \wedge q \equiv$ The workers will go on strike and there will be no salaries.

10. (a) For a continuous variable, y and some observed values x ,$y = ax^2 + bx$ with $a, b \in \mathbb{R}$,

$$\implies \frac{y}{x} = ax + b$$

which is the equation of a line with gradient a and y-intercept b. so we have the table A

x	2	6	10	14	16
y	62	270	580	994	1248
$\frac{y}{x}$	31	45	58	71	78

graph of $\frac{y}{x}$ against x is a straight line with gradient a and y-intercept b.
From the graph, $a =$ and $b =$

(b) The first term of an arithmetic progression is 7 and the last term is 51. If the sum of the terms of the progression is 348 then

$$T_n = a + (n-1)d = 7 + (n-1)d = 51 \qquad and \qquad S_n = \frac{n}{2}(a + l) = \frac{n}{2}(7 + 51) = 348$$

Thus, we have that $58n = 696 \implies n = 12$ and $51 = 7 + 10d \implies 10d = 44 \implies d = 11$

2.16 JUNE 2019 SOLUTION

1. (a) If $F(x) = 2x^3 + px^2 + qx - 30$ leaves a remainder of -28 when divided by $x-1$ and a remainder of 66 when divided by $x-3$ then by the remainder theorem, $F(1) = -28$ and $F(3) = 66$

$$\begin{aligned} F(1) &= -28 \implies 2+p+q-30 = -28 \implies p+q=0 && (2.20)\\ F(3) &= 66 \implies 2(3)^3 + p(3)^2 + q(3) - 30 = 66 \implies 2(27)+9p+3q-30 = 66\\ \implies & \quad 9p+3q = 42 \implies 3p+3q = 14 && (2.21) \end{aligned}$$

Solving the two above equations, we have $p = 7$ and $q = -7$

(b) Given that $x-2$ is a factor of $F(x)$, then $F(2) = 0$. To find the other factors we proceed by long division.

$$\begin{array}{r|llll} & 2x^2 & +11x & +15 & \\ \hline x-2) & 2x^3 & +7x^2 & -7x & -30\\ & -2x^3 & +4x^2 & & \\ \hline & & 11x^2 & -7x & \\ & & -11x^2 & +22x & \\ \hline & & & 15x & -30\\ & & & -15x & +30\\ \hline & & & & 0 \end{array}$$

Thus, $F(x) = 2x^3 + px^2 + qx - 30 = (x-2)(2x^2+11x+15) = (x-2)(x+3)(2x+5)$

2. (a) If α and β are roots of the equation $2x^2 - x + 6 = 0$, then we have that $\alpha+\beta = \frac{1}{2}$ and $\alpha\beta = 3$.

$$\begin{aligned} (\alpha-2\beta)+(\beta-2\alpha) &= -(\alpha+\beta) = -\frac{1}{2}\\ (\alpha-2\beta)(\beta-2\alpha) = \alpha\beta &- 2\alpha^2 - 2\beta^2 + 4\alpha\beta = 5\alpha\beta - 2(\alpha^2+\beta^2) = 9(3) - 2(\frac{1}{2})^2 = \frac{53}{2} \end{aligned}$$

Thus, the quadratic equation with integral coefficients whose roots are $\alpha - 2\beta$ and $\beta - 2\alpha$ is given by $2x^2 + x + 53$

(b) The quadratic equation $x^2 + (2-k)x + 2(2-k) = 0$ has imaginary roots if

$$\begin{aligned} \Delta &= (2-k)^2 - 4(1)(2)(2-k) < 0 \implies 4 - 4k + k^2 - 8(2-k) < 0\\ \implies & \quad 4 - 4k + k^2 - 16 + 8k < 0 \implies k^2 + 4k - 12 < 0 \implies (k+6)(k-2) < 0 \end{aligned}$$

We draw a sign table to determine where the solution lies.

		-6		2	
$k+6$	$-$		$+$		$+$
$k-2$	$-$		$-$		$+$
$(k+6)(k-2)$	$+$		$-$		$+$

The interval of solution for which $(k+6)(k-2) < 0$ is given by $-6 < k < 2$

3. (a) $\frac{x-1}{(x+2)(x+1)}$ is a type 1 partial fraction and we express it in the form

$$\frac{x-1}{(x+2)(x+1)} = \frac{A}{x+1} + \frac{B}{x+1} \qquad (2.22)$$

$\Longrightarrow x-1 = A(x+2)+B(x+1)$. When $x=-1$, we have that $-1-1 = A(-1+2)+0 \Longrightarrow A=-2$. When $x=-2$, we have $-2-1 = B(-2+1) \Longrightarrow B=3$. Thus, we have

$$\frac{x-1}{(x+2)(x+1)} = \frac{-2}{x+1} + \frac{3}{x+1} \tag{2.23}$$

(b) Of Peter's 13 friends, if 7 are older than him, then Peter can invite 6 friends with at least 4 older than him in the following ways

$$\begin{aligned} &{}^7C_4 \times^6 C_2 +^7 C_5 \times^6 C_1 +^7 C_6 \times^6 C_0 = \frac{7!}{4!3!} \times \frac{6!}{4!2!} + \frac{7!}{5!2!} \times \frac{6!}{5!1!} + \frac{7!}{6!1!} \times \frac{6!}{0!6!} \\ = &\frac{7\times6\times5\times4!}{4!3!} \times \frac{6\times5\times4!}{4!2!} + \frac{7\times6\times5!}{5!2!} \times \frac{6\times5!}{5!1!} + \frac{7\times6!}{6!1!} \times \frac{6\times5\times4!}{4!2!} \\ = &35\times15+21\times6+7 = 525+126+7 = 658 ways \end{aligned} \tag{2.24}$$

4. $\cos 3\theta - \sqrt{3}\sin 3\theta = R\cos(3\theta+\beta) = R(\cos 3\theta\cos\beta - \sin 3\theta \sin\beta)$. Equating coefficients, we have $R\cos\beta = 1$ and $R\sin\beta = \sqrt{3} \Longrightarrow \tan\beta = \sqrt{3}$. Thus, we have $R=\pm2$ and $\beta = 60^0$

Hence, we find the general solution of the $\cos 3\theta - \sqrt{3}\sin 3\theta = \sqrt{2} \Longrightarrow 3\theta + 60 = 360n \pm 45 \Longrightarrow \theta = 120n \pm 15 - 20$. To determine the minimum and maximum value of $\frac{5}{\cos 3\theta - \sqrt{3}\sin 3\theta + 7}$ we use the relation above to obtain the maximum value $\frac{5}{3}$ and minimum value $\frac{5}{11}$

5. Given that $\frac{z+1}{z-1} = i$

(a) We express z in the form $a+ib$

$$\begin{aligned} \frac{z+1}{z-1} &= i \Longrightarrow z+1 = i(z-1) \Longrightarrow z - iz = -i-1 \Longrightarrow iz - z = 1+i \\ \Longrightarrow \quad & z = \frac{1+i}{-1+i} = \frac{1+i}{i-1} \times \frac{-1-i}{-1-i} = \frac{-1-2i-i^2}{i^2-(-1)^2} = \frac{-2i}{2} = -i \end{aligned} \tag{2.25}$$

We have that $z=-i$

(b) The n^{th} term of an arithmetic progression is $T_n = a+(n-1)d$. If the ninth term is three times the third term, then we have

$$a+8d = 3(a+2d) \Longrightarrow 2a = 2d \Longrightarrow a = d \tag{2.26}$$

The sum of the first n^{th} terms is given by $S_n = \frac{n}{2}(2a+(n-1)d) \Longrightarrow S_4 = 2(2a+3d) = 30 \Longrightarrow 2a+3d = 15 \Longrightarrow 5a = 15 \Longrightarrow a = 3 = d$

6. (a) If $y = \frac{x^2-1}{x^2+1}$. To obtain $\frac{dy}{dx}$ we make the change of variable by letting $U = x^2-1$ and $V = x^2+1$ and use the quotient rule

$$\frac{d}{dx}\left(\frac{U}{V}\right) = \frac{V\frac{dU}{dx} - U\frac{dV}{dx}}{V^2} \tag{2.27}$$

But $\frac{dU}{dx} = 2x = \frac{dV}{dx}$ and we have

$$\frac{dy}{dx} = \frac{2x(x^2+1)-2x(x^2-1)}{(x^2+1)^2} = \frac{2x^3+2x-2x^3+2x}{(x^2+1)^2} \Longrightarrow \frac{dy}{dx} = \frac{4x}{(x^2+1)^2} \tag{2.28}$$

(b) If $y = \cos^4 x$, to obtain $\frac{dy}{dx}$ we make the change of variable $U = \cos x$ and use the chain rule

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 4u^3(-\sin x) = -4\sin x\cos^3 x \tag{2.29}$$

(c) If $y^2+x^2=16x$, then to obtain $\frac{dy}{dx}$ we differentiate implicitly (term by term with respect to its variable). Thus

$$2ydy+2xdx=16dx \implies dy=\left(\frac{16-2x}{2y}\right)dx \implies \frac{dy}{dx}=\frac{16-2x}{2y} \tag{2.30}$$

7. (a) To integrate $\int\limits_1^3 \frac{x^2}{1+x}dx$ we first put it in partial fraction. We have

$$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2 \\ & -x^2-x \\ \hline & -x \\ & x+1 \\ \hline & 1 \end{array}$$

and consequently,

$$\int\limits_1^3 \frac{x^2}{1+x}dx=\int\limits_1^3\left(x-1+\frac{1}{x+1}\right)dx=\left[\frac{x^2}{2}-x+\ln|1+x|\right]_1^3=\frac{8}{2}-2+\ln 4-\ln 2=2+\ln 2$$

(b) To evaluate $\int\limits_0^\pi e^{\cos x}\sin xdx$, we use an integration by part as follows: Let $I=\int\limits_0^\pi e^{\cos x}\sin xdx$, and $U=e^{\cos x}$ and $dV=\sin xdx$, then $dU=-\sin xe^{\cos x}dx$ and $V=-\cos x$

$$\begin{aligned} I &= -\cos xe^{\cos x}\Big|_0^\pi-\int\limits_0^\pi \sin x\cos xe^{\cos x}dx \\ &= e^{-1}+e-\int\limits_0^\pi \sin x\cos xe^{\cos x}dx \end{aligned}$$

We integrate again using an integration by part by letting $u'=\cos x$ and $dv'=\sin xe^{\cos x}dx \implies v'=I$ and $du'=-\sin xdx$. Hence,

$$\begin{aligned} I &= e+e^{-1}-\left[I\cos x\Big|_0^\pi+I\int\limits_0^\pi \sin xdx\right]=e+e^{-1}-\left[I\cos x\Big|_0^\pi-I\cos x\Big|_0^\pi\right] \\ &= e+e^{-1}-I\Big[(\cos\pi-\cos 0)+(-\cos\pi+\cos 0)\Big] \\ &= e+\frac{1}{e}-I\Big[(-1-1)+(1+1)\Big]=e+\frac{1}{e} \end{aligned}$$

Hence, $\int\limits_0^\pi e^{\cos x}\sin xdx=e+\frac{1}{e}$

8. (a) i. $\neg p\wedge q$: John is not sick and he will go to school

ii. $\neg(p\wedge\neg q)$: John is either not sick or he will go to school

(b) The relation R is defined by $(a,b)R(c,d)$ if and only if $a^2+b^2=c^2+d^2$. To prove that R is an equivalence relation, we show that R is reflexive, symmetric and transitive.

Reflexivity of R: Let $a,b\in\mathbb{R}$, then $a^2+b^2=a^2+b^2 \implies (a,b)R(a,b)$. Hence R is reflexive

Symmetry of R: Let $(a,b)\in\mathbb{R}^2$ and $(c,d)\in\mathbb{R}^2$ and suppose $(a,b)R(c,d)$ then $a^2+b^2=c^2+d^2 \implies c^2+d^2=a^2+b^2 \implies (c,d)R(a,b)$. Hence R is symmetric

Transitivity of R: Let $(a,b),(c,d),(x,y) \in \mathbb{R}^2$ and suppose $(a,b)R(c,d)$ and $(c,d)R(x,y)$. Then

$$(a,b)R(c,d) \implies a^2+b^2 = c^2+d^2 \tag{2.31}$$

and

$$(c,d)R(x,y) \implies c^2+d^2 = x^2+y^2 \tag{2.32}$$

$\implies a^2+b^2 = x^2+y^2 \implies (a,b)R(x,y)$. Hence, R is transitive.

9. (a) For the lines L_1 and L_2 to intersect, $L_1 = L_2$. That is

$$2i+j+\lambda(i+j+k) = 2i+2j+ak+\mu(i+2j+k) \implies \begin{cases} 2+\lambda = 2+\mu \\ 1+\lambda = 2+2\mu \\ \lambda = a+\mu \end{cases}$$

$\implies \lambda = -1$ and $\mu = -1 \implies \lambda = \mu$ and hence, $\lambda = a+\mu \implies a = 0$

(b)

$$\mathbf{AB} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 5 & 2 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 & 0 \\ 1 & 3 & 2 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2+1+0 & 4+3+0 & 0+2+0 \\ -1+5+4 & 2+15+0 & 0+10+2 \\ -2+-1+2 & 4-3+0 & 0-2+1 \end{pmatrix} = \begin{pmatrix} -1 & 7 & 2 \\ 8 & 17 & 12 \\ -1 & 1 & -1 \end{pmatrix}$$

$(AB)^{-1} = B^{-1}A^{-1}$, thus finding the inverse of (AB) will yield $B^{-1}A^{-1}$.

$$\begin{aligned} |AB| &= \begin{vmatrix} -1 & 7 & 2 \\ 8 & 17 & 12 \\ -1 & 1 & -1 \end{vmatrix} = -1\begin{vmatrix} 17 & 12 \\ 1 & -1 \end{vmatrix} - 7\begin{vmatrix} 8 & 12 \\ -1 & -1 \end{vmatrix} + 2\begin{vmatrix} 8 & 17 \\ -1 & 1 \end{vmatrix} \\ &= -1(-17-12) - 7(-8+12) + 2(8+17) = 29 - 28 + 50 = 51 \end{aligned}$$

Hence $|AB| = 51$. The Adjoint of the matrix AB is calculated as follows

$$Cof(AB) = \begin{pmatrix} \begin{vmatrix} 17 & 12 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 8 & 12 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 8 & 17 \\ -1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 7 & 2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ -1 & -1 \end{vmatrix} & -\begin{vmatrix} -1 & 7 \\ -1 & 1 \end{vmatrix} \\ \begin{vmatrix} 7 & 2 \\ 17 & 12 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ 8 & 12 \end{vmatrix} & \begin{vmatrix} -1 & 7 \\ 8 & 17 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -29 & -4 & 25 \\ 9 & 3 & -6 \\ 50 & 28 & -73 \end{pmatrix}$$

$Adj(AB) = Cof(AB)^T = \begin{pmatrix} -29 & 9 & 50 \\ -4 & 3 & 28 \\ 25 & -6 & -73 \end{pmatrix}$. Thus $(AB)^{-1} = \frac{1}{51}\begin{pmatrix} -29 & 9 & 50 \\ -4 & 3 & 28 \\ 25 & -6 & -73 \end{pmatrix}$

10. (a) i. If $f(x) = f(x \pm 4)$ then $f(27) = f(4(6)+3) = f(3) = 4(3) - 7 = 12 - 7 = 5$

ii. $f(-106) = f(-4(27)+2) = f(2) = 4(2) - 7 = 8 - 7 = 1$

(b) If $(x^2-1)\frac{dy}{dx} + xy = 3y$ then, rearranging, we have

$$\frac{dy}{y} = \left(\frac{3-x}{x^2-1}\right)dx \implies \int \frac{dy}{y} = \int \frac{3-x}{x^2-1}dx \tag{2.33}$$

Expressing the above into partial fraction, we have

$$\frac{3-x}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

But $3 - x = A(x+1) + B(x-1)$ and when $x = 1, A = 1$. When $x = -1, B = -2$ and so

$$\frac{3-x}{(x-1)(x+1)} = \frac{1}{x-1} - \frac{2}{x+1}$$

Thus, we have that

$$\ln y = \ln|x-1| - 2\ln|1+x| + C = \ln\left(\frac{|x-1|}{(1+x)^2}\right) + C \tag{2.34}$$

$y(0) = 1 \implies \ln 1 = C$ and hence, $y(x) = \frac{|x-1|}{(x+1)^2}$

2.17 JUNE 2020 SOLUTION

Chapter 3

STATISTICS

3.1 JUNE 2015 PAPER 3

1. (a) Three events A, B and C are such that $P(A) = 1/3, P(C) = 5/7$ and $P(A \cup B) = 1/2$ Find
    i. $P(B)$ if A and B are mutually exclusive
    ii. $P(B)$ if A and B are Independent
    iii. $P(A \cap C)$ if A and C are exhaustive

   (b) Two cards are drawn at random one at the time without replacement from a pack of 52 playing cards. Find the probability that
    i. the two cards drawn are picture cards
    ii. the two cards drawn are of the same suit.

---

2. The lengths, y , to the nearest cm of some 8 rods and their corresponding masses, x kg, produced from a certain factory are shown below.

Rod	A	B	C	D	E	F	G	H
Mass, X	44	79	72	67	64	52	58	76
Length, Y	51	73	65	68	62	64	67	75

   (a) Calculate, to 3 decimal places the product moment correlation coefficient bwtween the masses and the lengths of the rods. Interpret your results.

   (b) determine the least square regression line of the masses on the lengths of these rods

   (c) hence, estimate the mass to the nearest kg, of the rod whose length was measured as 70cm.
   $\left(\text{You may use } \sum x^2 = 33790, \sum y^2 = 34833, \sum xy = 34129 \right)$

---

3. The Probability distribution of a discrete random variable X, is as shown below.

X=x	1	2	3	4	5
P(X=x)	a	$\frac{3}{10}$	b	$\frac{1}{5}$	$\frac{1}{20}$

Given that $E(X) = \frac{14}{5}$, Find

   (a) The values of the constants a and b,

(b) $P(3 \leq X \leq 5)$

(c) $P(X \leq 3)$

(d) Var(X)

(e) Given that $Y = 3X - 4$, find Var(Y).

---

4. (a) A firm manufactures radios and the probability that any radio fails a quality control is 0.2. Find the probability that in a sample of 15 radios,

   i. all will pass the control

   ii. exactly one fails the quality control

   iii. at least 13 pass the quality control

   (b) a building has an automatic telephone exchange and the number of wrong connections in any one day follow a Poisson distribution with mean λ. Given that $\lambda = 1.5$, find, to 3 decimal places the probability that in any one day

   i. there will be exactly three wrong connections

   ii. there will be three or more wrong connections.

---

5. The probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} \frac{x+1}{4}, & 0 \leq x \leq c, \\ 0, & \text{else} \end{cases}$$

Find

(a) the value of the constant c

(b) the mean and variance of X

(c) the median of X

(d) $P(X < \frac{1}{2})$

---

6. A variable X is normally distributed with mean 24 and variance 144. Find correct to four significant figures

(a) $P(X < 0)$

(b) $P(6 < X < 42)$

(c) $P(|x| < 12)$

(d) the value of y for which $P(3 < X < y) = 0.6299$

---

7. (a) Explain briefly, the following as used in Hypothesis testing.

   i. Type I error and Type II error,

   ii. the null and the alternative hypothesis,

   (b) In a competition to grow the tallest orange plant, samples of plants were taken from two farms and their heights measured. Let X denote the heights of the plants from one farm and Y the heights of the plants from the other farm, n_1 and n_2 the number of plants taken from the two farms respectievly. Given that

$$n_1 = 35, \quad E(X) = 206.5cm \quad Var(X) = 40.4cm^2$$
$$n_2 = 40, \quad E(Y) = 211.5cm \quad Var(Y) = 43.3cm^2$$

i. Find to one decimal place, an estimate of the population variance from the two samples

ii. Assuming that the heights of the plant are normally distributed with a common variance, test at the 2% level of significance whether there is a difference in the growth rate of the plants from the two samples.

---

8. A factory has 50 workers, 30 females and 20 males. The ages of the female workers to the nearest year are shown in the table below,

Age	21 − 25	26−30	31−35	36−40	41−45
Female	6	12	8	2	2

(a) Find to two decimal places, the mean and standard deviation of the ages of the female workers.

(b) Given that the mean of the ages of the male workers is 31.75, find the mean of the ages of all the workers in the factory.

(c) Estimate the modal age of the female workers in this factory.

3.2 JUNE 2016 PAPER 3

1. The table below shows the distribution of marks obtained by some candidates in an examination.

Marks (x)	0 − 9	10−19	20−29	30−39	40−49	50−59	60−69	70−79	80−89	90−99
No. of cand(f)	10	26	42	66	83	71	52	30	14	6

(a) Calculate to two decimal places, the mean mark of the distribution.

(b) By using 1 cm to represent 20 candidates on the vertical axis and 2 cm to represent 10 marks on the horizontal axis, draw a cumulative frequency curve for this distribution.

(c) Using your curve, estimate the number of candidates who passed the examination, if the pass mark is at least 40.

(d) Estimate also the percentages of candidates who passed the examination.

---

2. There are 2500 male and 2000 female students in a certain university. It is known that 5% of the male students and 0.75% of the female students wear reading glasses. A student X is chosen at random from this university. Find the probability that X

(a) Wears reading glasses

(b) is a female student, given that X wears reading glasses

(c) is a male student or X wears reading glasses.

---

3. A bank auditor finds out that a particular bookkeeping department makes averagely 1 error per week. Find the probability that in a period of one week,

(a) exactly 4 errors will be made

(b) at most 3 errors will be made

(c) at least 5 errors will be made

Calculate also to 4 decimal places, the probability that in 7-week period, no error will be made.

4. A random sample of 120 measurements is taken from a normal population. From the sample data given below:

$$\sum x = 1008, \qquad \sum(x-\bar{x})^2 = 172.8$$

Calculate,

(a) the unbiased estimates for the population mean and variance

(b) a 97% confidence interval for the population mean,

5. The probability mass function f of a discrete random variable, X is defined by

$$f(x) = \begin{cases} kx, & for \quad x = 1,2,3,4,5, \\ k(10-x) & for \quad x = 6,7,8,9 \end{cases}$$

Calculate

(a) the value of the constant k

(b) the mean and variance of X

(c) $E(2X-3)$ and $Var(2X-3)$

6. (a) A random variable X is such that $X \sim Bin(10, \frac{1}{5})$. Find the values

i. $P(X = 2)$

ii. $P(X \geqslant 1)$

(b) The masses of mangoes from a certain orchard are normally distributed with a mean of 225.2 grams and standard deviation 16.89 grams. A random sample of 100 mangoes selected from this orchard had a mean mass of 220.8 grams.
Stating clearly, the null and alternative hypotheses, perform a hypothesis test at 5% level of significance to find out if this sample provides significant evidence of a reduction in the population mean mass.

7. The probability density function f of a continuous random variable X, is defined as

$$f(x) = \begin{cases} \frac{4x}{81}(k-x^2), & 0 \leqslant x \leqslant 3 \\ 0 & elsewhere \end{cases}$$

Find

(a) the value of the constant k

(b) the mean and variance of X

Given that the median of this distribution is m, show that $2m^4 - 36m^2 + 81 = 0$.

8. The table below shows the inflation rate, x percent and unemployment rate, y percent, for 10 different countries in the month of December 2010.
Calculate, correct to three decimal places

Inflation rate, x	13.9	21.4	9.6	1.5	31.7	23.1	18.4	34.4	27.6	5.6
Unemployment rate, y	2.9	11.3	5.2	6.1	9.0	8.8	5.9	15.6	9.8	3.7

(a) the product moment correlation coefficient for this data

(b) the least square regression line of inflation rate on unemployment rate.

(c) the Kendall coefficient of rank correlation between inflation rate and unemployment rate.

You may use $\sum x = 187.2, \sum x^2 = 4599.12, \sum y = 78.3, \sum y^2 = 746.69, \sum xy = 1766.18$

3.3 JUNE 2017 PAPER 3

1. The probability mass function of a random variable X is defined by

$$f(x) = \begin{cases} \frac{1+kx}{22} & x = 0, 1, 2, 3, \\ 0, & otherwise \end{cases} \tag{3.1}$$

(a) Obtain the value of the constant k

(b) Write down the probability distribution of X

(c) Find $E(X)$ and $Var(X)$

(d) Calculate the values of $E(11X - 4)$ and $Var(11X - 4)$

2. Three events A, B and C are defined in a sample space S. Given that $P(A) = \frac{2}{5}$, $P(B) = \frac{1}{5}$ $P(C) = \frac{1}{2}$ and $P(B \cap C) = \frac{1}{10}$. Find

(a) $P(A \cup B)$ if A and B are independent

(b) $P(C|B)$

(c) $P(C|\bar{B})$

(d) the probability that one and only one of the events B or C will occur.

3. The marks obtained by some 50 class six pupils in an aptitude test into secondary schools are shown in the frequency distribution below.

Marks (x)	1 – 5	6 – 10	11–15	16–20	21–25
Frequency (f)	14	9	11	10	6

Calculate to two decimal places,

(a) the mean and standard deviation of this distribution,

(b) the mode and the median of the marks distribution

4. The probability that a woman from a particular area has brown eyes is $\frac{2}{5}$. A random sample of 100 women is to be selected from this area. Using the normal distribution as an approximation to the binomial distribution, estimate to three decimal places, the probability that

(a) at least half of the women will have brown eyes.

(b) the number of women who will have brown eyes will be between 30 and 45 inclusive.

(c) exactly 35 women will have brown eyes

---

5. The continuous random variable X has probability density function f where

$$f(x) = \begin{cases} kx(2-x) & 0 \leqslant x \leqslant 2 \\ 0, & otherwise \end{cases}$$

Find

(a) the value of the constant k

(b) the mean and the variance of X

(c) the cumulative distribution function of X

---

6. State two properties of the Poisson distribution.
In a certain malaria ward, the mean number of malaria parasites detected per milliliter of blood from the patients is 4. Find to three decimal places, the probability that in a randomly extracted one milliliter of blood, there will be

(a) no malaria parasite

(b) four malaria parasites

(c) less than three malaria parasite

Find also the probability that in another randomly extracted one-third milliliter of blood, there will be no malaria parasite.

---

7. (a) The probability that a basketball player will make a basket in any trial is $\frac{1}{4}$. Find the probability that

i. she makes her first basket in her fourth trial

ii. She requires at most 5 trials to make her first basket.

(b) A large population has mean $E(X) = \mu$

i. Independent identical random samples X_1, X_2 and X_3 are drawn from this population. Show that $\hat{\mu}$ is an unbiased estimator of the population mean, where

$$\hat{\mu} = \frac{1}{2}X_1 + \frac{1}{5}X_2 + \frac{3}{10}X_3$$

ii. A random sample of size 27 from this population produced the following statistics: $\sum x = 1560$ and $\sum (x-\bar{x})^2 = 168900$

Calculate to two decimal places, the most efficient unbiased estimate of the population mean and the population variance.

---

8. For 10 towns in a certain country, the heights above sea level, x in hundreds of metres, and their temperatures, y in degree centigrade on a particular day were recorded as shown in the table below.
Calculate to three decimal places

(a) the Pearson product-moment correlation coefficient for this data and interpret your result.

(b) the least square regression line of temperatures on height

(c) the temperature of a city that is 1300m above sea level on that particular day

---

City	A	B	C	D	E	F	G	H	I	J
Height, x	18	11	3	5	8	11	4	15	16	5
Temperature, y	9	13	18	17	13	10	16	10	6	14

3.4 JUNE 2018 PAPER 3

1. The frequency distribution below shows the masses of 400 pupils measured to the nearest kilogram.

Mass (kg)	31 – 35	36 – 40	41 – 45	46 – 50	51 – 55	56 – 60	61 – 65	66 – 70	71 – 75
No. of pupils	0	15	42	65	92	75	67	37	7

For this distribution, calculate to two decimal places,

(a) the mean

(b) the standard deviation

(c) the median

(d) the mode

---

2. (a) Two events A and B are such that $P(A) = \frac{1}{3}$, $P(B) = \frac{2}{5}$ and $P(A|B) = \frac{1}{12}$. Find

i. $P(A \cap B)$

ii. $P(B|A)$

iii. $P(A \cap B')$

(b) In a certain village, one quarter of a population has a particular disease. If a person has the disease, the probability that a laboratory test will show positive results is $\frac{19}{20}$. If a person does not have the disease, the probability that a laboratory test will show a negative result is $\frac{9}{10}$. A person is selected at random from the village and tested. Find the probability that

i. the test result is positive,

ii. the person has the diseases or the test result is positive

---

3. A continuous random variable X has probability density function f defined by

$$f(x) = \begin{cases} kx(3-x), & 0 \leq x \leq 3 \\ 0, & otherwise \end{cases}$$

Calculate, for this distribution

(a) the value of the constant k

(b) the mean,

(c) the variance,

(d) the mode

---

4. (a) State two conditions under which the normal distribution may be used as an approximation to the binomial distribution

(b) In a particular community, the probability that a man has brown eyes is $\frac{1}{6}$. A random sample of 180 men is taken from the community. Using the normal distribution as an approximation to the binomial distribution, find the probability that the number of men with brown eyes is:

i. exactly 35

ii. less than 35

iii. between 29 and 32 inclusive.

---

5. The probability mass function of a discrete random variable X is given by

$$P(X=x) = \begin{cases} k(7-x)(x+1) & \text{for} \quad x = 0,1,2,3,4,5,6 \\ 0, & \textit{otherwise} \end{cases}$$

Calculate

(a) The value of the constant k

(b) the mean of X

(c) the variance of X

(d) $P(1 \leqslant X < 4)$

(e) The mean and variance of $3X - 4$

---

6. (a) On the basis of the results obtained from a random sample of 64 adult males taken from a population with standard deviation σ, the 95% confidence interval for the mean blood pressure in the adult male is found to be $(115.55Nm^{-2}, 120.45Nm^{-2})$. Find

i. the values of $\bar{x}$, the sample mean and σ, the population standard deviation.

ii. the 99% confidence interval for the mean blood pressure of the adult male

(b) Two independent random variables X and Y are such that $X \sim N(200, 144)$ and $Y \sim N(175, 81)$. Find the distributions of

i. $X + Y$

ii. $X - Y$

---

7. (a) In a certain busy airport, planes leave at an average rate of 4 per minute. Use the Poisson distribution to find to 4 decimal places, the probability that:

i. no plane leaves during a particular one minute period

ii. at least one plane leaves during a particular 30 second period

(b) Two independent random variables X and Y are such that $X \sim N(40, 25)$ and $Y \sim N(34, 18)$. If from each of the distribution, a random sample of size 10 is taken, find the distribution of $\bar{X} - \bar{Y}$.

---

8. The relationship between two variables x and y is as shown in the table below

x	15	16	18	20	25	27	30	32	36	40
y	9	3	8	10	1	5	7	2	6	4

(a) Calculate to 2 decimal places, the product moment correlation coefficient for the data.

(b) Determine the least squares regression line of y on x

(c) Estimate the value of y when $x = 38$.

3.5 JUNE 2019 PAPER 3

1. (a) Two events A and B are such that $P(A) = 0.85$, $P(B) = 0.90$ and $P(A \bigcup B) = 0.95$. Calculate

   i. $P(A \cap B)$

   ii. $P(A \cap B')$

   State with reasons, whether or not A and B are independent

   (b) A bag contains 2 red and 3 green balls. Two balls are drawn from the bag at random in succession without replacement

   i. If the first ball drawn is red, find the probability that the second ball is green.

   ii. Find the probability that the two balls are of same color.

---

2. (a) Given that f is a probability mass function of a discrete random variable X, where $f(x_i) = p(X = x_i)$. State a formula for calculating

   i. the expected value, $E(X)$ of X

   ii. the variance, $Var(X)$, of X

   (b) A function f defined on the set $\{1, m, 6, 8, 12\}$ is given by the table below If f is the probability

x	1	m	6	8	12
$f(x)$	0.1	0.3	0.2	n	0.15

   mass function of the random variable X and $E(X) = 6.0$, calculate

   i. the values of the constants m and n,

   ii. the variance of X

   iii. $P(X < 8)$

---

3. (a) The probability density function f of a random variable X is given by

$$f(x) = \begin{cases} k(x+3) & -3 < x < 3 \\ 0, & otherwise \end{cases} \tag{3.2}$$

   Calculate

   i. the value of the constant k

   ii. the expected value of X

   iii. $P(0 < X < 2)$

   (b) If the median of X is m, show that m satisfies the equation $m^2 + 6m - 9 = 0$.

---

4. (a) The cumulative distribution function F, of a random variable X is defined by

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{4}, & 1 \leqslant x < 2, \\ \frac{3}{4} & 2 \leqslant x < 3 \\ \frac{63}{64} & 3 \leqslant x < 4 \\ 1, & 4 \leq x \end{cases}$$

   Calculate

i. $P(X = 3)$

ii. $P(1 < X \leqslant 3)$

(b) The masses, M of cubes of soap produced by a certain machine are normally distributed with mean $400g$ and standard deviation $1.6g$. By using the cumulative distribution function of the standard normal distribution, find the percentage of the cubes that will weigh

i. less than $399g$

ii. between $400.9g$ and $402.9g$

(c) The machine develops a fault that affects only the mean mass of the cubes. On investigation, technicians discover that 95% of the cubes produced weigh less than $402g$. What is the new mean mass?

---

5. (a) Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from an infinite population with mean μ and variance σ^2

i. What do you understand by the sample mean $\bar{X}$?

ii. By calculating $E(\bar{X})$, show that $\bar{X}$ is an unbiased estimator of the population mean μ.

iii. if the poppulation sampled is a normal population, state the distribution of $\bar{X}$.

(b) The masses of cows in a certain grazing settlement are normally distributed with mean $500kg$ and standard deviation $20kg$. Grazers from this settlement usually sell their cows in the same cattle market.

i. A trader buys 50 cows from this market. What is the probability that the mean mass of these cows will be between $495kg$ and $515kg$

ii. How many cows must he buy in order to be 95% sure that the mean mass is at least $495kg$?

---

6. (a) Define the following terms as used in hypothesis testing

i. Null hypothesis

ii. Test statistic

iii. Critical region

iv. level of significance

(b) Outline the procedure of performing a test of statistical hypothesis

(c) The proprietor of Northern provision center (N.P.C.) complains that of recent, the average mass of sachets of detergent supplied by a certain company is less than the stated value of $100g$. In order to investigate the proprietors complaint, the company took a sample of 100 sachets and realized that their mean mass was $99.88g$ with a standard deviation of $0.7g$. By using this information and the above procedure outlined in (b), determine at the 5% level of significance if the proprietors complaint is justified.

---

7. The marks obtained by 100 candidates in an examination are summarized in the distribution table.

Marks (x)	$10-19$	$20-29$	$30-39$	$40-49$	$50-59$	$60-69$	$70-79$
Frequency (f)	3	20	35	25	7	5	5

(a) Calculate

i. The mean mark

ii. the standard deviation of the marks

(b) Draw the cumulative frequency curve for this distribution.

(c) Using the curve drawn in (b), determine

i. the number of candidates who will pass the exam if the pass mark is 45

ii. the least mark a candidate must earn to be awarded an A grade if the top 10% of the class is to earn an A grade.

---

8. The scores of 10 candidates in a mock exams and the final exams are presented below

Mock exam, M	22	41	46	53	62	65	72	80	91	92
Final exam (F)	27	30	38	35	44	41	50	44	62	68

These marks are summarized as follows

$$\sum M = 624, \quad \sum M^2 = 43488, \quad \sum F = 439 \quad \sum F^2 = 20819 \quad \sum MF = 29833$$

(a) Calculate to two decimal places, the Pearson product moment correlation coefficient for this data

(b) Find the least squares regression line of the final exam scores on the mock exam score

(c) If the pass mark for the final exam is 40, determine whether a candidate who scored 60 in the mock exams but missed the final exams would have passed.

3.6 JUNE 2020 PAPER 3

1. Two events A and B are such that $P(A) = 1/3 \quad P(B) = 2/9$ and $P(A|B) = 1/2$. Find

(a) $P(A \cap B)$,

(b) $P(A \cup B)$,

(c) $P(B|A')$.
State with reasons whether A and B are

(d) mutually exclusive

(e) independent

(f) collectively exhaustive.

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2. Some express bus services occasionally experience delays in their departure or arrival time. Over a period of 12 weeks, the delays, in minutes for a particular bus service were recorded in a grouped frequency distribution shown below.

Delays in minutes	0−5	6 − 11	12−17	18−23	14−29	30−35	36−41
Frequency (f)	9	16	16	18	5	5	1

Compute to one decimal place the

(a) mean

(b) variance

(c) standard deviation

(d) mode of the distribution.

---

3. The mean number of goals scored per match by a certain football team is 1.5. Calculate , to 4 decimal places, the probability that the team will score

   (a) no goal in a match

   (b) not more than three goals in a match,

   (c) less than three goals in two matches.

   The team is to play n matches. Find the least value of n if the probability that the team is to score at least one goal is to be more than 0.95.

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4. The masses of oranges from an orchard are normally distributed with mean μ and standard deviation σ. 10% of the oranges from this orchard have masses les than 25 grammes and 5% have masses greater than 85 grammes.
   Calculate to 4 decimal places, the value of the mean, the standard deviation and the variance of the masses of oranges.

---

5. A discrete random variable X has the following probability distribution.

x	1	3	6	n	12
P(X=x)	$\frac{1}{10}$	$\frac{3}{10}$	k	$\frac{1}{4}$	$\frac{3}{20}$

   (a) Find the value of the probability k.
   Given that $E(X) = 6$ Calculate the

   (b) value of n,

   (c) variance of X,

   (d) mean and variance of $5X - 4$.

---

6. The probability density function f of a continuous random variable X is defined by

$$f(x) = \begin{cases} kx(4 - x^2), & 0 \leq x \leq 3 \\ 0, & otherwise. \end{cases}$$

   (a) Show that the value of the constant k is $-4/9$.

   (b) Calculate the mean of X

   (c) If the upper quartile of the random variable X is q, show that $4q^4 - 32q^2 - 27 = 0$.

---

7. A shop manager complains that the average mass of chocolate bars of a certain type that he buys from a company is less than 8.5g as stated on each packet. The shop manager randomly selelcted a sample of 100 chocolate bars from a large delivery and found out that the sample has a mean mass of 8.36g and a variance of 0.5185g

   (a) Calculate to 4 decimal places, an estimate of the variance of the masses of the chocolate bars.

   (b) By stating clearly the null and alternative hypothesis, test the manager's claim at the 5% level of significance.

Time, x	4	8	22	30	44	56	70	80	85	88
Marks,y	20	27	33	40	45	50	53	58	65	70

8. The time x, in minutes spent by some 10 students to prepare for a test and the marks, y , which they socred are shown in the table below.
Given that $\sum x^2 = 32805,$ $\sum y^2 = 23641,$ and $\sum xy = 27037.$

   (a) Calculate, to 4 decimal places, the product moment correlation coefficient for these data and briefly explain what information your answer gives you about the relationship between x and y

   (b) Find the equation of the regression line of y on x in the form $y = a + bx$, where a and b are constants.

   (c) A student who spent 50 minutes preparing for the test could not write because of ill health. Estimate the marks that this student would have scored if she wrote the test.

End

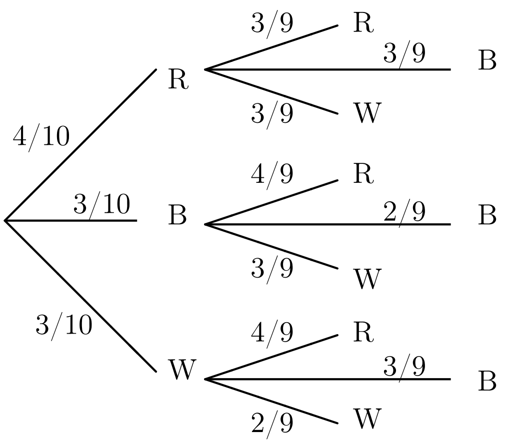
3.7 JUNE 2013 SOLUTION

1. (a) $P(A \cup B) = P(A) + P(B) - P(A \cap B) \implies P(A \cap B) = P(A) + P(B) - P(A \cup B) = \frac{1}{5} + \frac{3}{4} - \frac{7}{10} = \frac{1}{4}$

   (b) $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{4} \div \frac{3}{4} = \frac{1}{3}$

   (c) $P(A^c \cap B^c) = P(A \cup B)^c = 1 - P(A \cup B) = 1 - \frac{7}{10} = \frac{3}{10}$

   We draw the probability tree.(let Red=R, Blue=B and White=W)



   (d) $P(\text{same colour}) = P(WW) + P(BB) + P(RR) = 6/90 + 6/90 + 12/90 = 4/15$.

   (e) $P(\text{at least one white})$ =P(1 white) + $P(2 \text{ whites}) = P(RW) + P(WR) + P(WB) + P(BW) + P(WW) = 12/90 + 12/90 + 9/90 + 9/90 + 6/90 = 8/15$.

---

2. (a) If X is a Random variable then $\sum P(X = x) = 1$ i.e $3m + 3n = 1$...........(1)

   Now $P(X \geq 2) = P(X = 2) + P(X = 1) + P(X = 0) = m + 3n$ and

   $3P(X < 2) = 3[P(X = 0) + P(X = 1)] = 3(m + m) = 6m$

   Thus $P(X \geq 2) = 3P(X < 2) \implies m + 3n = 6m$.........(2). Solving (1) and (2) yields $m = 1/8$ and and $n = 5/24$.

   (b) $E(X) = \sum xP(X = x) = \frac{1}{8}(0 + 1 + 2) + \frac{5}{24}(3 + 4 + 5) = \frac{23}{8}$.

   (c) $E(X^2) = \sum x^2 P(X = x) = \frac{1}{8}(0^2 + 1^2 + 2^2) + \frac{5}{24}(3^2 + 4^2 + 5^2) = \frac{264}{24}$.

   Hence $Var(X) = E(X^2) - E(X)^2 = \frac{264}{24} - \frac{23}{8} = \frac{533}{192}$.

   (d) $E(Y) = E(5X - 3) = E(5X) - E(3) = 3E(X) - 3 = \frac{91}{8}$. Similarly,

   $Var(Y) = Var(5X - 3) = Var(5X) - Var(3) = 5^2 Var(X) - 0 = 69.401$

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3. (a) The class width is $\frac{35+30}{2} = 32.5$

$$E(X) = \frac{\sum xf}{\sum f} = \frac{32.5(18) + 38.5(34) + 44.5(58) + 50.5(42) + 56.5(24) + 62.5(10)}{18 + 34 + 58 + 42 + 24 + 10} = \frac{8577}{186}$$

   The modal class is $42 - 47$, Hence the mode is given by

$$mode = L + c\left(\frac{d_1}{d_1 + d_2}\right) = 41.5 + 6\left(\frac{24}{24 + 16}\right) = 45.1$$

   where L= lcb of modal class: C=class width: $d_1 = 58 - 34$, and $d_2 = 58 - 42$.

$$E(X^2) = \frac{\sum x^2 f}{\sum f} = \frac{32.5^2(18) + 38.5^2(34) + 44.5^2(58) + 50.5^2(42) + 56.5^2(24) + 62.5^2(10)}{18 + 34 + 58 + 42 + 24 + 10} = 2188.44$$

   The standard deviation is given by

   $\sigma = \sqrt{Var(X)} = \sqrt{E(X^2) - E(X)^2} = \sqrt{2188.44 - (45.1)^2} = 7.89$.

   %%%insert ogive here %%% The cummulative distribution table is given by

Mass x-kg(Ucb)	< 35.5	< 41.5	< 47.5	< 53.5	< 59.5	< 65.5
Cumulative freq	18	52	110	152	176	186

i. Median $= \left(\frac{n+1}{2}\right)^{th}$ Value $= \frac{186+1}{2} = 93.5^{th}$ value. From graph, median $=46$.
Alternatively: Without using graph, Median$= L + c\left(\frac{\frac{n}{2}-F_{m-1}}{fm}\right) = 41.5 + 6\left(\frac{\frac{186}{2}-52}{58}\right) =$ 45.7

ii. Quartile deviation $=$ semi-interquartile range $=\frac{1}{2}(Q_3 - Q_1)$
$Q_1 = \left(\frac{n+1}{4}\right)^{th}$ Value $\approx 47^{th}$ value, and $Q_3 = \left(\frac{3(n+1)}{4}\right)^{th}$ Value $\approx 140^{th}$ value. From graph, 47^{th} value $= 41$ and 140^{th} value $=52$, hence quartile deviation $= \frac{1}{2}(52-41) = 5.5$
Alternatively: $Q_1 = Lcb + c\left(\frac{P_v - F_{c-1}}{f}\right) = 35.5 + 6\left(\frac{47-18}{34}\right) = 40.62$
$Q_3 = Lcb + c\left(\frac{P_v - F_{c-1}}{f}\right) = 47.5 + 6\left(\frac{140-110}{42}\right) = 51.79$
hence quartile deviation $= \frac{1}{2}(51.79 - 40.62) \approx 5.6$

iii. Number of bags with mass $\leq 50kg = 128$. Number of bads with mass $> 50kg =$ $186 - 128 = 60$ bags. % of bags with mass $> 50kg = \frac{60}{186} \times 100 = 32.3\%$.

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4. (a) Since X is a continuous random variable, $\int_0^2 f(x)dx = 1 \Longrightarrow \int_0^1 f(x)dx + \int_1^2 f(x)dx = 1$

$$\Longrightarrow \int_0^1 xdx + \int_1^2 (k-x)dx = 1 \Longrightarrow \frac{x^2}{2}\bigg]_0^1 + \left(kx - \frac{x^2}{2}\right)\bigg]_1^2 = 1 \Longrightarrow k = 2.$$

(b) $E(X) = \int_0^2 xf(x)dx = \int_0^1 xf(x)dx + \int_1^2 xf(x)dx$

$$\Longrightarrow \int_0^1 x^2dx + \int_1^2 x(2-x)dx = \frac{x^3}{3}\bigg]_0^1 + \left(x^2 - \frac{x^3}{3}\right)\bigg]_1^2 = \frac{1}{3} + \frac{2}{3} = 1$$

(c) RTS that $\sigma = \frac{\sqrt{6}}{6}$. $E(X^2) = \int_0^2 x^2 f(x)dx = \int_0^1 x^3dx + \int_1^2(2x^2 - x^3)dx = \frac{1}{4} + \frac{11}{12} = \frac{7}{6}$.
$Var(X) = E(X^2) - E(X)^2 = \frac{7}{6} - (1)^2 = \frac{1}{6}$. But $\sigma = \sqrt{Var(X)} = \sqrt{\frac{1}{6}} = \frac{\sqrt{6}}{6}$.

(d) $F(t) = P(X \leq t) = \int_0^t xdx$.

Hence for $0 \leq x < 1$: $\quad F(t) = \int_0^t xdx = \frac{x^2}{2}\bigg]_0^t$. $\quad$ Note that $F(1) = \frac{1}{2}$

For $0 \leq x \leq 2$: $\quad F(t) = F(1) + \int_1^t (2-x)dx = \frac{1}{2} + (2x - \frac{x^2}{2})\bigg]_1^t = 2t - \frac{t^2}{2} - 1$. Note:$F(1) = 1$.

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{2} & 0 \leq x < 1 \\ 2x - \frac{x^2}{2} - 1 & 1 \leq x \leq 1 \\ 1 & x \geq 2 \end{cases}$$

Hence $P(\frac{3}{2} \leq x \leq 2) = F(2) - F(\frac{3}{2}) = 1 - \frac{7}{8} = \frac{1}{8}$.

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5. (a) Given the generating function $G(t) = e^{\lambda}e^{-\lambda t}$ we have that $E(X) = G'(1)$ and $Var(X) = G''(1) - \mu^2 + \mu$. RTS that $E(X) = Var(X)$.
$G(t) = e^{\lambda}e^{-\lambda t} \Longrightarrow G'(t) = \lambda e^{\lambda}e^{-\lambda t}$ and $G'(1) = \lambda e^0 = \lambda \Longrightarrow E(X) = \lambda = \mu$.......(1)

$G'(t) = \lambda e^{\lambda} e^{-\lambda t} \Longrightarrow G''(t) = \lambda^2 e^{\lambda} e^{-\lambda t} \Longrightarrow G''(1) = \lambda^2 e^{\lambda} e^{-\lambda} \Longrightarrow G''(1) = \lambda^2.$
$Var(X) = G''(1) - \mu^2 + \mu = \lambda^2 - \lambda^2 + \lambda = \lambda$.......(2)
From (1) and (2) we conclude that $E(X) = \lambda = Var(X)$.

(b) Let X be "the number of accidents that occur per week".Then $X \sim Poi(2)$. and $P(X = x) = \frac{e^{-2}2^x}{x!}$

i. P(exactly 3 accidents)= P(X=3)= $\frac{e^{-2}2^3}{3!} = 0.180$

ii. In a two weeks period, $X \sim Poi(4)$ and $P(X = x) = \frac{e^{-4}4^x}{x!}$
Hence P(at least 3 accidents)=$P(X \geq 3) = 1 - P(X < 3)$
$= 1 - \Bigg[P(X = 0) + P(X = 1) + P(X = 2)\Bigg] = 1 - \Bigg[\frac{e^{-4}4^0}{0!} + \frac{e^{-4}4^1}{1!} + \frac{e^{-4}4^2}{2!}\Bigg] = 0.762$

iii. see Junior

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6. (a) i. we take samples for the following reasons: (1) the population might be infinite (2) to reduce cost (3) the population might be infinite

ii. Population : 0,3,6,8 ; Samples (Without replacement): (0,3),(0,6),(0,8),(3,0),(3,6),(3,8), (8,0),(8,3),(8,6),(6,0),(6,3),(6,8). The frequency distribution of the means is given below

mean	2.5	3	4	4.5	5.5	7
frequency	2	2	2	2	2	2

iii. $E(\bar{X})$ (mean of sample means)= $\frac{2.5(2)+3(2)+4(2)+4.5(2)+5.5(2)+7(2)}{12} = \frac{53}{12}$
$\sigma^2 = \frac{\sum X^2}{n} - [E(\bar{X})]^2 = \frac{261.5}{12} - (\frac{53}{12})^2 = 2.28.$
Without replacement, $Var(\bar{X}) = \frac{\sigma^2}{n}\left(\frac{N-n}{N-1}\right) = \frac{2.28}{2}\left(\frac{4-2}{4-1}\right) = 0.78$

(b) The critical region reffers to the set of values of the parameter z for which the null hypothesis H_0 should be rejected. Howerver, the critical values are the boundries of the critical regions.

- If $X \sim N(450, 17^2)$ then $\bar{X} \sim N(450, \frac{17^2}{\sqrt{17}})$: We will let
$H_0 : \mu = 450$
$H_1 : \mu < 450$
- One tail test at 5% level of significance. $5\% level = 95\%$ confidence interval .This imples $\phi(a) = 0.95 \Longrightarrow a = \phi^{-1}(0.95) = 1.645$
- we reject H_0 if $Z < -1.645$
- test statistique: $z = \frac{\bar{x}-\mu}{\frac{\sigma^2}{\sqrt{n}}} = \frac{440-450}{\frac{17}{\sqrt{17}}} = -2.43$
- Since $z < -2.43 < -1.645$, we reject H_0 and conclude that at 5%-level, there is no significant evidence of a reduction in the population.

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7. (a) Let X, be the length of the diameter of the wheel drum. Then $X \sim N(28, 2^2)$.Then

i. $P(X < 26) = P(Z < \frac{26-28}{2}) = P(Z < -1.00) = 1 - \phi(1.00) = 1 - 0.8413 = 0.1587.$

ii. $P(X > 29) = P(Z > \frac{29-28}{2}) = P(Z > 0.5) = 1 - \phi(0.5) = 1 - 0.6915 = 0.3085.$

iii. $P(25 < X < 34) = P\left(\frac{25-28}{2} < Z < \frac{34-28}{2}\right) = P(-1.5 < Z < 3) = P(Z < 3) - P(Z < -1.5) = \phi(3.00) + \phi(1.5) + 1 = 0.9319$

(b) Let X, be the number of matches won. Then $X \sim B(4, \frac{2}{3})$. Then $P(X = x) = {}^{n}\mathrm{C}_x q^{n-x} p^x$.

i. $P(X=2) = {}^4\mathrm{C}_2(\frac{1}{3})^{4-2}(\frac{2}{3})^2 = 0.296.$

ii. P(at least one match)$= P(X \geq 1) = 1 - P(X \leq 0) = 1 - P(X = 0) = 1 - {}^4\mathrm{C}_0(\frac{1}{3})^{4-0}(\frac{2}{3})^0 = 0.988.$

iii. See junior

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8. (a) $\bar{X} = \frac{428}{8}, \quad \bar{Y} = \frac{354}{8}, \quad S_{xy} = \frac{\sum xy}{n} - \bar{X}\bar{Y} = \frac{20733}{8} - \left(\frac{428}{8} \times \frac{354}{8}\right) = 224.25$

$S_x^2 = \frac{\sum x^2}{n} - \bar{X}^2 = \frac{25588}{8} - \left(\frac{428}{8}\right)^2 = 336.25 \quad S_y^2 = \frac{\sum y^2}{n} - \bar{Y}^2 = \frac{18144}{8} - \left(\frac{354}{8}\right)^2 = 309.94$.

Hence the pearson's -Product moment correlation is given by

$$r = \frac{S_{xy}}{\sqrt{S_x^2 S_y^2}} = \frac{224.25}{\sqrt{336.25 * 309.94}} = 0.695.$$

(b) Regression line Physics on Maths reffer to Y on X.

$$y - \bar{y} = \frac{S_{xy}}{S_x^2}(x - \bar{x}) \Longrightarrow y - \frac{354}{8} = \frac{224.25}{336.25}\left(x - \frac{428}{8}\right) \Longrightarrow y = 0.6669x + 8.5709.$$

(c) $x = 67 \Longrightarrow y = 0.6669(67) + 8.5709 = 53.3$ Therefore the Physics score that corresponds to a Mathematics score of 67 is 53.3

(d) For Spearman's coefficient r_s we tabulate the ranks as follows

$Maths(x)$	8	7	3	4	6	2	1	5
$Phy(y)$	7	4	2	6	8	1	3	5
d^2	1	9	1	4	4	1	4	0

since $\sum d^2 = 24$ we have $\quad r_s = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6(24)}{8(64-1)} = 0.71$ to 2 d.p.

3.8 JUNE 2015 SOLUTION

1. (a) A and B are mutually exclusive if and only if $P(A \cup B) = P(A) + P(B) \Longrightarrow 1/2 = 1/3 + P(B) \Longrightarrow P(B) = 1/6.$

(b) A and B are independent if $P(A \cap B) = P(A) \cdot P(B) \Longrightarrow P(A \cup B) = P(A) + P(B) + P(A)P(B) \Longrightarrow P(A \cup B) = P(A) + P(B)[1 + P(A)] \Longrightarrow \frac{1}{2} = \frac{1}{3} + \frac{2}{3}P(B) \Longrightarrow P(B) = \frac{1}{4}.$

(c) A and B are ehaustive if $P(A \cup B) = 1 \Longrightarrow P(A \cup B) = P(A) + P(C) - P(A \cap B) = 1 \Longrightarrow 1 = \frac{1}{3} + \frac{5}{7} - P(A \cap C) \Longrightarrow P(A \cap C) = \frac{1}{21}.$

(d) P(Picture card)$= \frac{12}{52} \cdot \frac{11}{51} = \frac{11}{221.}$

(e) P(Same suit)$=$ p(2 spade)+p(2 club)+ p(2 diamond)+p(2 heart)$= 4 \cdot \left(\frac{13}{52} \cdot \frac{12}{51}\right) = 0.235$

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2. (a) $\bar{X} = \frac{512}{8}, \quad \bar{Y} = \frac{525}{8}, \quad S_{xy} = \frac{\sum xy}{n} - \bar{X}\bar{Y} = \frac{34129}{8} - \left(\frac{512}{8} \times \frac{525}{8}\right) = 66.125$

$S_x^2 = \frac{\sum x^2}{n} - \bar{X}^2 = \frac{33790}{8} - \left(\frac{512}{8}\right)^2 = 127.75 \quad S_y^2 = \frac{\sum y^2}{n} - \bar{Y}^2 = \frac{34833}{8} - \left(\frac{525}{8}\right)^2 = 47.484$.

Hence the pearson's -Product moment correlation is given by

$$r = \frac{S_{xy}}{\sqrt{S_x^2 S_y^2}} = \frac{66.125}{\sqrt{127.75 * 47.484}} = 0.849$$

We have a high positive correlation between the masses and the lengths of the rods produced by the machine.

(b) Regression line of masses on length (X on Y) is given by

$$x - \bar{x} = \frac{S_{xy}}{S_y^2}(y - \bar{y}) \Longrightarrow x - \frac{512}{8} = \frac{66.125}{47.4837}\left(y - \frac{525}{8}\right) \Longrightarrow x = 1.393y - 91.387$$

(c) when $y = 70$, the mass of the rod is given by $x = 1.393(70) - 27.387 = 70.092$ i.e mass $x = 70cm$ to the nearest cm.

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3. (a) If X is a discrete random variable, $\sum P(X = x) = 1, \Longrightarrow a + 3/10 + b + 1/5 + 1/20 = 1 \Longrightarrow a + b = 9/20$......(1). Also $E(X) = 14/5 \Longrightarrow a + \frac{6}{10} + 3b + \frac{4}{5} + \frac{5}{20} = \frac{14}{5} \Longrightarrow a + 3b = \frac{23}{20}$........(2). By solving (1) and (2), we have $a = \frac{1}{10}$ and $b = \frac{7}{20}$.

(b) $P(3 \leq X \leq 5) = P(X = 3) + P(X = 4) + P(X = 5) = \frac{7}{20} + \frac{1}{5} + \frac{1}{20} = \frac{3}{5}$

(c) $P(X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3) = \frac{1}{10} + \frac{3}{10} + \frac{7}{20} = \frac{3}{4}$

(d) $E(X^2) = \frac{1}{10} + \frac{12}{10} + \frac{63}{10} + \frac{16}{5} + \frac{25}{20} = \frac{89}{10}$. But $Var(X) = E(X^2) - E(X)^2 = \frac{89}{10} - \left[\frac{14}{5}\right]^2 = \frac{53}{50}$.

(e) $Var(Y) = Var(3X - 4) = 3^2 Var(X) - Var(4) = 9(\frac{53}{50}) - 0 = \frac{477}{50}$.

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4. Let X be "the number of radio that fail te test" then $X \sim Bin(15, 0.2)$. i.e $P(X = x) = {}^nC_x q^{n-x} p^x$. If 0.2 fail the test then 0.8 pass the same test and vice versa.

(a) P(all pass)$= P(X = 15) == {}^{15}C_{15}(0.2)^0(0.8)^{15} = 0.035$
Note: p and q have beem interchanged in the formula above since the variable X is defined wrt those who failed and the question is asked wrt those who pased.

(b) P(exactly 1 fail)$= P(X = 1) = {}^{15}C_1(0.8)^{14}(0.2)^1 = 0.132$

(c) P(at leat 13 pass) $P(X \geq 13) = P(X = 13) + P(X = 14) + P(X = 15) = {}^{15}C_{13}(0.2)^2(0.8)^{13} + {}^{15}C_{14}(0.2)^1($ 0.0398.
Let Let X be "the number of wrong connections per day" then $X \sim Poi(1.5)$.

(d) $P(X = 3) = \frac{e^{-1.5}1.5^3}{3!} =$

(e) $P(X \geq 3) = 1 - P(X < 3) = 1 - \left[P(X = 0) + P(X = 1) + P(X = 2)\right] = 1 - e^{-1.5}(1 + 3/2 + 9/8) = 0.1912$

---

5. (a) X is a continous random variable if $\int_0^c f(x)dx = 1. \Longrightarrow \frac{1}{4}\int_0^c (x + 1)dx = 1 \Longrightarrow c^2 + 2c - 8 = 0 \Longrightarrow c = 2$ or $c = -4$ choose $c = 2 > 0$.

(b) $E(X) = \int_0^2 xf(x)dx = \frac{1}{4}\int_0^2 x(x+1)dx = \frac{7}{6}$. Now $E(X^2) = \int_0^2 x^2 f(x)dx = \frac{1}{4}\int_0^2 x^2(x+1)dx = \frac{5}{3}$ Hence $Var(X) = \frac{5}{3} + \left(\frac{7}{6}\right)^2 = \frac{11}{36}$.

(c) Median is m such that $\int_0^m f(x)dx = \frac{1}{2} \Longrightarrow \frac{1}{4}\int_0^m x^2(x + 1)dx = \frac{1}{2} \Longrightarrow m^2 + 2m - 4 = 0 \Longrightarrow$ median $= \sqrt{5} - 1$.

(d) $P(X < \frac{1}{2}) =$????

---

6. $X \sim N(24, 12^2) \Longrightarrow P(X < x) = P(Z < \frac{x-\mu}{\sigma})$

(a) $P(X < 0) = P(Z < \frac{0-24}{12}) = P(Z < -2) = 1 - \phi(2) = 1 - \phi(1 - 2) = 1 - \phi(-1) = 1 - [1 - \phi(1)] = \phi(1) = 0.8413.$?????

(b) $P(6 < X < 42) = P(\frac{6-24}{12} < Z < \frac{42-24}{12}) = P(-1.5 < Z < 1.5) = P(|Z| < 1.5)$ which by symmetry is equal to $P(Z < 1.5) = \phi(1.5) = 0.9332$

(c) $P(|X| < 1.5) = P(-12 < X < 12) = P(\frac{-12-24}{12} < Z < \frac{12-24}{12}) = P(-3 < Z < -1) = 1 - \phi(1) - [1 - \phi(3)] = \phi(3) - \phi(1) =$???

---

7. (a) Type I error means rejecting a true null hypothesis. While Type II error means accepting a false null hypothesis.

   (b) The null hypothesis (H_0) makes an asertion about the value a parameter can take. it states the statu quo. mean while the alternative hypothesis (H_1) state how the value of a parameter will deviate from that specified by (H_0).

   (c) $\sigma hat^2 = \frac{n_1\sigma_1^2+n_2\sigma_2^2}{n_1+n_2-2} = \frac{35(40.4)+40(43.3)}{34+40-2} = 43.1$

   (d) ????

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8. The class width is $\frac{25+21}{2} = 23$

   (a) $E(X) = \frac{\sum xf}{\sum f} = \frac{23(6)+28(12)+33(8)+38(2)+43(2)}{30} = \frac{900}{30} = 30$
   $E(X^2) = \frac{\sum xf}{\sum f} = \frac{23^2(6)+28^2(12)+33^2(8)+38^2(2)+43^2(2)}{30} = \frac{27880}{30}$
   $\sigma = \sqrt{var(X)} = \sqrt{E(X^2) - E(X)^2} = 5.42$

   (b) $E(X_{men}) = 31.75 \implies \frac{\sum X_{men}}{20} = 31.75 \implies \sum X_{men} = 20(31.75)$
   the excpecteda age of all workers is given by $\frac{\sum_{men}+\sum_{women}}{50} = \frac{20(31.75)+900}{50} = 30.7$ which is thw average age of men and women.

   (c) The modal class is $26-30$, Hence the mode is given by

$$mode = L + c\left(\frac{d_1}{d_1+d_2}\right) = 25.5 + 5\left(\frac{4}{4+6}\right) = 27.5$$

3.9 JUNE 2016 SOLUTION

3.10 JUNE 2017 SOLUTION

1. (a) Since X is a discrete random variable, and

$$f(x) = \begin{cases} \frac{1+kx}{22} & x = 0, 1, 2, 3, \\ 0, & otherwise \end{cases}$$

   $\sum f(x) = 1$. That is

$$\frac{1}{22}+\frac{1+k}{22}+\frac{1+2k}{22}+\frac{1+3k}{22} = 1 \implies 1+1+k+1+2k+1+3k = 22 \implies 4+6k = 22 \implies k = 3$$

   Thus,

$$f(x) = \begin{cases} \frac{1+3x}{22} & x = 0, 1, 2, 3, \\ 0, & otherwise \end{cases}$$

   (b) The probability distribution of X is given by

$f(x)$	$\frac{1}{22}$	$\frac{4}{22}$	$\frac{7}{22}$	$\frac{10}{22}$
x	27	30	38	35

   (c) We find $E(X)$ and $Var(X)$

$$E(X) = \sum xf(x) = 0\left(\frac{1}{22}\right) + 1\left(\frac{4}{22}\right) + 2\left(\frac{7}{22}\right) + 3\left(\frac{10}{22}\right) = \frac{24}{11}$$

$$Var(X) = \sum x^2 f(x) = 0^2\left(\frac{1}{22}\right) + 1^2\left(\frac{4}{22}\right) + 2^2\left(\frac{7}{22}\right) + 3^2\left(\frac{10}{22}\right) = \frac{61}{11}$$

   Therefore, the variance of X is given by $Var(X) = E(X^2) - E(X)^2 = \frac{61}{11} - \left(\frac{24}{11}\right)^2 = 0.7851$

(d) $E(11X-4) = 11E(X) - E(4) = 11\left(\frac{24}{11}\right) - 4 = 20$ and $Var(X) = Var(11X) - Var(4) = (11)^2Var(X) - 0 = 121\left(\frac{95}{121}\right) - 0 = 95$

2. (a) If A and B are independent, then $P(A\bigcap B) = P(A).P(B)$

$$\begin{aligned} P(A\cup B) &= P(A)+P(B)-P(A\cap B) = P(A)+P(B)-P(A)P(B) \\ &= \frac{2}{5}+\frac{1}{5}-\frac{2}{5}\times\frac{1}{5} = \frac{3}{5}-\frac{2}{25} = \frac{15-2}{25} = \frac{13}{254} \end{aligned}$$

(b) $P(C/B) = \frac{P(B\cap C)}{P(C)} = \frac{\frac{1}{10}}{\frac{1}{2}} = \frac{2}{10} = \frac{1}{5}$

(c) $P(C/\bar{B}) = \frac{P(\bar{B}\cap C)}{P(C)} = \frac{P(C)-P(B\cap C)}{P(\bar{B})} = \frac{\frac{1}{2}-\frac{1}{10}}{\frac{4}{5}} = \frac{1}{2}$

(d) In this case, B abd C are mutually exclusive. That is, $P(B\cup C) = P(B)+P(C) = \frac{1}{5}+\frac{1}{2} = \frac{7}{10}$

3. (a) The class width is $\frac{1+5}{2} = 3$

$$E(X) = \frac{\sum_{i=1}^n f_ix_i}{\sum\limits_{i=1}^n f_i} = \frac{3(14)+8(9)+13(11)+18(10)+23(6)}{14+9+11+10+6} = \frac{575}{50} = \frac{23}{2}$$

$$E(X^2) = \frac{\sum_{i=1}^n f_ix_i^2}{\sum\limits_{i=1}^n f_i} = \frac{3^2(14)+8(9)+13^2(11)+18^2(10)+23^2(6)}{14+9+11+10+6} = \frac{8975}{50} = 179.5$$

We have that $Var(X) = E(X^2) - E(X)^2 = 179.5 - (11.5)^2 = 47.25$

(b) ????????

4. Let X be the number of women with brown eyes, then $X \approx Bin(100, \frac{2}{5})$. But we know that if $X \approx Bin(n,p)$, then $X \approx N(np, npq)$ for $n > 10$ and $np > 5$. That is $N \approx N(40, 24)$

(a) The probability that at least half of the women will have brown eyes is given by $P(X \geqslant 50) =$

(b) $P(30 \leqslant X \leqslant 45) =$?????

5. If the probability density function f is given by

$$f(x) = \begin{cases} kx(2-x) & 0 \leqslant x \leqslant 2 \\ 0, & otherwise \end{cases}$$

(a) Since X is a continuous random variable, $\int\limits_0^2 f(x)dx = 1$

$$k\int\limits_0^2 x(2-x)dx = k\left[x^2 - \frac{x^3}{3}\right]_0^2 = 1 \implies k\left[4-\frac{8}{3}\right] = 1 \implies k = \frac{3}{4}$$

(b) The mean, $E(X) = \int\limits_0^2 xf(x)dx = \frac{3}{4}\int\limits_0^2 x^2(2-x)dx = \frac{3}{4}\left[\frac{2x^3}{3}-\frac{x^2}{4}\right]_0^2 = \frac{3}{4}\left[\frac{16}{3}-\frac{16}{4}\right] = \frac{3}{4}\times\frac{4}{3} = 1$

$$Var(X) = E(X^2) - E(X)^2$$

But

$$E(X^2) = \frac{3}{4}\int\limits_0^2 x^3(2-x)dx = \frac{3}{4}\left[\frac{2x^2}{4}-\frac{x^5}{5}\right]_0^2 = \frac{3}{4}\times\frac{8}{5} = \frac{6}{5}$$

Hence, $Var(X) = \frac{6}{5} - 1^2 = \frac{1}{5}$

(c) Generally, $F(t) = \int\limits_a^t f(s)ds$. Hence, $F(t) = \frac{3}{4}\int\limits_0^t x(2-x)dx = \frac{3}{4}\left[x^2 - \frac{x^3}{3}\right]_0^t = \frac{3t^2-t^3}{4}$ Thus,

$$F(t) = \begin{cases} 0 & x < 0 \\ \frac{3x^2-x^3}{4} & 0 \leqslant x \leqslant 2 \\ 1 & x > 2 \end{cases}$$

6. (a) The properties of the Poisson distribution are as follows
   i. the events occur randomly
   ii. the events occur uniformly. that is the mean number of events occurring over a given interval of time is constant.
   iii. the events are independent.

   (b) Let X be the random variable " the number of malaria parasite per millimeter of blood", then $X \approx Poi(4)$ and $P(X = x) = \frac{e^{-4}4^x}{x!}$
   i. $P(X = 0) = \frac{e^{-4}4^0}{0!} = 0.018$
   ii. $P(X = 4) = \frac{e^{-4}4^4}{4!} = 0.195$
   iii. $P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{e^{-4}4^0}{0!} + \frac{e^{-4}4^1}{1!} + \frac{e^{-4}4^2}{2!} = 0.238$

7. (a) Let X be the random variable "First basket made", then $X \approx Geo(\frac{1}{4})$

   (b) $\hat{\mu}$ is an unbiased estimator of μ if $E(\hat{\mu}) = \mu$. Given that $E(X) = \mu$ and $\hat{\mu} = \frac{1}{2}X_1+\frac{1}{5}X_2+\frac{3}{10}X_3$ we have

$$E(\hat{\mu}) = E(\frac{1}{2}X_1 + \frac{1}{5}X_2 + \frac{3}{10}X_3) = \frac{1}{2}E(X_1) + \frac{1}{5}E(X_2) + \frac{3}{10}E(X_3) = \frac{\mu}{2} + \frac{\mu}{5} + \frac{3\mu}{10} = \mu$$

   The unbiased estimator for the population mean is $\hat{\mu} = \mu = \frac{\sum x}{n} = \frac{1560}{27} \implies \hat{\mu} = 57.78$
   The unbiased estimator for the population variance is $\hat{\sigma^2} = \frac{nS^2}{n-1} = \frac{27(\frac{168900}{27})}{26} = 6496.15$

8. (a) The Pearson product moment correlation coefficient is given by

$$r = \frac{S_{xy}}{S_xS_y} \qquad where \qquad S_{xy} = \frac{\sum xy}{n} - \bar{x}\bar{y} \qquad S_x^2 = \frac{\sum x^2}{n} - \bar{x}^2 \qquad S_y^2 = \frac{\sum y^2}{n} - \bar{y}$$

   But $\frac{\sum xy}{n} = \frac{1038}{10}, \frac{\sum x}{n} = \frac{96}{10}, \frac{\sum y}{n} = \frac{126}{10}, \frac{\sum y^2}{n} = \frac{1720}{10}, \frac{\sum x^2}{n} = \frac{1186}{10}$

$$S_{xy} = \frac{1038}{10} - \left(\frac{96}{10} \times \frac{126}{10}\right) = -17.16$$

$$S_x^2 = \frac{1038}{10} - \left(\frac{96}{10} \times \frac{126}{10}\right) = -17.16 \qquad S_y^2 = \frac{1038}{10} - \left(\frac{96}{10} \times \frac{126}{10}\right)$$

$$r = \frac{S_{xy}}{S_xS_y} = \frac{-17.16}{\sqrt{26.44 \times 13.24}} = \frac{-17.16}{18.71} = -0.917$$

   (b) If the temperature is y, and the height is x, then the least square regression line of temperature on height refers to the line y on x given by

$$y - \bar{y} = \frac{S_{xy}}{S_x^2}(x - \bar{x}) \implies y - 12.6 = \frac{103.8}{26.44}(x - 9.6) \implies y - 12.6 = 3.923(x - 9.6)$$

   Thus, the least square regression line y on x is $y = 3.923x - 25.088$

   (c) Given that the height is $1300m$, then $y = 3.923(13) - 25.088 = 25.911$. Hence, for a distance of 1300m, the temperature is 25.911^0C

3.11 JUNE 2018 SOLUTION

1. (a) Class size is $\frac{35.5+30.5}{2} = 33$

$$E(X) \quad = \quad \frac{\sum\limits_{i=1}^{n} f_i x_i}{\sum\limits_{i=1}^{n} f_i} = \frac{33(0) + 38(15) + 43(42) + ... + 63(67) + 68(37) + 73(7)}{15 + 42 + ... + 67 + 37 + 7} = \frac{2197}{40}$$

(b) The variance $\sigma = \sqrt{Variance}$ but $Var(X) = E(X^2) - E(X)^2$

$$E(X^2) \quad = \quad \frac{(33)^2(0) + (38)^2(15) + (43)^2(42) + ... + (63)^2(67) + (68)^2(37) + (73)^2(7)}{15 + 42 + ... + 67 + 37 + 7} = \frac{30853}{10}$$

Hence, $\sigma = \sqrt{Variance} = \sqrt{E(X^2) - E(X)^2} = 8.28$

(c) The median is given by the formula $Median = L + C\left(\frac{\frac{1}{2}n - F_{m-1}}{f_m}\right)$. The median class is $51 - 55$ so, the median is

$$50.5 + 5\left(\frac{200.5 - 122}{92}\right) = 54.77$$

(d) The mode is $Mode = L + C\left(\frac{d_1}{d_1+d_2}\right)$. the modal class is the class with the highest frequency and in this case it is $51 - 55$, $d_1 = |f_m - f_{m-1}|$ and $d_2 = |f_{m+1} - f_m|$. therefore, the mode is

$$50.5 + 5\left(\frac{27}{27 + 17}\right) = 53.57$$

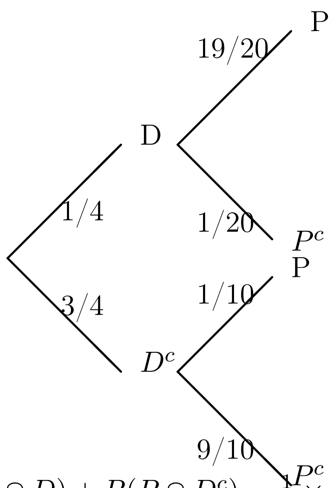
2. (a) $P(A) = \frac{1}{3}, P(B) = \frac{2}{5}$ and $P(A/B) = \frac{1}{12}$

   i. $P(A \cap B) = P(A/B)P(A) \implies P(A \cap B) = \frac{1}{12} \times \frac{2}{5} = \frac{1}{30}$

   ii. $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1}{30} \div \frac{1}{3} = \frac{1}{10}$

   iii. $P(A \cap B') = P(A) - P(A \cap B) = \frac{1}{3} - \frac{1}{30} = \frac{3}{10}$

(b)



   i. $P(P) = P(P \cap D) + P(P \cap D^c) = \frac{1}{4} \times \frac{19}{20} + \frac{3}{4} \times \frac{1}{10} = \frac{5}{16}$

   ii. $P(D \cup P) = P(D) + P(P) - P(D \cap P) = \frac{1}{4} + \frac{5}{16} - \frac{19}{80} = \frac{26}{80}$

3. (a) Since X is a continuous random variable, $\int\limits_0^3 f(x)dx = 1$. Hence,

$$\int\limits_0^3 kx(3 - x)dx = 1 \implies k\int\limits_0^3 (3x - x^2)dx = 1 \implies k\left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = 1 \implies k = \frac{2}{9}$$

(b) $E(X)=\int\limits_0^3 x^2 f(x)dx=\frac{2}{9}\int\limits_0^3(3x^2-x^3)dx=\frac{2}{9}\left[x^3-\frac{x^4}{4}\right]_0^3=\frac{3}{2}$

(c) $Var(X)=E(X^2)-E(X)^2$ but

$$E(X^2)=\int\limits_0^3 x^3(3-x)dx=\frac{2}{9}\left[\frac{3x^4}{4}-\frac{x^5}{5}\right]_0^3=\frac{27}{10}$$

Thus, $Var(X)=E(X^2)-E(X)^2=\frac{27}{10}-\left(\frac{3}{2}\right)^2=\frac{9}{20}$

(d) $\frac{df(x)}{dx}=k(3-2x)$ but if x is the mode then $\frac{df(x)}{dx}=0 \implies k(3-2x)=0 \implies x=\frac{3}{2}$

4. (a) To approximate a Binomial distribution using the Normal distribution, we require that $n>10$ and $p\approx 0.5$ and consequently, $np>5$. In that case, if $X\approx B(n,p)$ then $X\approx N(np,npq)$

(b) i. Let X be the number of men with brown eyes, then $X\approx Bin(180,\frac{1}{6})$. Using the normal distribution to approximate, we write $X\approx N(30,25)$ and
$P(X=35)=P(\frac{34.5-30}{5}<Z<\frac{35.5-30}{5}) \implies P(0.9<Z<1.1)=P(Z<1.1)-P(Z<0.9)=\Phi(1.1)-\Phi(0.9)=0.8643-0.8159=0.048$

ii. $P(X<35)=P(X<34.5)$ by the continuity correction. But $P(X<34.5)=P(Z<\frac{34.5-30}{5})=P(Z<0.9)=\Phi(0.9)=0.8159$

iii. $P(29\leqslant X\leqslant 32)=P(28,5\leqslant X\leqslant 32.5)$ by the continuity correction. Hence, $P(\frac{28.5-30}{5}\leqslant X\leqslant\frac{32,5-30}{5})=P(-0.3\leqslant Z\leqslant 0.5)=P(Z\leqslant)\Phi(0.5)-(1-\Phi(0.3))=\Phi(0.5)+\Phi(0.6179)-1=0.3094$

5. (a) If X is a discrete random variable, then $\sum\limits_{x_i=1}^{n}P(X=x_i)=1$ and

$$P(X=x)=\begin{cases} k(7-x)(x+1) & for \quad x=0,1,2,3,4,5,6\\ 0, & otherwise\end{cases}$$

$k\left[7+12+15+16+15+12+7\right]=1 \implies 84k=1 \implies k=\frac{1}{84}$

(b)

$P(X=x)$	$\frac{7}{84}$	$\frac{12}{84}$	$\frac{15}{84}$	$\frac{16}{84}$	$\frac{15}{84}$	$\frac{12}{84}$	$\frac{7}{84}$
x	0	1	2	3	4	5	6

$E(X)=\sum\limits_{x=0}^{6}xP(X=x)=\frac{7}{84}(0)+\frac{12}{84}(1)+\frac{15}{84}(2)+\frac{16}{84}(3)+\frac{15}{84}(4)+\frac{12}{84}(5)+\frac{7}{84}(6)=3$

(c) $Var(X)=E(X^2)-E(X)^2$. But $E(X^2)=\frac{7}{84}(0^2)+\frac{12}{84}(1^2)+\frac{15}{84}(2^2)+\frac{16}{84}(3^2)+\frac{15}{84}(4^2)+\frac{12}{84}(5^2)+\frac{7}{84}(6^2)=\frac{1008}{84}=12$. Thus $Var(X)=12-3^2=3$

(d) $P(1\leqslant X<4)=P(X=1)+P(X=2)+P(X=3)=\frac{12}{84}+\frac{15}{84}+\frac{16}{84}=\frac{43}{84}$

(e) $E(3X-4)=E(3X)-E(4)=3E(X)-4=3(3)-4=5$
$Var(3X-4)=Var(3X)-Var(3)=3^2Var(X)-0=3^2(3)=27$

6. (a) We construct the 95% confidence interval as follows $\Phi(a)=0.95 \implies a=\Phi^{-1}(0.95)=1.645$. Therefore the 95% confidence interval is given by $\bar{x}\pm 1.645\frac{\sigma}{\sqrt{n}}$ and hence $(\bar{x}-1.645\frac{\sigma}{\sqrt{n}},\bar{x}+1.645\frac{\sigma}{\sqrt{n}})=(115.55,120.45)$. Given that $n=64$ we have

$$\begin{aligned}\bar{x}-1.645\frac{\sigma}{\sqrt{164}} &= 115.55\\ \bar{x}+1.645\frac{\sigma}{\sqrt{164}} &= 120.45\end{aligned}$$

i. $\bar{x} = 117.98$ and $\sigma = 11.91$

ii. To find the 99% confidence interval, we write $\Phi(a) = 0.99 \implies a = \Phi^{-1}(0.99) = 2,575$. Therefore the 99% confidence interval is given by $\bar{x} \pm 2.575\frac{\sigma}{\sqrt{n}}$ and hence $(\bar{x} - 2,575\frac{\sigma}{\sqrt{n}}, \bar{x} + 2.575\frac{\sigma}{\sqrt{n}}) = (117.98 - 2.575(\frac{11.91}{8}), 117.98 + 2.575(\frac{11.91}{8})) = (114.00, 121.67)$ is the 99% confidence interval for the mean blood pressure.

(b) Let $X \approx N(200, 144)$ and $Y \approx N(175, 81)$, then

i. $X + Y \approx N(200 + 175, 144 + 81) \approx N(375, 225)$

ii. $X - Y \approx N(200 - 175, 144 + 81) \approx N(25, 225)$

7. (a) Let X be the number of planes that leave the airport every minute. Then $X \approx Poi(\lambda)$ where $\lambda = 4$ and $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$

i. $P(X = 0) = \frac{e^{-4}4^0}{0!} = e^{-4} = 0.018$

ii. In a period of one minute, 4 planes leave. In 30 seconds, two planes will leave. Hence, $P(X \geqslant 1) = 1 - P(X < 1) = 1 - P(X = 0) = 1 - \frac{e^{-2}2^0}{0!} = 1 - 0.018 = 0.86$

(b) $X \approx N(40, 25)$ and $Y \approx N(34, 18)$, then for a sample of size $n = 10$, $\bar{X} \approx N(40, \frac{25}{10})$ and $\bar{Y} \approx N(34, \frac{18}{10})$, then the distribution of $\bar{X} - \bar{Y} \approx N(40 - 34, \frac{25}{10} + \frac{18}{10}) \approx N(6, \frac{43}{10})$

8. (a) The product moment correlation coefficient is given by $r = \frac{S_{xy}}{S_x S_y}$ where $S_{xy} = \frac{\sum xy}{n} - \bar{x}\bar{y}$ and $S_x = \sqrt{\frac{\sum x^2}{n} - (\bar{x})^2}$ and $S_y = \sqrt{\frac{\sum y^2}{n} - (\bar{y})^2}$. Hence, $S_{xy} = \frac{1337}{10} - \frac{259}{10} \times \frac{55}{10} = -8.75$, $S_x = \sqrt{\frac{7379}{10} - (\frac{259}{10})^2} = \sqrt{67.09}$, $S_y = \sqrt{\frac{385}{10} - (\frac{55}{10})^2} = \sqrt{8.25}$. Hence, $r = \frac{S_{xy}}{S_x S_y} = \frac{-8.75}{\sqrt{67.09 \times 8.25}}$

(b) The least square regression line y on x is given by $y - \bar{y} = \frac{S_{xy}}{S_x S_y}(x - \bar{x}) \implies y - \frac{55}{10} = \frac{-875}{6709}(x - \frac{259}{10}) \implies y - 5.5 = -0.1304(x - 25.9) \implies y - 5.5 = \frac{-875}{6709}(x - 25.9) \implies y = \frac{-875}{6709}x + 2.122$

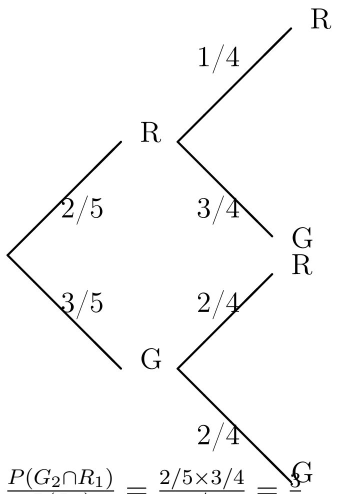
(c) When $x = 38$, $y = \frac{-875(38)}{6709} + 2.122 = -2.834$

3.12 JUNE 2019 SOLUTION

1. (a) i. $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.85 + 0.90 - 0.95 = 0.8$

ii. $P(A \cap B^c) = P(A - P(A \cup B) = 0.85 - 0.8 = 0.05$
A and B are independent if $P(A \cap B) = P(A) \times P(B)$. Since $P(A) \times P(B) = 0.765 \neq 0.8 = P(A \cap B)$. We conclude that A and B are not independent

(b) We use a tree diagram to answer the question



i. $P(G_2/R_1) = \frac{P(G_2 \cap R_1)}{P(R_1)} = \frac{2/5 \times 3/4}{2/5} = \frac{3}{4}$

ii. $P(R \cap R) + P(G \cap G) = \frac{2}{5} \times \frac{1}{4} + \frac{3}{4} \times \frac{2}{4} = \frac{8}{20} = \frac{2}{5}$

2. (a) i. $E(X) = \sum\limits_{i=1}^{n} x_i P(X = x_i)$

ii. $Var(X) = E(X^2) - E(X)^2 = \sum\limits_{i=1}^{n} x_i^2 P(X = x_i) - \left[\sum\limits_{i=1}^{n} x_i P(X = x_i)\right]^2$

(b) $E(X) = 6.0 \implies 1(0.1) + m(0.3) + 6(0.2) + 8(n) + 12(0.15) = 6 \implies 0.3m + 8n = 2.9$ Also, since X is a discrete random variable, $\sum\limits_{i=1}^{n} f(x_i) = 1$, then $0.1 + 0.3 + 0.2 + n + 0.15 = 1$

i. Thus, $n = 0.25 = \frac{1}{4}$. substituting in the previous equation, we have $m = 3$ and $n = \frac{1}{4}$

ii. $Var(X) = E(X^2) - E(X)^2$ But then,
$E(X) = 1^2(0.1) + 3^2(0.3) + 6^2(0.2) + 8^2(0.25) + 12^2(0.15) = 47.6$. Thus,
$Var(X) = 47.6 - (6.0)^2$

iii. $P(X < 8) = P(X = 1) + P(X = 3) + P(X = 6) = 0.1 + 0.3 + 0.2 = 0.6$

3. (a) Since X is a continuous random variable, $\int\limits_{-3}^{3} f(x)dx = 1$. That is

$$k\int_{-3}^{3}(x+3)dx = 1 \implies k\left[\frac{x^2}{2} + 3x\right] = 1 \implies k\left[\frac{9}{2} + 9 - (\frac{9}{2} - 9)\right] = 1 \implies 18k = 1 \quad (3.3)$$

i. We have that $k = \frac{1}{18}$

ii. $Var(X) = E(X^2) - E(X)^2$
$E(X) = \frac{1}{18}\int\limits_{-3}^{3} x(x+3)dx = \frac{1}{18}\int\limits_{-3}^{3}(x^2+3x) = \frac{1}{18}\left[\frac{x^3}{3} + \frac{3x^2}{2}\right]_{-3}^{3} = \frac{1}{18}\left[9 + \frac{27}{2} - (-9 + \frac{27}{2})\right] = 1$
and $E(X^2) = \frac{1}{18}\int\limits_{-3}^{3}(x^3 + 3x^2) = \frac{1}{18}\left[\frac{x^4}{4} + x^3\right]_{-3}^{3} = \frac{1}{18}\left[\frac{81}{4} + 27 - (\frac{81}{4} - 27)\right] = \frac{54}{18}$. Hence, the variance of X is $Var(X) = \frac{54}{18} - 1^2 = \frac{36}{18} = 2$

iii. $P(0 < X < 2) = \int\limits_{0}^{2} f(x)dx = k\int\limits_{0}^{2}(x+3)dx = \frac{1}{18}\left[\frac{x^2}{2} + 3x\right]_{0}^{2} = \frac{1}{18}(2+6) = \frac{8}{18}$

(b) The median is a number m such that $\int\limits_{-3}^{m} f(x)dx = \frac{1}{2} \implies \int\limits_{-3}^{m} \frac{1}{18}(x+3)dx = \frac{1}{2}$

$$\implies \frac{1}{18}\left[\frac{x^2}{2} + 3x\right]_{-3}^{m} = \frac{1}{2} \implies \frac{1}{18}\left[\frac{m^2}{2} + 3m - (\frac{9}{2} - 9)\right] = \frac{1}{2} \implies \frac{m^2 - 9}{2} + 3m = 0 \quad (3.4)$$

We thus have $m^2 + 6m - 9 = 0$

4. (a) i. $F(t) = P(X \leqslant t) = \sum\limits_{x=x_1}^{t} P(X = x)$. Hence, $P(X = 3) = P(X \leqslant 3) - P(X \leqslant 2) = F(3) - F(2) = \frac{63}{64} - \frac{3}{4} = \frac{15}{64}$

ii. $P(1 < X \leqslant 3) = F(3) - F(0) = \frac{63}{64}$

(b) i. $X \approx N(400, (1.6)^2)$

ii. $P(400.9 < Z < 402.9) = P(\frac{400.9-400}{1.6} < Z < \frac{402.9-400}{1.6}) = P(0.5625 < Z < 1.8125) = \Phi(1.8125) - \Phi(0.5625) = 0.965 - 0.7133 = 0.2517$. Hence the percentage of cubes with weight between 400.9 and 402.9 is 25.17%

(c) $P(X < 402) = 0.95 \implies P(Z < \frac{402-\mu}{1.6}) \implies \frac{402-\mu}{1.6}$ is to the right of 0 since $0.95 > 0.5$. In this case, $\Phi(\frac{402-\mu}{1.6}) = 0.95 \implies \frac{402-\mu}{1.6} = \Phi^{-1}(0.95) \implies \frac{402-\mu}{1.6} = 1.645 \implies \mu = 399.4$

5. (a) i. The sample mean $\bar{X}$ is the mean of the sample

ii. $\bar{X}$ is an unbiased estimator of μ if and only if $E(X) = \mu$.

$$\begin{aligned} E(\bar{X}) &= E(\frac{1}{n}(X_1 + X_2 + \cdots + X_n)) = \frac{1}{n}\Big[E(X_1) + E(X_2) + \cdots + E(X_n)\Big] \\ &= \frac{1}{n}\Big[\mu + \mu + \cdots + \mu\Big] = \frac{1}{n} \times n\mu = \mu \end{aligned} \tag{3.5}$$

Therefore $E(X) = \mu$ and hence, $\bar{X}$ is an unbiased estimator of the population mean μ

iii. For a normal sample, $\bar{X} \approx N(\mu, \frac{\sigma^2}{n})$ is the sample distribution for $\bar{X}$

(b) i. $X \sim N(500, 20^2)$ sample size 20. $\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) = N(500, \frac{400}{20})$ and $P(495 < X < 515) = P(\frac{495-500}{\sqrt{20}} < Z < \frac{515-500}{\sqrt{20}}) = P(\frac{-5}{2\sqrt{5}} < Z < \frac{15}{2\sqrt{5}}) = P(Z < \frac{15}{2\sqrt{5}}) - P(Z < \frac{-5}{2\sqrt{5}}) = \Phi(\frac{15}{2\sqrt{5}}) - \Big[1 - \Phi(\frac{5}{2\sqrt{5}})\Big] = 0.8680$

ii. $P(\bar{X} > 495) = 0.95 \implies P(Z > \frac{X-\mu}{\sigma}) = 0.95$ But $X \sim N(500, \frac{400}{n})$, thus $P(Z > \frac{495-500}{\sqrt{\frac{400}{n}}}) \implies P(Z > \frac{-5\sqrt{n}}{20}) = 0.95 \implies P(Z > \frac{-\sqrt{n}}{4}) = 0.95$ $\frac{-\sqrt{n}}{4}$ is on the left of 0 since $0.95 > 0.5$. Hence, $\Phi(\frac{\sqrt{n}}{4}) = 0.95 \implies \frac{\sqrt{n}}{4} = 0.95 = \Phi^{-1}(0.95) = 1.645 \implies \sqrt{n} = 4(1.645)$. Hence, $n = 43.3$. In order to be 95% sure that the mean mass is at least $495kg$ he should buy approximately 43 cows.

6. (a) i. The null hypothesis states

ii. A statistic is a quantity calculated from a sample for example mean, variance, standard deviation etc.

iii. Critical regions refers to the set of values for which a set of values for which the null hypothesis H_0 is to be rejected.

iv. Level of significance refers to the maximum probability of committing a type 1 error.

(b) In order to perform a statistical hypothesis, we do the following

i. State the null and alternate hypotheses H_0 and H_1

ii. Consider an appropriate distribution and decide the level of significance

iii. Determine the rejection criterion

iv. Calculate the value of the test statistic

v. Make a conclusion which consist of accepting or rejecting H_0

(c) $H_0 : \mu = 100$ and
$H_1 : \mu < 100$ (Proprietor's complain) One tail test
Under H_0, $X \sim N(99.88, 0.7^2)$ at a 5% level of significance

(d) We determine the rejection region as follows. 5% interval level on one side implies $\Phi(a) = 95\% = 0.95 \implies a = \Phi^{-1}(0.95) \implies a = -1.645$, that is $P(Z < -1.645) = 0.05$. We reject H_0 if $Z < -1.645$
$Z = \frac{X-\mu}{\sigma} = \frac{100-99.88}{0.1714} \implies Z = 0.1714$. Since $Z = 0.1714 \neq -1.645$, we do not reject H_0. We conclude that at a 5% level of significance, there is evidence that the proprietor's claim is not true.

7. (a) i. The mean mark obtained by 100 students is given by

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} = \frac{14(3) + 24(20) + 34(35) + 44(25) + 54(7) + 64(5) + 74(5)}{3 + 20 + 35 + 25 + 7 + 5 + 5} = \frac{3880}{100} = 38.8 \tag{3.6}$$

ii. The standard deviation $(SD) = \sqrt{Var(X)}$. But $Var(X) = E(X^2) - E(X)^2$

$$\begin{aligned} E(X^2) &= \frac{\sum_{i=1}^{n} f_i x_i^2}{\sum_{i=1}^{n} f_i} = \frac{14^2(3) + 24^2(20) + 34^2(35) + 44^2(25) + 54^2(7) + 64^2(5) + 74^2(5)}{3 + 20 + 35 + 25 + 7 + 5 + 5} \\ &= \frac{169240}{100} = 1692.4 \end{aligned}$$

$Var(X) = 1692.4 - (38.8)^2 = 186.96$. Hence, the standard deviation $\sigma = \sqrt{186.96} = 13.67$

(b) The cumulative frequency curve is drawn using the cumulative frequency table.

Marks (x)(Ucb)	< 19.5	< 29.5	< 39.5	< 49.5	< 59.5	< 69.5	< 79.5
Cum. Frequency (f)	3	23	58	83	90	95	100

(c) i. From the graph, if the pass mark is 45, then we have that the number of candidates who will pass the exam is

ii. The least mark a candidate must earn to be awarded an A grade is

8. (a) Pearson product moment correlation is given by $r = \frac{S_{xy}}{S_x S_y}$. In this case, $r = \frac{S_{MF}}{S_M S_F}$. $S_{MF} = \frac{\sum MF}{n} - \bar{M}\bar{F} = \frac{29833}{10} - \frac{624}{10} \times \frac{439}{10} = 243.94$ and $S_M = \sqrt{S_M^2}$ but $S_M^2 = \frac{\sum M^2}{n} - (\bar{M})^2 = \frac{43488}{10} - (62.4)^2 = 455.04$. Thus

$$S_M = 21.33$$

$S_F = \sqrt{S_F^2}$, but $S_F^2 = \frac{\sum F^2}{n} - (\bar{F})^2 = \frac{20819}{10} - (43.9)^2 = 154.69 \implies S_F = 12.44$. Hence, we have that $r = \frac{S_{MF}}{S_M S_F} = \frac{243.94}{12.44 \times 21.33} = 0.919$. Therefore, $r = 0.919$

(b) The least square regression line F on M is given by

$$y - \bar{y} = \frac{S_{xy}}{S_x^2}(x - \bar{x})$$

In this case, $F - \bar{F} = \frac{S_{MF}}{S_M^2}(M - \bar{M}) \implies F - 43.9 = \frac{243.94}{455.04}(M - 62.4) \implies F = 0.54M + 10.20$

(c) $F = 0.54M + 10.20$ and when $M = 60$, $F = 0.54(60) + 10.20 = 42.6$. Since $42.6 > 40$ the student will pass the exam

3.13 JUNE 2020 SOLUTION

1. (a) $P(A|B) = \frac{P(A \cap B)}{P(B)} \implies P(A \cap B) = P(A|B)P(B) = \frac{1}{2} \times \frac{2}{9} = \frac{1}{9}$

(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B) \implies P(A \cup B) = \frac{1}{3} + \frac{2}{9} - \frac{1}{9} = \frac{4}{9}$.

(c) $P(B|A') = \frac{P(B \cap A')}{P(A')} = \frac{P(B) - P(B \cap A)}{1 - P(A)} = \frac{\frac{2}{9} - \frac{1}{9}}{1 - \frac{1}{3}} = \frac{1}{6}$.

(d) A and B are mutually exclusive if and only if $P(A \cap B) = 0$. Since $P(A \cap B) = \frac{1}{9} \neq 0$, Hence A and B are not mutually exclusive.

Alternatively, A and B are mutually exclusive if and only if $P(A \cup B) = P(A) + P(B)$. Since $\frac{4}{9} \neq \frac{1}{3} + \frac{2}{9}$ Hence A and B are not mutually exclusive.

(e) A and B are independent if $P(A \cap B) = P(A) \cdot P(B)$. Since $\frac{1}{9} \neq \frac{1}{3} \cdot \frac{2}{9}$, we conclude that A and B are not independent.

(f) A and B are collectively exhaustive if $P(A)+P(B)=1$. Since, $P(A)+P(B)=\frac{1}{3}+\frac{2}{9}=\frac{5}{9}\neq 1$. Hence A and B are not collectively exhaustive.

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2. (a) The class width is $\frac{0+5}{2}=2.5$.

$$E(X)=\frac{\sum xf}{\sum f}=\frac{2.5(9)+8.5(16)+...+32.5(5)+38.5(1)}{9+16+...+5+1}=\frac{1043}{70}=14.9$$

(b)

$$E(X^2)=\frac{\sum x^2 f}{\sum f}=\frac{2.5^2(9)+8.5^2(16)+...+32.5^2(5)+38.5^2(1)}{9+16+...+5+1}=\frac{20265.5}{70}=289.5$$

$Var(X)=E(X^2)-E(X)^2=289.5-(14.9)^2=67.5$

(c) $\sigma=\sqrt{Var(X)}=\sqrt{67.5}=8.2$.

(d) The modal class is $18-23$, Hence the mode is given by

$$mode=L+c\left(\frac{d_1}{d_1+d_2}\right)=17.5+6\left(\frac{2}{2+13}\right)=18.3$$

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3. Let X be "the number of goals scored per match", then $X\sim Poi(1.5)$. i.e $P(X=x)=\frac{e^{-1.5}(1.5)^x}{x!}$.

(a) $P(X=0)=\frac{e^{-1.5}(1.5)^0}{0!}=0.2231$

(b) $P(x\leq 3)=P(x=0)+P(x=1)+P(x=2)+P(x=3)=e^{-1.5}\left[1+\frac{3}{2}+\frac{9}{8}+\frac{9}{16}\right]=0.9344$.

(c) In two matches, $X\sim Poi(2\times 1.5)\Longrightarrow X\sim Poi(3)$. i.e $P(X=x)=\frac{e^{-3}3^x}{x!}$.
$P(X<3)=P(x=0)+P(x=1)+P(x=2)=e^{-3}\left[1+3+\frac{9}{2}\right]=0.4232$.

(d) see junior kkkkkkkkkkkkkkkkkkkkkkkkkkkkkkkkkk

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4. Let X be the mass of oranges. Then $X\sim N(\mu,\sigma^2)$.
Now , $P(X<25)=10\%\Longrightarrow P(X<25)=0.1\Longrightarrow P\big(Z<\frac{25-\mu}{\sigma}\big)=0.1$ Thus from the normal distribution graph, $\frac{25-\mu}{\sigma}$ is negative. i.e $-\big(\frac{25-\mu}{\sigma}\big)=\phi^{-1}(0.1)=\phi^{-1}(1-0.1)=\phi^{-1}(0.9)=1.281$ hence we have the equation $\big(\frac{25-\mu}{\sigma}\big)=-1.281$.........(1). Similarly, $P(X>85)=5\%\Longrightarrow P(X>85)=0.05\Longrightarrow P\big(Z>\frac{85-\mu}{\sigma}\big)=0.05$ Thus from the normal distribution graph, $\frac{85-\mu}{\sigma}$ is positive. i.e $\big(\frac{85-\mu}{\sigma}\big)=\phi^{-1}(0.05)=\phi^{-1}(1-0.05)=\phi^{-1}(0.95)=1.645$
hence we have the equation $\big(\frac{85-\mu}{\sigma}\big)=1.645$.........(2). Solving equations (1) and (2) simultaneously, we have that $\sigma=20.5058$ and $\mu=51.2679$ and thus Variance $=\sigma^2=420.4878$ to 4 decimal places.

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5. (a) X is a discrete random variable implies $\sum P(X=x)=1$. i.e $1/10+3/10+k+1/4+3/20=1\Longrightarrow k=1/5$.

(b) $E(X)=6\Longrightarrow\sum xP(X=x)=6\Longrightarrow 1(\frac{1}{10})+3(\frac{3}{10})+6(k)+n(\frac{1}{4})+12(\frac{3}{20})=6\Longrightarrow 120k+5n=64$ but $k=1/5,\Longrightarrow n=8$.

(c) $E(X^2)=1(\frac{1}{10})+3^2(\frac{3}{10})+6^2(\frac{1}{5})+8^2(\frac{1}{4})+12^2(\frac{3}{20})=47.6$
But $Var(X)=E(X^2)-E(X)^2=47.6-6^2=\frac{58}{5}$

(d) $E(5X-4)=5E(X)-E(4)=5(58/5)-4=26$. also , $Var(5X-4)=5^2Var(X)-Var(4)=25(58/5)-0=290$.

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6. (a) X is a continuous random variable implies $\int_0^3 f(x)dx=1\Longrightarrow\int_0^3 kx(4-x^2)dx=1\Longrightarrow k\int_0^3(4x-x^3)dx=1\Longrightarrow-\frac{9k}{4}=1\Longrightarrow k=-\frac{4}{9}$.

(b) $E(X) = \int_0^3 xf(x)dx = k\int_0^3 x^2(4-x^2)dx = k\int_0^3(4x^2-x^4)dx = -\frac{63}{5}k$ but $k = -\frac{4}{9}$, thus $E(X) = \frac{28}{5}$.

(c) q is the upper quartile mean $\int_0^q f(x)dx = \frac{3}{4}$ i.e $k\int_0^3(4x-x^3)dx = \frac{3}{4} \Longrightarrow k\left[2x^2 - \frac{x^2}{4}\right]_0^q = \frac{3}{4} \Longrightarrow 4q^4 - 32q^2 - 27 = 0.$

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7. (a) $n = 100$, $\mu = 8.36g$ and $\sigma^2 = 0.5158g$ but $\hat{}\sigma^2 = \frac{nS^2}{(n-1)} = \frac{100(0.5185)^2}{99} = 0.2716.$

(b) If X is "the mass of chocolate bars" then we have that $X \sim N(8.36, 0.5185)$.

- If $X \sim N(8.36, 0.5185)$ then $\bar{X} \sim N(8.36, \frac{0.5185^2}{\sqrt{100}})$: We will let
  $H_0 : \mu = 8.5g$ (manager's claim is not justified)
  $H_1 : \mu < 8.5g$ (manager's claim is justified)
- One tail test at 5% level of significance. $5\%level = 95\%$ confidence interval .This imples $\phi(a) = 0.95 \Longrightarrow a = \phi^{-1}(0.95) = 1.645$
- we reject H_0 if $Z < -1.645$
- test statistique: $z = \frac{\bar{x}-\mu}{\frac{\sigma^2}{\sqrt{n}}} = \frac{8.5-8.36}{\frac{0.5185}{\sqrt{100}}} = 2.7142$
- Since $z = 2.7124 > -1.645$, we accept H_0 and conclude that at 5%-level, the manager's claim is not justified.

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8. (a) $\bar{X} = \frac{487}{10}$, $\bar{Y} = \frac{461}{10}$, $S_{xy} = \frac{\sum xy}{n} - \bar{X}\bar{Y} = \frac{27037}{10} - \left(\frac{487}{10} \times \frac{461}{10}\right) = 458.63$
$S_x^2 = \frac{\sum x^2}{n} - \bar{X}^2 = \frac{32805}{10} - \left(\frac{487}{10}\right)^2 = 908.81$ $S_y^2 = \frac{\sum y^2}{n} - \bar{Y}^2 = \frac{23641}{10} - \left(\frac{461}{10}\right)^2 = 238.89$.
Hence the pearson's -Product moment correlation is given by

$$r = \frac{S_{xy}}{\sqrt{S_x^2 S_y^2}} = \frac{458.63}{\sqrt{908.81 * 238.89}} = 0.9843.$$

(b) Regression line Y on X is given by

$$y - \bar{y} = \frac{S_{xy}}{S_x^2}(x - \bar{x}) \Longrightarrow y - \frac{461}{10} = \frac{458.63}{908.81}\left(x - \frac{487}{10}\right) \Longrightarrow y = 0.5046x + 21.5236.$$

where $a = 21.5236$ and $b = 0.5046$

(c) $x = 50 \Longrightarrow y = 0.5046(67) + 21.5236 = 46.75$ If the student wrote the test , he would have scored 46.75

END